

Intriguing Minimization

Problem with Many

Solutions: Geometry to Calculus

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The Mast Problem

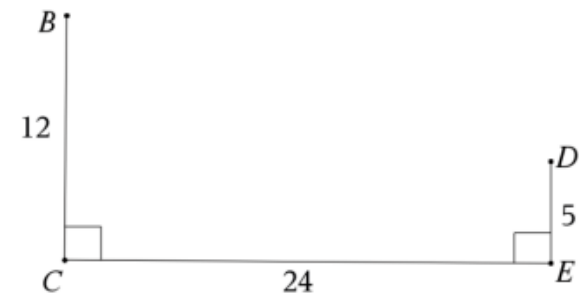
(revised)

Locate point A on segment CE such that $BA + AD$ is a minimum.

What is that minimum value?

Justify your answer.

Attempt to solve in as many ways as possible.



All materials can be downloaded at: bit.ly/mnctm2024

"I'm so glad I learned about parallelograms instead of how to do taxes.

It's really handy this parallelogram season."

- Unknown

The Mast Problem

(revised)

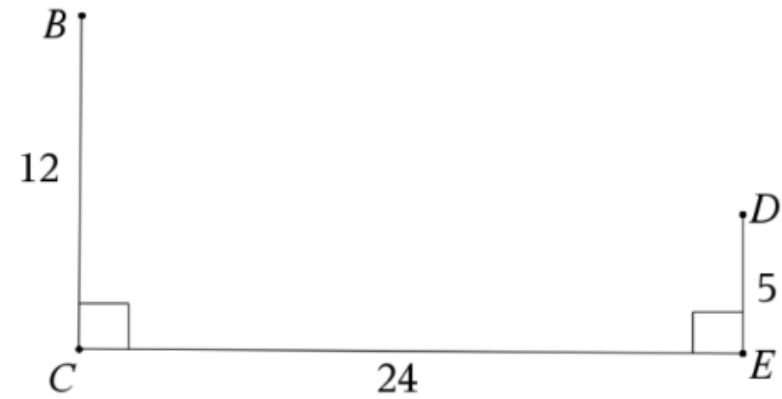
Locate point A on segment CE such that

$BA + AD$ is a minimum.

What is that minimum value?

Justify your answer.

Attempt to solve in as many ways as possible.



The solution paths are the emphasis

of this session - NOT the final answer!

It's the journey!!

Hold off on the geometry solution until LAST.

Table of Contents to the Various Solutions

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p.22 TI-84 CE solution withOUT calculus

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p.31 Use TI-Nspire CAS to solve WITH calculus

p.39 1st Geometry solution

p.68 2nd Geometry solution

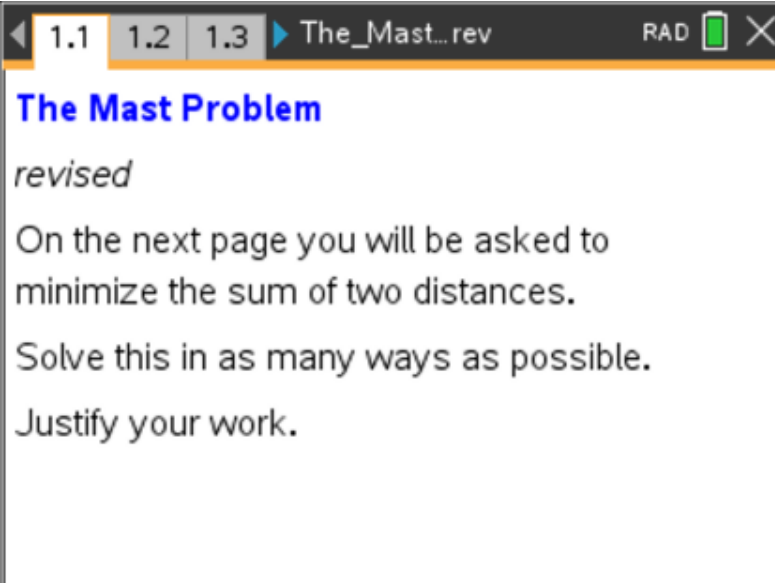
p.74 Angle of Incidence = Angle of Reflection

p.78 TI-84 CE solution with a form of data capture

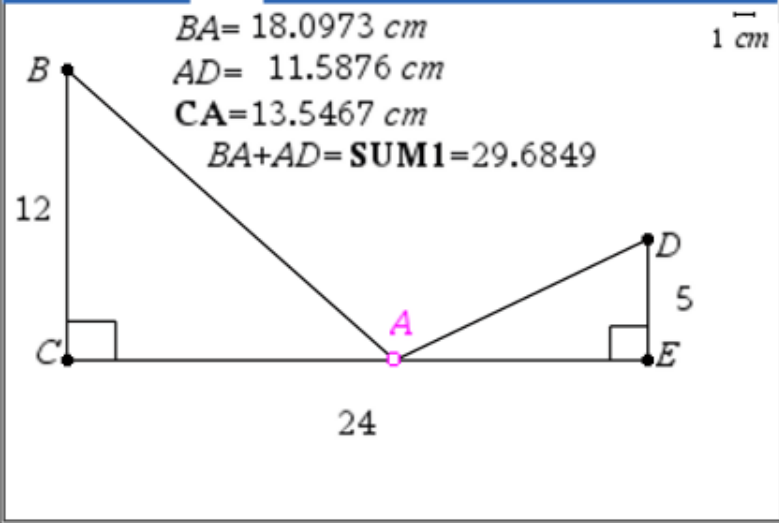
Open the file "The_Mast_Problem..." using

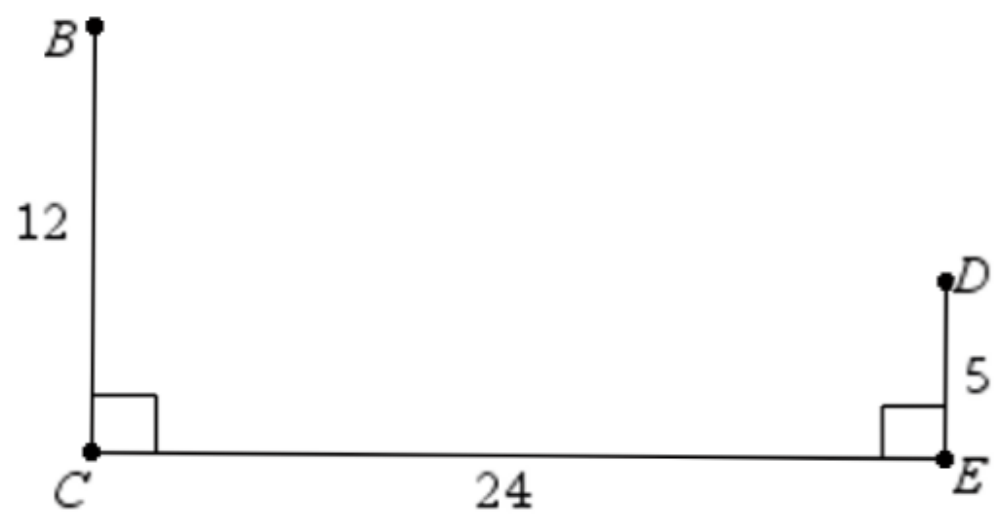
TI-Nspire CAS.

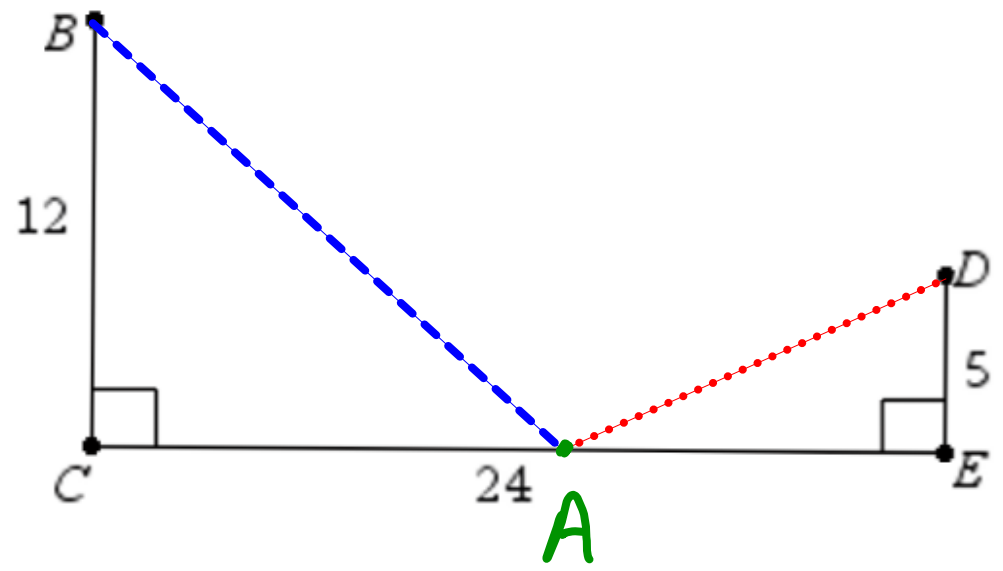
Do pages 1.1, 1.2, 1.3.

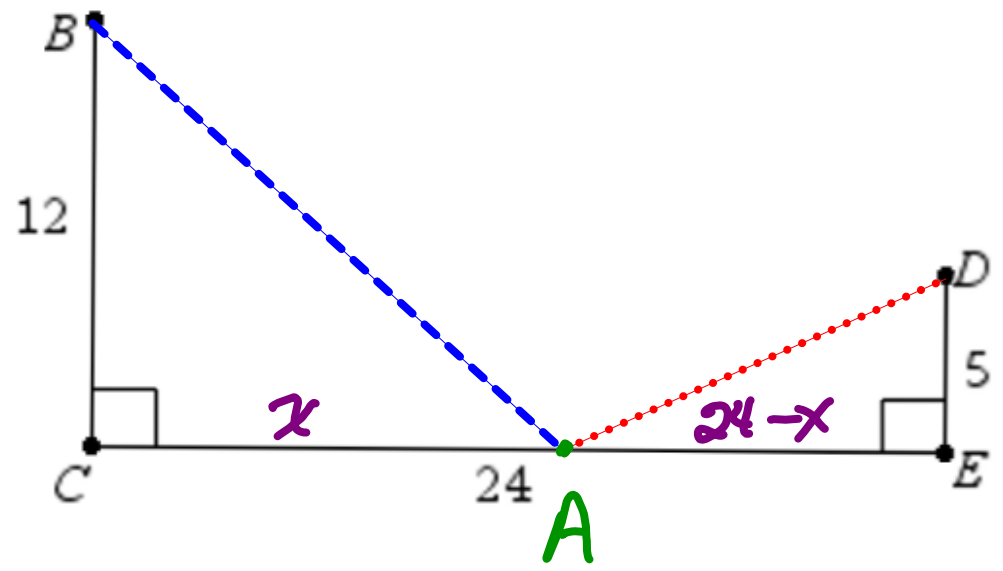


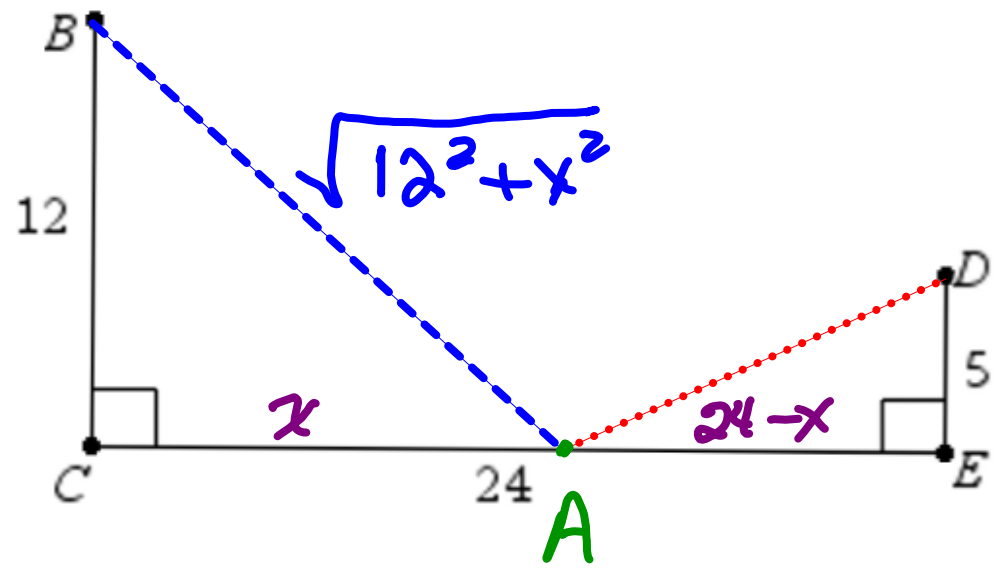
The screenshot shows a TI-Nspire CAS document window. The title bar at the top contains navigation buttons for pages 1.1, 1.2, and 1.3, followed by the document name "The_Mast... rev", a "RAD" indicator, a battery icon, and a close button. The main content area displays the title "The Mast Problem" in blue, followed by the word "revised" in italics. Below this, the text reads: "On the next page you will be asked to minimize the sum of two distances. Solve this in as many ways as possible. Justify your work."

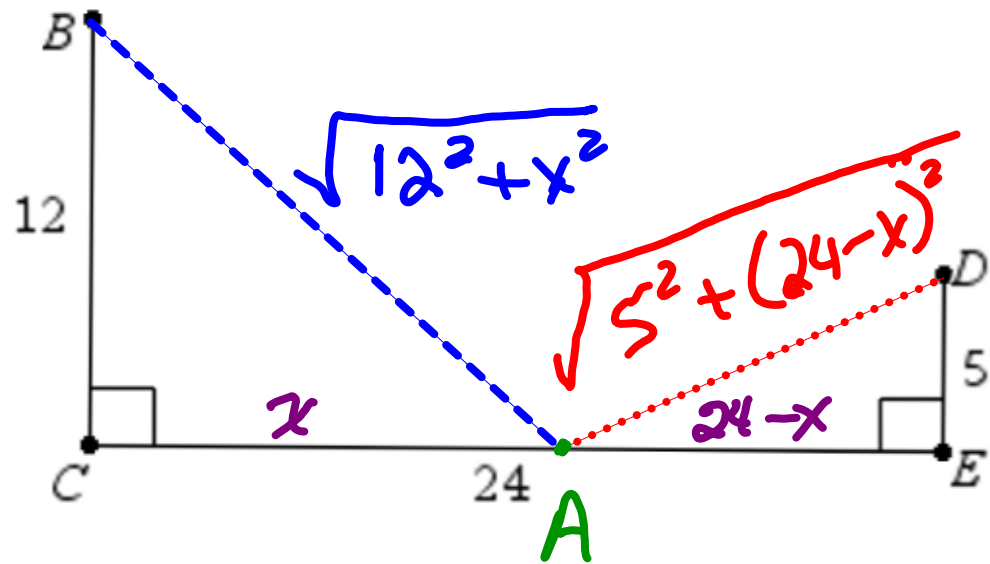


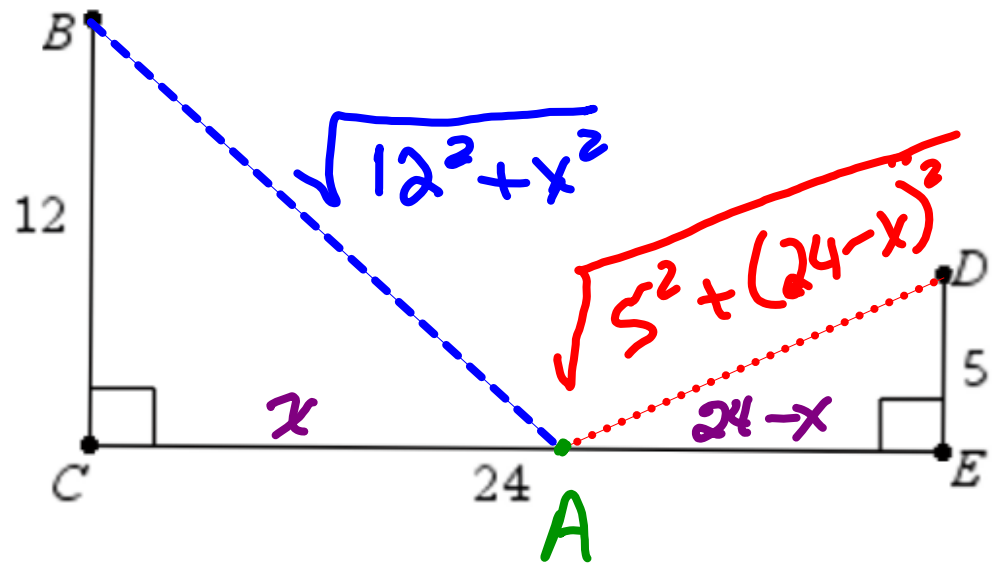






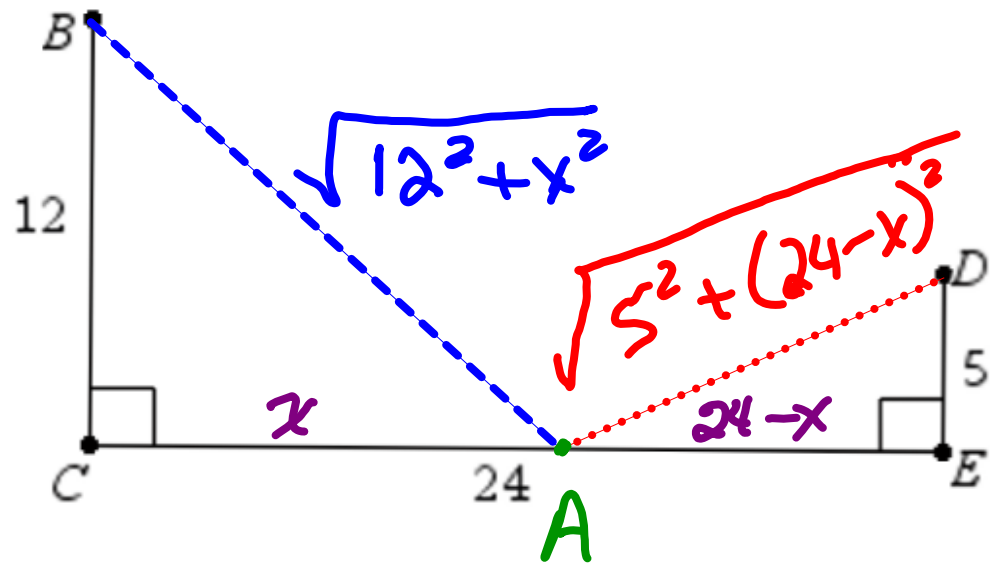






minimize.

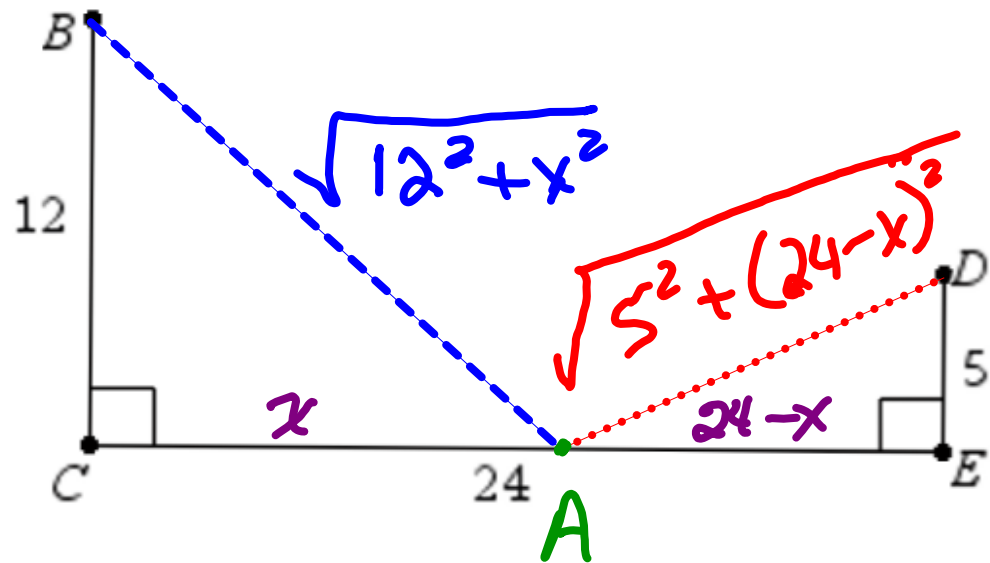
$$S(x) = \sqrt{12^2 + x^2} + \sqrt{5^2 + (24-x)^2}$$



minimize.

$$S(x) = \sqrt{12^2 + x^2} + \sqrt{5^2 + (24-x)^2}$$

How can we be sure that this is THE correct modeling equation?





minimize.

$$S(x) = \sqrt{12^2 + x^2} + \sqrt{5^2 + (24-x)^2}$$

How can we be sure that this is THE correct modeling equation?

TI-Nspire page 1.4, ...

1.1 1.2 1.3 ▶ The_Ma...ONS RAD  



The Mast Problem

revised 2

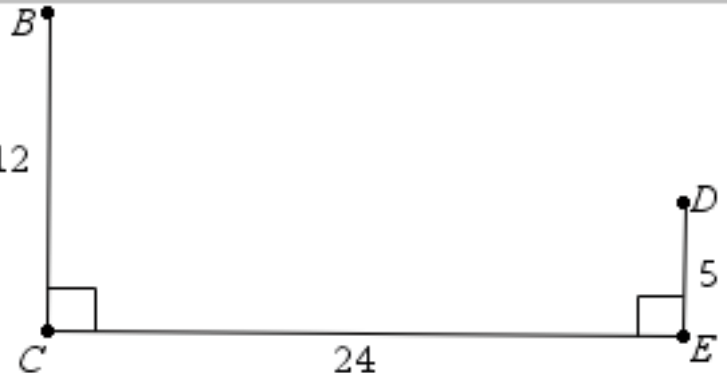
On the next page you will be asked to minimize the sum of two distances.

Solve this in as many ways as possible.

Justify your work.

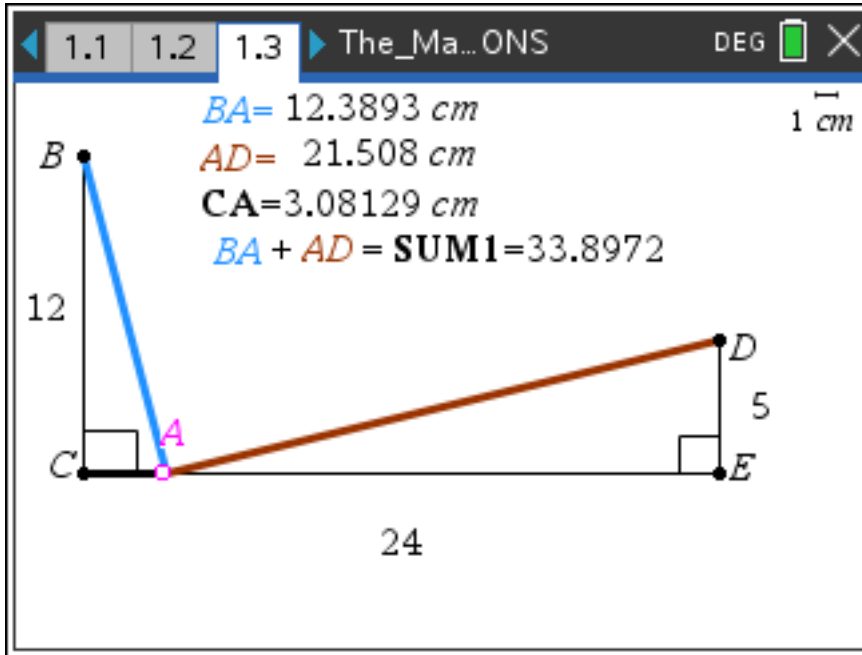
1.1 1.2 1.3 ▶ The_Ma...ONS RAD  

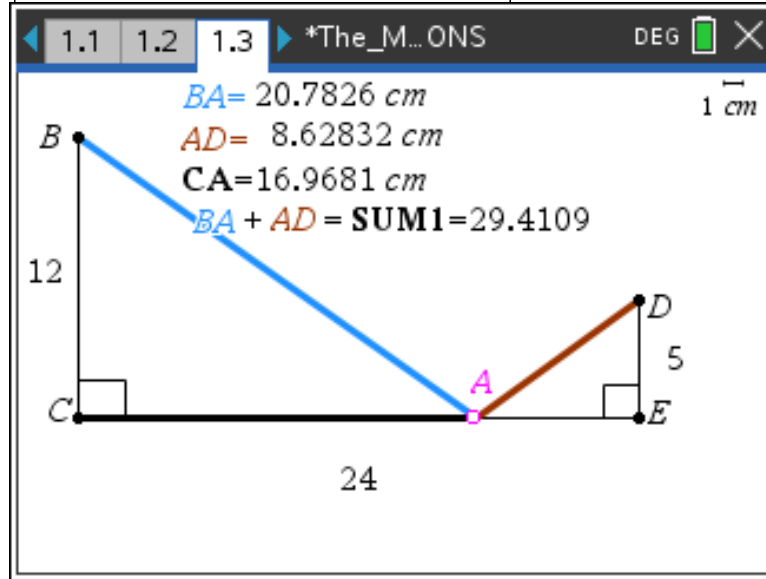
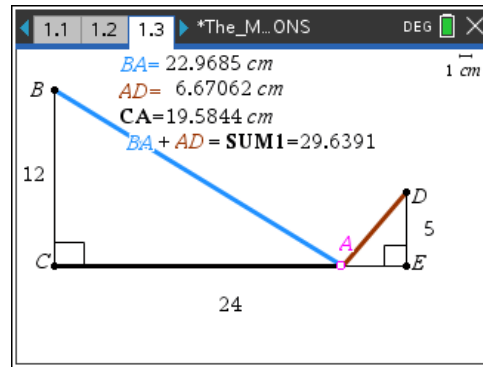
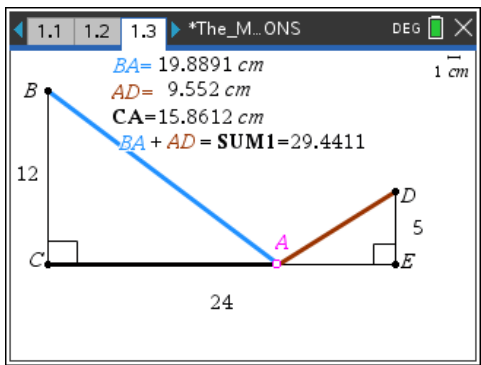
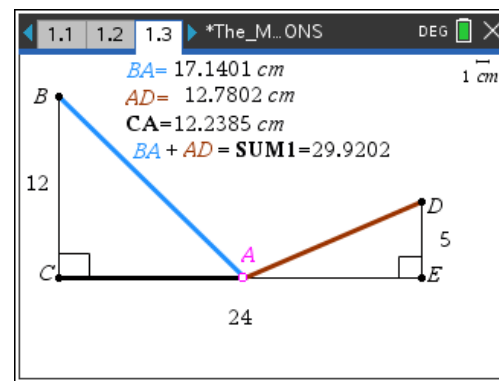
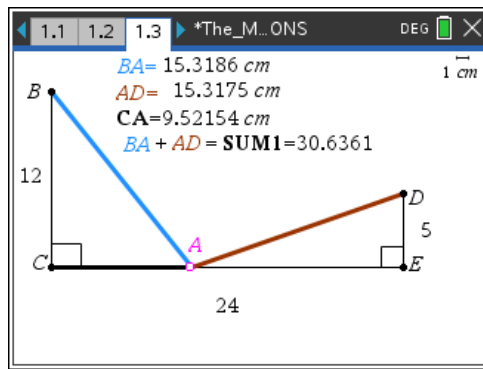
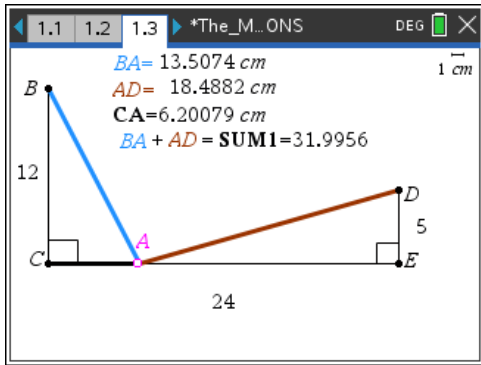
Locate point A on CE such that $BA + AD$ is a minimum.



The diagram shows a horizontal line segment CE with a length of 24. Point B is vertically above C at a distance of 12. Point D is vertically above E at a distance of 5. Right angle symbols are shown at C and E .

Pull point A back and forth on segment CE



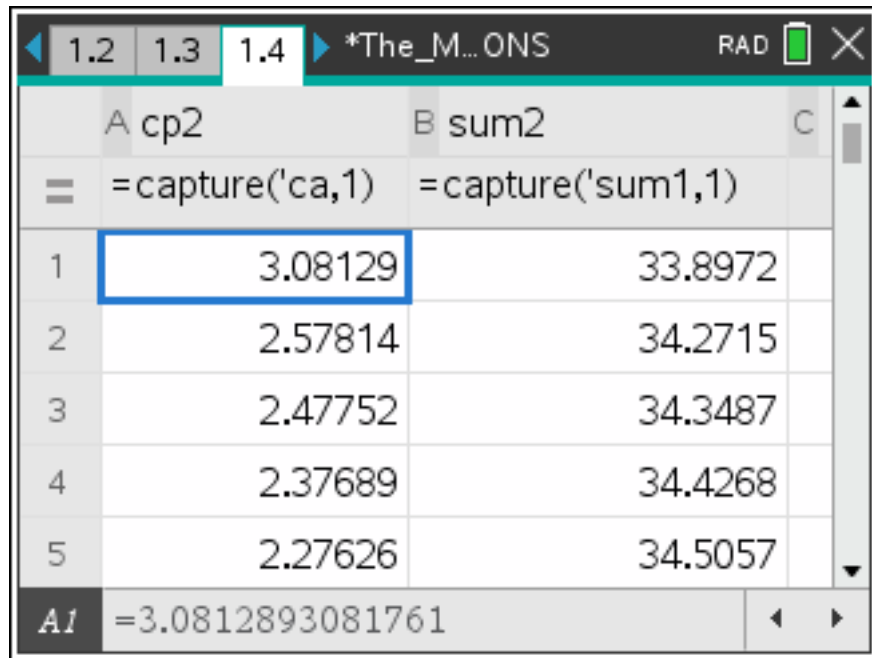


appears to be the minimum when
 $CA = 16.968$, minimum sum is 29.411

CA

While we were pulling point A, the calculator was "capturing" the data:

the distance CA (cp2) and the sum (sum2) into a lists and spreadsheet

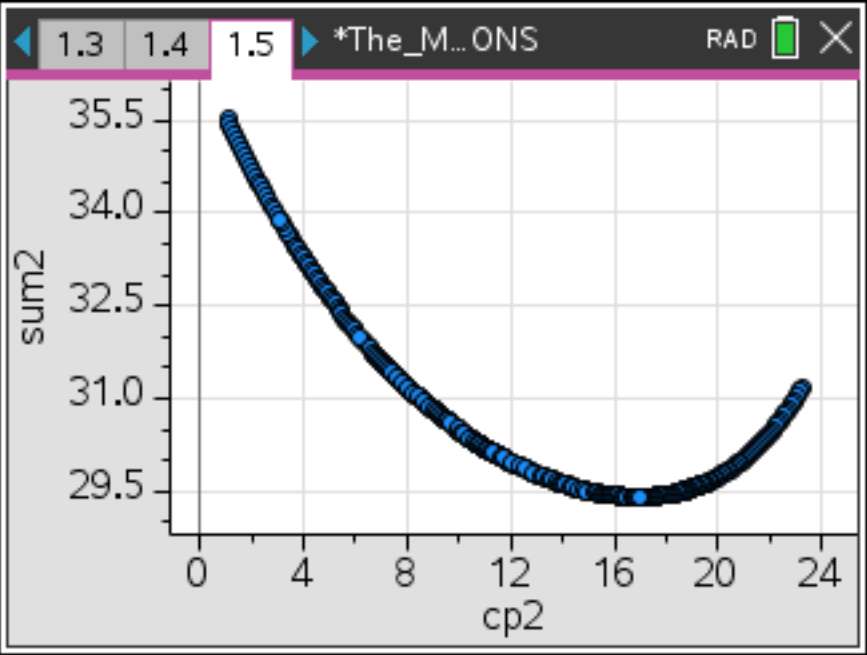


	A cp2	B sum2	C
=	=capture('ca,1)	=capture('sum1,1)	
1	3.08129	33.8972	
2	2.57814	34.2715	
3	2.47752	34.3487	
4	2.37689	34.4268	
5	2.27626	34.5057	

A1 =3.0812893081761

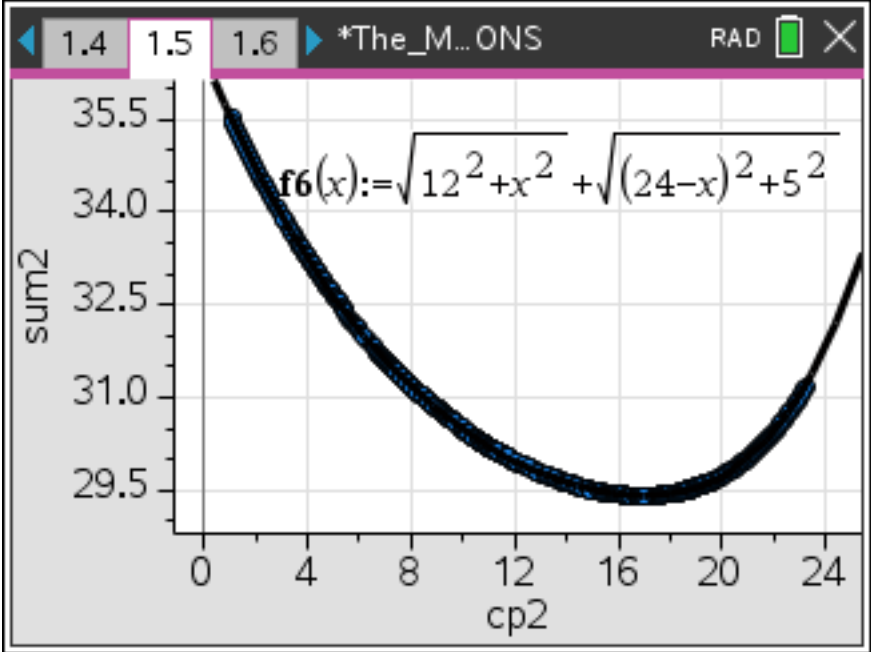
Now we can plot this data: sum2 versus cp2

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Then we can test to see if our modeling equation exactly matches the collected data

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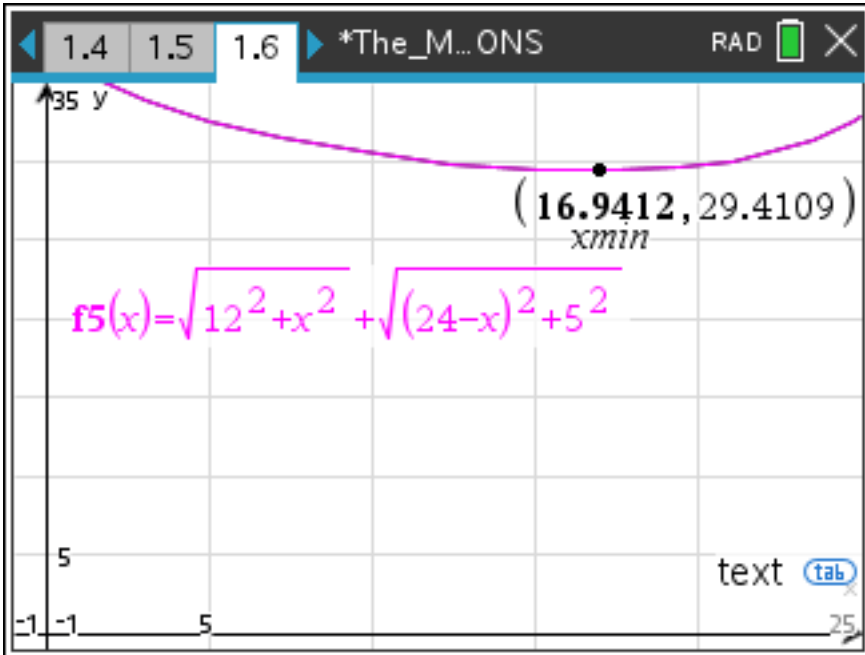


it's a perfect match! So we know our modeling equation can be used to answer the questions.

it's a perfect match! So we know our modeling equation

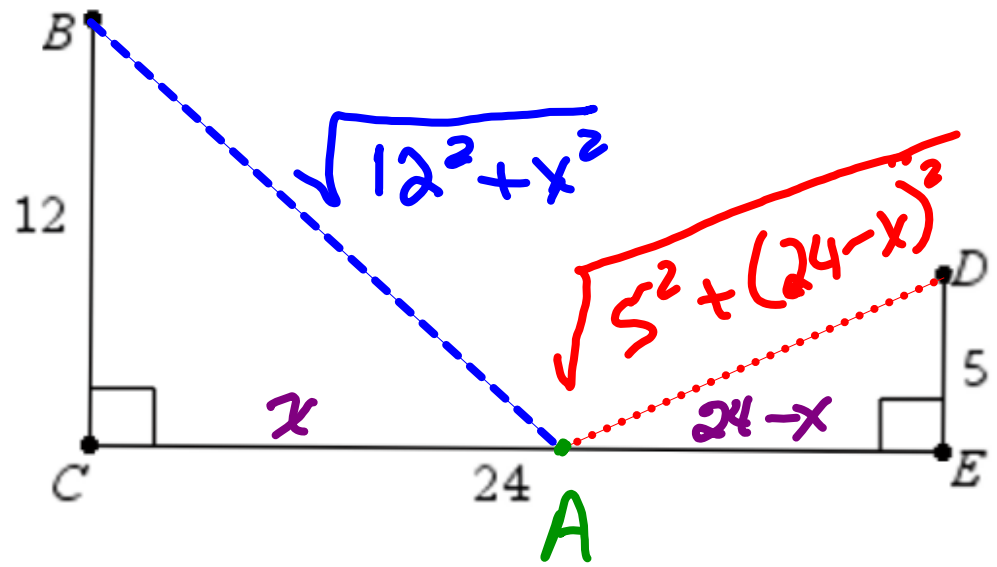
can be used to answer the questions.

using the minimum tool:



TI-84 CE solve graphically

withOUT Calculus



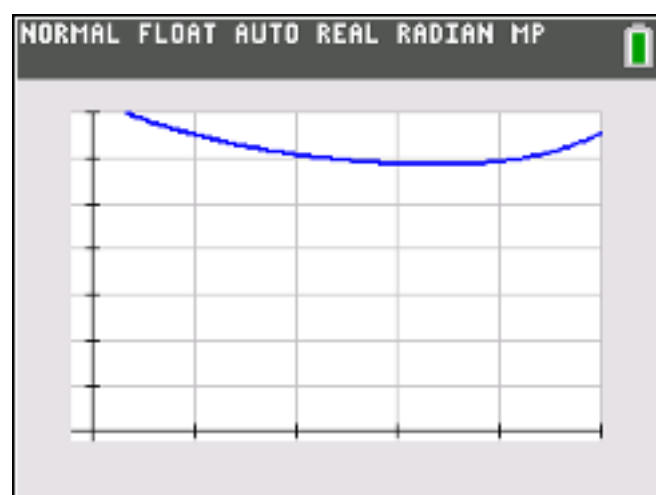
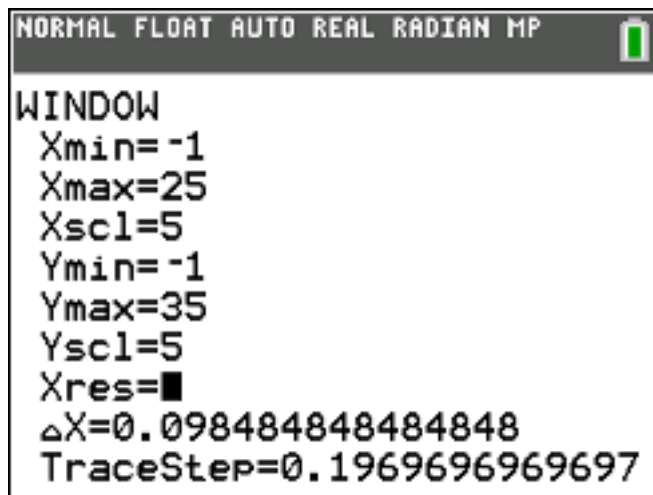
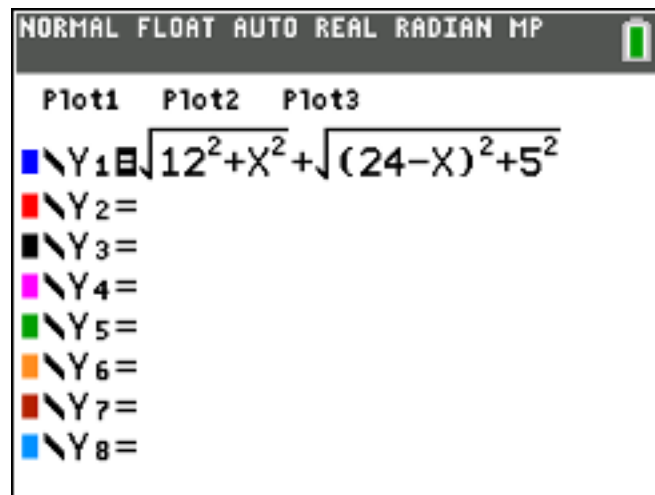
minimize.

$$S(x) = \sqrt{12^2 + x^2} + \sqrt{5^2 + (24-x)^2}$$

How to solve without calculus?

minimize:

$$S(x) = \sqrt{12^2 + x^2} + \sqrt{5^2 + (24-x)^2}$$



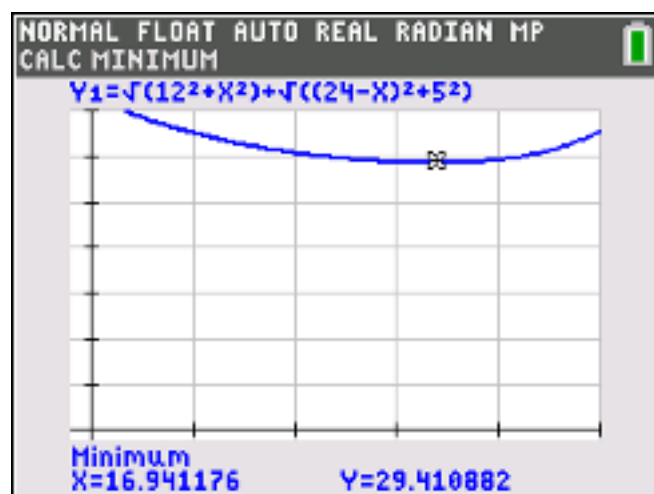
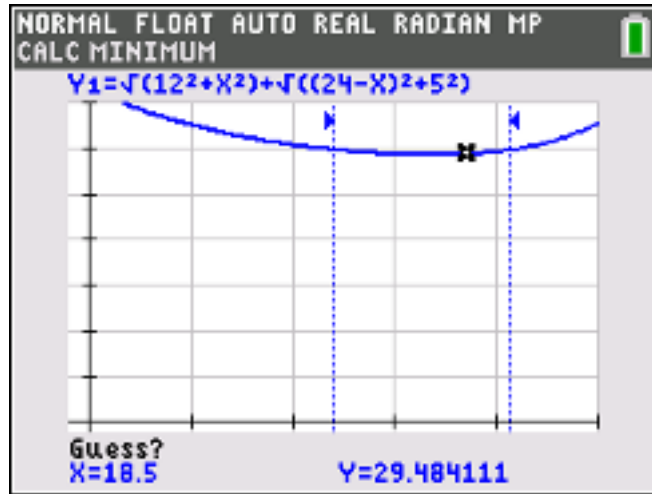
minimize:

$$S(x) = \sqrt{12^2 + x^2} + \sqrt{5^2 + (24-x)^2}$$

NORMAL FLOAT AUTO REAL RADIAN MP

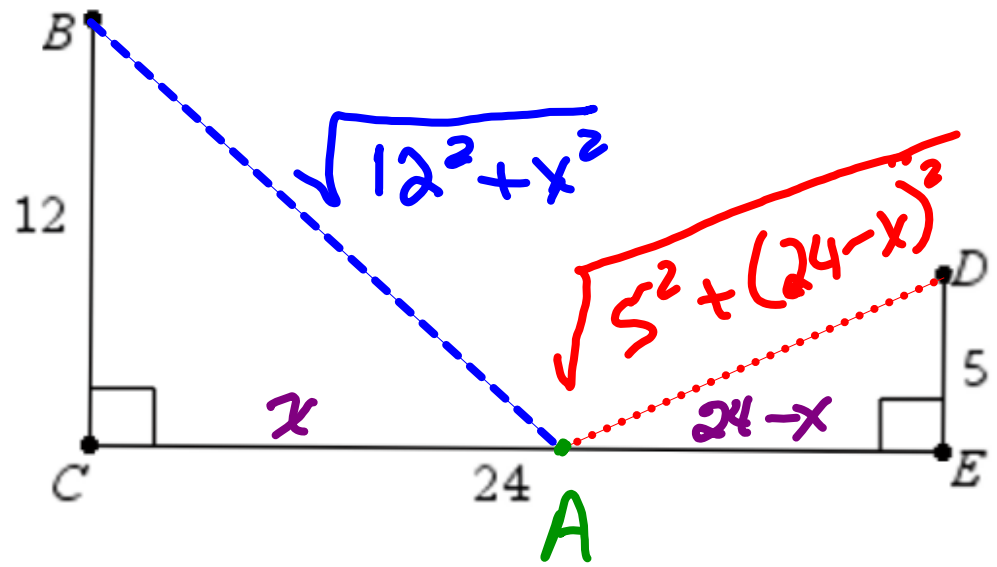
CALCULATE

- 1: value
- 2: zero
- 3: minimum**
- 4: maximum
- 5: intersect
- 6: dy/dx
- 7: $\int f(x)dx$



TI-84 CE solve graphically

WITH Calculus

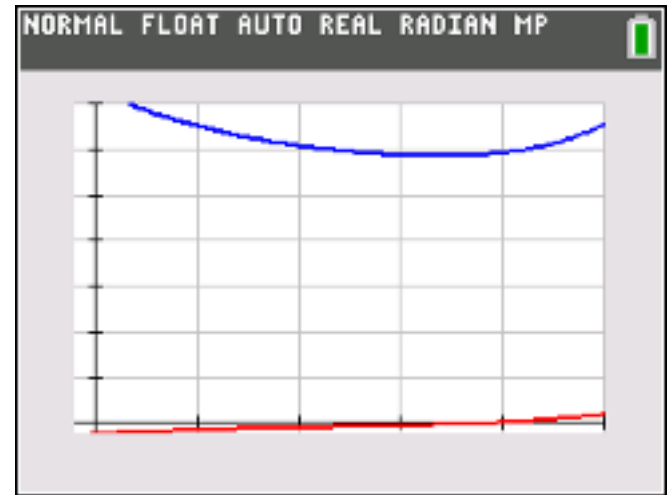
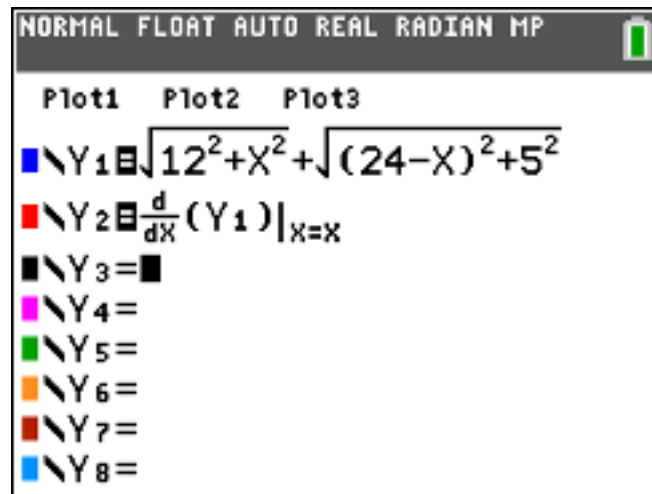
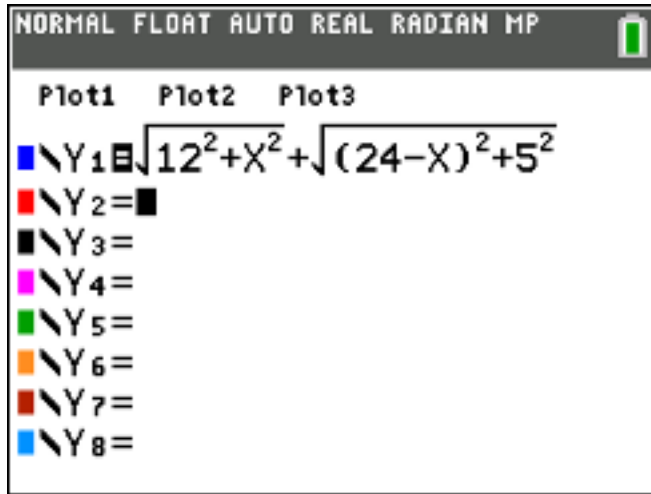


minimize.

$$S(x) = \sqrt{12^2 + x^2} + \sqrt{5^2 + (24-x)^2}$$

How to solve **WITH** calculus?

minimize:
 $S(x) = \sqrt{12^2 + x^2} + \sqrt{5^2 + (24-x)^2}$

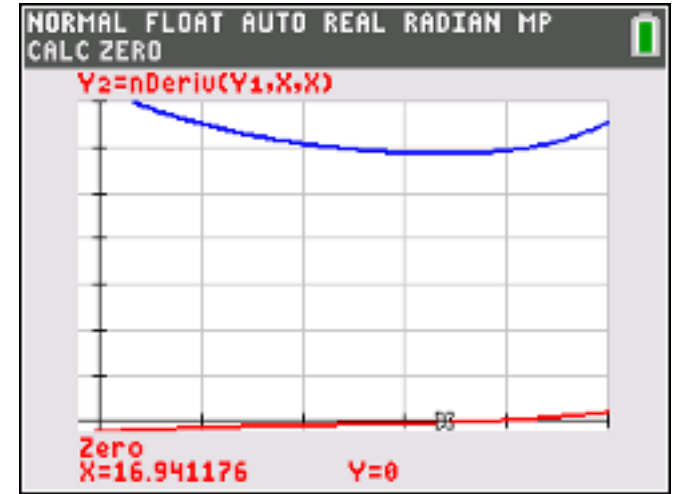
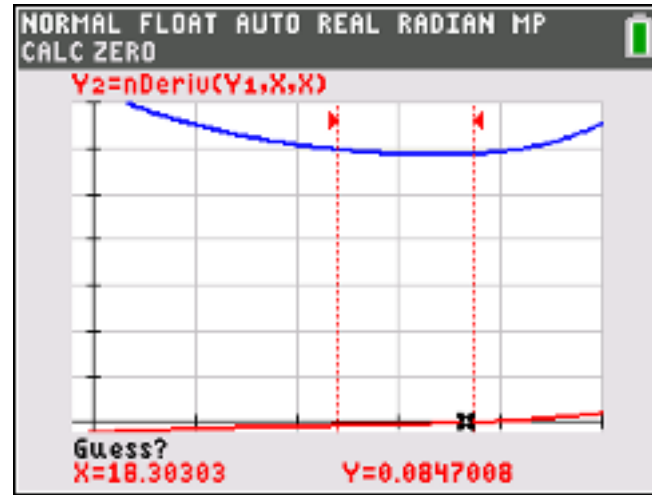
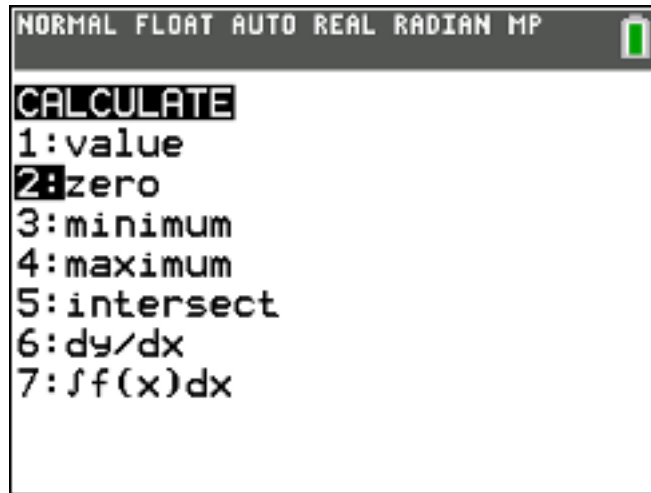


Y2 has the derivative

Look to see where the derivative is = 0

minimize:
 $S(x) = \sqrt{12^2 + x^2} + \sqrt{5^2 + (24-x)^2}$

Look to see where the derivative is = 0



derivative is = 0 at

x = 16.941

derivative is = 0 at

$x = 16.941$

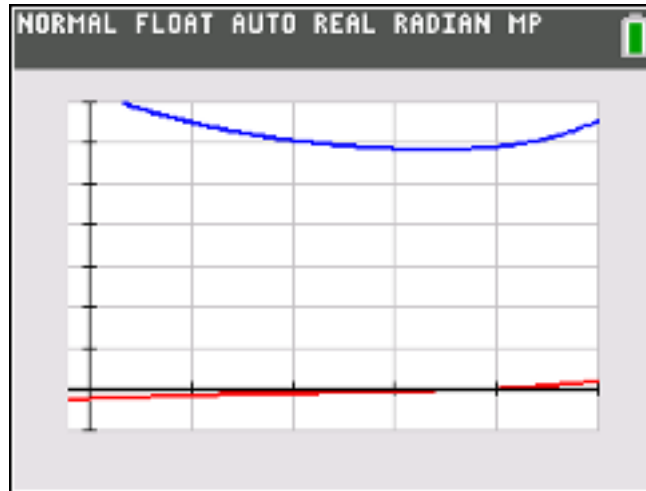
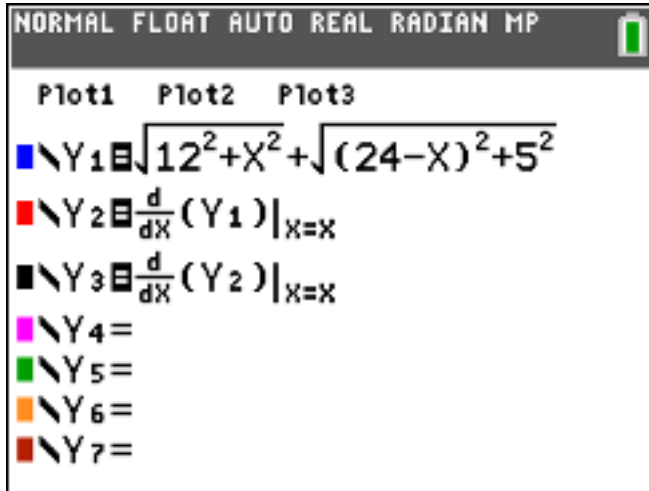
Check to ensure that this is

a relative minimum and not

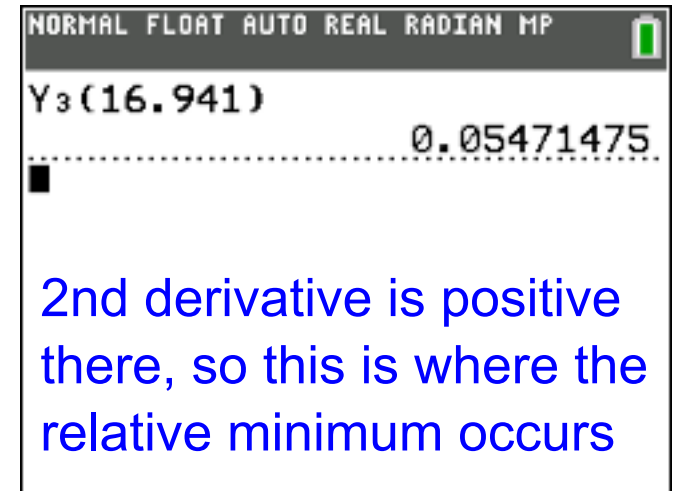
a maximum

minimize:
 $S(x) = \sqrt{12^2 + x^2} + \sqrt{5^2 + (24-x)^2}$

evaluate 2nd derivative at
the critical number found



2nd derivative is along x-axis



Use TI-Nspire CAS to find the solutions using
Calculus (could also be done "by hand")

Use TI-Nspire CAS to find the solutions using
Calculus (could also be done "by hand")

$$\frac{d}{dx}(f_3(x)) = \frac{x-24}{\sqrt{x^2-48x+601}} + \frac{x}{\sqrt{x^2+144}}$$

Use TI-Nspire CAS to find the solutions using
Calculus (could also be done "by hand")

$$\frac{d}{dx}(f_3(x)) = \frac{x-24}{\sqrt{x^2-48 \cdot x+601}} + \frac{x}{\sqrt{x^2+144}}$$

$$\text{zeros}\left(\frac{x-24}{\sqrt{x^2-48 \cdot x+601}} + \frac{x}{\sqrt{x^2+144}}, x\right) = \left\{ \frac{288}{17} \right\}$$

Use TI-Nspire CAS to find the solutions using Calculus (could also be done "by hand")

$$\frac{d}{dx}(f_3(x)) \quad \frac{x-24}{\sqrt{x^2-48 \cdot x+601}} + \frac{x}{\sqrt{x^2+144}}$$

The image shows a TI-Nspire CAS interface. At the top, there are navigation tabs for pages 1.8, 1.9, and 1.10. The current page is 1.10, titled '*The_Mast...try'. The mode is set to 'RAD'. The main display area shows the second derivative of $f_3(x)$ with respect to x , $\frac{d^2}{dx^2}(f_3(x))$. Below this, a yellow warning icon is visible. The expression is displayed as a sum of two fractions: $\frac{25}{(x^2-48 \cdot x+601)^2} + \frac{144}{(x^2+144)^2}$. A mouse cursor is pointing at the first fraction. The bottom of the screen shows a vertical bar.

$$\left\{ \frac{288}{17} \right\}$$

Use TI-Nspire CAS to find the solutions using Calculus (could also be done "by hand")

$$\frac{d}{dx}(f_3(x)) = \frac{x-24}{\sqrt{x^2-48 \cdot x+601}} + \frac{x}{\sqrt{x^2+144}}$$

The screenshot shows the TI-Nspire CAS interface. At the top, the derivative of $f_3(x)$ is displayed as $\frac{(x^2-48 \cdot x+601)^{-1/2}}{(x^2+144)^{-1/2}}$. Below this, the expression is evaluated at $x = \frac{288}{17}$, resulting in the fraction $\frac{83521 \cdot \sqrt{865}}{44893500}$. The interface also shows navigation buttons (1.8, 1.9, 1.10) and a window title '*The_Mast...try'.

$$\left\{ \frac{288}{17} \right\}$$

Since the second derivative is positive at this critical number, this critical number yields a relative minimum.

Use TI-Nspire CAS to find the solutions using
Calculus (could also be done "by hand")

$$\left\{ \frac{288}{17} \right\}$$

The image shows a TI-Nspire CAS calculator window titled '*The_Mast...try'. The window has tabs for pages 1.7, 1.8, and 1.9, with 1.9 selected. The mode is set to RAD. The calculator displays a table with three rows of data:

$\frac{288.}{17}$	<u>16.9412</u>
$f_3\left(\frac{288}{17}\right)$	$\sqrt{865}$
$f_3\left(\frac{288}{17}\right)$	<u>29.4109</u>

The first and third rows have red underlines under the numerical values. The second row shows the square root of 865. The bottom of the window shows a vertical cursor bar.

Use TI-Nspire CAS to find the solutions using
Calculus (could also be done "by hand")

$$\left\{ \frac{288}{17} \right\}$$

But wait...

there's more!

The screenshot shows a TI-Nspire CAS calculator window titled '*The_Mast...try' in RAD mode. It displays three rows of calculations for the fraction $\frac{288}{17}$:

$\frac{288.}{17}$	16.9412
$f3\left(\frac{288}{17}\right)$	$\sqrt{865}$
$f3\left(\frac{288}{17}\right)$	29.4109

Use TI-Nspire CAS to find the solutions using
Calculus (could also be done "by hand")

$$\left\{ \frac{288}{17} \right\}$$

But wait...

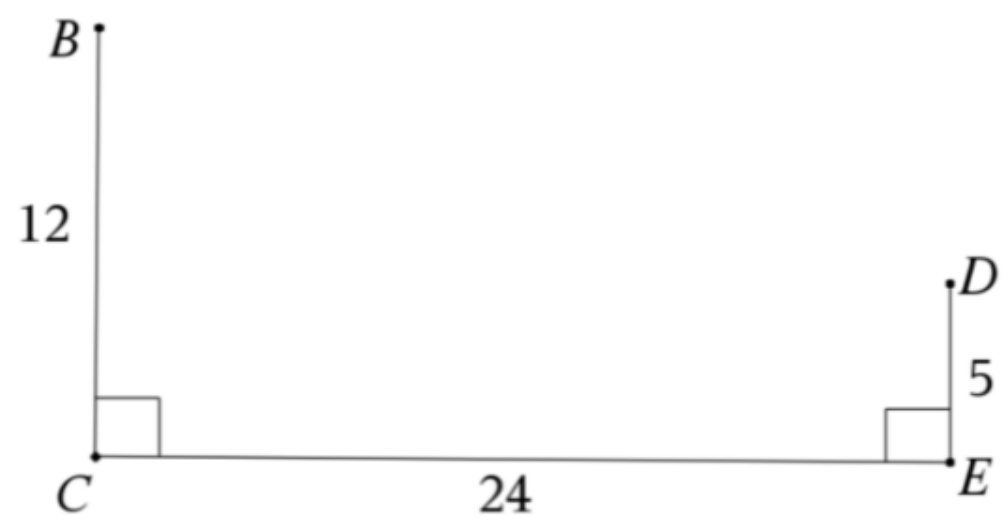
there's more!!

A geometric solution!

The image shows a TI-Nspire CAS calculator interface. The window title is "*The_Mast...try" and the mode is set to "RAD". The calculator displays the fraction $\frac{288}{17}$ and its decimal value 16.9412. Below this, the cube root of the fraction is shown as $\sqrt[3]{\frac{288}{17}}$ with a value of $\sqrt{865}$. A second entry shows the cube root of the fraction as $\sqrt[3]{\frac{288}{17}}$ with a value of 29.4109. The interface includes a navigation bar with buttons for 1.7, 1.8, and 1.9, and a close button (X).

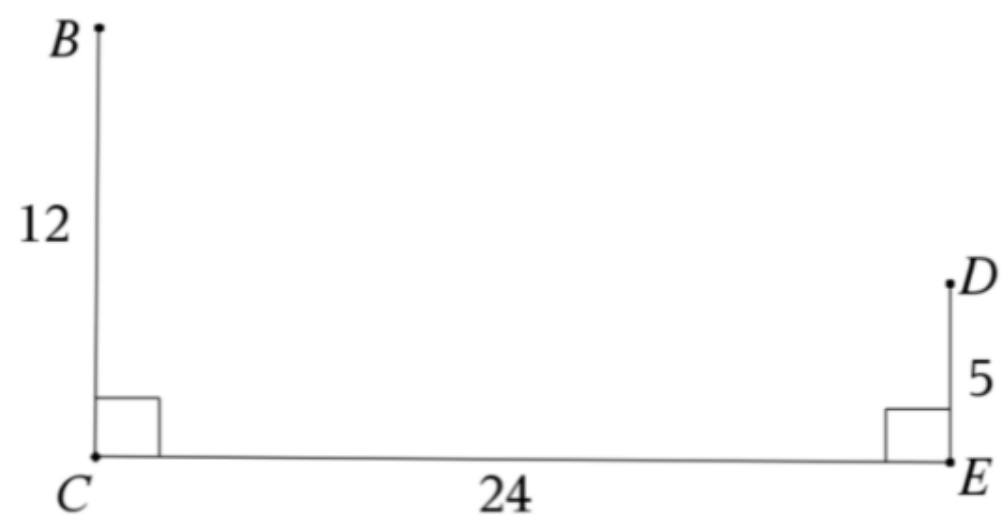
$\frac{288}{17}$	16.9412
$\sqrt[3]{\frac{288}{17}}$	$\sqrt{865}$
$\sqrt[3]{\frac{288}{17}}$	29.4109

Geometry solution



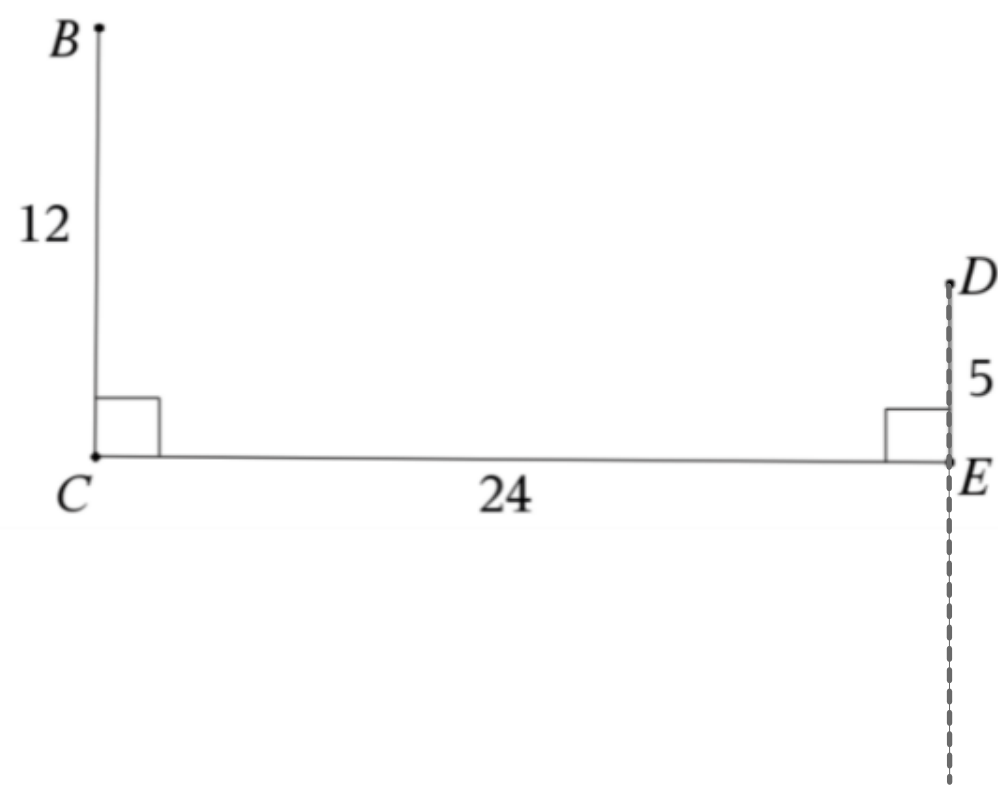
Geometry solution

Reflect DE over CE



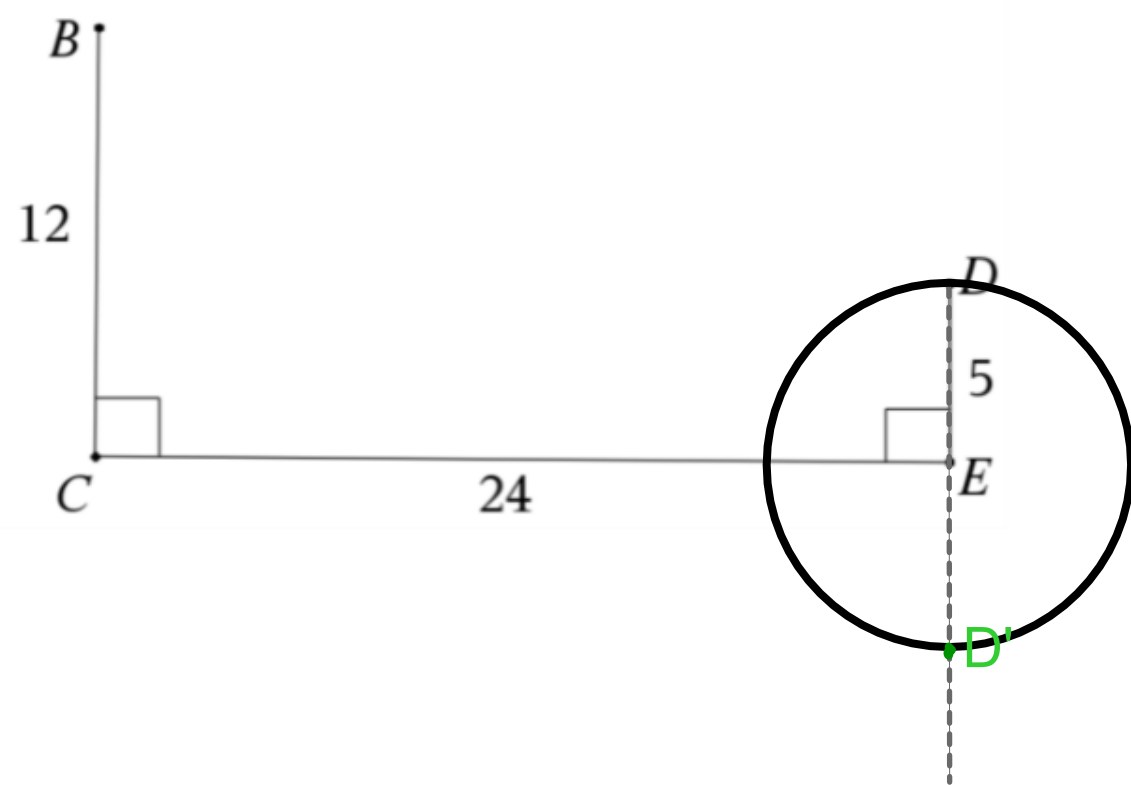
Geometry solution

Reflect DE over CE



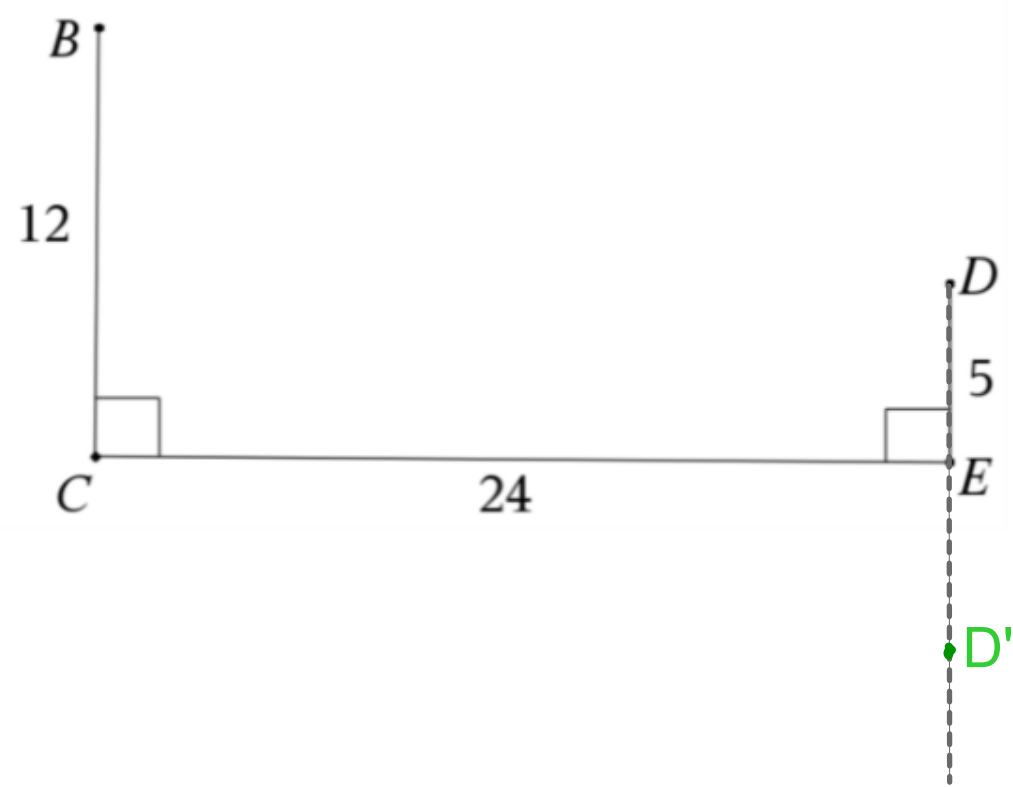
Geometry solution

Reflect DE over CE



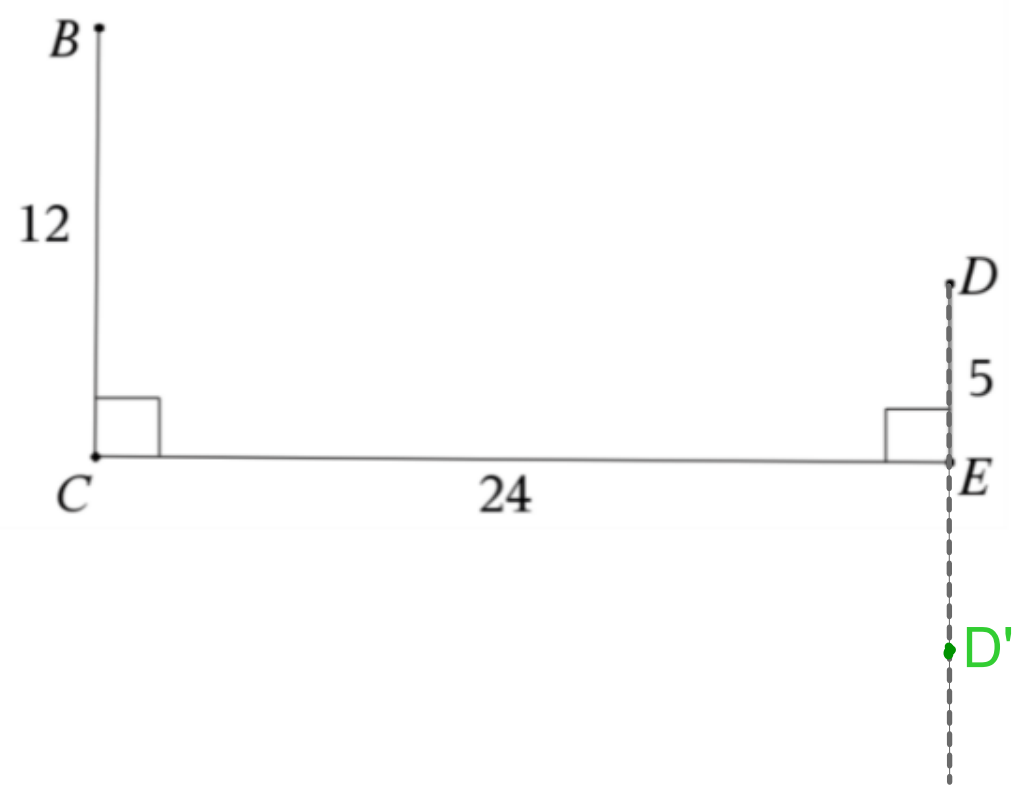
Geometry solution

Reflect DE over CE



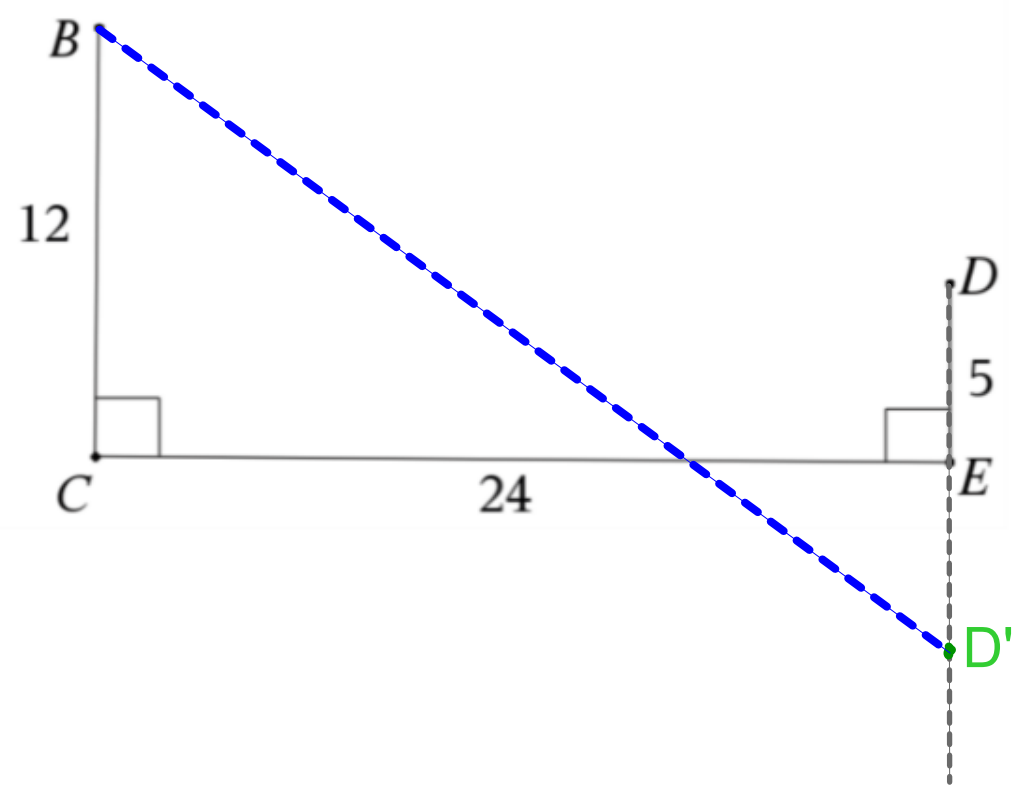
Geometry solution

Draw line BD'

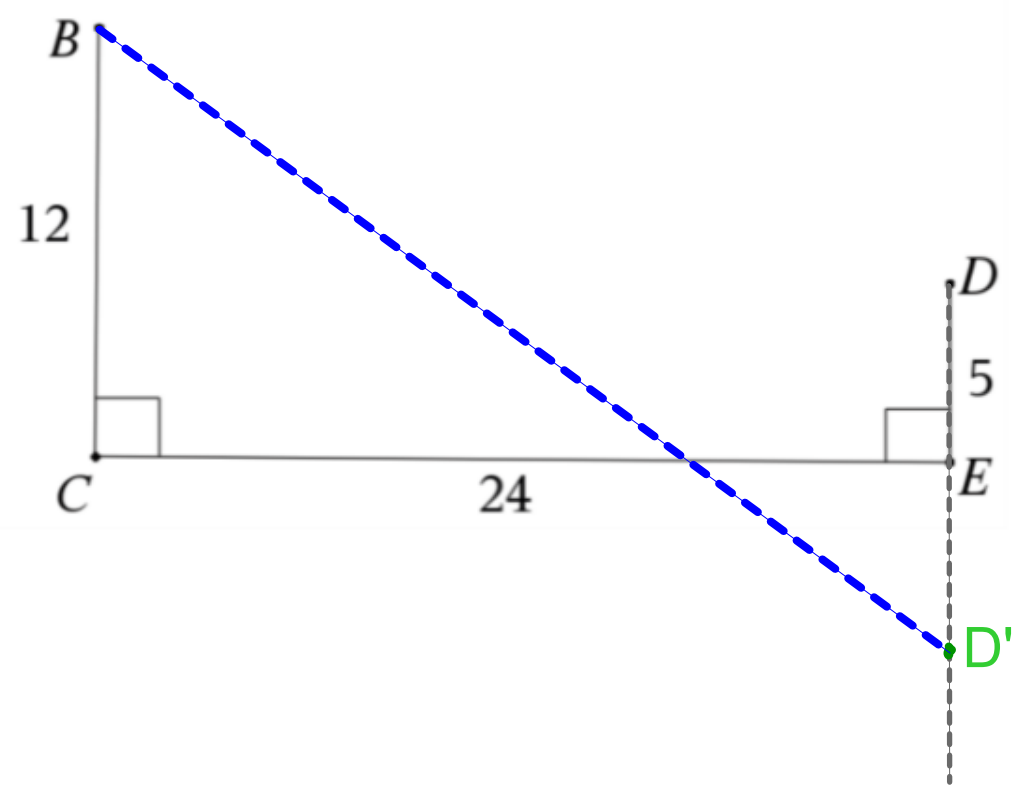


Geometry solution

Draw line BD'



Geometry solution

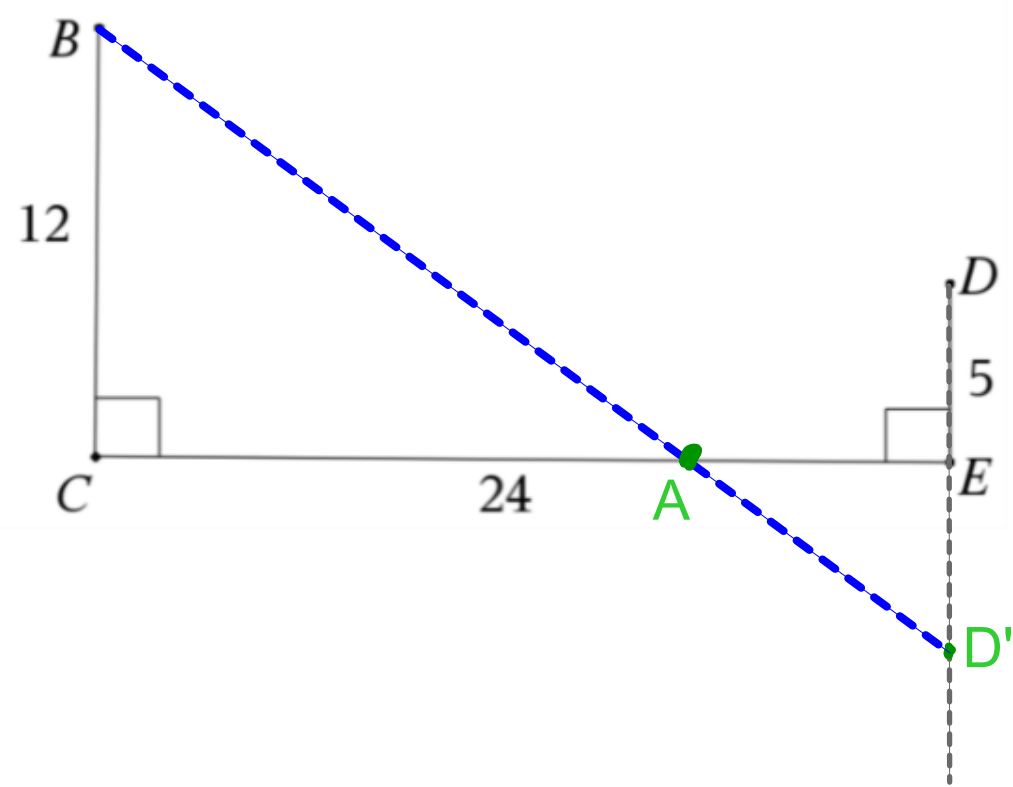


Draw line BD'

Label the point of intersection with

CE , point A .

Geometry solution

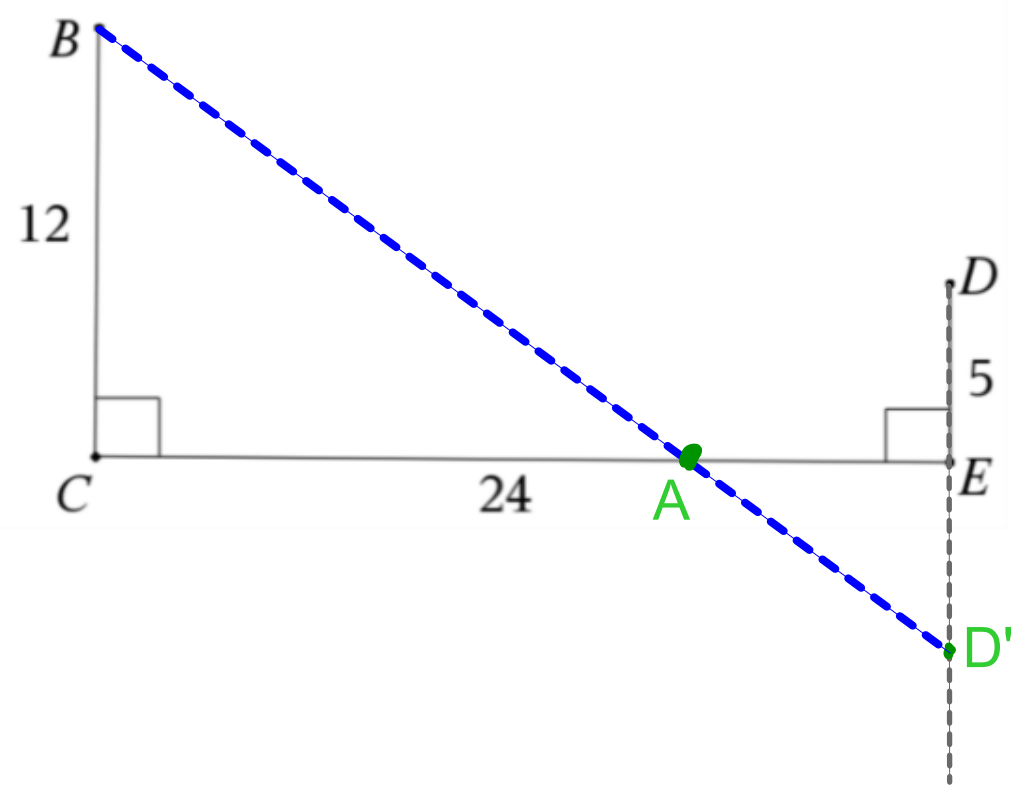


Draw line BD'

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Geometry solution



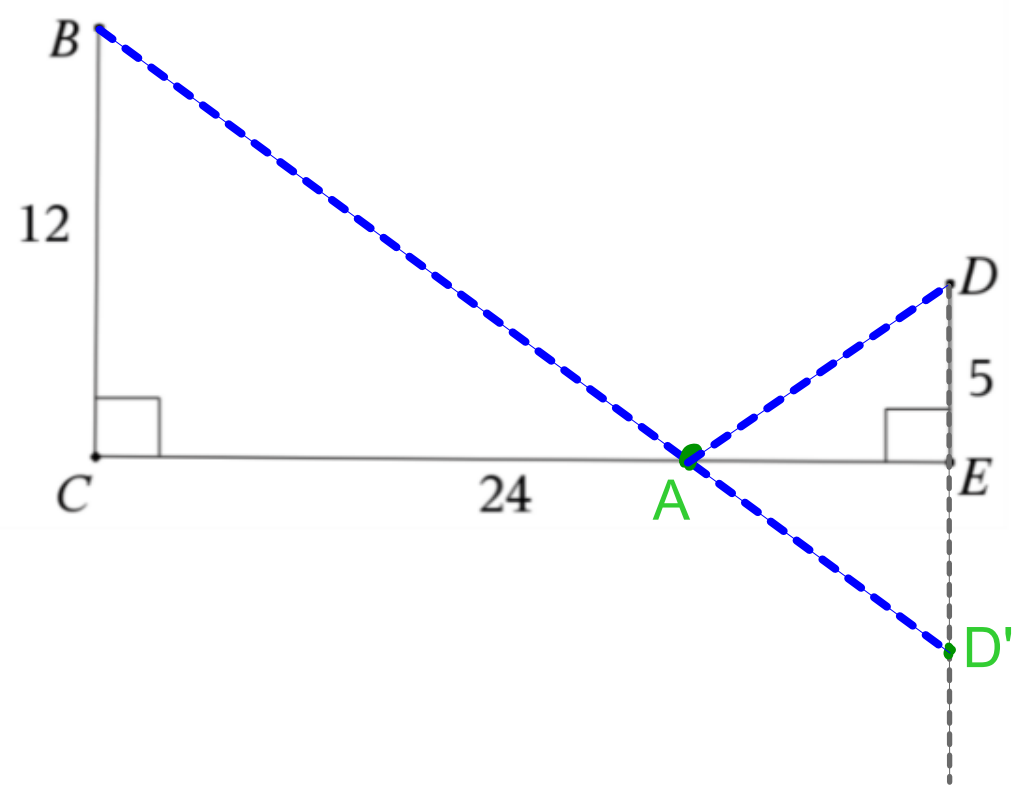
Draw line BD'

Label the point of intersection with

CE , point A .

Draw segment AD .

Geometry solution



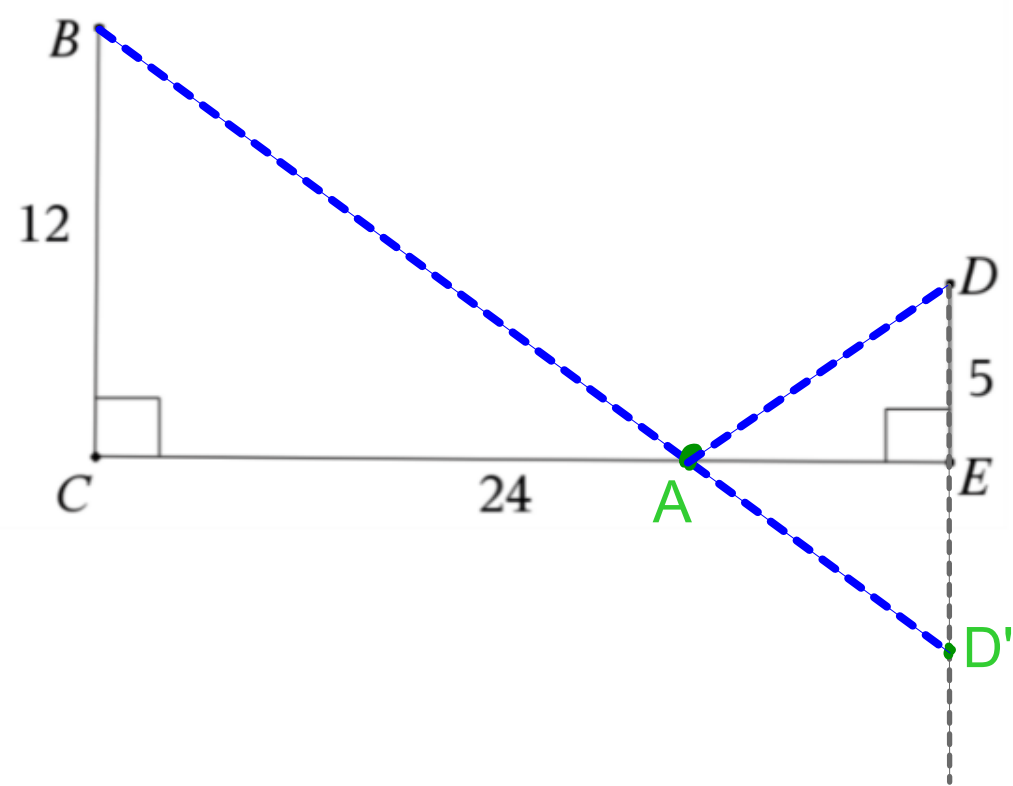
Draw line BD'

Label the point of intersection with

CE , point A .

Draw segment AD .

Geometry solution



Draw line BD'

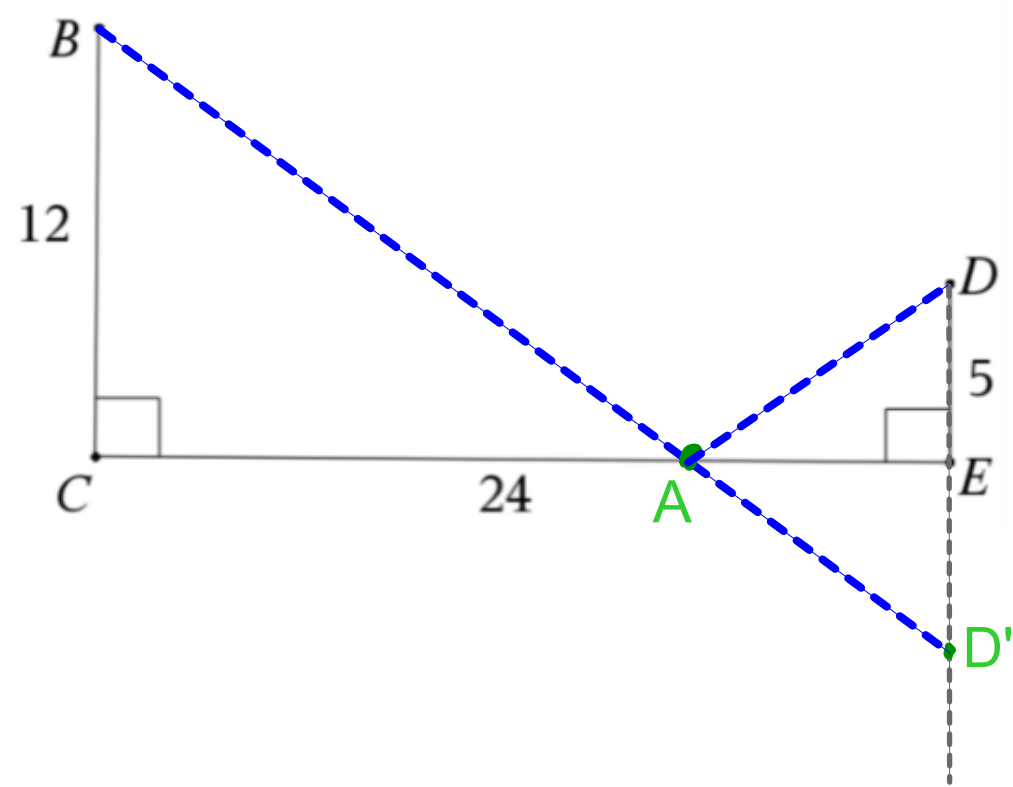
Label the point of intersection with

CE , point A .

Draw segment AD .

Notice?

Geometry solution



Draw line BD'

Label the point of intersection with

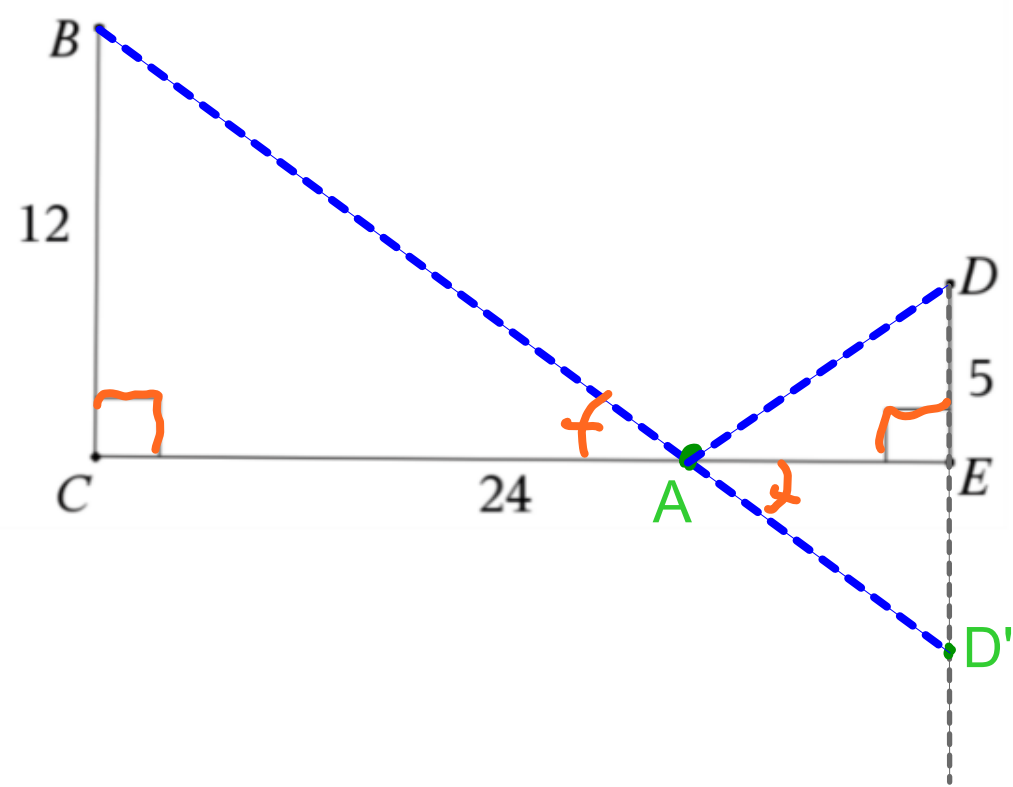
CE , point A .

Draw segment AD .

Notice?

Triangles AED and AED' are congruent. (SAS)

Geometry solution



Draw line BD'

Label the point of intersection with

CE , point A .

Draw segment AD .

Notice?

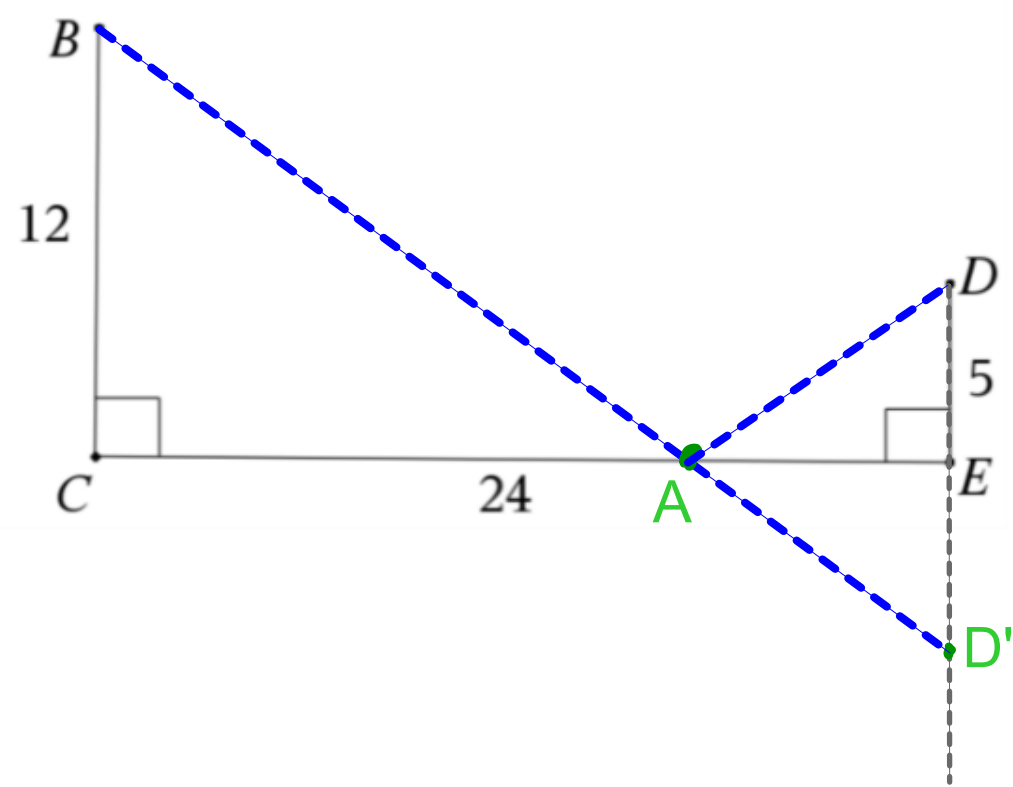
Triangles AED and AED' are congruent.

Also, triangles BCA and $D'EA$ are similar triangles.

(AA)

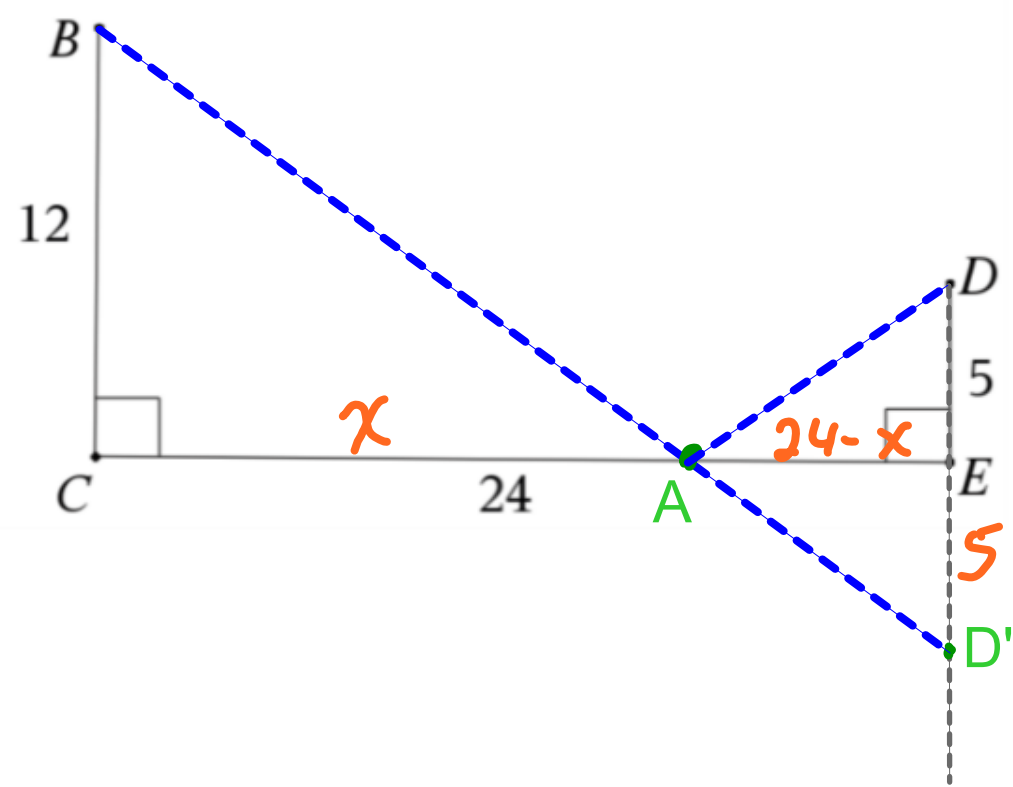
Geometry solution

Triangles BCA and $D'EA$ are similar triangles.



Geometry solution

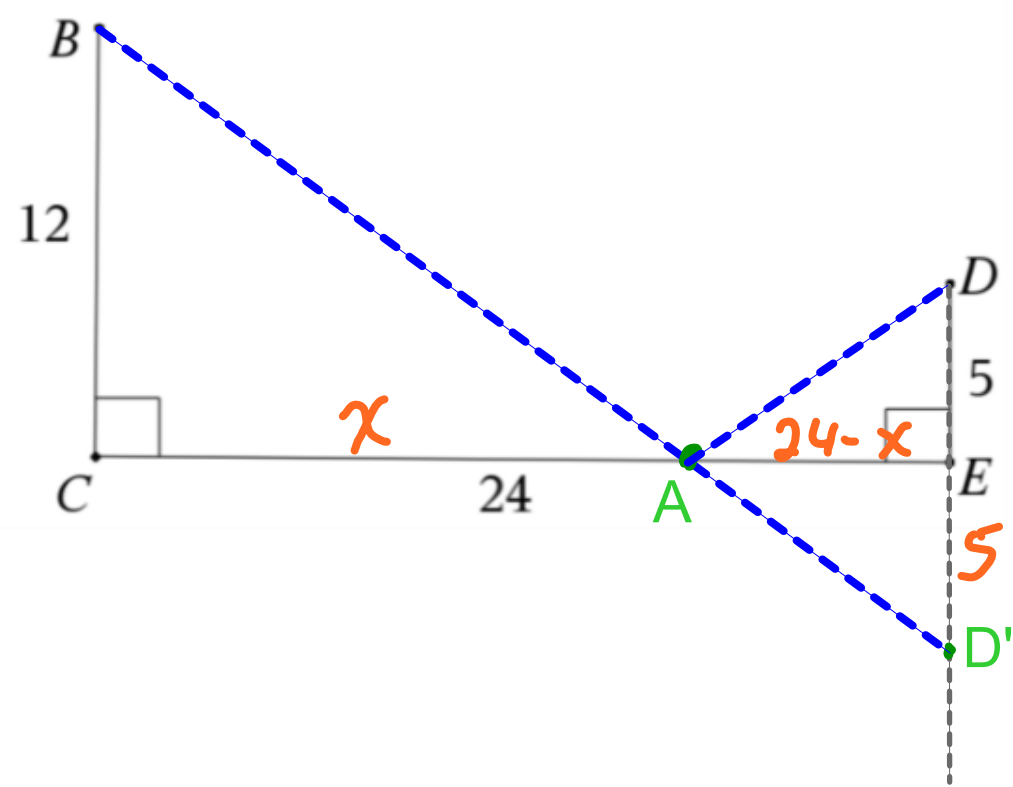
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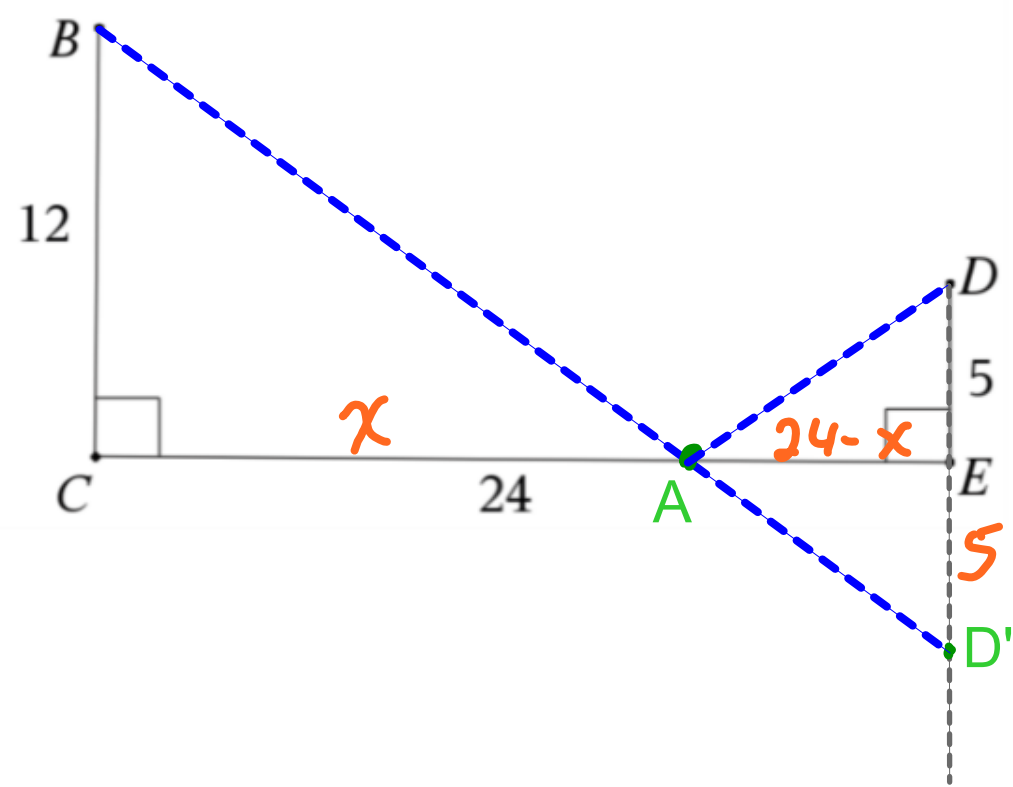
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$$\frac{BC}{D'E} = \frac{CA}{EA}$$



Geometry solution

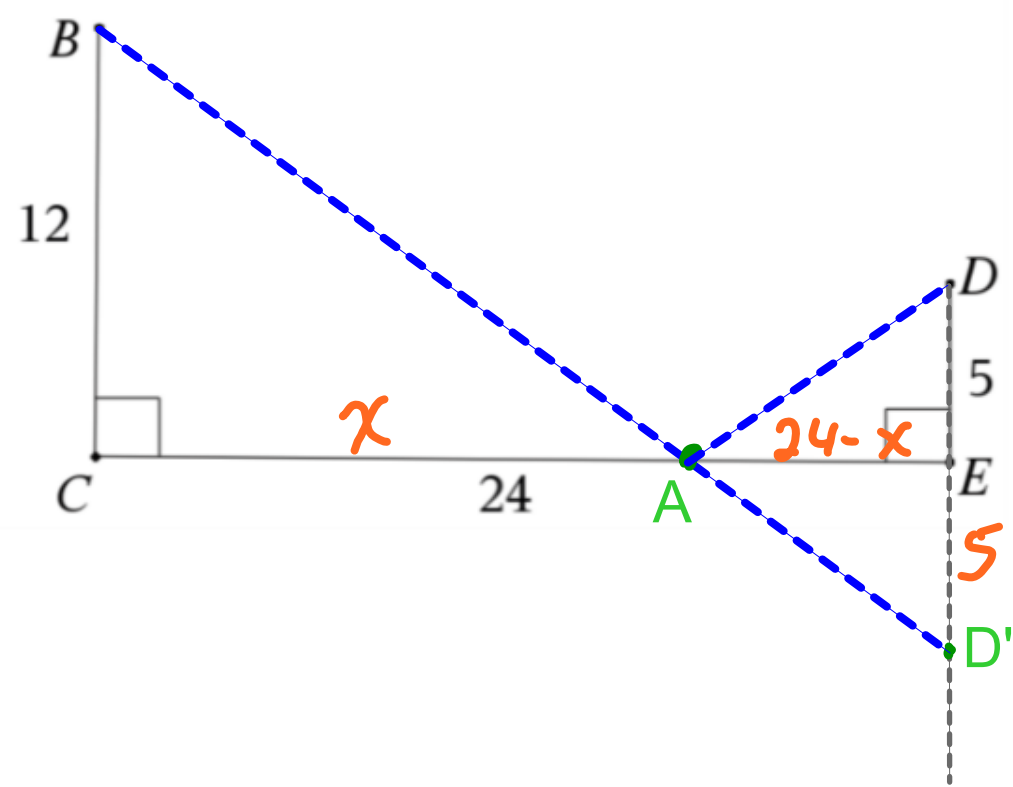


Triangles BCA and D'EA are similar triangles.

$$\frac{BC}{D'E} = \frac{CA}{EA}$$

$$\frac{12}{5} = \frac{x}{24-x}$$

Geometry solution



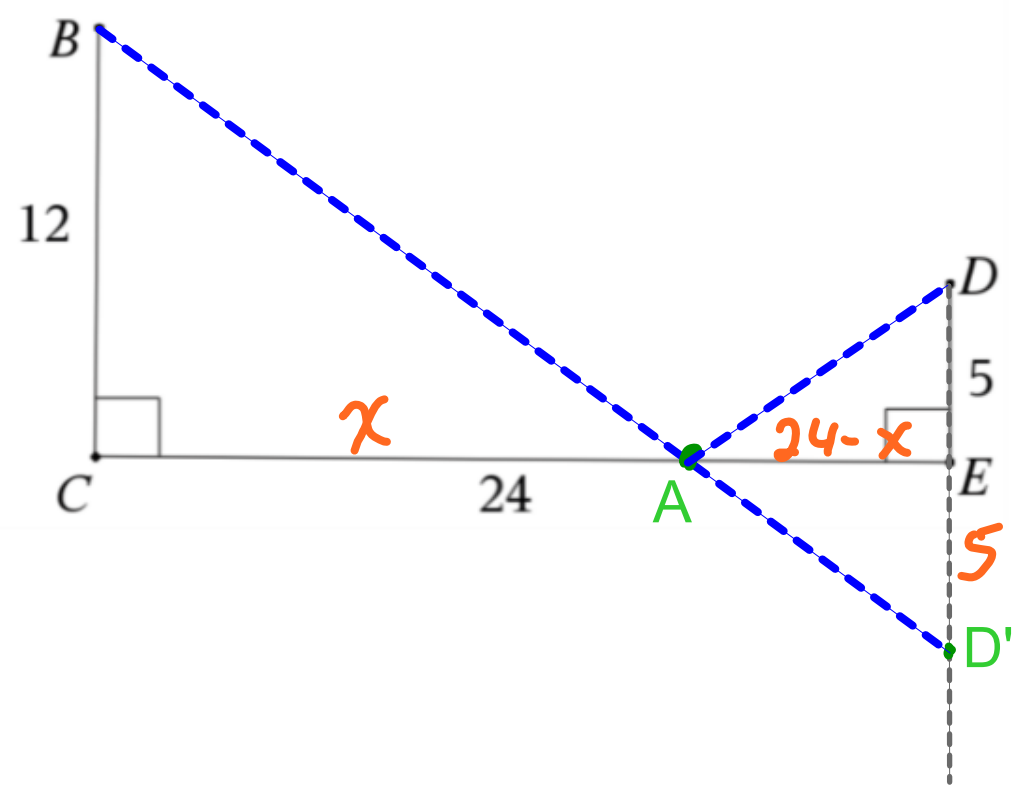
Triangles BCA and $D'EA$ are similar triangles.

$$\frac{BC}{D'E} = \frac{CA}{EA}$$

$$\frac{12}{5} = \frac{x}{24-x}$$

$$12(24-x) = 5x$$

Geometry solution



Triangles BCA and $D'EA$ are similar triangles.

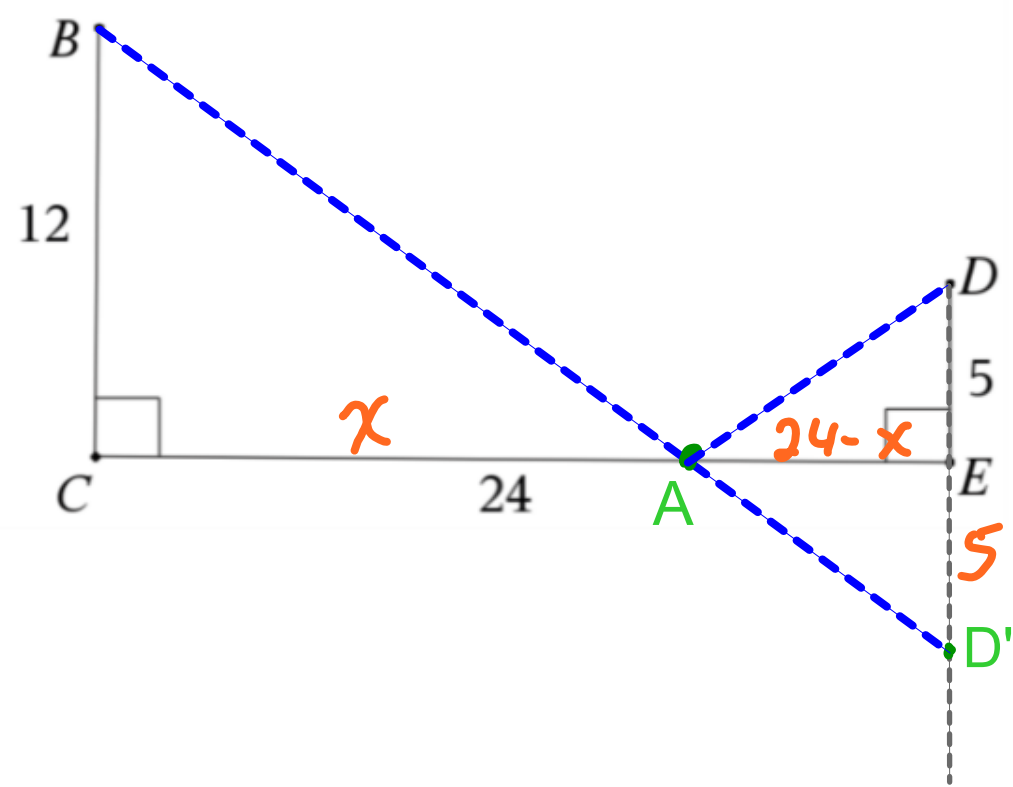
$$\frac{BC}{D'E} = \frac{CA}{EA}$$

$$\frac{12}{5} = \frac{x}{24-x}$$

$$12(24-x) = 5x$$

$$288 - 12x = 5x$$

Geometry solution



Triangles BCA and $D'EA$ are similar triangles.

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$$288 - 12x = 5x$$

$$288 = 17x$$

Geometry solution

Triangles BCA and $D'EA$ are similar triangles.

$$\frac{BC}{D'E} = \frac{CA}{EA}$$

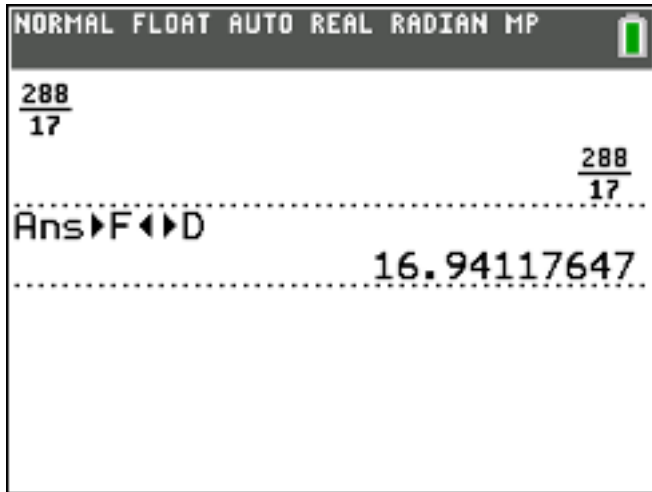
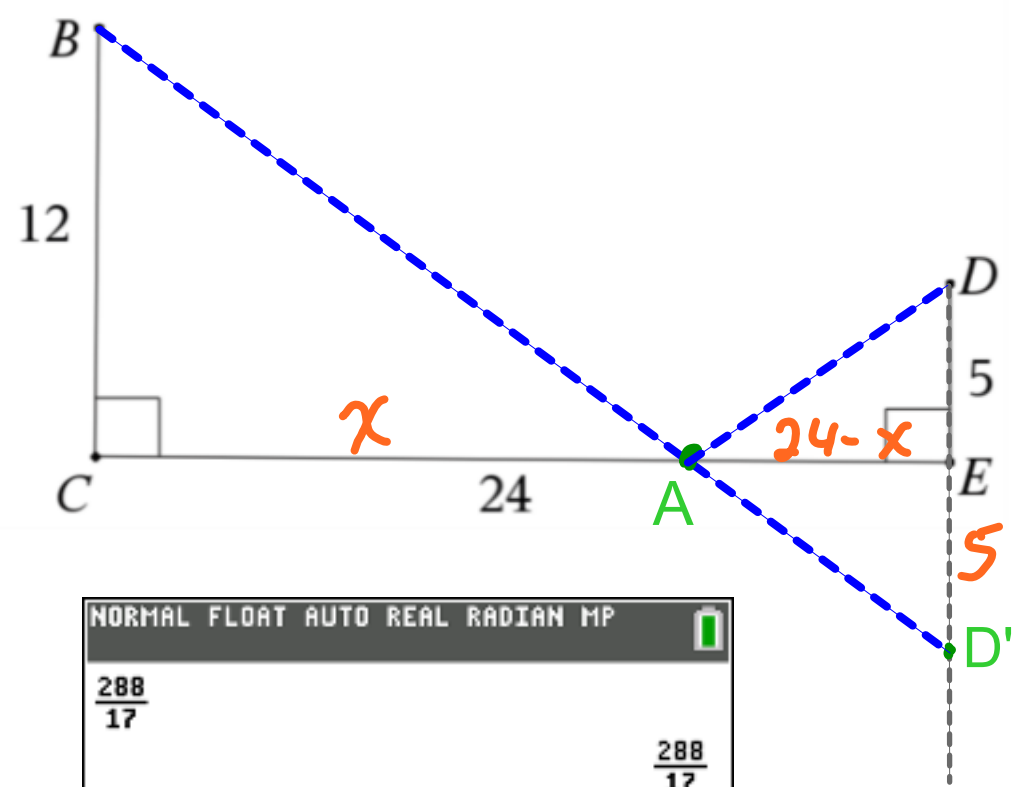
$$\frac{12}{5} = \frac{x}{24-x}$$

$$12(24-x) = 5x$$

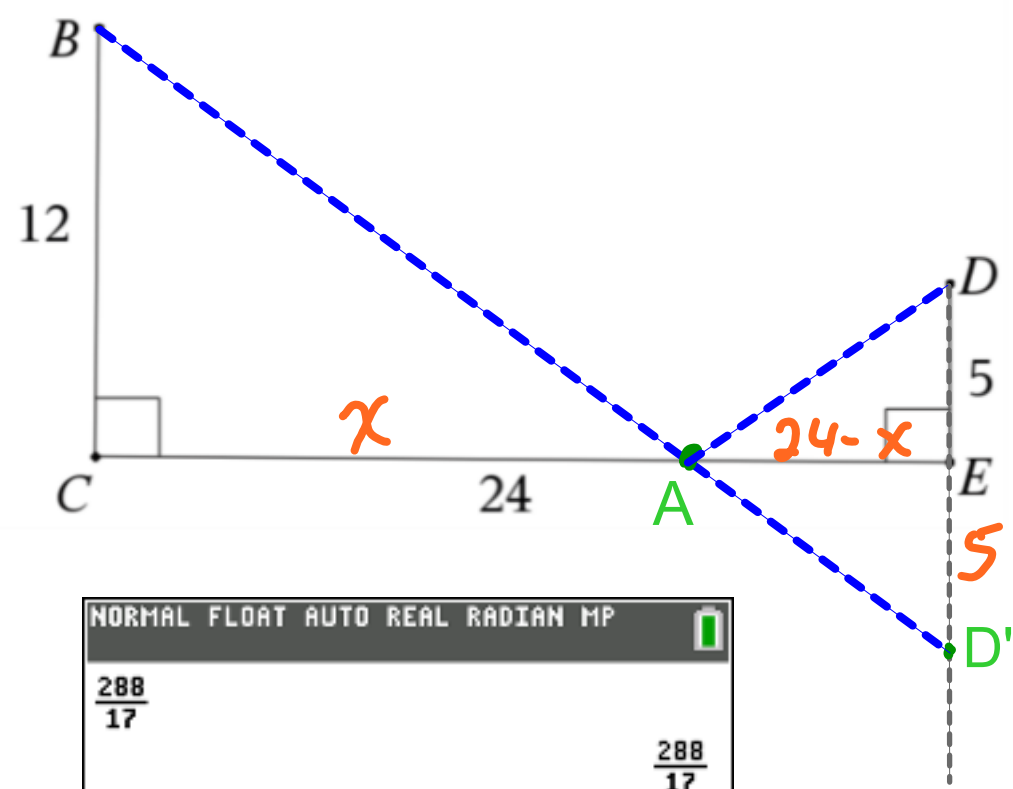
$$288 - 12x = 5x$$

$$288 = 17x$$

$$\frac{288}{17} = x$$

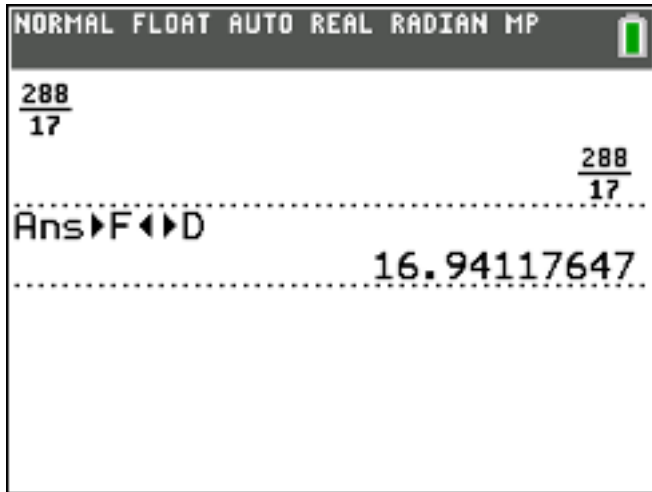


Geometry solution

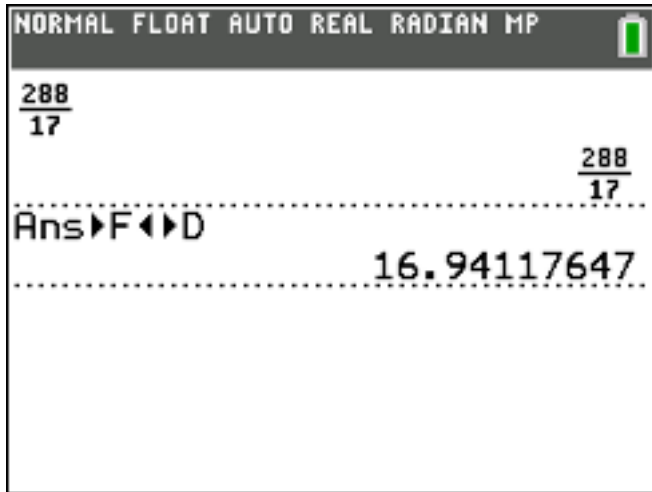
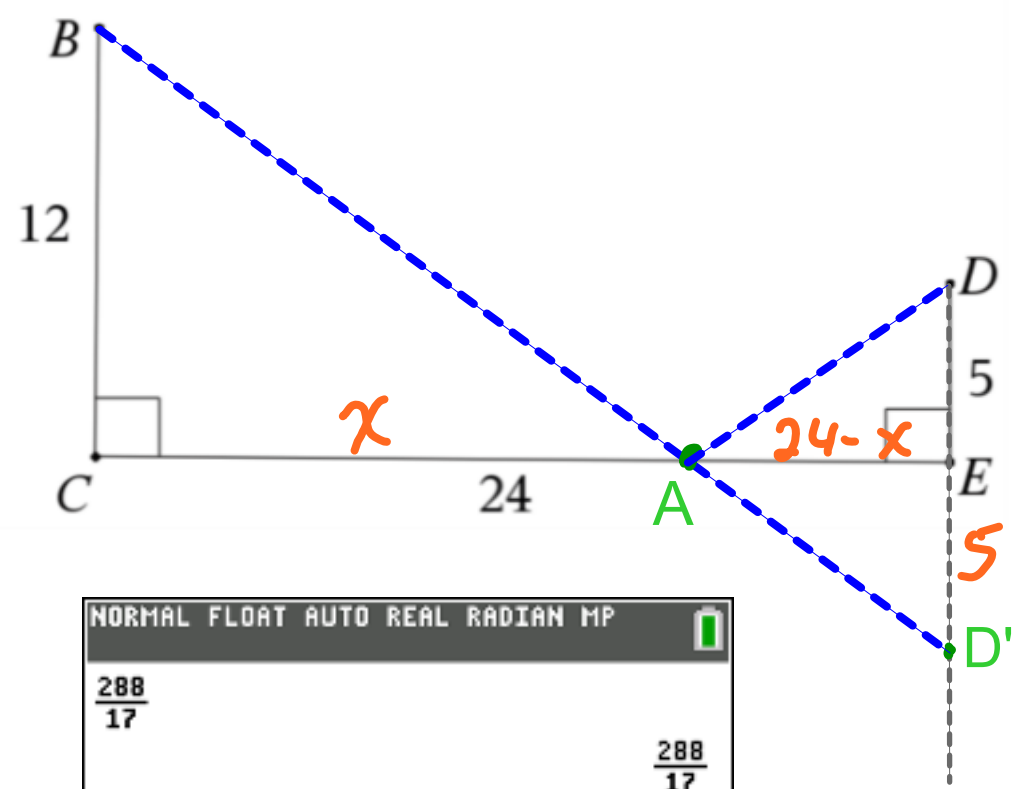


Since triangles AED and AED' are congruent:

$$BA + AD = BA + AD'$$



Geometry solution

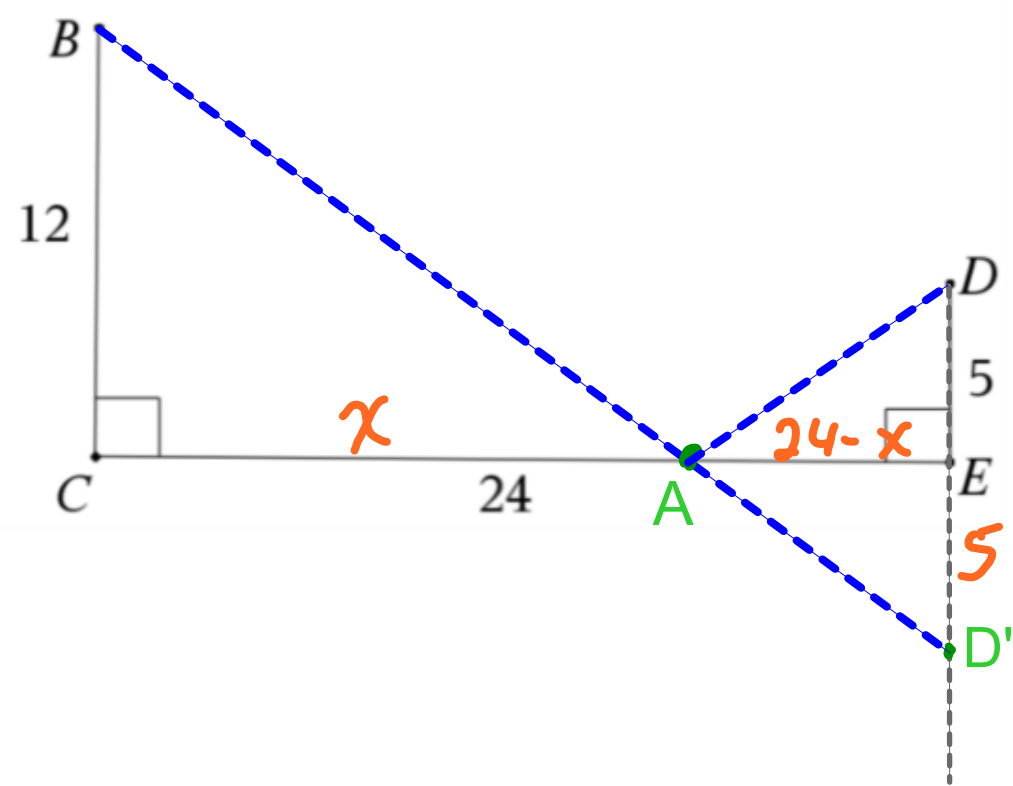


Since triangles AED and AED' are congruent:

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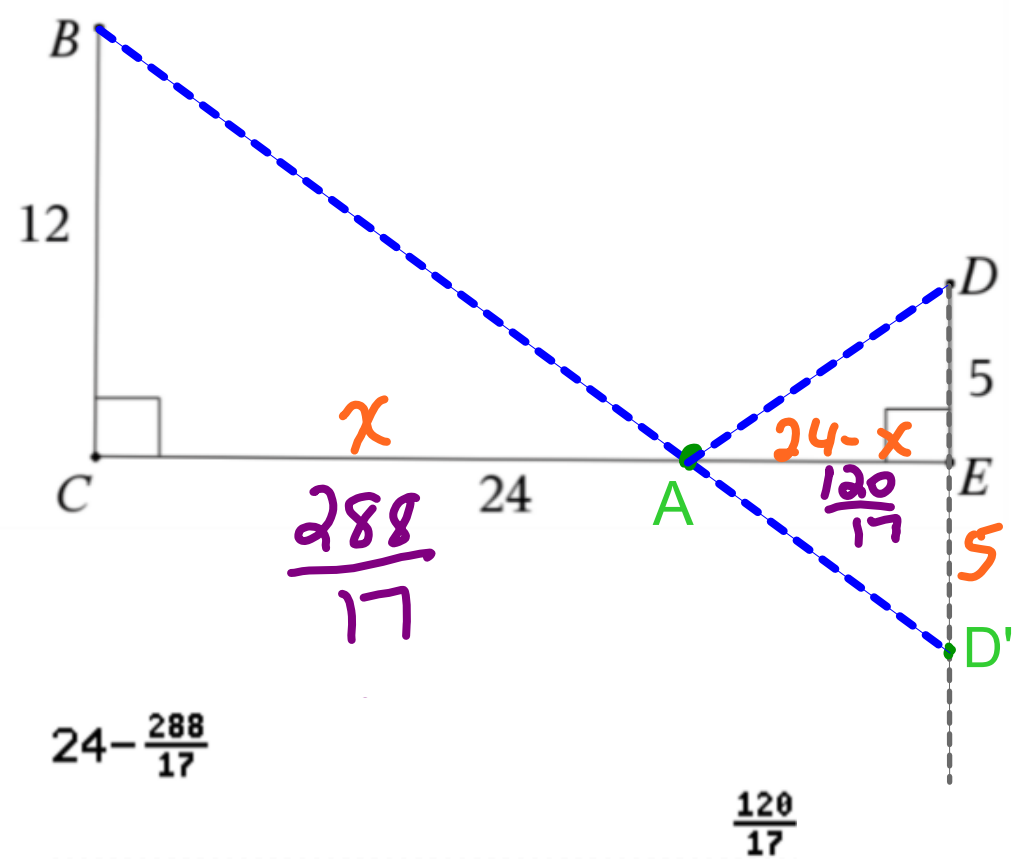
The shortest distance between two points is a line (segment):

BA + AD is the shortest distance, occurring at point A.



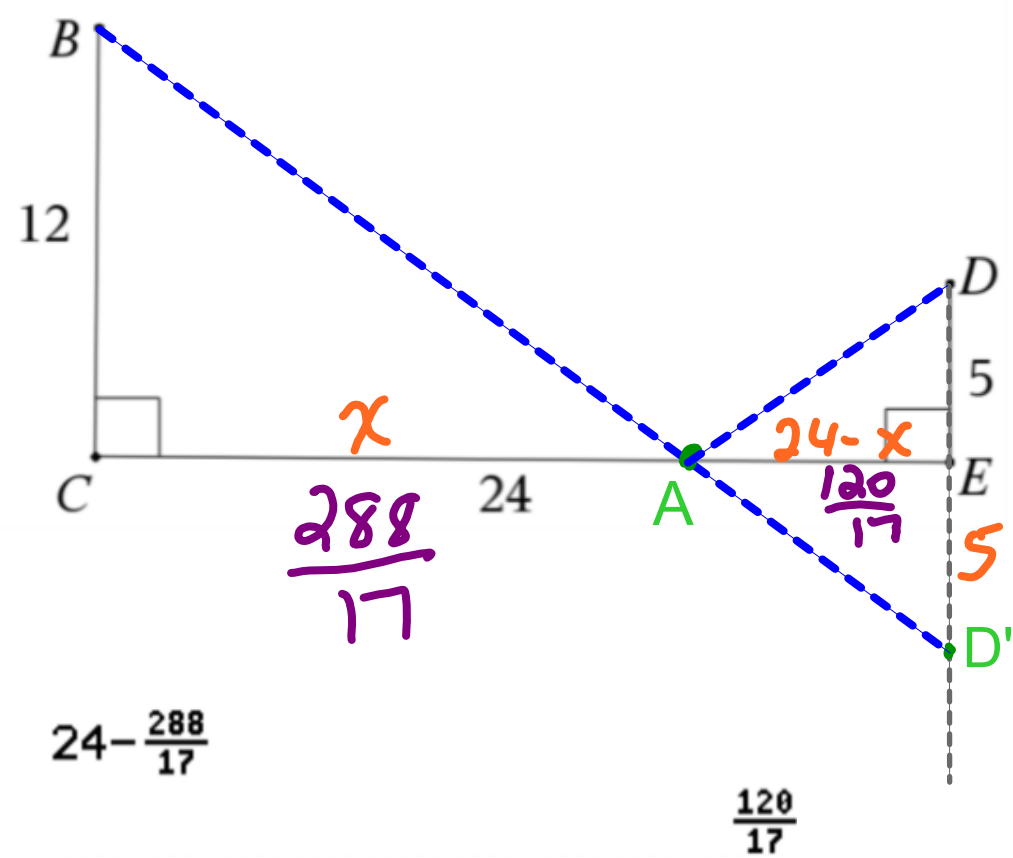
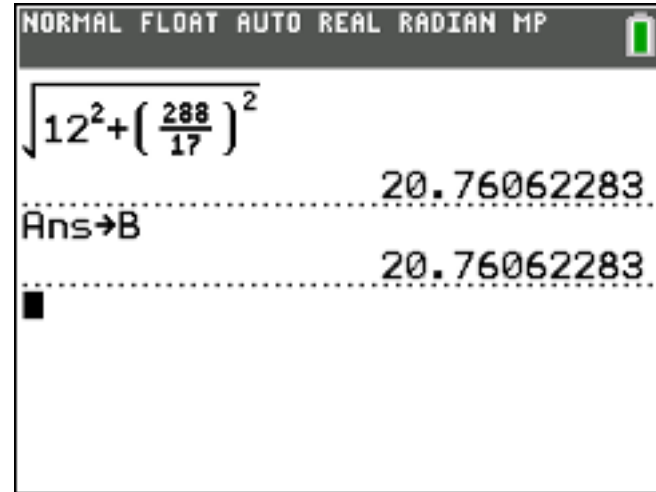
BA + AD is the shortest distance, occurring at point A.

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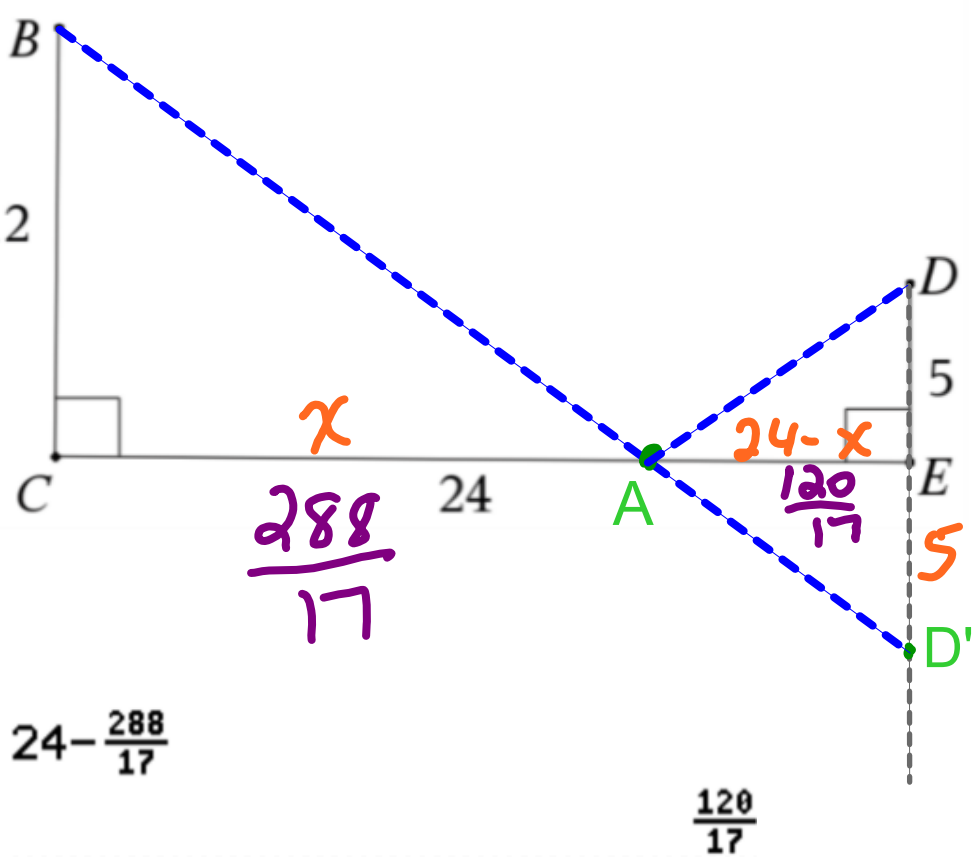
BA + AD is the shortest distance, occurring at point A.

$$BA = \sqrt{(12)^2 + \left(\frac{288}{17}\right)^2}$$



BA + AD is the shortest distance, occurring at point A.

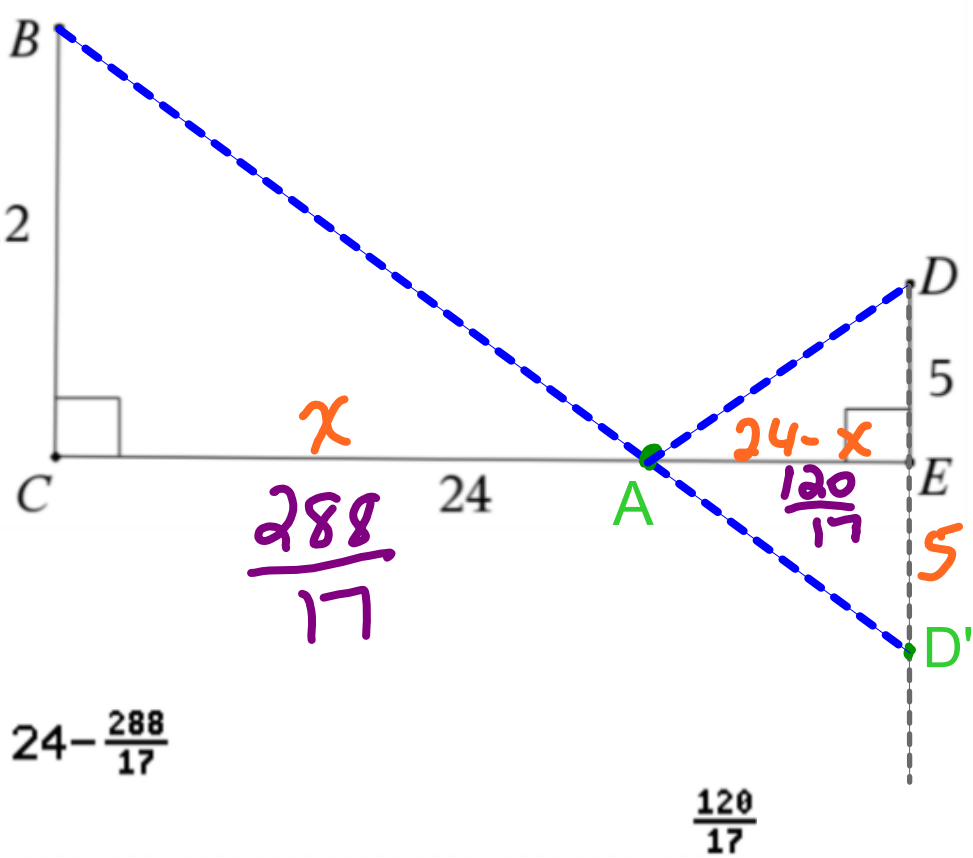
$$AD = \sqrt{\left(\frac{120}{17}\right)^2 + 5^2}$$



NORMAL FLOAT AUTO REAL RADIAN MP	
	20.76062283
Ans→B	20.76062283
$\sqrt{\left(\frac{120}{17}\right)^2 + 5^2}$	8.650259512
Ans→D	8.650259512

BA + AD is the shortest distance, occurring at point A.

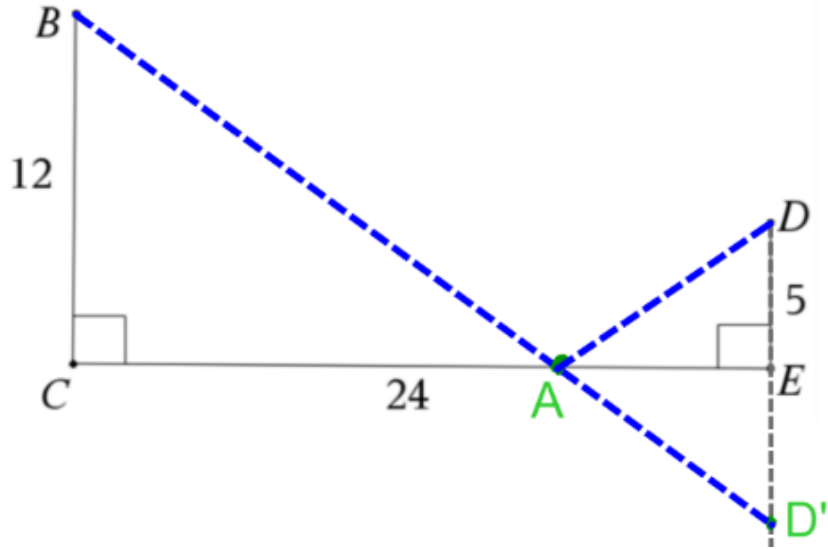
$$BD = BA + AD = 29.411$$



NORMAL FLOAT AUTO REAL RADIAN MP	
$\sqrt{\left(\frac{120}{17}\right)^2 + 5^2}$	20.76062283
Ans \rightarrow D	8.650259512
B+D	8.650259512
	29.41088234

Newly added ideas from participants on the Zoom - thank you!

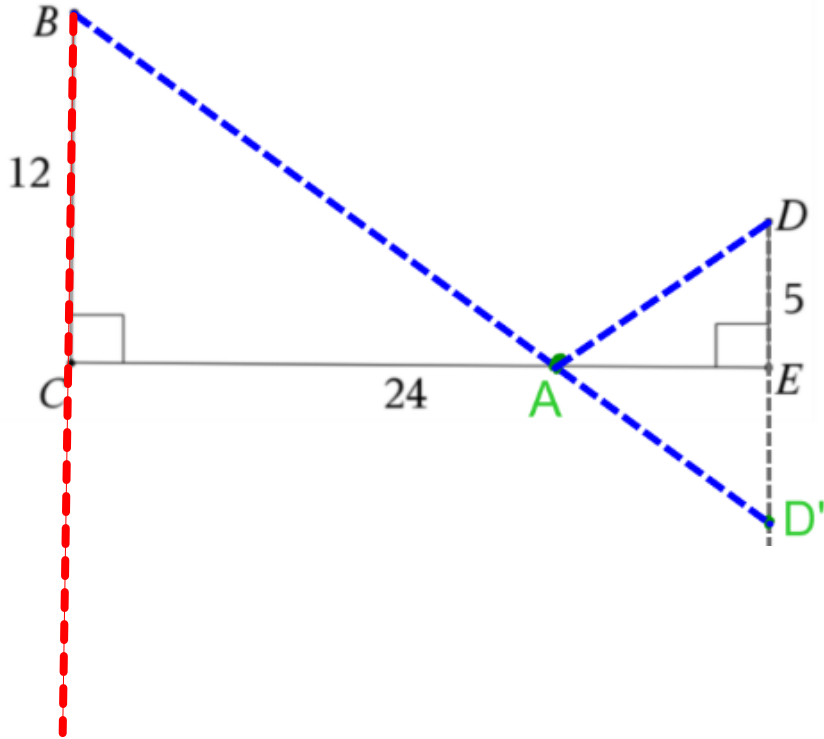
Alternative way to find the length of BD'



Newly added ideas from participants on the Zoom - thank you!

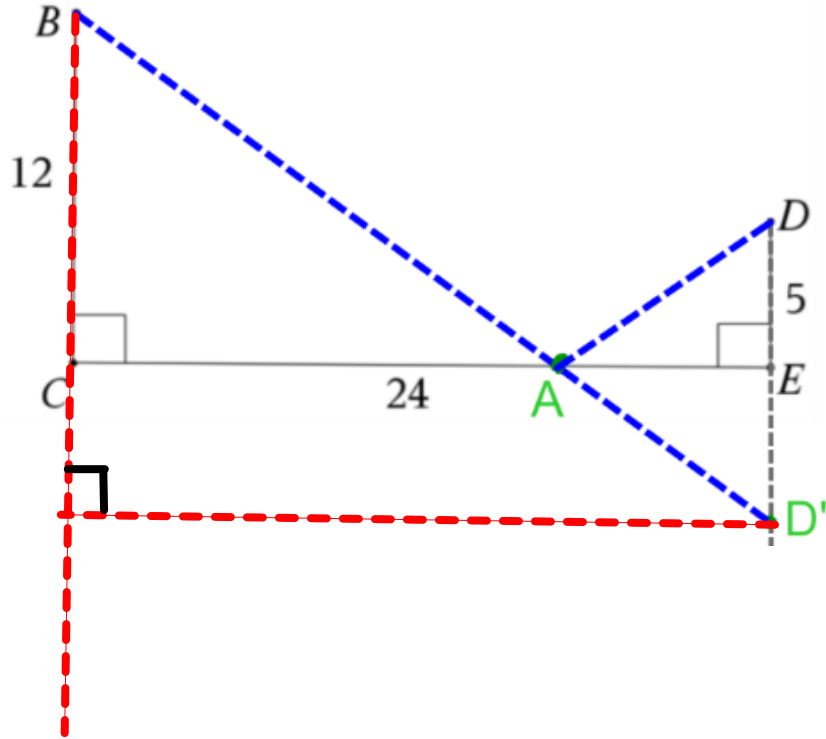
Alternative way to find the length of BD'

Extend BC



Newly added ideas from participants on the Zoom - thank you!

Alternative way to find the length of BD'



Extend BC

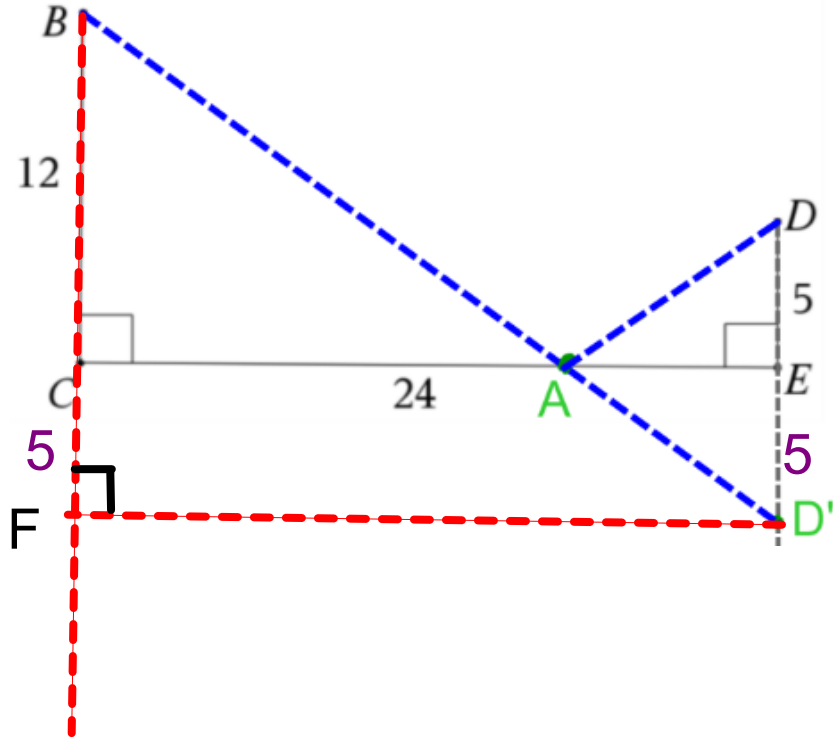
Perpendicular from D' to line BC

Newly added ideas from participants on the Zoom - thank you!

Alternative way to find the length of BD'

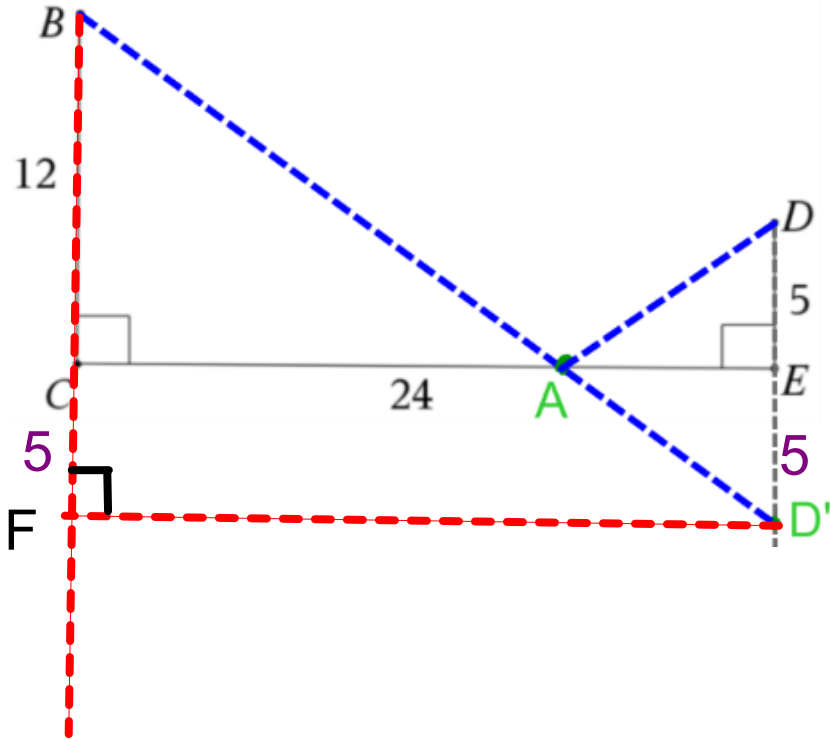
Extend BC

Perpendicular from D' to line BC



Newly added ideas from participants on the Zoom - thank you!

Alternative way to find the length of BD'

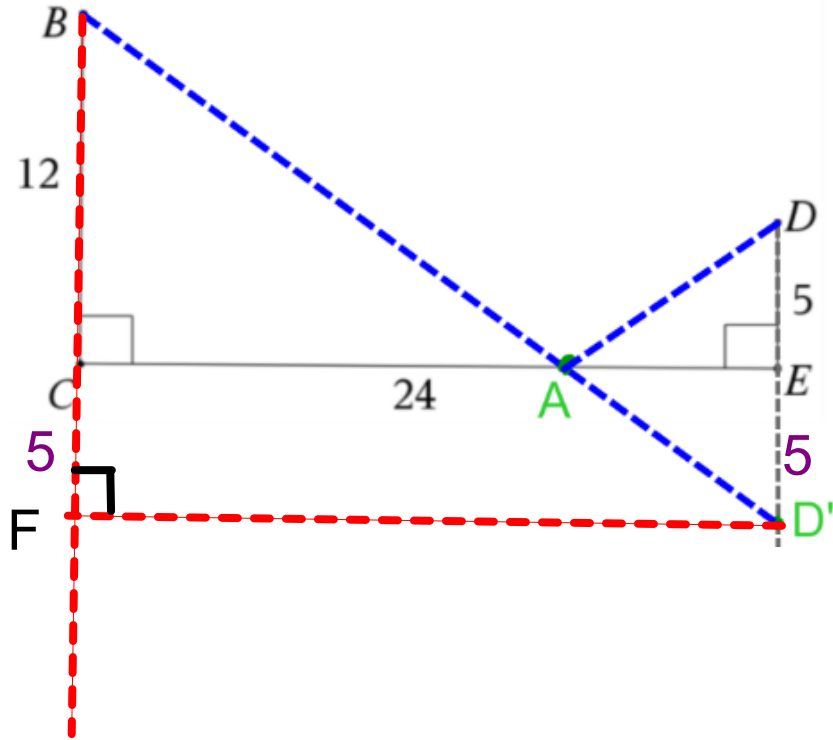


Extend BC

Perpendicular from D' to line BC

$$(BF)^2 + (FD')^2 = (BD')^2$$

Newly added ideas from participants on the Zoom - thank you!



Alternative way to find the length of BD'

Extend BC

Perpendicular from D' to line BC

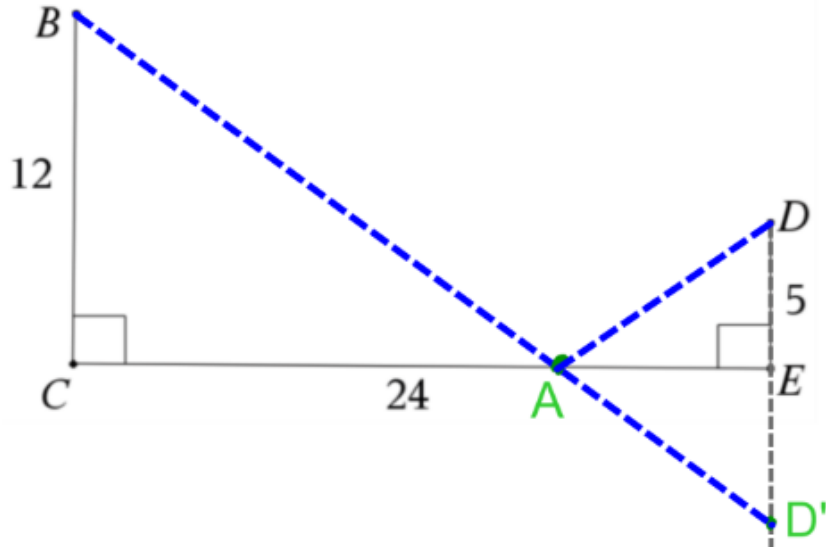
$$(BF)^2 + (FD')^2 = (BD')^2$$

$$17^2 + 24^2 = (BD')^2$$

$$865 = (BD')^2$$

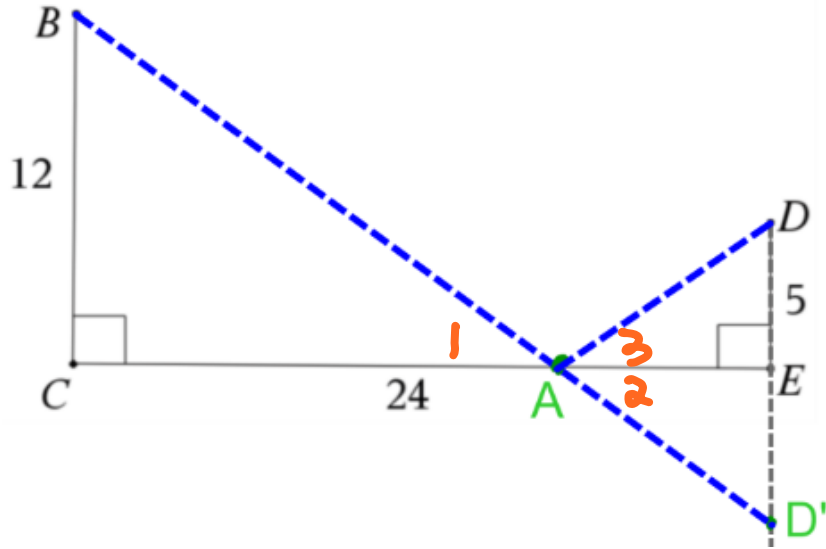
$$\underline{\underline{\sqrt{865} = BD' = BA + AD}}$$

Newly added ideas from participants on the Zoom - thank you!



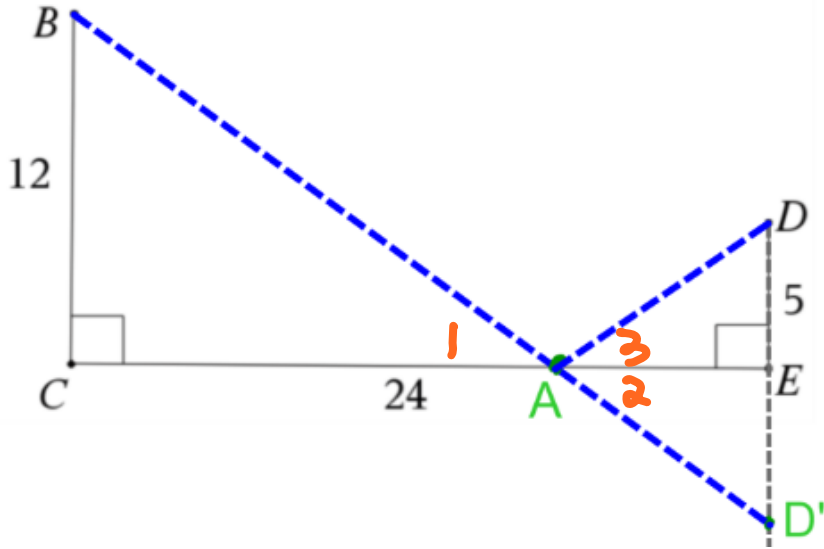
What about the angle of incidence is equal to the angle of reflection - when the minimum distance is found?

Newly added ideas from participants on the Zoom - thank you!



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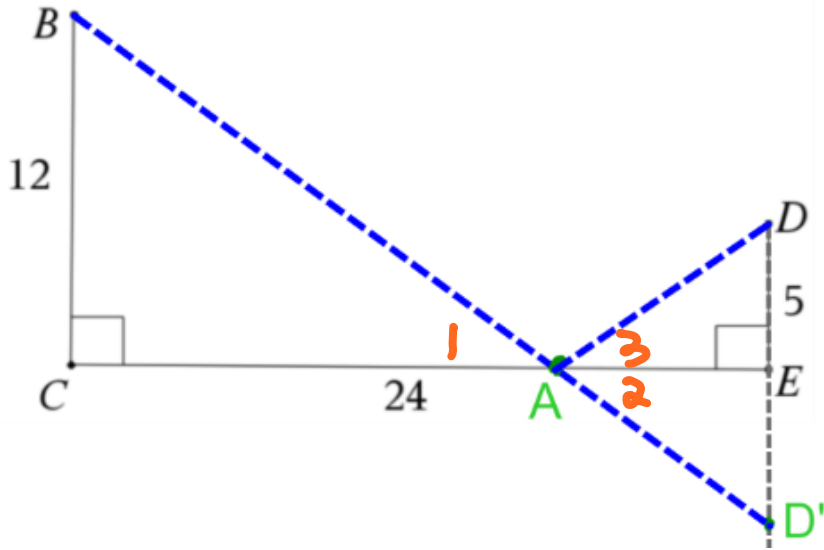
What about the angle of incidence is equal the angle of reflection - when the minimum distance is found?

$$\angle 1 = \angle 2 \text{ (VERTICAL } \angle \text{s)}$$

$$\angle 2 = \angle 3 \text{ (} \cong \Delta \text{s)}$$

$$\angle 1 = \angle 3 \text{ (TRANSITIVE)}$$

Newly added ideas from participants on the Zoom - thank you!



What about the angle of incidence is equal to the angle of reflection - when the minimum distance is found?

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$$\angle 1 = \angle 3 \text{ (TRANSITIVE)}$$

The Mast TI-84 CE

Solved using a form of data capture with the TI-84 CE

NORMAL FLOAT AUTO REAL RADIAN MP 

The Mast Problem

BA graphed in blue Y1
AD graphed in brown Y2
Y4 contains the sum

Press ENTER to continue

NORMAL FLOAT AUTO REAL RADIAN MP 

Right arrow key increases
the value of A

Left arrow key decreases
the value of A

Press ENTER to continue

NORMAL FLOAT AUTO REAL RADIAN MP 

Run the TRANSFRM app

Press graph twice!

The step should be 1
in the SETUP of the app

.....Done.
■

NORMAL FLOAT AUTO REAL RADIAN MP
TRANSFORMATION GRAPHING APP

Plot1 Plot2 Plot3 QUIT-APP

Y1 = $-\frac{12}{A} * X + 12$

Y2 = $\frac{5}{24-A} * X + \frac{5A}{A-24}$

Y3 =

Y4 = $\sqrt{A^2 + (12^2)} + \sqrt{(24-A)^2 + (5^2)}$

Y5 =

Y6 =

Y7 =

NORMAL FLOAT AUTO REAL RADIAN MP
TRANSFORMATION GRAPHING APP

Y1 = $-(12/A)*X + 12$ Y2 = $(5/(24-A))*X + \dots$



A=10

SETUP

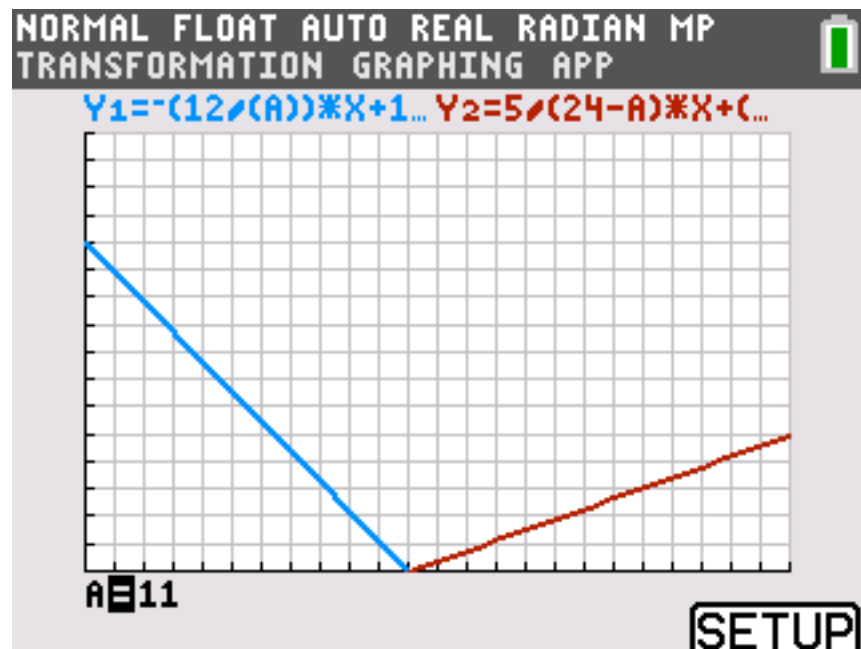
NORMAL FLOAT AUTO REAL RADIAN MP

Y4

30.4865681

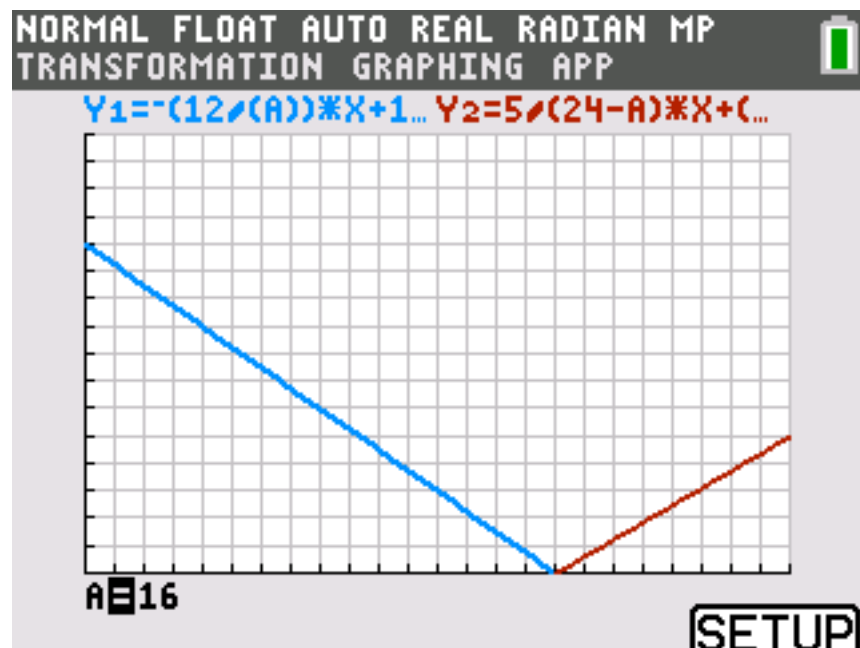
A

10



NORMAL FLOAT AUTO REAL RADIAN MP

Y4
 30.4865681
 A
 10
 Y4
 30.20720887
 A
 11
 ■



NORMAL FLOAT AUTO REAL RADIAN MP

..... 10

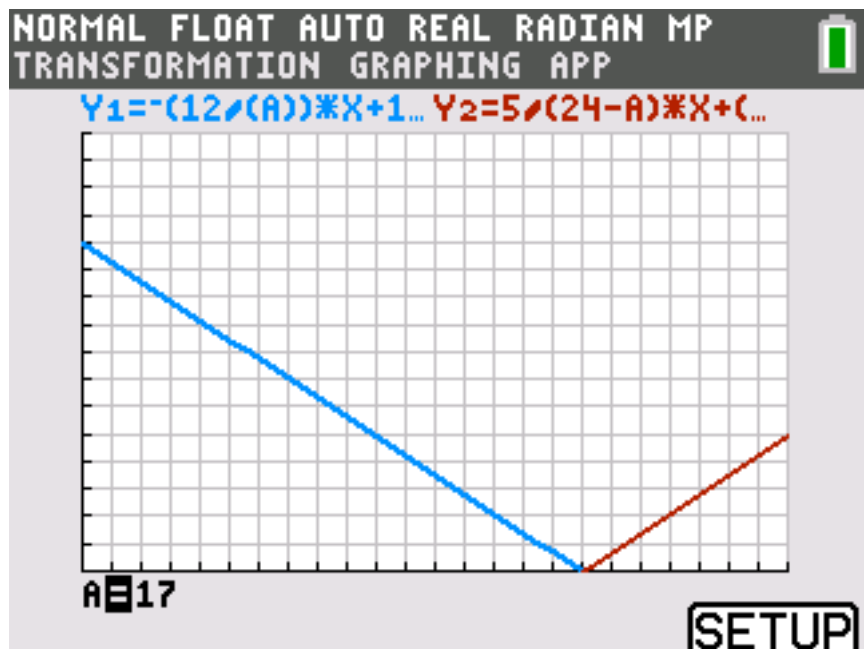
Y4 30.20720887

A 11

Y4 29.43398113

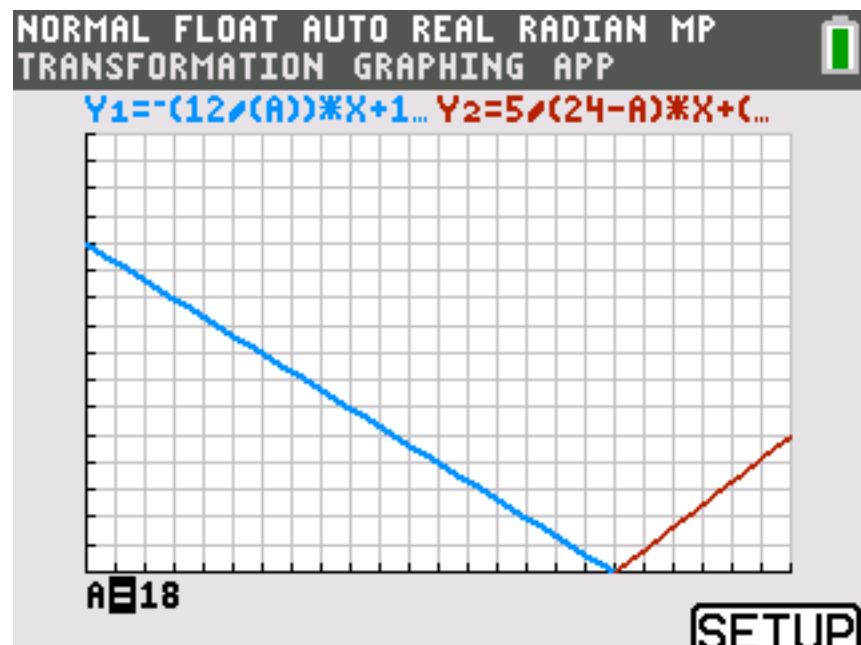
A 16

■



NORMAL FLOAT AUTO REAL RADIAN MP

Y4	11
A	29.43398113
Y4	16
A	29.41097731
Y4	17



NORMAL FLOAT AUTO REAL RADIAN MP

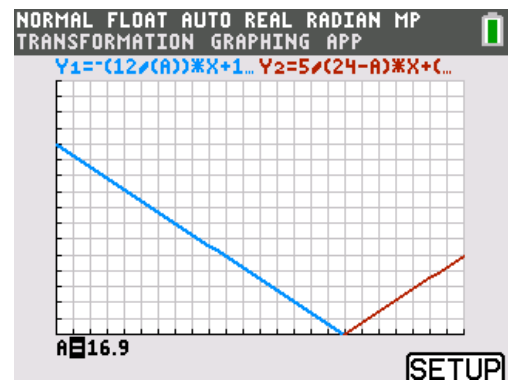
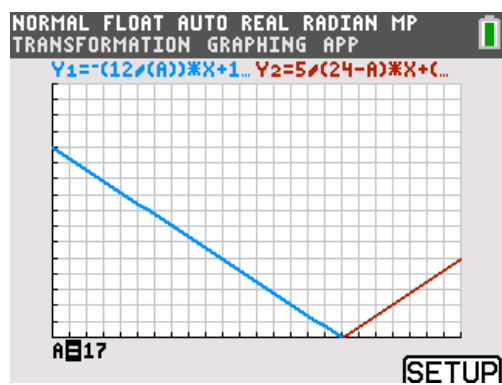
..... 16

Y4 29.41097731

A 17

Y4 29.44355733

A 18



NORMAL FLOAT AUTO REAL RADIAN MP
TRANSFORMATION GRAPHING APP

WINDOW SETTINGS

TrailOff TrailOn

▶▶▶ ▶▶▶

A=17

Step=0.1

GRAPH

NORMAL FLOAT AUTO REAL RADIAN MP

Y4	1/
A	29.44355733
Y4	18
A	29.41092862
Y4	16.9

HISTORY

Y4	29.41097731
A	17
Y4	29.44355733
A	18
Y4	29.41092862

"Mathematics is the garment that we continuously alter with our students and technology should be seamlessly interwoven throughout its fabric."

- Tom Reardon

Math teacher from Ohio