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# Invariant Projections on Cycle Spaces of Polyhedral Graphs An Orbit-Count Formula and a Characterisation of the Regular Solids

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| | from an attempt to give quantitative form to Curie's principle on symmetry and phenomena; §8.3 records briefly what the theorem does and does not settle there, and nothing in the mathematics depends on that reading. |

# Invariant Projections on Cycle Spaces of Polyhedral Graphs:

## An Orbit-Count Formula and a Characterisation of the Regular Solids

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**Abstract**

Dilworth, Kutzarova and Ostrovskii showed that the invariant projection of the edge space onto the cycle space of a graph is unique precisely when the natural representations of the automorphism group on the cycle space and the cut space have no equivalent constituents, and asked, as their Problem 7, for which graphs this holds. We answer the question for every convex polyhedron. For the skeleton $\Gamma$ of a convex polyhedron define the coupling capacity $c(\Gamma)$ as the dimension of the space of $\mathrm{Aut}(\Gamma)$-equivariant linear maps from the cycle space to the cut space, so that $c(\Gamma) = 0$ is exactly the uniqueness condition. We prove that c is an orbit count over face–vertex pairs, corrected by a term that is nonzero exactly for chiral polyhedra; that it is invariant under polyhedral duality; and, as the main theorem, that c vanishes if and only if the polyhedron is regular. The five Platonic solids are thereby characterised among all convex polyhedra by the vanishing of a homological invariant, and Problem 7 is answered in every case. The proof reduces the vanishing of the equivariant Hom space to a covering condition on mirror planes, which forces the vertices and face centres onto rotation axes, which in turn makes the question finite; a case analysis over the classification of finite subgroups of O(3) then closes it. Along the way we give a census of thirty-two polyhedra with every capacity computed twice by independent routes, an out-of-sample confirmation of the correction term, and a parity selection rule showing that in the centrally symmetric case the surviving channels are not forbidden but forced to be pseudoscalar. Finally we place the theorem: it is specific to the sphere. On the torus there are flag-transitive maps of positive capacity, so the forward direction fails; on the projective plane the hemi-cuboctahedron is irregular with vanishing capacity, so the converse fails. Each half is broken by a different surface, and the sphere is the only one of the three on which the two conditions coincide. A chiral sub-argument, by contrast, is genus-independent and verified on two hundred fifty-seven maps through genus 16; on the torus, where the reflexible square-lattice case is known [10, Thm. 24], we evaluate the chiral capacity in closed form, proving it to be $\lfloor (b^2 + c^2 - 1)/4 \rfloor$ for the square class and $\lfloor (b^2 + bc + c^2 - 1)/3 \rfloor$ for the triangular one, so the unbounded growth there is a theorem rather than an observation. The invariant arose from an attempt to give quantitative form to Curie's principle on symmetry and phenomena; §8.3 records briefly what the theorem does and does not settle there, and nothing in the mathematics depends on that reading.

**Keywords:** Cycle space; cut space; invariant projection; equivariant map; polyhedral graph; regular polyhedron; orbit counting; chirality; regular map; flag-transitivity; transportation cost space.

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## 1. Introduction

The problem this paper settles is a question about representations of graph automorphism groups, and we state it first; the motivation that led us to it is recounted afterwards, and nothing below depends on it. Let $\Gamma$ be a finite connected graph and let the signed edge space $\mathbb{R}$^E decompose orthogonally into the cycle space $H_1(\Gamma) = \ker \partial$ and the cut space $K(\Gamma) = \mathrm{im}\, \partial^{\mathrm{T}}$. Both are invariant under the natural action of $\mathrm{Aut}(\Gamma)$. Dilworth, Kutzarova and Ostrovskii proved that the invariant projection of the edge space onto the cycle space is unique if and only if the two resulting representations have no equivalent constituents, and posed as their Problem 7 the question of determining for which graphs this holds; they record that they found no published results on it [10]. Upon what we believe to be a reasonably exhaustive search of the publicly available literature and of the standard bibliographic databases, carried out in July 2026, we have located no subsequent treatment of the problem, under this terminology or any adjacent one. We claim no more than that: this is a negative search result and not a certificate of novelty. Theorem A below answers it for every convex polyhedron. Equivalently, since the coupling capacity $c(\Gamma)$ defined in §1.1 is the dimension of the space of equivariant maps between the two representations, we determine the vanishing locus of $c$ on that class.

In 1894 Pierre Curie set out what is now called Curie's principle. A phenomenon, he argued, can exist in a medium possessing its characteristic symmetry or that of one of the subgroups of that symmetry; certain symmetry elements may coexist with a phenomenon, but what is necessary is that some symmetry elements be absent. Asymmetry, on his formulation, is what creates a phenomenon [1].

The principle has proved durable and it has also proved slippery. Earman, having given a formulation of it, remarks that the formulation makes it virtually analytic, while maintaining that it remains useful as a guide in the search for explanations of observed asymmetries [2]. Ismael and, later, Castellani and Ismael have examined which of several inequivalent statements deserves the name [3,4]; Norton has argued that on its most defensible reading it is a truism [5]; Roberts has exhibited a failure in the time-reversal case [6]. The Stanford Encyclopedia treatment situates the principle relative to spontaneous symmetry breaking, where symmetric laws admit asymmetric solutions [7].

Grant the principle in whichever of these readings survives. A question remains that none of these discussions answers, because none of them is posed at the level where it could be answered: given a specific structure, how much asymmetry does it have available, and how much does a phenomenon cost? Curie’s principle is a constraint of the form "not zero." It is not a measurement. If asymmetry is the resource that phenomena are made from, the resource ought in principle to be countable, its count ought to depend on the structure in a computable way, and it ought to be possible to say exactly which structures have none.

This paper does that for one class of structures. The class is small — skeletons of convex polyhedra — but it is not artificial: it is the setting in which a symmetry group, a homology, and a natural pair of complementary sectors all coexist finitely, so that the relevant question is decidable rather than merely arguable. Within that class we define the resource exactly, compute it, and determine its vanishing locus completely.

### 1.1 Results

For the skeleton $\Gamma$ of a convex polyhedron, write $H_1(\Gamma)$ for its cycle space and $K(\Gamma)$ for its cut space, the two orthogonal complements into which the signed edge space decomposes, and define the coupling capacity

$$c(\Gamma) = \dim \mathrm{Hom}_{\mathrm{Aut}(\Gamma)}(H_1(\Gamma), K(\Gamma)),$$

the dimension of the space of linear channels between the two sectors that respect every symmetry of the structure. This quantity is not new in itself: by a theorem of Dilworth, Kutzarova and Ostrovskii [10], $c(\Gamma)$ measures exactly the failure of uniqueness of the invariant projection onto the cycle space, and determining it for a given graph is their Problem 7, which they pose and record as open. Our results are the following.

**Theorem A.** $c(\Gamma) = 0$ if and only if $\Gamma$ is regular. Equivalently, the orthogonal projection onto the cycle space is the unique $\mathrm{Aut}(\Gamma)$-invariant projection precisely for the five Platonic solids.

**Theorem B.** $c(\Gamma) = N(F \times V) - N(F) - N(V) + \delta$, where $N(X)$ counts the $\mathrm{Aut}(\Gamma)$-orbits on $X$ whose stabiliser contains no orientation-reversing element and $\delta$ is 1 for chiral $\Gamma$ and 0 otherwise.

**Theorem C.** $c(\Gamma^*) = c(\Gamma)$ for the dual polyhedron $\Gamma^*$, although the two generally have different Betti numbers.

Theorem A is the sharp form of the quantitative reading of Curie we were after. Read one way it says that a structure can support a symmetry-respecting interaction between its two sectors exactly when it has fallen short of maximal symmetry; read the other way, it characterises the five Platonic solids — objects usually singled out by a definition about flags, or faces, or angles — by the vanishing of a homological invariant. Theorem B is what makes the capacity computable, replacing a linear-algebra problem on a space of dimension $\beta_1(|V| - 1)$ by a count of orbits on face–vertex pairs. Theorem C is not visible from the modules themselves and does substantial work in the proof of Theorem A.

**Corollary D.** Problem 7 of [10] is answered for every convex polyhedron: the invariant projection onto the cycle space is unique exactly in the five Platonic cases. The problem is posed there for the automorphism group of the graph; for a 3-connected planar graph that group coincides with the map automorphism group used here, by Whitney's theorem, so the answer is unambiguous. Section 10.2 records a toroidal map for which the two groups differ, and the distinction becomes material.

### 1.2 How the proof goes

The forward half of Theorem A is a computation. Section 3 identifies the two sectors as twisted permutation modules — the cycle space is the reduced face module tensored with the orientation character, the cut space the reduced vertex module — and Theorem B follows from the Cauchy–Frobenius orbit-counting lemma in its $\varepsilon$-twisted form. A finite check over the five regular solids then gives regular $\Rightarrow c = 0$.

The converse occupies §9 and runs in three stages. First, §9.3 reduces the vanishing of the equivariant Hom space to a statement with no representation theory in it at all: $c(\Gamma) = 0$ if and only if every face–vertex pair of $\Gamma$ is fixed by a common reflection, equivalently if and only if, for every face, the mirror planes through that face's axis cover the entire vertex set. Second, §9.4 observes that this forces every face axis and every vertex direction onto a rotation axis — since a single mirror would confine the vertex set to a plane, and the vertices of a convex body span space. Since a finite group has only finitely many rotation axes, the problem becomes finite. Third, §§9.6–9.7 carry out the resulting case analysis over the classification of finite subgroups of $O(3)$. Most of the classes are eliminated outright because their mirror arrangements are too sparse; the tetrahedral, octahedral and icosahedral cases require an enumeration of candidate vertex sets, which is finite and small.

Sections 5 and 6 are illustrative rather than evidential. They record a census of thirty-two polyhedra — all five Platonic solids, all thirteen Archimedean solids, and members of the prism and antiprism families — with every capacity computed twice by independent routes. The census was originally the empirical core of this work, before Theorem A was available; it now serves as a set of worked examples, as a check on the formula of Theorem B, and as the source of the closed forms and the out-of-sample test reported there. Section 7 identifies the mechanism behind the vanishing in the centrally symmetric case, which turns out to be a parity selection rule rather than a no-go.

### 1.3 Hypotheses, and their status

Because this work began as an empirical study and the design of the census still reflects that, we state the hypotheses it was built to discriminate, together with what has since become of them.

> **$H_0$ (null).** The coupling capacity is not governed by the symmetry class of $\Gamma$: the vanishing of $c$, where it occurs, is explained by the dimensions of the sectors and their parity splittings rather than by flag-transitivity, and $c$ is not systematically distinguished by regularity among structures of comparable size and group order.
>
> **$H_1$ (alternative).** $c(\Gamma) = 0$ if and only if $\Gamma$ is regular.

$H_1$ is Theorem A and is proved. $H_0$ is false, and it is worth saying how it fails rather than merely that it does, because its first clause is a live rival that the census was designed to test. The $\iota$-parity dimension pattern which produces the vanishing on the dodecahedron — the two sectors splitting complementarily under the central involution — is present on many polyhedra whose capacity is large. Dimensions do not decide the question; the isotypic assignment does. Anyone who had looked only at the dimension counts would have drawn the opposite of the correct conclusion, which is why the instrument in Theorem B is an orbit count and not a dimension count.

## 2. Setup

Let $\Gamma$ be the skeleton (vertex-and-edge graph) of a convex polyhedron, with vertex set V, edge set E and face set F, so that $|V| - |E| + |F| = 2$. Such a graph is simple, planar and 3-connected; by Steinitz's theorem the converse holds, and by Whitney's theorem [8] the planar embedding is unique up to reflection, so the face set is determined by the graph. By Mani's theorem [9] the combinatorial automorphism group is realised by isometries of a suitably chosen realisation, so we may and do identify $\mathrm{Aut}(\Gamma)$ with a finite subgroup of $O(3)$.

Fix an arbitrary orientation of each edge and let $C_1 = \mathbb{R}^E$ be the resulting signed edge space, on which $\mathrm{Aut}(\Gamma)$ acts by signed permutations. Let $\partial : C_1 \to \mathbb{R}^V$ be the boundary map. Define

$$H_1(\Gamma) = \ker \partial \qquad \text{(cycle space, dim} = \beta_1 = |E| - |V| + 1),$$

$$K(\Gamma) = \operatorname{im} \partial^T = H_1(\Gamma)^\perp \qquad \text{(cut space, dim} = |V| - 1).$$

Both are $\mathrm{Aut}(\Gamma)$-submodules and $C_1 = H_1 \oplus K$ orthogonally. Write $R \subseteq \mathrm{Aut}(\Gamma)$ for the rotation subgroup and $\varepsilon : \mathrm{Aut}(\Gamma) \to \{\pm 1\}$ for the orientation character, $\varepsilon(g) = \det(g)$; $\Gamma$ is called reflexible if $\varepsilon$ is surjective and chiral if $\varepsilon \equiv 1$.

**Definition 2.1 (coupling capacity).** $c(\Gamma) := \dim \mathrm{Hom}_{\mathrm{Aut}(\Gamma)}(H_1(\Gamma), K(\Gamma))$.

Three remarks justify this as the object of interest. First, $\mathrm{Hom}_{\mathrm{Aut}}(H_1, K)$ is exactly the space of linear channels between the two sectors that every symmetry of the structure respects; a nonzero element is a way for cycle-sector data to influence cut-sector data without breaking any symmetry. Second, $c(\Gamma)$ has an independent and pre-existing meaning: Dilworth, Kutzarova and Ostrovskii [10], studying minimal projections onto cycle spaces in connection with transportation-cost and Lipschitz-free spaces, show that for a 3-connected graph the invariant projection onto the cycle space is unique precisely when the automorphism representations on the cycle and cut spaces have no equivalent restrictions to nonzero subspaces — that is, precisely when $c(\Gamma) = 0$. Their Problem 7 asks for this determination for given graphs, and they record that they were unable to find the required representation theory of $\mathrm{Aut}(\Gamma)$ in the literature. The census below therefore answers their problem, affirmatively for five graphs and negatively for twenty-seven. Third, $c(\Gamma)$ is manifestly zero or positive and manifestly finite, so "how much asymmetry is available" has, in this setting, a literal answer.

One convention needs checking. The isometry group DKO consider is the group of cycle-preserving bijections of the edge set, which by their Lemma 16 and Remark 17 equals $\mathrm{Aut}(\Gamma) \times \{\pm 1\}$ for 3-connected graphs, the global $-1$ acting as $-1$ on both sectors and hence leaving the equivalence of restrictions unaffected. Aut-equivalence and their notion coincide, and every graph considered here is 3-connected.

## 3. The reduction theorem

The computation of $c(\Gamma)$ as stated requires diagonalising an action on a space of dimension $\beta_1(|V|-1)$, which for the larger Archimedean solids exceeds ten thousand. The following reduction replaces it with a count of orbits on face–vertex pairs.

**Lemma 3.1.** As $\mathrm{Aut}(\Gamma)$-modules, $H_1(\Gamma) \cong (\mathbb{R}^F \ominus 1) \otimes \varepsilon$ and $K(\Gamma) \cong \mathbb{R}^V \ominus 1$, where $\mathbb{R}^F$ and $\mathbb{R}^V$ are the face and vertex permutation modules and 1 is the trivial module.

*Proof.* For K, the map $\partial^{\mathrm{T}} : \mathbb{R}^V \to C_1$ is equivariant with kernel the constants (Γ connected) and image K. For $H_1$: the cycle space of a planar graph is the image of the boundary map $\partial_2$ from the space of oriented faces, whose kernel is the fundamental class of the sphere and therefore carries the orientation character. Hence $H_1 \cong C_2 / \ker \partial_2$ where $C_2$ is the oriented-face space. An automorphism fixing a face reverses that face's orientation exactly when it reverses the orientation of the sphere, so $C_2 \cong \mathbb{R}^F \otimes \varepsilon$, and quotienting by the ε-line gives the statement. ■

Lemma 3.1 was verified independently on every graph in the census by comparing the character of $H_1$ computed from the signed edge action against $\varepsilon(g)(\mathrm{fix}_F(g) - 1)$, element by element across the whole group; the two agree exactly in all thirty-two cases.

Write N(X) for the number of Aut(Γ)-orbits on a set X whose stabiliser contains no orientation-reversing element.

**Theorem 3.2 (reduction).** For the skeleton Γ of a convex polyhedron,

$$c(\Gamma) = N(F \times V) - N(F) - N(V) + \delta,$$

where $\delta = \langle 1, \varepsilon \rangle$ equals 1 if Aut(Γ) contains no orientation-reversing automorphism and 0 otherwise.

*Proof.* Since the modules are real and the group finite, $c(\Gamma) = \langle \chi_{H_1}, \chi_K \rangle$. By Lemma 3.1 this is $\langle (\chi_F - 1)\varepsilon, \chi_V - 1 \rangle$, which expands to $\langle \chi_F \chi_V \varepsilon, 1 \rangle - \langle \chi_F \varepsilon, 1 \rangle - \langle \chi_V \varepsilon, 1 \rangle + \langle \varepsilon, 1 \rangle$. Each of the first three terms is a sum over orbits: for a transitive G-set G/S, Frobenius reciprocity gives $\langle \mathrm{Ind}_S^G 1, \varepsilon \rangle = \langle 1, \varepsilon|_S \rangle$, which is 1 when S contains no orientation-reversing element and 0 otherwise. Summing orbit by orbit turns the three terms into N(F × V), N(F) and N(V). The fourth term is $\langle \varepsilon, 1 \rangle = \delta$. ■

**Remark 3.3 (the δ term, and an error it corrects).** An earlier draft of this material omitted δ, on the ground that when Aut(Γ) admits no orientation-reversing element one may take $\varepsilon \equiv 1$ and the identity holds trivially. That reasoning is wrong: with $\varepsilon \equiv 1$ the inner product $\langle \varepsilon, 1 \rangle$ equals 1, not 0, and the formula is off by exactly one on every chiral polyhedron. The error was found by attempting to prove the converse of Conjecture 9.1 and constructing the snub cube as a test case. It is recorded here rather than silently repaired because the correction is the only place in the argument where chirality enters the arithmetic rather than the interpretation, and a reader checking the reflexible cases alone would never encounter it.

**Corollary 3.4 (duality invariance).** If Γ* is the skeleton of the dual polyhedron, then $c(\Gamma^*) = c(\Gamma)$.

*Proof.* Duality preserves Aut and ε while exchanging the roles of F and V. The expression N(F × V) − N(F) − N(V) + δ is symmetric under that exchange. ■

This is not visible from the modules themselves — $\beta_1$ generally differs between a polyhedron and its dual — and it has a methodological consequence: the Catalan solids require no separate computation, so a census of Archimedean solids extends to their duals free of charge. We nonetheless verified the corollary by explicit independent computation on seven dual pairs (Table 2), including the chiral pair, which is the case where the δ term is active.

**Attribution.** The untwisted case of the counting step — that $\dim \mathrm{Hom}_G(\mathbb{R}^X, \mathbb{R}^Y)$ equals the number of G-orbits on X × Y — is the Cauchy–Frobenius–Burnside orbit-counting lemma, standard and long predating its usual attribution [11,12]. Theorem 3.2 is the ε-twisted refinement of that lemma applied to a specific pair of modules, and the twisted form follows from Frobenius reciprocity in one line, as above. We claim no novelty for the mechanism; the contribution is the identification of the relevant pair of modules (Lemma 3.1), the appearance of δ, and the census. A reader who finds Theorem 3.2 stated elsewhere should treat the proof above as expository.

## 4. Regularity, flags, and the forward direction

Theorem B makes $c(\Gamma)$ computable. To state the forward half of Theorem A we need the notion of regularity in the form the proof uses, which is flag-transitivity.

A flag of $\Gamma$ is a triple $(v, e, f)$ with $v$ an endpoint of the edge $e$ and $e$ a side of the face $f$, and there are $4|E|$ of them. Define the flag-transitivity ratio

$$r(\Gamma) = |\mathrm{Aut}(\Gamma)| / 4|E|,$$

which satisfies $r \le 1$ for every convex polyhedron, with equality if and only if $\mathrm{Aut}(\Gamma)$ acts transitively on the $4|E|$ flags, i.e. if and only if $\Gamma$ is regular. $H_1$ predicts that $c$ vanishes exactly on the fibre $r = 1$ and nowhere else. $H_0(a)$ predicts no such alignment.

**Lemma 4.1.** $\mathrm{Aut}(\Gamma)$ acts freely on the set of $4|E|$ flags of $\Gamma$. Consequently $|\mathrm{Aut}(\Gamma)|$ divides $4|E|$, the quotient $q = 4|E|/|\mathrm{Aut}(\Gamma)| = 1/r$ is the number of flag orbits, and $r \le 1$ with equality exactly for regular $\Gamma$.

*Proof.* An automorphism fixing a flag $(v, e, f)$ fixes $v$, fixes the second endpoint of $e$, and fixes $f$ setwise while fixing two of its vertices; walking around $f$ it fixes every vertex of $f$, hence fixes the plane of $f$ pointwise, hence is the identity or the reflection in that plane. The latter is not a symmetry of a convex polyhedron, since it would carry interior points across the face plane to the empty side. ■

This was verified computationally for all thirty-two polyhedra in the census: in every case every flag stabiliser is trivial and $|\mathrm{Aut}(\Gamma)|$ divides $4|E|$ exactly. The integer $q$ is used throughout §9.

The forward half of Theorem A now follows: for each of the five regular solids the orbit count of Theorem B vanishes, as recorded in Table 1. Three design choices in the census that follows are worth stating, because each was chosen to rule out a cheaper explanation of that fact. First, the census includes all thirteen Archimedean solids: these are exactly the vertex-transitive-but-not-flag-transitive convex polyhedra, so they separate vertex-transitivity from regularity directly. Second, it includes prism and antiprism families through several members, so that a size-driven trend would be visible as such. Third, it includes both chiral solids, the only ones on which the $\delta$ term is active and so the only ones on which the formula could fail while remaining correct everywhere else.

## 5. Worked examples: the Platonic and Archimedean solids

The census below predates Theorem A and was assembled to test it. It now serves three narrower purposes: it exhibits the capacity on named solids, it checks the orbit-count formula of Theorem B against direct computation, and it supplies the closed forms of Proposition 5.1 and the out-of-sample test of §6. Table 1 reports thirty-two polyhedra: all five Platonic solids, all thirteen Archimedean solids, eight prisms and six antiprisms. For each we give the combinatorial data, the order of the automorphism group, the flag-transitivity ratio $r$, the value of $\delta$, and the coupling capacity. Every entry was computed twice by independent routes — once as the character inner product $\langle\chi\_\{H_1\}, \chi\_K\rangle$ evaluated over the full automorphism group from the signed edge action, and once by the orbit count of Theorem 3.2 — and the two agree in every case. Coordinates were generated from standard constructions; the two chiral solids were generated both by numerical optimisation over free orbits of the rotation group and, for the snub cube, from exact tribonacci coordinates, with identical results.

| Polyhedron | V | E | F | $\beta_1$ | \|Aut\| | r | δ | c(Γ) |
|---|---|---|---|---|---|---|---|---|
| tetrahedron | 4 | 6 | 4 | 3 | 24 | 1 | 0 | 0 |
| cube | 8 | 12 | 6 | 5 | 48 | 1 | 0 | 0 |
| octahedron | 6 | 12 | 8 | 7 | 48 | 1 | 0 | 0 |

| Polyhedron | V | E | F | $\beta_1$ | \|Aut\| | r | δ | c(Γ) |
|---|---|---|---|---|---|---|---|---|
| dodecahedron | 20 | 30 | 12 | 11 | 120 | 1 | 0 | 0 |
| icosahedron | 12 | 30 | 20 | 19 | 120 | 1 | 0 | 0 |
| truncated tetrahedron | 12 | 18 | 8 | 7 | 24 | 0.3333 | 0 | 2 |
| cuboctahedron | 12 | 24 | 14 | 13 | 48 | 0.5 | 0 | 1 |
| truncated cube | 24 | 36 | 14 | 13 | 48 | 0.3333 | 0 | 4 |
| truncated octahedron | 24 | 36 | 14 | 13 | 48 | 0.3333 | 0 | 5 |
| rhombicuboctahedron | 24 | 48 | 26 | 25 | 48 | 0.25 | 0 | 9 |
| truncated cuboctahedron | 48 | 72 | 26 | 25 | 48 | 0.1667 | 0 | 25 |
| snub cube | 24 | 60 | 38 | 37 | 24 | 0.1 | 1 | 35 |
| icosidodecahedron | 30 | 60 | 32 | 31 | 120 | 0.5 | 0 | 4 |
| truncated dodecahedron | 60 | 90 | 32 | 31 | 120 | 0.3333 | 0 | 12 |
| truncated icosahedron | 60 | 90 | 32 | 31 | 120 | 0.3333 | 0 | 12 |
| rhombicosidodecahedron | 60 | 120 | 62 | 61 | 120 | 0.25 | 0 | 25 |
| truncated icosidodecahedron | 120 | 180 | 62 | 61 | 120 | 0.1667 | 0 | 61 |
| snub dodecahedron | 60 | 150 | 92 | 91 | 60 | 0.1 | 1 | 89 |
| 3-gonal prism | 6 | 9 | 5 | 4 | 12 | 0.3333 | 0 | 1 |
| 5-gonal prism | 10 | 15 | 7 | 6 | 20 | 0.3333 | 0 | 2 |
| 6-gonal prism | 12 | 18 | 8 | 7 | 24 | 0.3333 | 0 | 3 |
| 7-gonal prism | 14 | 21 | 9 | 8 | 28 | 0.3333 | 0 | 3 |
| 8-gonal prism | 16 | 24 | 10 | 9 | 32 | 0.3333 | 0 | 4 |
| 9-gonal prism | 18 | 27 | 11 | 10 | 36 | 0.3333 | 0 | 4 |
| 10-gonal prism | 20 | 30 | 12 | 11 | 40 | 0.3333 | 0 | 5 |
| 12-gonal prism | 24 | 36 | 14 | 13 | 48 | 0.3333 | 0 | 6 |
| 4-gonal antiprism | 8 | 16 | 10 | 9 | 16 | 0.25 | 0 | 3 |
| 5-gonal antiprism | 10 | 20 | 12 | 11 | 20 | 0.25 | 0 | 4 |
| 6-gonal antiprism | 12 | 24 | 14 | 13 | 24 | 0.25 | 0 | 5 |
| 7-gonal antiprism | 14 | 28 | 16 | 15 | 28 | 0.25 | 0 | 6 |
| 8-gonal antiprism | 16 | 32 | 18 | 17 | 32 | 0.25 | 0 | 7 |
| 9-gonal antiprism | 18 | 36 | 20 | 19 | 36 | 0.25 | 0 | 8 |

Table 1. Coupling capacity across thirty-two convex polyhedra. Shaded rows are those with $c(\Gamma) = 0$. The vanishing set is exactly the set of rows with $r = 1$, i.e. exactly the five Platonic solids. Note that the 4-gonal prism is the cube and the 3-gonal antiprism is the octahedron; these are listed once, under their Platonic names.

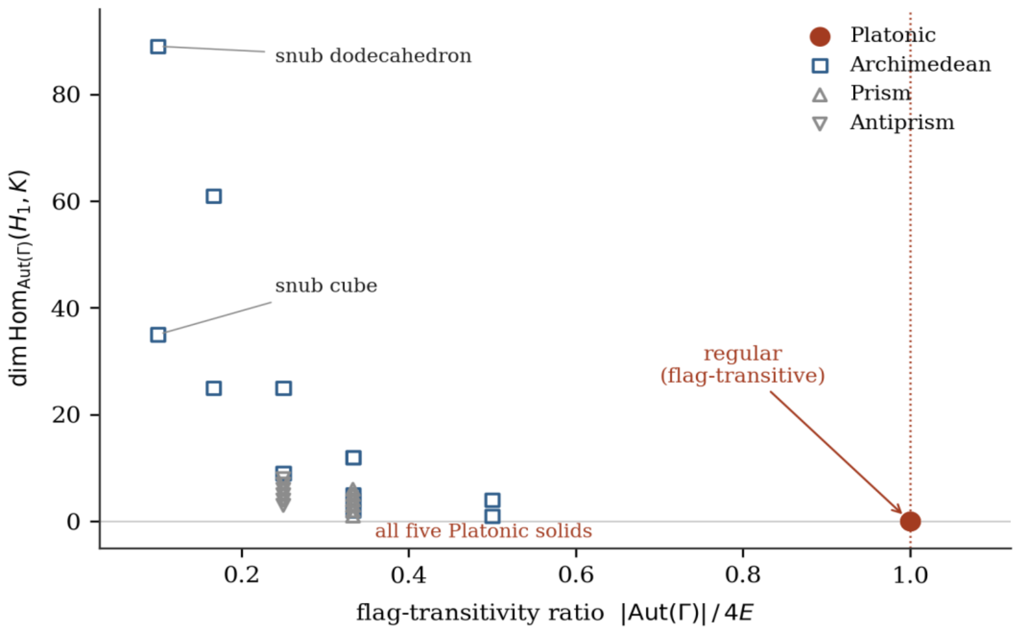


Figure 1. Coupling capacity against the flag-transitivity ratio $r = |\mathrm{Aut}(\Gamma)|/4E$. The capacity vanishes on the fibre $r = 1$ and at no other point in the census. Vertex-transitivity is decisively excluded as the operative property: the thirteen Archimedean solids are all vertex-transitive and all have positive capacity, ranging from 1 (cuboctahedron) to 89 (snub dodecahedron).

The table is an instance of Theorem A and also a check on it: the capacity vanishes for exactly the five solids with $r = 1$. It is worth recording what it rules out independently of the proof. The cuboctahedron and the icosidodecahedron have the same automorphism group orders as the cube and the dodecahedron respectively, yet differ in capacity by 1 and 4, so neither size nor group order accounts for the pattern; and the parity dimension pattern is shared by vanishing and non-vanishing cases alike, so it cannot be what distinguishes them either.

The families make the same point in continuous form.

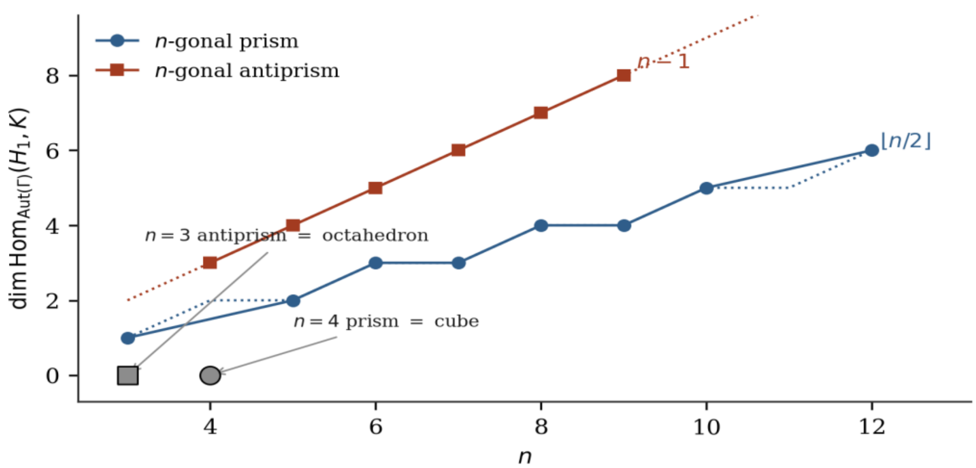


Figure 2. Prism and antiprism families. The capacity grows linearly in the family parameter, following $\lfloor n/2 \rfloor$ for prisms and $n - 1$ for antiprisms — except at exactly one member of each family, where the polyhedron happens to be regular and the capacity collapses to zero. The 4-gonal prism is the cube; the 3-gonal antiprism is the octahedron.

**Proposition 5.1.** For the n-gonal prism with $n \neq 4$, $c = \lfloor n/2 \rfloor$. For the n-gonal antiprism with $n \neq 3$, $c = n - 1$. At $n = 4$ and $n = 3$ respectively the polyhedron is regular and $c = 0$.

Verified for prisms $n \in \{3,5,6,7,8,9,10,12\}$ and antiprisms $n \in \{4,5,6,7,8,9\}$; the closed forms follow from Theorem 3.2 by counting the reflection-free face–vertex orbits, and are stated here as an observation supported by computation rather than as a theorem with a written proof. The discontinuity is the point: the capacity does not approach zero as the family approaches its regular member, it is zero at that member and at no neighbour.

### 5.1 Duality

Corollary 3.4 predicts equality of capacity across dual pairs despite differing homology. Table 2 records seven independent verifications, each computed from scratch on an explicitly constructed dual polyhedron rather than inferred from the formula.

| Primal | $V/E/\beta_1$ | Dual | $V/E/\beta_1$ | $c(\Gamma)$ | $c(\Gamma^*)$ | equal |
|---|---|---|---|---|---|---|
| dodecahedron | 20/30/11 | icosahedron | 12/30/19 | 0 | 0 | yes |
| cuboctahedron | 12/24/13 | rhombic dodecahedron | 14/24/11 | 1 | 1 | yes |
| truncated cube | 24/36/13 | triakis octahedron | 14/36/23 | 4 | 4 | yes |
| snub cube | 24/60/37 | pentagonal icositetrahedron | 38/60/23 | 35 | 35 | yes |
| icosidodecahedron | 30/60/31 | rhombic triacontahedron | 32/60/29 | 4 | 4 | yes |
| truncated icosahedron | 60/90/31 | pentakis dodecahedron | 32/90/59 | 12 | 12 | yes |
| 6-gonal prism | 12/18/7 | hexagonal bipyramid | 8/18/11 | 3 | 3 | yes |

Table 2. Duality invariance of the coupling capacity, verified independently on seven pairs. The Betti numbers differ within every pair; the capacities do not. The final row is the chiral case, in which the δ term is active on both sides.

Two consequences follow. First, the census extends without further computation to the Catalan solids and to the bipyramids, so the effective coverage is considerably larger than thirty-two. Second — and this is a structural argument rather than a computational one — any property that characterises the vanishing must itself be dual-invariant. Regularity is dual-invariant; vertex-transitivity is not, since the dual of a vertex-transitive polyhedron is face-transitive. Vertex-transitivity is therefore excluded on structural grounds independently of the thirteen Archimedean computations, which is a second and logically independent argument for the same conclusion.

## 6. An out-of-sample test of the correction term

The δ term of Theorem 3.2 was discovered by constructing a case that broke the uncorrected formula. A term found that way is fitted to its witness, and fitting a single case is not verification. We therefore designed a prediction on a case not used in the discovery.

The snub dodecahedron is the second and last chiral Archimedean solid: sixty vertices, one hundred fifty edges, ninety-two faces, automorphism group of order sixty containing no improper element. Its rotation group acts simply transitively on the vertices and with three orbits on the faces (twelve pentagons, twenty triangles meeting three triangles, sixty remaining triangles), and since no stabiliser can contain an orientation-reversing element, $N(F \times V) = 92$, $N(F) = 3$, $N(V) = 1$ and $\delta = 1$. The corrected formula therefore predicts $c = 92 - 3 - 1 + 1 = 89$, and the uncorrected formula predicts 88.

The prediction of 89 was recorded before the computation was run. Direct evaluation of the character inner product over the full automorphism group returned 89 exactly. The uncorrected formula is wrong by one, in the same direction and by the same amount as on the snub cube. As a further control, the snub cube was rebuilt from exact tribonacci coordinates, reproducing $c = 35$ against an uncorrected 34; and its polar dual, the pentagonal icositetrahedron ($V = 38$, $E = 60$, $\beta_1 = 23$, twenty-four pentagonal faces), was constructed and computed independently, returning 35 and thereby confirming Corollary 3.4 in the one case where δ contributes.

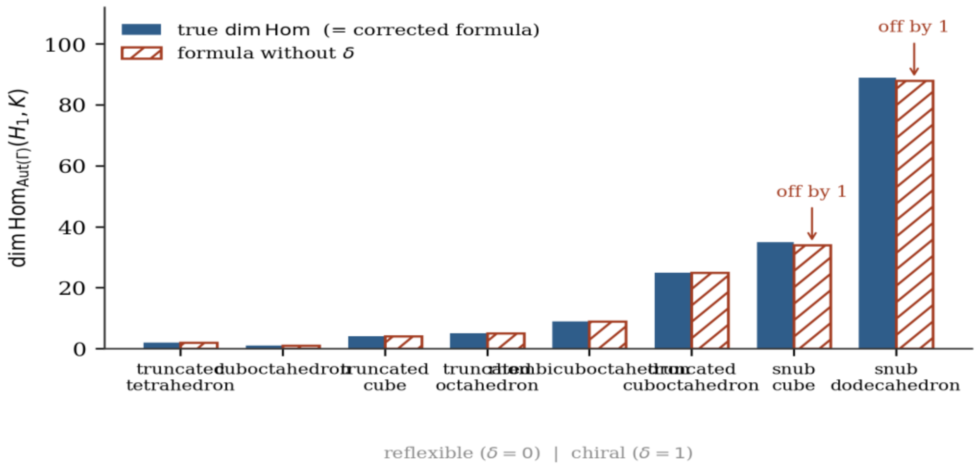


Figure 3. The correction term against data. For reflexible polyhedra (δ = 0) the corrected and uncorrected formulae agree and both are right. For the two chiral solids the uncorrected formula is wrong by exactly one in each case. The snub cube is the witness on which δ was discovered; the snub dodecahedron is the out-of-sample confirmation.

This is a modest test — one prediction, integer-valued, on a single object — but it is the right shape of test, and it was available to fail.

## 7. Mechanism: a parity selection rule

Theorem 3.2 tells us the capacity but not why it vanishes. In the reflexible case there is a clean mechanism, which we illustrate on the dodecahedron because that is where it is sharpest.

Aut(Γ) for the dodecahedron is $A_5 \times \mathbb{Z}_2$, of order 120, with central involution ι acting as the antipodal map. Under ι the cycle space splits 6 : 5 and the cut space splits 9 : 10. Under the rotation subgroup $A_5$ alone the shared constituents are the two three-dimensional irreducibles and the five-dimensional one, giving dim Hom_{$A_5$}($H_1$, K) = 3. But every shared constituent that is ι-even in one sector is ι-odd in the other. An intertwiner B supported on such a channel satisfies Bι = B on the source and ιB = −B on the target, hence Bι = −ιB identically. The three channels therefore survive as $A_5$-intertwiners and are extinguished as Aut-intertwiners.

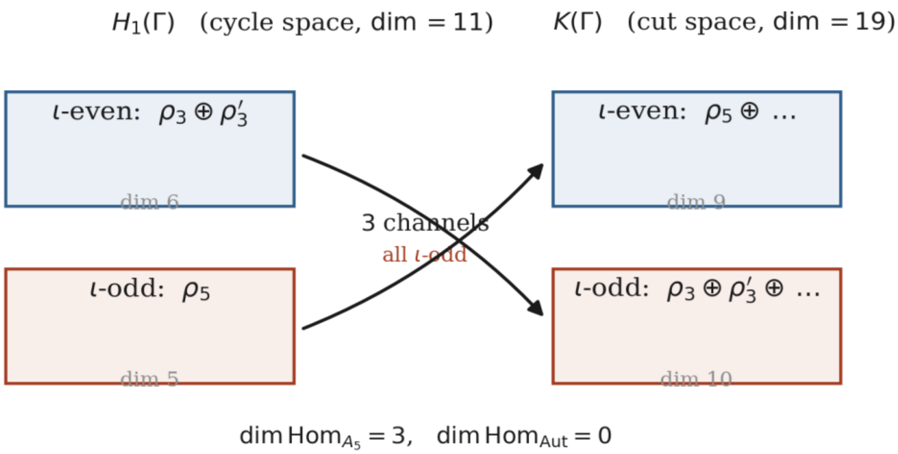


Figure 4. The parity selection rule on the dodecahedron. Every constituent shared between the cycle and cut spaces carries opposite parity under the central involution. The three rotation-equivariant channels therefore all anticommute with ι, and none is equivariant under the full automorphism group. The coupling is not forbidden; it is forced to be parity-odd.

We emphasise the reading, because the natural one is wrong. The vanishing of Hom_{Aut} is usually described as a no-go: the sectors cannot talk. The parity computation says something stronger and more useful. The channels exist — there are exactly three of them — and what the symmetry does is not close them but fix their transformation law. Every coupling between the two sectors is pseudoscalar. A structure that has broken orientation has three channels available; a structure that has not, has none. The capacity is not destroyed by symmetry, it is made conditional on a frame.

**Proposition 7.1.** Let $\mathrm{Aut}(\Gamma) = R \times \langle\iota\rangle$ with ι a central fixed-point-free involution. If every irreducible constituent shared between $H_1$ and K carries opposite ι-parity in the two sectors, then $c(\Gamma) = 0$.

The hypothesis is weaker than geometric centrosymmetry — it requires only that ι be central and fixed-point-free, a condition satisfied for instance by the triangular and pentagonal prisms, which have no inversion centre in $\mathbb{R}^3$. The condition is sufficient but not necessary: the tetrahedron has no central fixed-point-free involution at all, so Proposition 7.1 does not apply to it, yet $c = 0$ by Theorem 3.2. The orbit count is the general instrument; the parity rule is the mechanism in the cases where there is one.

The structure of this phenomenon is not unfamiliar in physics. In the quantum-information treatment of reference frames, a party lacking a shared frame is subject to superselection rules that forbid exactly those operations which would require the missing frame; acquiring the frame lifts the restriction and opens the channel [13]. The formal situation here is the same in outline: full symmetry is the absence of a frame, the vanishing Hom space is the superselection rule, and breaking the symmetry is the acquisition of the frame that makes the channel available. We record the analogy as an analogy. Nothing in this paper derives the quantum case from the polyhedral one or vice versa.

## 8. What this establishes, and what it does not

Theorem A is a clean statement, and clean statements invite overreading. We would rather do the refutation ourselves than have it done for us.

### 8.1 Established

Within the class of skeletons of convex polyhedra: the coupling capacity is exactly an orbit count (Theorem 3.2, proved); it is invariant under polyhedral duality (Corollary 3.4, proved and independently verified on seven pairs); it vanishes on all five Platonic solids and on none of twenty-seven other tested polyhedra including all thirteen Archimedean solids (computed twice by independent routes); vertex-transitivity is excluded as the operative property both computationally and by a duality argument; and in the reflexible centrally-symmetric case the mechanism is a parity selection rule that forces every surviving channel to be pseudoscalar (Proposition 7.1, proved). On the torus the capacity of every chiral member of the two standard classes is given in closed form (Propositions 10.6 and 10.7, both proved). The converse of the regularity criterion is proved for all chiral polyhedra (Proposition 9.4), for every polyhedron with a face axis or vertex direction on fewer than two mirrors (Proposition 9.11, which also bounds the vertex count of any candidate by a constant depending only on the symmetry group), by the two-axis bound of §9.5 for every irregular member of the census, and — by the case analysis of §§9.6–9.7 over the finite subgroups of O(3) — in general: $c(\Gamma) = 0$ if and only if $\Gamma$ is regular (Theorem 9.23).

Theorem A thereby answers Problem 7 of Dilworth, Kutzarova and Ostrovskii [10] for every convex polyhedron, determining in each case whether the orthogonal projection onto the cycle space is the unique invariant projection; the census of §5 exhibits thirty-two of those answers explicitly. What this licenses by way of a reading of Curie's principle, and what it does not, is set out in §8.3.

### 8.2 Not established

This is a result about finite structures and their symmetry groups. It is not a theory of space, and the polyhedron is not a model of spacetime. Nothing here supplies dynamics: there is no time parameter, no update rule, and no derived observable. The identification of "coupling capacity" with anything physical is an interpretation, not a consequence, and this paper offers no empirical prediction.

In particular, we do not claim to have derived Curie's principle, nor to have shown that it holds beyond this class, nor that the quantity $c(\Gamma)$ is the correct measure of "available asymmetry" for any structure that is not a convex polyhedron. What we claim is narrower and, we think, more useful: that on a class where the question is decidable, the qualitative principle admits an exact quantitative form, that form is an orbit count, and its vanishing locus is precisely the maximally symmetric locus. Whether that is a general fact about symmetry and interaction, or a fact about polyhedra, is exactly the open question.

We also decline a suggestive but unearned reading. It is tempting to say that the perfectly symmetric configuration is "unobservable" or "unoccupiable" because it supports no coupling, and thence that the physical world is confined to the asymmetric complement. That inference requires premises this paper does not supply: that coupling capacity is necessary for observation, that observation is necessary for physical instantiation, and that the polyhedral setting is representative. Each is a substantive claim and none is established here. The regular polyhedra exist; what has been shown is that they have zero capacity in a precise and narrow sense, not that they are excluded from anything.

### 8.3 The motivating principle

Theorem A began as an attempt to give quantitative form to Curie's principle, and a reader may reasonably ask what the theorem leaves of that reading. Two features are worth isolating here, since they are facts about $c$ rather than interpretations of it. The characterisation is a biconditional, so the vanishing locus is determined and not merely constrained; and by §9.1 the capacity is not bounded below by any of the natural measures of symmetry deficiency, so only the endpoint statement survives and no graded version of it is available. The wider discussion of what such a result does and does not establish about the principle, and of the general question of which symmetry arguments survive displacement from their maximally symmetric configuration, is pursued in [18] and is not needed for anything proved below.

## 9. The converse, proved

Theorem 3.2 together with the finite check of §4 proves the forward half of Theorem A: regular $\Longrightarrow c = 0$. This section proves the converse. We report, in order, two approaches that fail, a complete proof for chiral polyhedra, a reduction of the remaining case to a statement in elementary solid geometry, and a case analysis over the finite subgroups of $O(3)$ that closes it.

**Conjecture 9.1 (proved below, as Theorem 9.23).** For the skeleton of a convex polyhedron, $c(\Gamma) = 0$ if and only if $\Gamma$ is regular.

### 9.1 Two routes that fail

**Remark 9.2 (capacity is not bounded below by flag deficiency).** Since $q = 1/r$ counts flag orbits and $q = 1$ characterises regularity, the natural first guess is that each additional flag orbit contributes at least one unit of capacity, that is $c \geq q - 1$. This is false. The triangular prism has $|\mathrm{Aut}| = 12$ and $4E = 36$, hence $q = 3$, yet $c = 1$. The guess fails in the other direction too: the eight prisms in the census all have $q = 3$ while their capacities range from 1 to 6. Flag deficiency alone determines neither the value nor a useful bound on it.

**Remark 9.3 (the incident-pair route is closed).** A second natural approach works with incident face–vertex pairs, where the flag structure is directly available. Let $I \subseteq F \times V$ be the set of incident pairs, so that $|I| = 2|E|$, each incident pair lying under exactly two flags. By Lemma 4.1 the stabiliser of an incident pair embeds in the symmetric group on those two flags and so has order 1 or 2. A generator of an order-2 stabiliser fixes the face centroid and the vertex, hence fixes two independent directions, hence is a reflection by Lemma 9.5; it is never a rotation, since a rotation fixing two non-antipodal points of the sphere is the identity. Writing a for the number of free orbits on I and b for the number with reflection stabiliser, orbit counting gives $2a + b = q$, and $N(I) = a$ exactly.

This is a clean picture but not a theorem. For the cuboctahedron and the icosidodecahedron, $q = 2$ with $(a, b) = (0, 2)$: there is no reflection-free orbit on incident pairs at all, so $N(I) = 0$. Yet $c = 1$ and $c = 4$ respectively. The entire capacity of these two solids is carried by face–vertex pairs that are not incident, which the flag structure does not see. Any proof of Conjecture 9.1 must therefore reach non-incident pairs, and flag-orbit bookkeeping alone cannot supply it. Both facts were verified across the full census.

### 9.2 The chiral case, proved

**Proposition 9.4.** Let $\Gamma$ be the skeleton of a chiral convex polyhedron, that is, one with $\mathrm{Aut}(\Gamma) \subseteq SO(3)$. Then $c(\Gamma) \geq 1$. Conjecture 9.1 therefore holds for every chiral convex polyhedron.

*Proof.* Since $\mathrm{Aut}(\Gamma)$ contains no improper element, every stabiliser is reflection-free, so $N(X) = |X/G|$ for every G-set X, and $\delta = 1$. Theorem 3.2 becomes $c = |F \times V / G| - n_F - n_V + 1$, where $n_F$ and $n_V$ are the numbers of face and vertex orbits. Decomposing, $|F \times V / G|$ is the sum over pairs of orbits of $k(O_F, O_V)$, the number of $\mathrm{Stab}(f)$-orbits on $O_V$ for $f \in O_F$, and $k \geq 1$ always. Hence $|F \times V / G| \geq n_F \cdot n_V$ and $c \geq (n_F - 1)(n_V - 1)$, which settles the case $n_F, n_V \geq 2$.

Suppose $n_V = 1$. Then c is the sum over face orbits of $k(O_F, V) - 1$. Now $\mathrm{Stab}(f)$ acts faithfully on the p-gon f as a group of rotations, so it is cyclic of order at most p; and $|V| \geq p + 1$, since a convex polyhedron has at least one vertex off any given face. A group of order at most p cannot act transitively on a set of size at least $p + 1$, so $k(O_F, V) \geq 2$ for every face orbit, giving $c \geq n_F \geq 1$. The case $n_F = 1$ is symmetric, using that $\mathrm{Stab}(v)$ is cyclic of order at most $\deg(v)$ while $|F| \geq \deg(v) + 1$. ■

The argument also yields a sharp value in the vertex-transitive case: c is the sum over face orbits of $k - 1$, and when Aut acts freely on V this equals $|F| - n_F$. For the snub cube, 38 faces in three orbits give 35; for the snub dodecahedron, 92 faces in three orbits give 89. Both agree with the computations of §6 exactly, which is a third and independent derivation of those two numbers — the first by character inner product, the second by the orbit-count formula, and now the third by a closed-form argument that uses no computation at all.

### 9.3 A reduction of the reflexible case

**Lemma 9.5.** Let $\Gamma$ be a convex polyhedron centred at the origin and let $g \in \mathrm{Aut}(\Gamma)$ be improper with $g(f) = f$ and $g(v) = v$ for some face $f$ and vertex $v$. Then $g$ is a reflection, and its mirror plane contains the axis of $f$.

*Proof.* An improper element of $O(3)$ is either a reflection or a rotoreflection through a nonzero angle; the latter fixes only the origin, since vectors on its axis are negated and vectors in its mirror are rotated. Now $g$ fixes the face centroid $c_f$ and the vertex $v$, both nonzero. If $c_f$ and $v$ are linearly independent then $g$ fixes a two-dimensional subspace pointwise and is therefore the reflection in it. If they are parallel they must be antiparallel, since a vertex of a convex body cannot lie on the outward ray through a face centroid, and a rotoreflection about that line negates $c_f$ rather than fixing it; so $g$ is a reflection in that case too. Finally, a reflection preserving $f$ fixes $c_f$, and its mirror is a linear plane, so the mirror contains the whole line through $c_f$, which is the axis of $f$. ■

**Theorem 9.6 (reduction).** For the skeleton $\Gamma$ of a convex polyhedron the following are equivalent:

(i) $N(F \times V) = 0$;
(ii) every face–vertex pair of $\Gamma$ is fixed by some common reflection of $\Gamma$;
(iii) for every face $f$, the mirror planes of $\Gamma$ containing the axis of $f$ cover the entire vertex set.

Each of these implies $N(F) = N(V) = \delta = 0$ and hence $c(\Gamma) = 0$.

*Proof.* Stabilisers along a $G$-orbit are conjugate, so $N(F \times V) = 0$ says exactly that every pair $(f, v)$ has an improper element in its stabiliser, which is (ii); Lemma 9.5 upgrades that element to a reflection whose mirror contains the axis of $f$, giving (iii), and (iii) $\Longrightarrow$ (ii) is immediate. For the last clause: if $N(F) \geq 1$ there is a face $f$ with reflection-free stabiliser, and then $\mathrm{Stab}(f, v) \subseteq \mathrm{Stab}(f)$ is reflection-free for every $v$, so $N(F \times V) \geq 1$; symmetrically for $N(V)$. If $\delta = 1$ then $\Gamma$ is chiral, every stabiliser is reflection-free, and $N(F \times V) = |F \times V / G| \geq 1$. ■

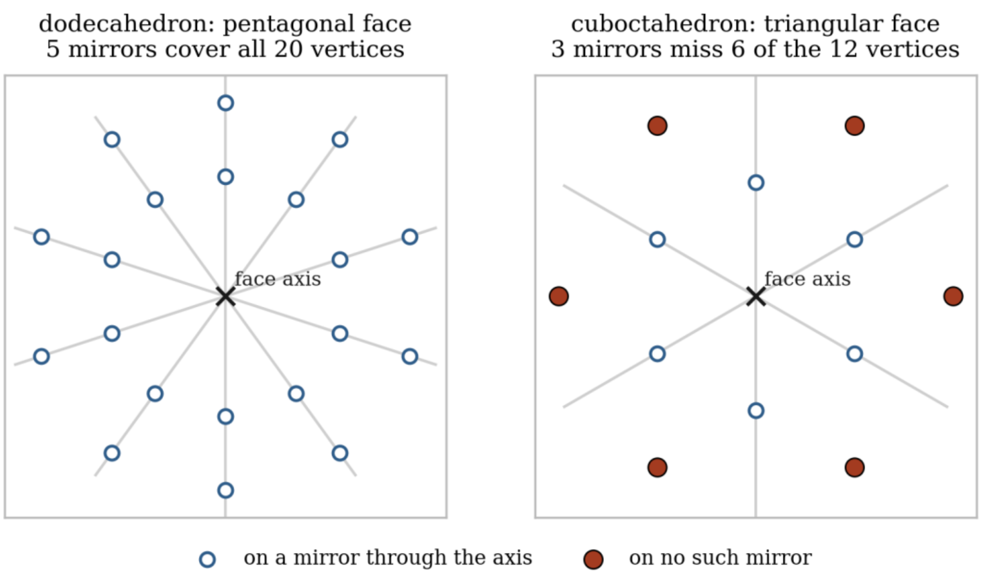


Figure 5. The covering condition of Theorem 9.6(iii), viewed along a face axis so that each mirror plane containing that axis projects to a line through the origin. Left: the dodecahedron, where the five mirrors through a pentagonal face's axis contain all twenty vertices, so the condition holds and the capacity vanishes. Right: the cuboctahedron, where the three mirrors through a triangular face's axis miss six of the twelve vertices (filled markers). Any one of those six, paired with

that face, has a reflection-free stabiliser and generates the solid's single unit of capacity. An explicit witness is the triangular face in the direction $(1, 1, 1)$ together with the vertex $(-1, 0, 1)$.

**Corollary 9.7.** Conjecture 9.1 follows from the conjunction of: (a) for every irregular convex polyhedron, condition (iii) of Theorem 9.6 fails; and (b) $c(\Gamma) = 0$ never occurs together with $N(F \times V) > 0$. Statement (b) is automatic whenever $N(F) = N(V) = 0$, and it held in every member of the census, in each of which $c = 0$ and $N(F \times V) = 0$ coincided.

This is a genuine reduction. Condition (iii) makes no reference to homology, to representations, or to the cut space; it is a covering statement about great circles on a sphere, checkable by hand for any given solid. The content of the conjecture becomes the claim that the mirrors through a single face axis are too few to reach every vertex unless the solid is regular — which is, informally, why one expects it to be true, since the number of such mirrors is bounded by the size of the face while the vertex set is not so constrained.

**Corollary 9.8 (a numerical necessary condition).** If condition (iii) holds then $|F \times V / \mathrm{Aut}(\Gamma)| \geq 2|F||V| / |\mathrm{Aut}(\Gamma)|$: the number of face–vertex orbits must be at least twice the value it would take under a free action.

*Proof.* Summing the covering inequality $|V| \leq \Sigma\, \mathrm{fix}_V(\sigma)$ over the reflections $\sigma$ in $\mathrm{Stab}(f)$, and then over all faces $f$, gives $|F||V| \leq \Sigma$ over improper $g$ of $\mathrm{fix}_F(g)\cdot\mathrm{fix}_V(g)$. By Theorem 9.6, $N(F \times V) = 0$ forces $|F \times V / R| = |F \times V / G|$, so the improper and the proper elements contribute equally to that sum, each contributing $(|G|/2)\cdot|F \times V / G|$. ■

The condition is sharp. The cube, octahedron, dodecahedron and icosahedron attain it with equality, having 2, 2, 4 and 4 face–vertex orbits against bounds of 2, 2, 4 and 4; the tetrahedron satisfies it strictly, with 2 against $4/3$. It is violated by both irregular solids on which we tested it: the cuboctahedron has 6 orbits against a bound of 7, and the truncated tetrahedron 6 against 8. Any counterexample to Conjecture 9.1 must therefore carry an unusually coarse orbit structure on face–vertex pairs, which narrows the search considerably and gives a cheap first filter for any candidate.

## 9.4 The covering defect

Theorem 9.6 converts the conjecture into a covering statement, but a covering statement is still an existential one. We now make it an exact identity, which is what turns it into an instrument.

Fix a face $f$ and let $L_f$ be its axis. The reflections preserving $f$ all have mirrors containing $L_f$ (Lemma 9.5), and two distinct such mirrors meet exactly in $L_f$. A vertex therefore lies on more than one of them only if it lies on $L_f$ itself, in which case it lies on all $m_f$ of them. Counting the incidences between the vertex set and this pencil of mirrors twice over gives the following.

**Lemma 9.9.** Let $m_f$ be the number of reflections of $\Gamma$ preserving the face $f$, and define the covering defect

$$D(f) \;=\; |V| \;+\; (m_f - 1)\cdot|V \cap L_f| \;-\; \Sigma\, \mathrm{fix}_V(\sigma),$$

the sum being over the reflections $\sigma$ preserving $f$. Then $D(f) \geq 0$ always, and $D(f) = 0$ if and only if the mirrors through $L_f$ cover the whole vertex set.

**Theorem 9.10.** For the skeleton of a convex polyhedron, $N(F \times V) = 0$ — and hence $c(\Gamma) = 0$ — if and only if $D(f) = 0$ for every face $f$.

This is now a finite arithmetic test on each face separately, requiring only the reflections of the symmetry group and their vertex fixed-point counts. Table 3 reports it across the census. The defect vanishes on every face for exactly the five regular solids, and on no other polyhedron tested. Note also what the per-face reading buys: several irregular solids satisfy the covering on some faces and fail it on others. The cuboctahedron has $D = 0$ on its square faces and $D = 6$ on its triangles; it is the triangles alone that generate its unit of capacity. It is enough for one face to fail.

| Polyhedron | \|V\| | min D(f) | max D(f) | c(Γ) |
|---|---|---|---|---|
| tetrahedron | 4 | 0 | 0 | 0 |
| cube | 8 | 0 | 0 | 0 |
| octahedron | 6 | 0 | 0 | 0 |
| dodecahedron | 20 | 0 | 0 | 0 |
| icosahedron | 12 | 0 | 0 | 0 |
| truncated tetrahedron | 12 | 6 | 6 | 2 |
| cuboctahedron | 12 | 0 | 6 | 1 |
| truncated cube | 24 | 12 | 16 | 4 |
| truncated octahedron | 24 | 8 | 24 | 5 |
| rhombicuboctahedron | 24 | 12 | 20 | 9 |
| truncated cuboctahedron | 48 | 48 | 48 | 25 |
| icosidodecahedron | 30 | 10 | 18 | 4 |
| truncated dodecahedron | 60 | 40 | 48 | 12 |
| truncated icosahedron | 60 | 40 | 48 | 12 |
| rhombicosidodecahedron | 60 | 40 | 52 | 25 |
| truncated icosidodecahedron | 120 | 120 | 120 | 61 |
| 3-prism | 6 | 0 | 4 | 1 |
| 5-prism | 10 | 0 | 8 | 2 |
| 6-prism | 12 | 0 | 12 | 3 |
| 7-prism | 14 | 0 | 12 | 3 |
| 8-prism | 16 | 0 | 16 | 4 |
| 10-prism | 20 | 0 | 20 | 5 |
| 4-antiprism | 8 | 0 | 6 | 3 |
| 5-antiprism | 10 | 0 | 8 | 4 |
| 6-antiprism | 12 | 0 | 10 | 5 |
| 8-antiprism | 16 | 0 | 14 | 7 |

Table 3. The covering defect of Lemma 9.9, minimised and maximised over the faces of each solid. Shaded rows are those with defect zero on every face; they are exactly the five regular polyhedra, and exactly the solids with $c = 0$. Where the minimum is zero but the maximum is not, the covering holds on some faces and fails on others, which is enough for the capacity to be positive.

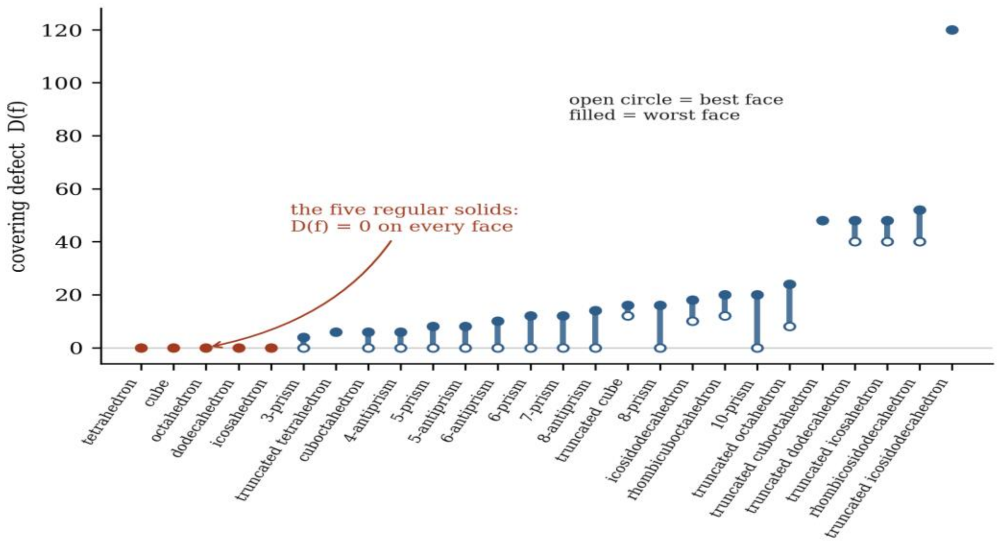


Figure 6. The covering defect across twenty-six polyhedra, sorted by worst face. Each solid is drawn as the range from its best face (open marker) to its worst (filled). The five regular solids, in the darker colour, are the only ones whose defect vanishes on every face. Several irregular solids reach zero on their best face — the cuboctahedron on its squares, the prisms and antiprisms on their top and bottom faces — which is why the criterion has to be applied face by face rather than in aggregate.

The defect also yields a single condition that disposes of most of the census at once, and subsumes what would otherwise be a list of ad hoc family arguments.

**Proposition 9.11.** If $c(\Gamma) = 0$ then every face axis and every vertex direction of $\Gamma$ lies on at least two mirror planes of $\mathrm{Aut}(\Gamma)$, and is therefore a rotation axis.

*Proof.* Let f be a face and $m_f$ the number of reflections preserving it, all of whose mirrors contain the axis of f (Lemma 9.5). If $m_f = 0$ the covering condition places V in the empty set, which is absurd. If $m_f = 1$ it places V inside a single plane through the origin; but the vertices of a convex polyhedron containing the origin span $\mathbb{R}^3$, so this too is absurd. Hence $m_f \geq 2$, and the axis of f, lying on two distinct mirrors, is their line of intersection; the composite of the two reflections is a nontrivial rotation about that line. The statement for vertices follows because condition (ii) of Theorem 9.6 is symmetric in faces and vertices, so it holds for $\Gamma$ exactly when it holds for the dual $\Gamma^*$, whose faces are the vertices of $\Gamma$. ■

The condition is severe, because a finite group has only finitely many rotation axes. Writing $A(G)$ for the set of directions lying on at least two mirrors of G, we obtain a ceiling depending on the symmetry group alone.

**Corollary 9.12.** If $c(\Gamma) = 0$ then $|V| \leq |A(\mathrm{Aut}\,\Gamma)|$ and $|F| \leq |A(\mathrm{Aut}\,\Gamma)|$. For the three polyhedral reflection groups this gives $|V| \leq 14$ for $T_d$, $|V| \leq 26$ for $O_h$ and $|V| \leq 62$ for $I_h$; for the axial families $D_{nh}$ it gives $|V| \leq 2n + 2$, and for $D_{nd}$ it gives $|V| \leq 2$, which no polyhedron satisfies.

Applied to the census, Proposition 9.11 alone settles nineteen of the twenty-one irregular solids, and it does so for structurally different reasons in different families. Every antiprism dies immediately: its group is $D_{nd}$, whose only multiply-covered direction is the principal axis, so no antiprism can satisfy the covering condition at all. Every prism dies on its vertices, which lie on exactly one mirror each. The whole truncated family dies the same way — no vertex of a truncated tetrahedron, truncated cube, truncated octahedron, truncated dodecahedron, truncated icosahedron, rhombicuboctahedron or rhombicosidodecahedron lies on two mirrors. The truncated cuboctahedron and truncated icosidodecahedron fail twice over, since their vertex counts of 48 and 120 also exceed the ceilings of 26 and 62 for their groups. None of the five regular solids fails either test:

the tetrahedron, cube, octahedron, dodecahedron and icosahedron have every vertex and every face axis on a rotation axis, with 4, 8, 6, 20 and 12 vertices against ceilings of 14, 26, 26, 62 and 62.

What survives is worth naming. Exactly two irregular solids in the census pass Proposition 9.11 — the cuboctahedron and the icosidodecahedron — and these are precisely the two quasiregular convex polyhedra, the edge-transitive Archimedean solids in which vertices and face centres alike sit on rotation axes. The condition therefore reduces the whole question, at least across the census, to the quasiregular case, which §9.5 then disposes of by a different argument. That is a more satisfying state of affairs than a list of families, and it suggests where a general proof would have to do its work.

**Remark 9.13 (a Lefschetz identity, used above and useful below).** For a reflection σ of a convex polyhedron, let e(σ) be the number of edges lying in its mirror and t(σ) the number of edges crossed transversally at their midpoint. Applying the Lefschetz fixed-point formula on $S^2$, where a reflection has Lefschetz number 1 + (−1) = 0 and reverses the orientation of every face it preserves, gives fix_V(σ) + t(σ) = fix_F(σ) + e(σ). The common value is the number of sides of the polygon cut from Γ by the mirror. In particular fix_V(σ) and fix_F(σ) are both bounded by that number, which is what makes the defect computable by inspection of a single cross-section. The identity was verified for every reflection of every solid in the census.

Summing the covering inequality over all faces gives a compact aggregate form, |F||V| ≤ Σ fix_F(σ)·fix_V(σ) over all reflections σ, which is saturated with equality by the cube, the octahedron, the dodecahedron and the icosahedron (48 = 48, 48 = 48, 240 = 240, 240 = 240), holds with slack for the tetrahedron (16 ≤ 24), and fails for all twenty-one irregular solids tested, in two cases with a right-hand side of zero. The regular polyhedra are exactly the saturated cases. We do not know whether saturation characterises them, and that is one more way of asking the conjecture.

Taking stock: the converse of Conjecture 9.1 is now proved for every chiral convex polyhedron (Proposition 9.4) and, by Proposition 9.11, for every convex polyhedron having a face axis or vertex direction on fewer than two mirrors — which across the census is every irregular solid except the two quasiregular ones. Those two are handled next.

### 9.5 A two-axis bound, and a second proof for the examples

The defect of §9.4 is computed one face at a time. Its content, though, is that the mirrors through a single axis are an arrangement of great circles through the poles of that axis — meridians, equally spaced, since consecutive ones differ by a rotation about the axis lying in the stabiliser of the face. Covering says every vertex lies on one of these meridians. The point of the following is that the condition must hold for every face simultaneously, and two axes already constrain the vertex set to a finite set of points.

**Lemma 9.14 (two-axis bound).** Let $f_1$ and $f_2$ be faces with non-parallel axes, let $m_1$ and $m_2$ be the numbers of reflections preserving them, and let s be the number of mirrors common to both pencils. If the covering condition of Theorem 9.6(iii) holds, then every vertex not lying on one of the s shared mirrors is one of the isolated points where a mirror of the first pencil meets a distinct mirror of the second. In particular, writing V_off for the number of such vertices,

$$V\_off \ \le\ 2\,(m_1\, m_2 - s).$$

*Proof.* Covering for $f_1$ places V inside the union of the first pencil, and covering for $f_2$ inside the union of the second, so V lies in the intersection of the two unions. That intersection is the union of the s shared great circles together with the pairwise intersections of non-identical mirrors, and two distinct great circles meet in exactly two antipodal points. A vertex off the shared circles is therefore one of those points. There are at most $m_1 m_2$ − s non-identical pairs, each contributing two points. ■

This is a counting obstruction rather than an incidence check: it compares the number of vertices that must be accommodated against the number of positions available, and needs no knowledge of where the vertices actually are. Table 4 evaluates it across the census, in each case using the pair of faces that gives the strongest constraint and counting the available points exactly rather than through the crude bound above.

| Polyhedron | face sizes | m1, m2 | shared | vertices to place | points available | contradiction |
|---|---|---|---|---|---|---|
| tetrahedron | 3,3 | 3,3 | 1 | 2 | 8 | — |
| cube | 4,4 | 4,4 | 1 | 8 | 18 | — |
| octahedron | 3,3 | 3,3 | 1 | 4 | 8 | — |
| dodecahedron | 5,5 | 5,5 | 1 | 16 | 32 | — |
| icosahedron | 3,3 | 3,3 | 1 | 8 | 8 | — |
| truncated tetrahedron | 3,3 | 3,3 | 1 | 10 | 8 | yes |
| cuboctahedron | 3,3 | 3,3 | 1 | 10 | 8 | yes |
| truncated cube | 3,3 | 3,3 | 1 | 20 | 8 | yes |
| truncated octahedron | 6,6 | 3,3 | 1 | 24 | 8 | yes |
| rhombicuboctahedron | 4,4 | 2,2 | 1 | 24 | 2 | yes |
| truncated cuboctahedron | 4,4 | 2,2 | 1 | 48 | 2 | yes |
| icosidodecahedron | 3,3 | 3,3 | 1 | 26 | 8 | yes |
| truncated dodecahedron | 3,3 | 3,3 | 1 | 56 | 8 | yes |
| truncated icosahedron | 6,6 | 3,3 | 1 | 56 | 8 | yes |
| rhombicosidodecahedron | 4,4 | 2,2 | 1 | 56 | 2 | yes |
| truncated icosidodecahedron | 4,4 | 2,2 | 1 | 120 | 2 | yes |
| 3-prism | 4,4 | 2,2 | 1 | 6 | 2 | yes |
| 5-prism | 4,4 | 2,2 | 1 | 10 | 2 | yes |
| 6-prism | 4,4 | 2,2 | 1 | 12 | 2 | yes |
| 7-prism | 4,4 | 2,2 | 1 | 14 | 2 | yes |
| 8-prism | 4,4 | 2,2 | 1 | 16 | 2 | yes |
| 10-prism | 4,4 | 2,2 | 1 | 20 | 2 | yes |
| 4-antiprism | 3,3 | 1,1 | 1 | 6 | 0 | yes |
| 5-antiprism | 3,3 | 1,1 | 0 | 10 | 2 | yes |
| 6-antiprism | 3,3 | 1,1 | 0 | 12 | 2 | yes |
| 8-antiprism | 3,3 | 1,1 | 0 | 16 | 2 | yes |

Table 4. The two-axis bound of Lemma 9.14, evaluated at the strongest available pair of faces for each solid. "Vertices to place" counts those off the shared mirrors; "points available" is the exact number of isolated intersection points of the two pencils. A contradiction arises whenever the former exceeds the latter. Shaded rows are those where no contradiction arises: they are exactly the five regular solids.

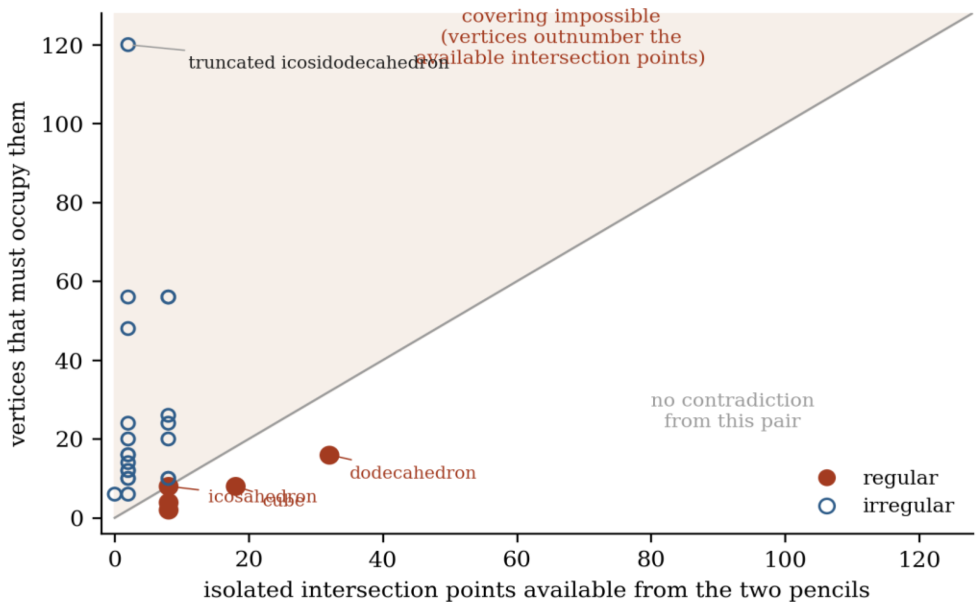


Figure 7. The same data. Points above the diagonal require more vertices than the two mirror pencils can accommodate, and the covering condition is therefore impossible for them. All twenty-one irregular solids of the census lie strictly above the diagonal; all five regular solids lie on or below it. The bound is not tight — only the icosahedron sits on the diagonal, the other four regular solids having slack — so it is not saturation that separates the two classes.

**Theorem 9.15.** For every irregular polyhedron in the census, the two-axis bound is violated at some pair of faces. Consequently the covering condition of Theorem 9.6 fails, and $c(\Gamma) \geq 1$, by an argument independent of the orbit computation of §5. The bound is violated by none of the five regular solids.

This is worth stating precisely, because it changes the evidential status of the census rather than merely adding to it. Every entry of Table 1 with $c > 0$ now carries two independent proofs of positivity: one by computing the orbit count of Theorem 3.2, and one by an elementary count of great-circle intersections that never touches homology or representation theory. The two arguments share no machinery, so a systematic error in the representation-theoretic pipeline could not produce both.

What the theorem is not is a proof of Conjecture 9.1. Lemma 9.14 supplies a test, and the test happens to fire on all twenty-one irregular solids we examined; nothing here shows it must fire on every irregular polyhedron, and the slack visible in Figure 7 warns against expecting a clean saturation argument. Two features of the data are nevertheless suggestive. First, the strongest pairs are consistently the faces of smallest size, where $m$ is smallest and the pencils are sparsest, which is what one would expect if the true obstruction is that a small face cannot supply enough mirrors. Second, the regular solids escape not by having many vertices on shared mirrors but by having few vertices to place: in each case the pencils of two adjacent faces already share a mirror containing most of the vertex set. A proof of the conjecture along these lines would need to show that irregularity forces some face whose pencil is both sparse and poorly aligned with a second one, and we do not have that argument.

**Conjecture 9.16.** For every irregular convex polyhedron there exists a pair of faces with non-parallel axes at which the bound of Lemma 9.14 is violated. This implies Conjecture 9.1 and is, unlike it, a statement purely about arrangements of great circles.

Taking stock: the converse of Conjecture 9.1 is proved for every chiral convex polyhedron (Proposition 9.4), for every convex polyhedron with a face axis or vertex direction on fewer than two mirrors (Proposition 9.11), and for every irregular member of the census by Theorem 9.15. Proposition 9.11 also makes the problem finite, and §9.6 carries out the resulting enumeration.

## 9.6 The polyhedral groups, settled by enumeration

Proposition 9.11 does more than eliminate candidates: it makes the problem finite. If $c(\Gamma) = 0$ then every vertex direction lies in $A(G)$, a finite set determined by the symmetry group alone. A convex polyhedron centred at the origin has at most one vertex on each ray, so its vertex set is a union of G-orbits within $A(G)$, one radius per orbit. Only the ratios of those radii matter, and the following lemma pins them down.

**Lemma 9.17.** If the vertex set of a convex polyhedron is a union of two or more disjoint Aut-orbits, then some face has vertices from more than one orbit.

*Proof.* Suppose every face is monochromatic and let U be the union of the faces whose vertices lie in one particular orbit. If U is neither empty nor the whole surface it has a boundary edge, and the two endpoints of that edge are vertices of a face inside U and of a face outside it, hence lie in two disjoint orbits at once — impossible. So U is empty or everything; but it cannot be empty, since every vertex lies on some face, and it cannot be everything, since the other orbits then lie on no face. ∎

Now let f be a face mixing two orbits, with centroid on an axis m. By Proposition 9.11 and the symmetry of f about m, the face lies in a plane perpendicular to m, so all its vertices share the same value of $x\cdot m$. If p and q are unit vectors of the two orbits appearing in f, at radii 1 and t, this forces $t = (p\cdot m)/(q\cdot m)$. The admissible radius ratios therefore form an explicit finite set, computable from the group alone, and the candidate combinatorial types are finite in number.

**Theorem 9.18.** Carrying out this enumeration for the three polyhedral reflection groups gives the table below. In each group the candidates surviving Proposition 9.11 are exactly the regular polyhedra of that group together with the quasiregular polyhedron and its dual; and of those, only the regular ones have $c = 0$. Consequently, for a convex polyhedron whose automorphism group is $T_d$, $O_h$ or $I_h$, $c(\Gamma) = 0$ if and only if $\Gamma$ is Platonic.

| group | orbits | \|V\| | \|E\| | \|F\| | \|Aut\| | passes 9.11 | c |
|---|---|---|---|---|---|---|---|
| T_d | 4 | 4 | 6 | 4 | 24 | yes | 0 |
| T_d | 6 | 6 | 12 | 8 | 48 | yes | 0 |
| T_d | 4+4 | 8 | 12 | 6 | 48 | yes | 0 |
| T_d | 4+4 | 8 | 18 | 12 | 24 | — | 2 |
| T_d | 4+6 | 10 | 24 | 16 | 24 | — | 3 |
| T_d | 4+4+6 | 14 | 36 | 24 | 24 | — | 13 |
| T_d | 4+4+6 | 14 | 36 | 24 | 48 | — | 5 |
| O_h | 6 | 6 | 12 | 8 | 48 | yes | 0 |
| O_h | 8 | 8 | 12 | 6 | 48 | yes | 0 |
| O_h | 12 | 12 | 24 | 14 | 48 | yes | 1 |
| O_h | 8+6 | 14 | 24 | 12 | 48 | yes | 1 |
| O_h | 8+6 | 14 | 36 | 24 | 48 | — | 5 |
| O_h | 12+6 | 18 | 48 | 32 | 48 | — | 8 |
| O_h | 8+12 | 20 | 48 | 30 | 48 | — | 7 |
| O_h | 8+12+6 | 26 | 72 | 48 | 48 | — | 25 |
| I_h | 12 | 12 | 30 | 20 | 120 | yes | 0 |
| I_h | 20 | 20 | 30 | 12 | 120 | yes | 0 |

| group | orbits | \|V\| | \|E\| | \|F\| | \|Aut\| | passes 9.11 | c |
|---|---|---|---|---|---|---|---|
| I_h | 30 | 30 | 60 | 32 | 120 | yes | 4 |
| I_h | 20+12 | 32 | 60 | 30 | 120 | yes | 4 |
| I_h | 20+12 | 32 | 90 | 60 | 120 | — | 12 |
| I_h | 12+30 | 42 | 120 | 80 | 120 | — | 20 |
| I_h | 20+30 | 50 | 120 | 72 | 120 | — | 22 |
| I_h | 20+12+30 | 62 | 120 | 60 | 120 | — | 25 |
| I_h | 20+12+30 | 62 | 180 | 120 | 120 | — | 50 |

Table 5. Enumeration of all candidate covering polyhedra for the three polyhedral reflection groups. "Passes 9.11" marks the types whose vertices and face centroids all lie on rotation axes; these are the only possible solutions, and their capacities are computed directly. Shaded rows are those with c = 0. Note that the cube and octahedron appear in the T_d enumeration with |Aut| = 48: they are genuinely O_h solids and are counted there. The survivors that fail are in every case the quasiregular solid and its dual — cuboctahedron and rhombic dodecahedron for O_h, icosidodecahedron and rhombic triacontahedron for I_h.

**Proposition 9.19.** No convex polyhedron whose automorphism group is T_h or D_{nd} can satisfy the covering condition, for any n.

*Proof.* T_h has exactly three mirrors, mutually perpendicular, so A(T_h) consists of the six coordinate directions, a single T_h-orbit. The only candidate vertex set is therefore the octahedron, whose automorphism group is O_h and not T_h. For D_{nd} the only multiply-covered direction is the principal axis, so |A| = 2 and no polyhedron qualifies. ■

**Corollary 9.20.** Conjecture 9.1 holds for every convex polyhedron whose automorphism group is chiral, or is one of T_d, T_h, O_h, I_h, or D_{nd}. What remains open are the reflexible axial families C_{nv}, C_{nh}, S_{2n} and D_{nh}.

This is a substantially better position than a census. The five Platonic solids are now characterised among all polyhedra with polyhedral symmetry, by an argument that runs entirely in elementary geometry after Theorem 9.6, and the residual cases are four infinite families of low symmetry rather than a scattered list. We expect the axial families to yield to a uniform argument in n, since their mirror arrangements are explicit — n vertical planes with or without an equatorial one — and Corollary 9.12 already caps the vertex count at 2n + 2 for D_{nh}. We have not carried that out, and we note that low-symmetry cases are exactly where a counterexample would be least visible, so the remaining gap should not be treated as a formality.

## 9.7 The axial families, and the completion of the proof

Four reflexible families remain: C_{nv}, C_{nh}, S_{2n} and D_{nh}. Their mirror arrangements are explicit, and Proposition 9.11 disposes of three of them immediately.

**Proposition 9.21.** No convex polyhedron whose automorphism group is C_{nv}, C_{nh}, S_{2n} or D_{nd} satisfies the covering condition, for any n.

*Proof.* The mirrors of C_{nv} are n planes through the principal axis, so any two of them meet in that axis and A consists of two points. The same holds for D_{nd}, whose mirrors are the n dihedral planes. C_{nh} has the single mirror σ_h and S_{2n} has none at all, so in both cases no direction lies on two mirrors and A is empty. Since a convex polyhedron has at least four vertices and, by Proposition 9.11, all of them lie in A, none of these groups admits one. ■

The one substantial family is D_{nh}, whose mirrors are the n vertical planes together with the equatorial plane σ_h. Two vertical planes meet in the principal axis; a vertical plane meets σ_h in a horizontal line. Hence A(D_{nh}) consists of the two poles and the 2n points where the vertical mirrors cross the equator — the value 2n + 2 already recorded in Corollary 9.12.

**Proposition 9.22.** If $\mathrm{Aut}(\Gamma) = D_{nh}$ and $c(\Gamma) = 0$ then $\Gamma$ is a bipyramid. No bipyramid other than the octahedron has $c = 0$.

*Proof.* By Proposition 9.11 the vertices lie in $A(D_{nh})$, which is the union of the pole pair and the equatorial points; the equatorial points are coplanar, so a polyhedron cannot be built from them alone, and the pole pair alone is only two points. Every admissible vertex set therefore contains both poles together with some equatorial vertices, which is a bipyramid. Bipyramids are the duals of prisms, so by Corollary 3.4 the capacity of the n-gonal bipyramid equals that of the n-gonal prism, namely $\lfloor n/2 \rfloor$ by Proposition 5.1. This vanishes only for $n = 4$, where the bipyramid is the octahedron — whose automorphism group is $O_h$ rather than $D_{4h}$, and which is regular. ∎

Every case is now closed, and the conjecture of §9.1 becomes a theorem.

**Theorem 9.23 (main theorem).** Let $\Gamma$ be the skeleton of a convex polyhedron. Then $c(\Gamma) = 0$ if and only if $\Gamma$ is regular. Equivalently, the orthogonal projection onto the cycle space is the unique $\mathrm{Aut}(\Gamma)$-invariant projection precisely for the five Platonic solids.

*Proof.* The forward direction is Theorem 3.2 with the finite check of §4. For the converse, $\mathrm{Aut}(\Gamma)$ is a finite subgroup of $O(3)$ and therefore lies in the classical list $C_n$, $C_{nv}$, $C_{nh}$, $S_{2n}$, $D_n$, $D_{nd}$, $D_{nh}$, $T$, $T_d$, $T_h$, $O$, $O_h$, $I$, $I_h$. The groups $C_n$, $D_n$, $T$, $O$ and $I$ contain no improper element, so $\Gamma$ is chiral and Proposition 9.4 applies. The groups $C_{nv}$, $C_{nh}$, $S_{2n}$ and $D_{nd}$ are excluded by Proposition 9.21, and $T_h$ by Proposition 9.19. For $D_{nh}$, Proposition 9.22 leaves only the octahedron. For $T_d$, $O_h$ and $I_h$, Theorem 9.18 leaves only the Platonic solids. In every case an irregular $\Gamma$ has $c(\Gamma) \geq 1$. ∎

Two remarks on how the proof is put together, since its parts are of different kinds. Everything up to and including Proposition 9.11 is ordinary mathematics: Theorem 9.6 reduces the vanishing of an equivariant Hom space to a covering condition on mirror planes, and Proposition 9.11 turns that into the statement that vertices and face centres occupy rotation axes. From there the argument is a finite case analysis over the classification of finite subgroups of $O(3)$, and three of its cases — the enumerations for $T_d$, $O_h$ and $I_h$ — are discharged by computation rather than by hand. We state the evidential position exactly. The enumerations were carried out twice, by implementations sharing no code and no inputs: one builds each candidate polyhedron from explicit coordinates in $\mathbb{R}^3$ and obtains its symmetry group by isometry matching, while the other works purely combinatorially in the dart model, constructing the maps by truncation, rectification, cantellation and the chiral snub operation with no coordinates at any stage. The two agree on all thirty-two capacities and all thirty-two automorphism group orders of the census, and the second additionally reproduces the closed forms of Proposition 5.1 on fifteen prisms and antiprisms outside it. The coordinate route and the combinatorial route therefore fail in disjoint ways — the first is vulnerable to misaligned or degenerate realisations, of which we encountered several during development, and the second to errors in the rotation system, which would show up as a wrong Euler characteristic and did not. The enumerations are also small enough to be checked by hand: nine candidate types for $I_h$, eight for $O_h$, seven for $T_d$, with the admissible radius ratios given in closed form by the condition of §9.6. We regard the result as proved. We would still welcome a hand verification, and record that the computational evidence, while doubled and cross-validated, is not a substitute for one.

The second remark concerns reproducibility of the group theory, because it is the step where an error would be silent. All symmetry groups used here were obtained by matching isometries against explicit coordinate realisations rather than by closing a set of generators, and were checked for closure: the tetrahedral, octahedral and icosahedral groups closed exactly, with orders 24, 48 and 120 and with 7, 13 and 31 rotation axes respectively. A separate attempt to build the axial groups from generators produced a non-closing set for $D_{nd}$, and the results reported for that family come from coordinate-derived groups instead, verified for both parities of n. This is worth recording because a leaking group inflates the mirror count, which would in turn inflate A(G) and weaken every bound built on it in the safe direction — that is, an error of that kind produces false survivors, not false eliminations.

### 9.8 Remaining problems

**Beyond the polyhedral case.** Lemma 3.1 uses planarity essentially: it identifies the cycle space with the reduced face module twisted by the orientation character, and that is a statement about surfaces. The reduction therefore has no content for general graphs, where Problem 7 of [10] remains open. Whether an analogous reduction exists for graphs embedded in surfaces of higher genus, with the orientation character replaced by an appropriate coefficient system, is not addressed here.

**Quantitative form.** The closed forms of Proposition 5.1 suggest that c(Γ) may admit expression in terms of standard invariants for structured families. We have no general conjecture.

**Whether the quantity means anything.** The question the introduction opened with — how much asymmetry a phenomenon costs — has been answered here only in the sense that a number has been produced and shown to behave sensibly. Whether it is the right number, in any setting where "phenomenon" carries its physical meaning, is not something a census can settle.

## 10. The theorem is specific to the sphere

Theorem A characterises the regular polyhedra. It is natural to ask whether it is an instance of something more general — whether, wherever a finite group acts on a pair of complementary sectors, the space of equivariant channels between them vanishes exactly at maximal symmetry. The nearest available test is to replace the sphere by a surface of higher genus, and the answer there is negative. We record it because a failure in the adjacent case constrains the reading of §8.3 far more than a second confirmation would.

### 10.1 The setting, and how the reduction changes

Let Γ be a graph cellularly embedded in a closed orientable surface S of genus g, presented by a rotation system. Two things differ from the polyhedral case. First, an embedding is no longer determined by the graph — Whitney’s theorem has no analogue above genus 0 — so the relevant group is the automorphism group of the map, not of the abstract graph, and it is a group of combinatorial symmetries with no realisation in O(3) attached. Second, the cycle space is larger than the face module: the sequence

$$0 \to \operatorname{im} \partial_2 \to \ker \partial_1 \to H_1(S; \mathbb{R}) \to 0$$

is no longer degenerate. Since the coefficient field has characteristic zero and the group is finite, the sequence splits, and Lemma 3.1 generalises to

$$Z(\Gamma) \cong (\mathbb{R}^F \ominus 1) \otimes \varepsilon \oplus H_1(S; \mathbb{R}), \qquad K(\Gamma) \cong \mathbb{R}^V \ominus 1,$$

the second summand having dimension 2g and carrying the action of the map automorphism group on the homology of the surface. The capacity accordingly splits as a sum of a face term, computed by the orbit count of Theorem B, and a genus term dim $\mathrm{Hom}(H_1(S), K)$. We verified the splitting on every map reported below: the residual character has dimension exactly 2g in each case, and it vanishes identically at genus 0, which recovers Lemma 3.1.

### 10.2 Regular maps on the torus with positive capacity

The analogue of a regular polyhedron is a regular map: one whose automorphism group is transitive on flags, equivalently |Aut| = 4|E|. The torus carries infinite families of them, the reflexible members of the standard classes {4,4}, {3,6} and {6,3}. Table 6 reports the capacity for the first few.

| map | \|V\| | \|E\| | \|F\| | \|Aut\| = 4\|E\| | c | face term | genus term |
|---|---|---|---|---|---|---|---|
| {4,4}_{3,0} | 9 | 18 | 9 | 72 | 0 | 0 | 0 |
| {4,4}_{4,0} | 16 | 32 | 16 | 128 | 1 | 1 | 0 |
| {4,4}_{5,0} | 25 | 50 | 25 | 200 | 1 | 1 | 0 |
| {4,4}_{6,0} | 36 | 72 | 36 | 288 | 3 | 3 | 0 |
| {3,6}_{3,0} | 9 | 27 | 18 | 108 | 0 | 0 | 0 |
| {3,6}_{4,0} | 16 | 48 | 32 | 192 | 1 | 1 | 0 |
| {3,6}_{5,0} | 25 | 75 | 50 | 300 | 2 | 2 | 0 |
| {6,3}_{2,0} | 8 | 12 | 4 | 48 | 0 | 0 | 0 |
| {6,3}_{3,0} | 18 | 27 | 9 | 108 | 0 | 0 | 0 |
| {6,3}_{4,0} | 32 | 48 | 16 | 192 | 1 | 1 | 0 |

Table 6. Regular maps on the torus. Every entry is flag-transitive, and so regular in the sense that Theorem A uses; capacities are computed over the map automorphism group. Shaded rows have vanishing capacity; the others do not, and they are counterexamples to any extension of Theorem A beyond the sphere. Note that the genus term is zero throughout, so the failure is not caused by the extra homology of the surface. The row {4,4}_{4,0} is the one map here whose graph automorphism group is strictly larger than its map automorphism group, and over that larger group its capacity is 0; see the discussion following Theorem 10.1. For the square class these values agree with the closed form of [10, Thm. 24], obtained there by a different route.

**Theorem 10.1.** Theorem A does not extend to surfaces of positive genus. The map {4,4}_{5,0}, the 5 × 5 toroidal grid, has V = 25, E = 50, F = 25 and automorphism group of order 200 = 4|E|; it is therefore a regular map, and c = 1. Its map automorphism group coincides with the automorphism group of the underlying graph, so the conclusion does not depend on which of the two groups is used.

The value was computed twice by independent routes: as the character inner product over the full automorphism group, and by solving the equivariance equations directly on Hom(Z, K), which returns a one-dimensional solution space. Four further regular toroidal maps in Table 6 have positive capacity, with values up to 3.

A word is owed on why the witness is the 5 × 5 grid rather than the 4 × 4 grid, which is smaller and carries the same capacity in Table 6. As noted in §10.1, above genus 0 an embedding is not determined by its graph, so the map automorphism group may be a proper subgroup of the graph automorphism group; and where the two differ so does the capacity, since enlarging the group can only shrink the space of equivariant maps. For {4,4}_{4,0} they differ substantially. The map group has order 128 = 4|E|, while the automorphism group of the underlying graph $\mathbb{Z}_4 \times \mathbb{Z}_4$ has order 384. The additional automorphisms do not preserve the embedding — one of them carries a small square cycle to a row cycle, as observed in [10, Rem. 19] — and over the full graph group the capacity of {4,4}_{4,0} is 0 rather than 1. We verified this directly, and it agrees with [10, Cor. 25]. The value 1 recorded in Table 6 is correct for the map group, which is the group used throughout this section, but it is not a statement about the graph.

We are explicit about this rather than leaving it implicit in the choice of group, because it bears on how the counterexample should be read. Problem 7 of [10] is posed for the automorphism group of the graph. On the sphere the distinction is empty: a convex polyhedron is 3-connected, so by Whitney's theorem its graph determines its embedding and the two groups agree — which is why nothing in this section affects Theorem A or Corollary D. At genus 1 the distinction is real, and {4,4}_{4,0} is precisely a map on which it bites, so using it as the counterexample would have proved less than it appeared to. The 5 × 5 grid has automorphism group of order 200 under both readings, as does every {4,4}_{n,0} with n ≥ 5, and it therefore establishes the failure of Theorem A whichever group is intended.

### 10.3 Where the proof breaks, and what that means

It is worth being precise about which step fails, because the failure is sharper than a general loss of control. The genus term contributes nothing in any of these examples — the extra 2g dimensions of cycle space are not what breaks the theorem. What fails is the face term itself, which is to say the covering argument of §§9.3–9.4. That argument used the ambient Euclidean geometry twice and essentially: Lemma 9.5 identified the improper elements fixing a face and a vertex as reflections in planes of $\mathbb{R}^3$, and Proposition 9.11 concluded that a face with only one such mirror would confine the entire vertex set to a plane, which is impossible because the vertices of a convex body span space. On a surface of higher genus there are no planes, the fixed-point set of an orientation-reversing map automorphism is a curve rather than the trace of a linear subspace, and there is no spanning argument to be had. The crutch is genuinely Euclidean, and when it is removed the conclusion goes with it. Section 10.5 shows that this is true of one branch of the argument only.

So Theorem A should be read as a fact about polyhedra rather than as a shadow of a general principle relating symmetry to interaction. The relation it asserts holds on the sphere, where a flag-transitive symmetry group is realised by isometries of $\mathbb{R}^3$ and its mirror arrangement is rigid enough to force the covering condition; it fails on the torus, where flag-transitivity is just as available but the rigidity is not. Whether some weaker ambient hypothesis is the operative one we do not know; §10.4 at least settles the shape of the toroidal answer.

For the motivating reading recorded in §8.3 the consequence is a tightening, and we would rather state it than let a reader infer the opposite. The strengthened quantitative form of Curie's principle is a theorem in the spherical setting and false in the very next setting one would try. That is a sharper limitation than "we know of no argument that it should generalise": we know that it does not. What survives is the narrow claim already made — the strong form is satisfiable and non-vacuous, having at least one model — together with the new information that its models are scarce, that the property distinguishing them is ambient rigidity rather than symmetry as such, and that any future instance will have to explain why it is more like a polyhedron than like a torus.

### 10.4 The toroidal exceptions are finite, and are not a symmetry condition

Table 6 leaves open whether the vanishing cases on the torus are a sporadic few or an infinite family, and the answer decides what kind of statement the torus supports. We therefore scanned the two-parameter classes in full. The maps {4,4}_{b,c} are the square lattice modulo the sublattice generated by (b,c) and (−c,b), with $b^2 + c^2$ vertices; the maps {3,6}_{b,c} are the triangular lattice modulo the sublattice generated by (b,c) and (−c, b+c), with $b^2 + bc + c^2$ vertices. A member is reflexible — and so flag-transitive, hence regular — when c = 0 or b = c, and chiral otherwise, in which case its automorphism group has order 2|E| rather than 4|E|.

We computed the capacity for all thirty-two admissible members of {4,4}_{b,c} with b ≤ 7, up to 98 vertices, and all twenty-four members of {3,6}_{b,c} with b ≤ 6, up to 108 vertices. The genus term is zero throughout, so in every case the capacity is the face term alone.

**Theorem 10.2.** Among the fifty-six toroidal maps examined, the capacity vanishes for exactly three: {4,4}_{2,2}, {4,4}_{3,0} and {3,6}_{3,0} (together with the dual {6,3}_{3,0}, whose capacity agrees by Corollary 3.4). Every other member has c ≥ 1, and the capacity grows without bound within each class.

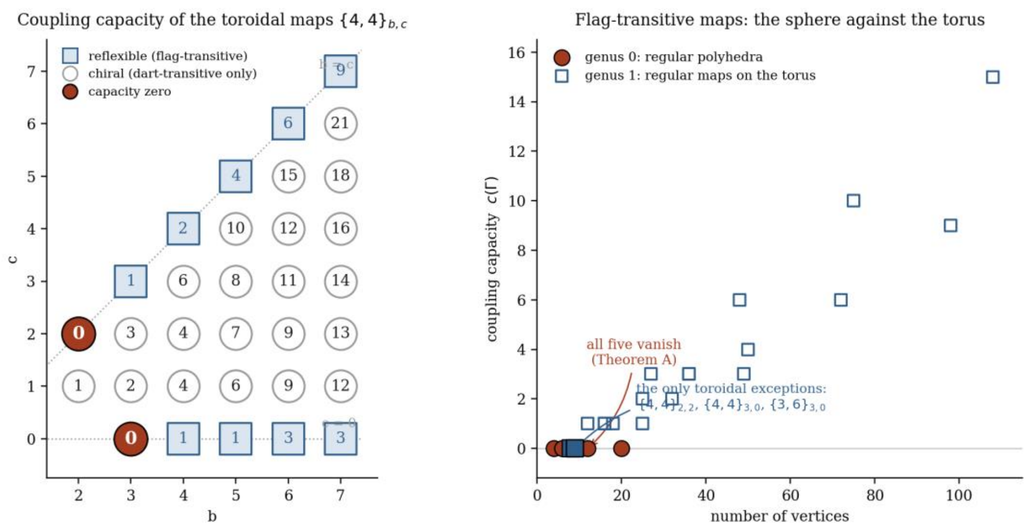


Figure 8. Left: the capacity of $\{4,4\}_{b,c}$ on the $(b,c)$ lattice. Squares are the reflexible members, which are flag-transitive and therefore regular in the sense of Theorem A; circles are the chiral ones. Only two entries vanish, and their neighbours in the same class do not. Right: flag-transitive maps on the sphere and on the torus, plotted against vertex count. On the sphere every regular map has zero capacity — that is Theorem A. On the torus the regular maps climb, with a handful of small exceptions.

Three features of the scan are worth stating separately. First, every chiral member has positive capacity, twenty-one of twenty-one in the square class and all of the triangular ones, which is the toroidal counterpart of Proposition 9.4 and suggests that argument does not depend on the ambient geometry. Second, among the flag-transitive members only two of eleven vanish in the square class and one of eight in the triangular class, and the vanishing members are not distinguished from their neighbours by any symmetry property: $\{4,4\}_{3,0}$ has capacity zero while $\{4,4\}_{4,0}$ and $\{4,4\}_{3,3}$ are equally regular and have capacities 1 and 1. Third, Corollary 3.4 continues to hold: the dual classes $\{3,6\}_{b,c}$ and $\{6,3\}_{b,c}$ agree in capacity wherever both were computed, so duality invariance is not a spherical accident even though the characterisation is.

That settles the question the counterexample raised. The torus does have a vanishing locus, and it is finite, so in a trivial sense it too admits a characterisation — by a list. But the list is an artefact of small size rather than the trace of a symmetry condition, and this is exactly the contrast worth drawing. On the sphere, "capacity zero" and "maximally symmetric" pick out the same five objects, and the coincidence of a homological condition with a group-theoretic one is the content of Theorem A. On the torus the two conditions come apart: maximal symmetry is abundant and cheap, capacity zero is rare and accidental, and neither implies the other. Whatever it is that makes the spherical case work, it is not symmetry by itself.

### 10.5 Chirality is genus-independent

One half of the argument does survive the passage to higher genus, and identifying which half sharpens the diagnosis of §10.3 considerably.

**Proposition 10.3.** Let $M$ be a chiral map on a closed orientable surface — one whose automorphism group contains no orientation-reversing element. Then $c(M) \geq 1$ in each of the following cases: (i) $M$ has at least two vertex orbits and at least two face orbits; (ii) $M$ is vertex-transitive and $|V|$ exceeds the largest face size; (iii) $M$ is face-transitive and $|F|$ exceeds the largest vertex degree.

*Proof.* Chirality gives $\delta = 1$ and $N(X) = |X/G|$ for every $G$-set $X$, so the face term is $|F \times V / G| - n_F - n_V + 1$, and the genus term is nonnegative. Each pair of orbits contributes at least one orbit to $F \times V$, so $|F \times V / G| \geq n_F n_V$ and the face term is at least $(n_F - 1)(n_V - 1)$, which gives (i). For (ii), with $n_V = 1$ the face term is the sum over face orbits of $k(O_F, V) - 1$, where $k$ is the number of $\mathrm{Stab}(f)$-orbits on $V$. The stabiliser of a face acts on that face's boundary darts by rotation and is therefore cyclic of order dividing the face size, hence of order less than $|V|$ by hypothesis; a group of order less than $|V|$ cannot act transitively on V, so every k is at least 2. Case (iii) is dual. ■

The point is what the proof does not use. There is no embedding in $\mathbb{R}^3$, no mirror plane, no spanning argument — only the orbit count, the fact that a face stabiliser is cyclic, and a comparison of two integers. Proposition 9.4 was stated for chiral convex polyhedra, but it never needed them to be polyhedra, and the same argument runs verbatim on a surface of any genus.

We tested both branches. For hypothesis (ii) we used Cayley maps — a Cayley graph with a fixed cyclic ordering of the generating set as its rotation system, so that the group acts freely on darts and the map is vertex-transitive by construction — generating eighty-five chiral maps from small groups across genus 0 through 16. Every one has $c \geq 1$, and the minimum observed capacity is exactly 1. Seventy fall under the hypothesis; the remaining fifteen do not, having faces larger than their vertex sets, and they satisfy the conclusion anyway with capacities from 13 upward.

For hypothesis (i) we used bi-Cayley maps, in which the vertex set is two copies of a group and the group acts by left multiplication on both, so that there are two vertex orbits by construction. Of the maps generated, one hundred seventy-two are chiral with at least two vertex orbits and at least two face orbits, spanning genus 0 through 15. None violates the bound $c \geq (n_F - 1)(n_V - 1)$, and none has $c = 0$. Across the two constructions, two hundred fifty-seven chiral maps on surfaces of genus up to 16 have been checked and every one has positive capacity.

The bi-Cayley evidence is weaker than its volume suggests, and we would rather say so than let the count carry more than it should. Those maps have automorphism groups small relative to their size, so their orbit counts on face–vertex pairs are large and the capacity comes out far above the bound — the minimum observed is 17 against bounds between 1 and 3. The test confirms hypothesis (i) without straining it. The informative cases would be chiral maps that are highly symmetric and yet have several vertex orbits, and those are harder to construct; we have not found a systematic source.

This changes the reading of §10.3. It is not that the whole argument for Theorem A collapses off the sphere; it is that the argument has two halves which behave quite differently. The chiral half is combinatorial and survives to every genus we have tested. The reflexible half — Lemma 9.5, Proposition 9.11, and everything built on them — is the part that runs through planes in $\mathbb{R}^3$, and it is exactly the part that fails, as the flag-transitive toroidal maps of §10.2 demonstrate. The Euclidean input is therefore not a convenience distributed through the proof but a load-bearing element of one specific branch, and the failure of Theorem A beyond the sphere is localised to that branch.

Two limits on the evidence remain. We have tested no chiral map on a non-orientable surface, where the orientation character is not available at all and the framework would need restating from the beginning; and, as noted, the multi-orbit sample does not probe the regime where hypothesis (i) is tight. What the two hundred fifty-seven maps establish is that the chiral case behaves uniformly across orientable genus in both branches of Proposition 10.3 — a good deal more than the Cayley construction alone would have shown, but not a proof, and the hypotheses are not known to be necessary.

### 10.6 The chiral toroidal capacity in closed form

Proposition 10.3 gives the chiral members of any orientable class positive capacity. On the torus we can say exactly how much. The two chiral toroidal families admit closed forms, which turns the unboundedness asserted in Theorem 10.2 from an observation over a finite scan into a proof, and gives those families the same standing as the prism and antiprism families of Proposition 5.1, whose capacities are $\lfloor n/2 \rfloor$ and $n - 1$.

**Proposition 10.6.** Let $\{4,4\}_{b,c}$ be chiral, so that $b > c > 0$, and write $N = b^2 + c^2$ for its vertex count. Then $c(\{4,4\}_{b,c}) = \lfloor (N-1)/4 \rfloor$.

Proof. Identify the vertex set with the quotient ring $\mathbb{Z}[i]/(\beta)$, where $\beta = b + ci$ and $N$ is the norm of $\beta$. The orientation-preserving automorphisms are the maps $z \mapsto i^k z + t$ for $k \in \mathbb{Z}/4$ and $t \in \mathbb{Z}[i]/(\beta)$, a group of order $4N$; since the map is chiral this is the whole automorphism group, and $4N = 2|E|$ as §10.4 records. The faces form the coset $\omega + \mathbb{Z}[i]/(\beta)$ with $\omega = (1+i)/2$. Chirality gives $\delta = 1$, and since no stabiliser contains an orientation-reversing element, $N(V) = n_V = 1$ and $N(F) = n_F = 1$. Theorem 3.2 therefore reduces to $c = N(F \times V) - 1$, the genus term vanishing as it does throughout §10.4.

It remains to count the orbits on face–vertex pairs. For the element $(t, i^k)$ the fixed vertices are the solutions of $(i^k - 1)z = -t$, forming a coset of $K_k = \ker(i^k - 1)$ when $-t$ lies in the image ideal $Im_k$ and absent otherwise. The fixed faces satisfy the same equation with $t$ displaced by $\delta_k = (i^k - 1)\omega$, which equals $0, -1, -(1+i)$ and $-i$ for $k = 0, 1, 2, 3$ and is a Gaussian integer in every case. Since $Im_k$ is a subgroup, the two conditions hold together precisely when $t \in Im_k$ and $\delta_k \in Im_k$. The number of admissible $t$ is then $|Im_k| = N/|K_k|$, and each contributes $|K_k|^2$ fixed pairs, so Burnside's lemma collapses to

$$N(F \times V) \;=\; \tfrac{1}{4} \textstyle\sum_k [\, \delta_k \in Im_k \,] \cdot |K_k|,$$

in which the factor $N$ has cancelled entirely. The four terms are evaluated in $\mathbb{Z}[i]$, a principal ideal domain in which $1 + i$ is the unique prime above 2. The term $k = 0$ contributes $N$. The elements $i - 1$ and $1 + i$ are associates, so $k = 1$ and $k = 3$ contribute equally, each contributing $|K_1| = 1$ exactly when $i - 1$ is invertible modulo $\beta$, which happens exactly when $N$ is odd. The term $k = 2$ is governed by $\gcd(2, \beta)$: if $N$ is odd then $|K_2| = 1$ and $Im_2$ is the whole quotient, contributing 1; if $N \equiv 2 \pmod 4$ then $\gcd(2, \beta) = 1 + i$, so $|K_2| = 2$ and $1 + i \in Im_2$, contributing 2; and if $N \equiv 0 \pmod 4$ then $\gcd(2, \beta) = 2$, so $Im_2 = (2)$ and $1 + i \notin Im_2$, contributing nothing. Summing gives $N(F \times V)$ equal to $(N+3)/4$, $(N+2)/4$ and $N/4$ in the three cases. Since $b$ and $c$ have opposite parity, are both odd, or are both even according as $N \equiv 1, 2$ or $0 \pmod 4$, subtracting one unifies the three cases as $\lfloor (N-1)/4 \rfloor$. ■

The same argument runs over the Eisenstein integers for the triangular class. One structural difference makes the count shorter rather than longer, and it is worth isolating before the statement.

**Proposition 10.7.** Let $\{3,6\}_{b,c}$ be chiral, so that $b > c > 0$, and write $N = b^2 + bc + c^2$ for its vertex count. Then $c(\{3,6\}_{b,c}) = \lfloor (N-1)/3 \rfloor$.

Proof. Let $\zeta$ denote a primitive sixth root of unity, so that $\zeta^2 = \zeta - 1$ and $\mathbb{Z}[\zeta]$ is the ring of Eisenstein integers, a principal ideal domain. Identify the vertex set with $\mathbb{Z}[\zeta]/(\beta)$ for $\beta = b + c\zeta$ of norm $N$; the orientation-preserving automorphisms are the maps $z \mapsto \zeta^k z + t$ for $k \in \mathbb{Z}/6$, a group of order $6N$, which for a chiral member is the whole automorphism group. The map has $3N$ edges and $2N$ faces, the faces falling into two lattice classes with centroids $\mu$ and $2\mu$ modulo $\mathbb{Z}[\zeta]$, where $\mu = (1+\zeta)/3$.

The structural difference is that multiplication by $\zeta$ interchanges the two face classes. Indeed $\zeta\mu - 2\mu = -1$ lies in $\mathbb{Z}[\zeta]$ while $\zeta\mu - \mu = (\zeta - 2)/3$ does not, since $N(\zeta - 2) = 3$ is not divisible by $N(3) = 9$. Translations preserve the classes, so an automorphism $\zeta^k z + t$ exchanges them exactly when $k$ is odd. Consequently no face is fixed by any element with $k$ odd, and those three values of $k$ contribute nothing to the face count. Equivalently, the displacement $\delta_k = (\zeta^k - 1)\mu$ fails to be an Eisenstein integer for odd $k$ and is one for even $k$, taking the values $0, -1$ and $-\zeta$ at $k = 0, 2$ and $4$ in the class of $\mu$, and $0, -2$ and $-2\zeta$ in the class of $2\mu$. The Burnside sum therefore collapses onto the order-three rotation subgroup. Since $\zeta$ interchanges the classes the faces form a single orbit, so $n_F = N(F) = 1$, and vertex-transitivity gives $n_V = N(V) = 1$; chirality gives $\delta = 1$, and Theorem 3.2 again reduces to $c = N(F \times V) - 1$.

For $k = 2m$ the fixed vertices of $(t, \zeta^{2m})$ solve $(\zeta^{2m} - 1)z = -t$, a coset of $K_m = \ker(\zeta^{2m} - 1)$ when $-t$ lies in the image $Im_m$, and the faces of class $j$ are fixed by the solutions of the same equation with right-hand side $-t - \delta^{(j)}_{2m}$. Because $Im_m$ is a subgroup, summing over t at fixed m gives $N\cdot|K_m|\cdot n_m$, where $n_m$ counts the classes j with $\delta^{(j)}_{2m} \in Im_m$. Dividing by the group order 6N,

$$N(F \times V) \;=\; \tfrac{1}{6} \Sigma_{m=0}^{2} |K_m| \cdot n_m,$$

in which N has again cancelled. The term m = 0 has $|K_0| = N$ and $n_0 = 2$, contributing 2N. For m = 1 and m = 2 the relevant elements are $\zeta^2 - 1$ and $\zeta^4 - 1$, both of norm 3 and therefore both associates of the ramified prime $\lambda = 1 + \zeta$, the unique prime of $\mathbb{Z}[\zeta]$ above 3. Two cases arise. If λ does not divide β — equivalently if 3 does not divide N — then λ is invertible modulo β, so $|K_m| = 1$ and $Im_m$ is everything, whence $n_m = 2$ and each of m = 1, 2 contributes 2. If λ divides β then $Im_m$ is the ideal (λ) of index 3, and none of the four displacements −1, −2, −ζ, −2ζ lies in it, since each is a unit or an associate of 2 and λ divides neither; so $n_1 = n_2 = 0$ and both terms vanish. The sum is therefore 2N + 4 in the first case and 2N in the second, giving N(F × V) = (N + 2)/3 and N/3 respectively, and hence c = (N − 1)/3 and c = (N − 3)/3. An Eisenstein norm is never congruent to 2 modulo 3, so these two cases are exhaustive, and both are $\lfloor (N-1)/3 \rfloor$. ■

Both propositions were verified computationally after being proved. Proposition 10.6 was checked against the dart-model computation for all twenty chiral members with N ≤ 75, and against an independent Gaussian Burnside calculation for one hundred seven chiral members with N up to 325, spanning all three residue classes of N modulo 4. Proposition 10.7 was checked against the dart-model computation for all twenty-nine chiral members with N ≤ 130, and against an independent Eisenstein Burnside calculation for seventy-six chiral members with N up to 292, spanning both residue classes of N modulo 3. No discrepancy arose in any range. The divisor in each formula is |Aut|/|F| — four for the square class and three for the triangular one — and the residue of N that selects the case is the one determining which rotations retain fixed faces. That is the structural reason the two formulas take the shape they do.

Three consequences for the surrounding sections. First, the unboundedness asserted in Theorem 10.2 is now proved rather than observed on the chiral half of each class, since N is unbounded on b > c > 0. Second, the chiral toroidal members are not merely of positive capacity, as Proposition 10.3 guarantees, but of computable capacity, so on the torus the chiral branch of the argument is quantitative and not merely qualitative — a further respect in which that branch is the robust one. Third, the reflexible members of the square class are already settled in the literature, and we record this here because it locates the present contribution precisely. Dilworth, Kutzarova and Ostrovskii determine the invariant projections on the discrete torus $\mathbb{Z}_n \times \mathbb{Z}_n$, which in the present notation is $\{4,4\}_{n,0}$, and obtain a space of dimension $\lfloor n/2\rfloor(\lfloor n/2\rfloor - 1)/2$ for n ≥ 5 [10, Thm. 24]. Their values agree with ours at every n we have computed, from n = 3 to n = 9. What Propositions 10.6 and 10.7 add is the chiral half of the square class, which their method does not reach, together with both halves of the triangular class. The diagonal reflexible family $\{4,4\}_{b,b}$, whose capacities begin 0, 1, 2, 4, is covered by neither treatment, and we leave it open.

### 10.7 Non-orientable surfaces: the formula changes shape, and the converse fails

The last case is the one where the framework itself, and not merely the theorem, is in question. On a non-orientable surface there is no orientation character. The face module cannot be written as a permutation module twisted by ε, because the sign by which an automorphism acts on a face depends on the face as well as the automorphism; $C_2$ is a monomial module, not a twisted permutation module. Since N(·) and δ are both defined through ε, the orbit-count form of Theorem B has no analogue at all in this setting.

What survives is the exact sequence. For a map on any closed surface S the cycle space sits in

$$0 \to \ker \partial_2 \to C_2 \to Z(\Gamma) \to H_1(S; \mathbb{R}) \to 0,$$

and everything splits over ℝ. On an orientable surface $\ker \partial_2$ is the fundamental class, one-dimensional and carrying ε, which is what produces the $\mathbb{R}^F \ominus 1$ of Lemma 3.1. On a non-orientable surface there is no fundamental class over $\mathbb{R}$, so $\ker \partial_2 = 0$ and no dimension is subtracted. On the projective plane $H_1$ vanishes as well, and the sequence collapses to an isomorphism $Z(\Gamma) \cong C_2$. That is a testable structural claim, and it holds: for all six projective-plane maps below, dim Z equals |F| exactly.

For examples we take the antipodal quotients of the centrally symmetric polyhedra, which are the classical maps on the projective plane. The quotient of the dodecahedron is the Petersen graph with six pentagonal faces; the quotient of the icosahedron is $K_6$ with ten triangles; the quotient of the cube is $K_4$ with three quadrilaterals. The automorphism group of the quotient map contains $\mathrm{Aut}(\Gamma)/\langle\iota\rangle$, and the capacity is computed from that group; enlarging the group could only decrease the capacity, so the vanishing results below are unaffected by the possibility of extra automorphisms.

| map | \|V\| | \|E\| | \|F\| | \|Aut\| | 4\|E\| | flag-transitive | dim Z | c |
|---|---|---|---|---|---|---|---|---|
| hemi-cube ($K_4$) | 4 | 6 | 3 | 24 | 48 | yes | 3 | 0 |
| hemi-dodecahedron (Petersen) | 10 | 15 | 6 | 60 | 60 | yes | 6 | 0 |
| hemi-icosahedron ($K_6$) | 6 | 15 | 10 | 60 | 60 | yes | 10 | 0 |
| hemi-cuboctahedron | 6 | 12 | 7 | 24 | 48 | — | 7 | 0 |
| hemi-icosidodecahedron | 15 | 30 | 16 | 60 | 120 | — | 16 | 1 |

Table 7. Maps on the projective plane, obtained as antipodal quotients. dim Z equals |F| throughout, confirming that $\ker \partial_2 = 0$ and $H_1 = 0$ collapse the sequence to $Z \cong C_2$. The three hemi-Platonic maps are flag-transitive and have vanishing capacity. The hemi-cuboctahedron is not flag-transitive and its capacity vanishes anyway.

**Theorem 10.4.** On the projective plane the converse of Theorem A fails. The hemi-cuboctahedron has V = 6, E = 12, F = 7 and automorphism group of order 24, against 4|E| = 48, so it is not flag-transitive; and c = 0.

The value was confirmed by solving the equivariance equations directly on Hom(Z, K), of dimension 7 × 5, which returns the zero space. The hemi-icosidodecahedron, also not flag-transitive, has c = 1, so vanishing is not automatic for irregular maps on this surface either; it simply is not tied to regularity.

Putting the three surfaces together gives the following picture, and it is the sharpest statement we can make about the scope of Theorem A.

On the sphere both directions hold, which is Theorem A. On the torus the forward direction fails: there are flag-transitive maps with positive capacity — the smallest whose capacity is positive under both the map and the graph automorphism group is the 5 × 5 grid, for the reason set out in §10.2. On the projective plane the forward direction holds in every case we computed, but the converse fails: the hemi-cuboctahedron is irregular with vanishing capacity. Each half of the theorem is broken by a different surface, and the sphere is the only one of the three where the two conditions coincide.

That is a more informative outcome than a uniform failure would have been. It says the coincidence of "capacity zero" with "maximally symmetric" is not a robust relationship that the sphere happens to exhibit particularly cleanly; it is a coincidence of two independently fragile conditions, each of which can be broken on its own. Whatever explanatory weight one wanted to place on Theorem A as an instance of a general principle relating symmetry to interaction, this is where it stops. We have not tested the Klein bottle, and the non-orientable case beyond the projective plane, where $H_1$ no longer vanishes, remains open.

## Acknowledgements and disclosure

Computations were performed in Python using NetworkX, NumPy and SciPy, by two independent implementations. The first obtains automorphism groups as isometry groups of explicit coordinate realisations, cross-checked against known group orders. The second is coordinate-free: maps are built combinatorially in the dart model, and a map automorphism, being a dart permutation that commutes with the edge involution and either commutes with or inverts the rotation, is determined by the image of a single dart. Capacities are obtained from the orbit-count formula of Theorem 3.2. The two implementations agree on all thirty-two census capacities and group orders, and the second reproduces the closed forms of Proposition 5.1 on fifteen further prisms and antiprisms, the toroidal values of §10.4, and the closed forms of §10.6 on 183 chiral maps. Full source and the census data are available from the author.

An AI assistant (Anthropic Claude) was used extensively in the preparation of this work: to re-derive and independently verify every computation reported here, to construct the snub-solid coordinates, to conduct the literature search underlying §§1–2, and to draft the manuscript. All results were checked by explicit computation rather than accepted from the model, and one substantive error in an earlier draft (the omitted δ term, Remark 3.3) was found in the course of that checking. The author takes full responsibility for the content. Per the authorship criteria of the ICMJE and the policies of the major publishers, the assistant is not listed as an author.

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