Combinatorics

Combinatorics deals with listing the arrangement of objects according to some pattern and counting the number of ways it can be done.

It may deal with finite number of objects or infinite number of objects. But it mainly deals with finite number of objects.

The 26 letters of the English alphabet can be arranged in finite ways and infinite ways. When we consider the number of words formed with 3 letters of English alphabet, we deal with the condition of arranging finite number of objects arranged in finite number of ways. But, when the number of letters in the word is not defined then we deal with infinite number of objects arranged in infinite number of ways.

The multiplication principle and addition principle

The multiplication principle

Suppose there are n tasks and each task can be performed in different number of ways.

T_1 can be performed in n_1 ways.
T_2 can be performed in n_2 ways after task T_1 is performed.
T_3 can be performed in n_3 ways after task T_1 and T_2 are performed.
Similarly other tasks are performed.

Then the total number of ways in which the whole task can be performed is
$n_1 \times n_2 \times n_3 \times \ldots . \times n_n$ ways.

This principle is called **the multiplication principle**.

The addition principle

Suppose there are n subtasks (T_i , i = 1, 2, 3, 4,..., n) of a task. Each subtask is mutually exclusive. One and only one subtask can be performed at a time.

T_1 can be performed in n_1 ways.
T_2 can be performed in n_2 ways.
T_3 can be performed in n_3 ways.
Similarly other tasks are performed.

Then the total number of ways in which the whole task can be performed is
$n_1 + n_2 + n_3 + \ldots . + n_n$ ways.

Permutations

Permutations are ordered arrangement of objects.

Example:

The permutation of a,b and c taken two at a time are ab, ac, ba, bc, ca and cb. They are six in number. ab and ba are considered different even though they consist of two same objects.

We write

1×2 = 2!
1×2×3 = 3!
1×2×3×4 = 4!
1×2×3×...×n = n! Read as n! For every positive integer n.

Value of P(n,r)

Suppose n and r are two positive integers r ≤ n. The the number of permutations of n distinguishable objects taken r at a time is denoted by P^{n}_{r}, P(n,r) or ${}^{n}P_{r}$.

$$P(n,r) = \frac{n!}{(n-r)!} = n(n-1)(n-2)(n-3)...(n-r+1).$$

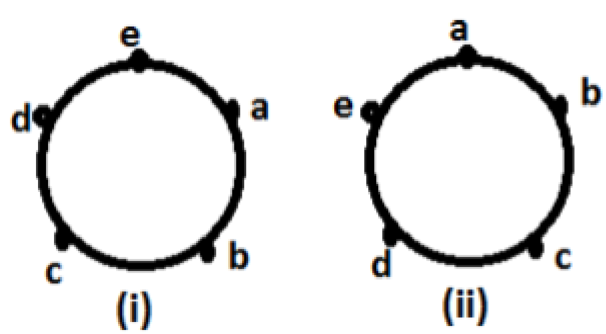
Value of 0!

0! = 1 as

$$P(n,n) = \frac{n!}{(n-n)!} = n!$$

$$0! = \frac{n!}{n!} = 1$$

Circular permutations



In circular permutation the objects are arranged along the circumference of a circle. We observe the objects along the clockwise direction. If we consider the n! permutations of n things then each permutation will be indistinguishable from the (n-1) earlier permutations obtained from transferring the object at first position to the last position, if arrangements are considered the same when one has been obtained from other by rotation. Thus as circular permutation we will have exactly n!/n = (n-1)! Permutations.

Permutation of objects not necessarily distinct

Suppose there are n things of which m_1 identical objects belong to category 1, m_2 identical objects belong to category 2, etc. m_k identical objects belong to category k, with the categories mutually exclusive and exhaustive so that $m_1 + m_2 ... + m_k = n$. Then the number of distinct permutations of these n things is

$$\frac{n!}{m_1!m_2!...m_n!}$$

m_1 objects of category 1 can be permutated among themselves in $m_1!$ Ways. As they are of same type they are considered one. Similarly for others.

Combinations

Combination refers to the selection of a specified number of objects from a number of distinguishable objects.

Suppose there are n distinct objects and we want a selection of r objects, where r ≤ n, and order of the objects in the selection does not matter. This is called the combination of n things taken r at a time. It is denoted by C^n_r, C(n,r) or ^nC_r.

Formula for C(n,r)

For every combination of r things, there are r! ways of arranging objects. These total ways give us the permutation.

i) $^nP_r = r!\ ^nC_r$

$^nC_r = \frac{P(n,r)}{r!}$

As, $^nP_r = \frac{n!}{(n-r)!}$

So, $^nC_r = \dfrac{n!}{(n-r)!r!}$

ii) As n = (n - r) + r

Substituting (n - r) for r
We get,

$^nC_{n-r} = \dfrac{n!}{(n-(n-r))!(n-r)!} = \dfrac{n!}{r!(n-r)!}$

As the result is the same so , $^nC_r = {}^nC_{n-r}$

$^nC_n = {}^nC_0 = {}^nP_0 = 1$
$^nC_1 = {}^nC_{n-1} = {}^nP_n = n$

Combinations with repetition

Let n and r be natural numbers. Then the number of solutions in natural numbers to the equation $x_1 + x_2 + ... + x_n = r$ or equivalently the number of ways to choose r objects from a collection of n objects with repetition allowed is $^{n+r-1}C_r$.

The Binomial Expansion

Sum of two distinct symbols like a + b, p + q, x + y, etc. is called a binomial, the binomial expansion refers to the expansion of a positive integral power of such a binomial assuming that the symbols stand for real numbers or complex numbers.

Some expansions are

$(a+b)^2 = a^2 + 2ab + b^2$
$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$
$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$
$(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$

We can get the coefficient of a^rb^{n-r} from the expansion of $(a+b)^n$ by taking r a's and (n-r) b's from n parentheses containing (a+b). Which can be done in ^nC_r ways. Thus, the coefficient of a^rb^{n-r} in the expansion of $(a+b)^n$ is ^nC_r.

As, $^nC_r = {}^nC_{n-r}$
So, the coefficient of a^rb^{n-r} = the coefficient $a^{n-r}b^r$

$(a+b)^n = a^n + {}^nC_1\, a^{n-1}b + {}^nC_2\, a^{n-2}\, b^2 + \ldots + {}^nC_r a^{n-r}b^r + \ldots + {}^nC_{n-1}ab^{n-1} + {}^nC_n b^n$

Pascal's Triangle

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

1 8 28 56 70 56 28 8 1

1 9 36 84 126 126 84 36 9 1

1 10 45 120 210 252 210 120 45 10 1

1 11 55 165 330 462 462 330 165 55 11 1

Pascal's triangle.

Some Identities Involving Binomial Coefficients

- When a = b = 1 in the expansion of $(a + b)^n$, we get

${}^nC_0 + {}^nC_1 + {}^nC_2 + \ldots + {}^nC_r + \ldots + {}^nC_{n-1} + {}^nC_n = 2^n$

- When a = 1, b = -1 in the expansion of $(a + b)^n$, we get

${}^nC_0 - {}^nC_1 + {}^nC_2 - {}^nC_3 + \ldots + {}^nC_r (-1)^r + \ldots + {}^nC_{n-1} (-1)^{n-1} + {}^nC_n (-1)^n = 0$

$$\sum_{r \text{ is even}} C(n,r) = \sum_{r \text{ is odd}} C(n,r) = 2^{n-1}$$

The Multinomial Expansion

$(a_1 + a_2 + a_3 + \ldots + a_n)^n = \sum \frac{n!}{r_1! r_2! r_3! \ldots r_n!} a_1^{r_1} a_2^{r_2} a_3^{r_3} \ldots a_n^{r_n}$

$\frac{n!}{r_1! r_2! r_3! \ldots r_n!}$ is called multinomial coefficient.