PINBALL

Using Root Locus to Factor a Polynomial

By Professor Robert H. Cannon, Jr.

Author's Note: This explanation of the Pinball Method is based upon a 2001 description by Robert H. Cannon, Jr. in response to a request from Don Bently, founder of the Bently Corporation.

Dear Mr. Bently,

It is a pleasure to send you an example of the pinball method. The easy way for me to tell you about the method I remember is to use it to solve a simple, typical problem. You will then find it easy to generalize use of the method to any polynomial whose roots you would like to know. But you do have to put this in an historical context. There were at that time no digital computers. The Spirule is an ingenious analog substitute with which we were designing control systems at that time for inertial navigation stable platforms. Suppose you have the polynomial.

$$s^5 + 3s^4 + 7s^3 + 10s^2 + 18s + 80 = 0$$

I've used s because I'm used to working with polynomials that are the equations derived by assuming a solution to its differential equations of the form x = C e^(st); and because it takes only one stroke to make an s.

You simply rewrite it in the form:

$$s (s (s (s(s+3) + 7) + 10) + 18) + 80 = 0$$

Then you factor each bracketed term in turn, starting with the inside one, and using the root-locus method for each successive factoring, as we'll now demonstrate. I did all the original plots in this appendix with a Spirule! The accuracy is about two and a half significant figures. Today you would have a computer program to plot them for you in seconds with however many significant figures you wanted. In real life your data are typically good to only two or three significant figures anyway.

Step 1.

Factoring the first bracket is of course trivial; but let's plot its locus of roots anyway,

$$s(s+3) = -7$$

and then check the result algebraically:

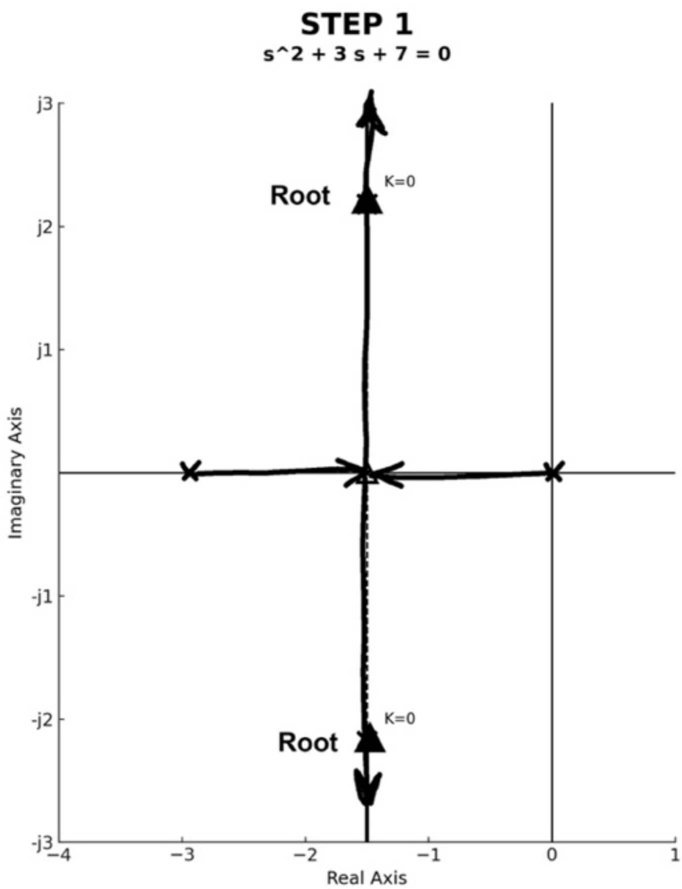
Roots: $s = -1.5 + j2.18.$ $s = -1.5 - j2.18$

or

$$\left(s + 1.5 + j\sqrt{7 - 1.5^2}\right)\left(s + 1.5 - j\sqrt{7 - 1.5^2}\right)$$

$$s^2 + \left(1.5 + j\sqrt{7 - 1.5^2} + 1.5 - j\sqrt{7 - 1.5^2}\right)s + 1.5^2 + \left(\sqrt{7 - 1.5^2}\right)^2$$

$s^2 + 3s + 7$



STEP 1
s^2 + 3 s + 7 = 0
Root
K=0
Root
K=0
j3
j2
j1
-j1
-j2
-j3
Imaginary Axis
−4
−3
−2
−1
0
1
Real Axis

Step 2

Factoring the second bracket is a classical third-order locus plot. The roots are found to be:

$$s = -2, \quad s = -0.5 + j\,2.18, \quad s = -0.5 - j\,2.18$$

which, of course, means that the second bracket in factored form is:

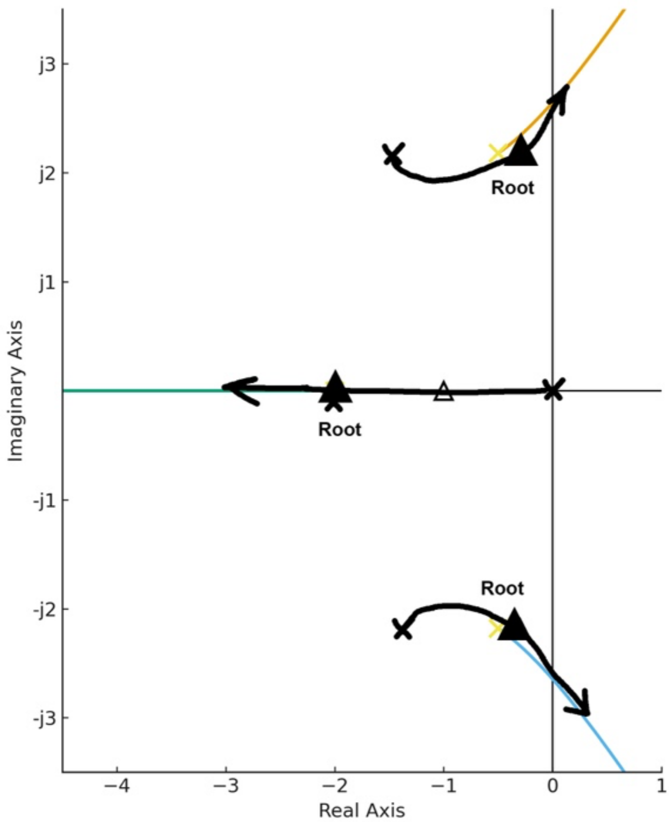
$$(s + 2)\,(s^2 + 0.5 \pm j\,2.18)$$

Again, it's easy to check by multiplying the factors out.

$$(s + 2)(\,s^2 + s + 5.0) \qquad = s^3 + 3s^2 + 7.0s + 10$$

STEP 2

s^3 + 3 s^2 + 7 s + 10 = 0



Step 3.
Factoring the third bracket is a fourth-order locus plot, which is a little more complex. The resulting set of roots are:

$$s = -1.75 + j\,1.45, \quad s = -1.75 - j\,1.45, \quad s = 0.2 + j\,1.85, \quad s = 0.2 - j\,1.85$$

So that the third bracket in factored form is:

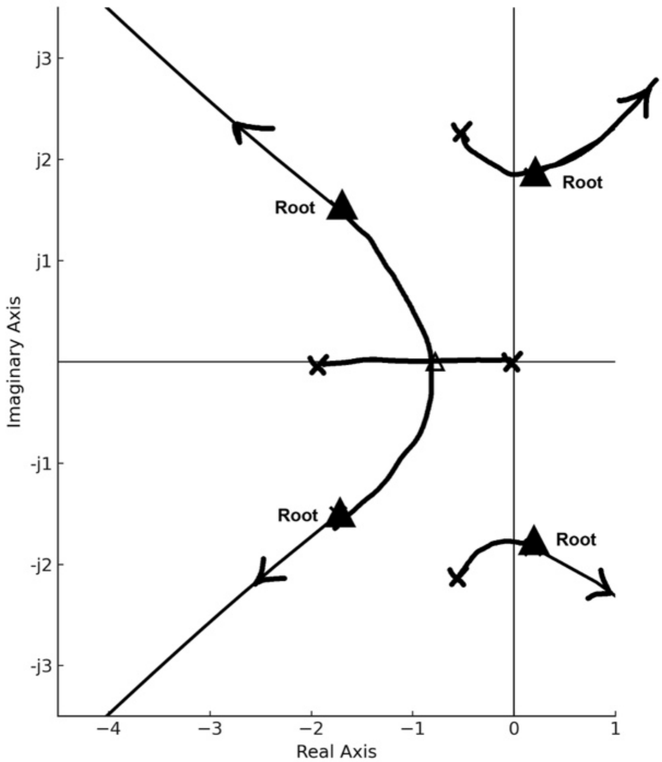
$$(s^2 + 3.5s + 5.3)\,(s^2 -.4\,s + 3.46)$$

When I multiply these factors out I begin to see a bit of error creeping in:

$$s^4 + 3.1s^3 + 7.4s^2 + 9.8s + 18.3$$

STEP 3

s^4 + 3.1 s^3 + 7.4 s^2 + 9.8 s + 18 = 0



Step 4.

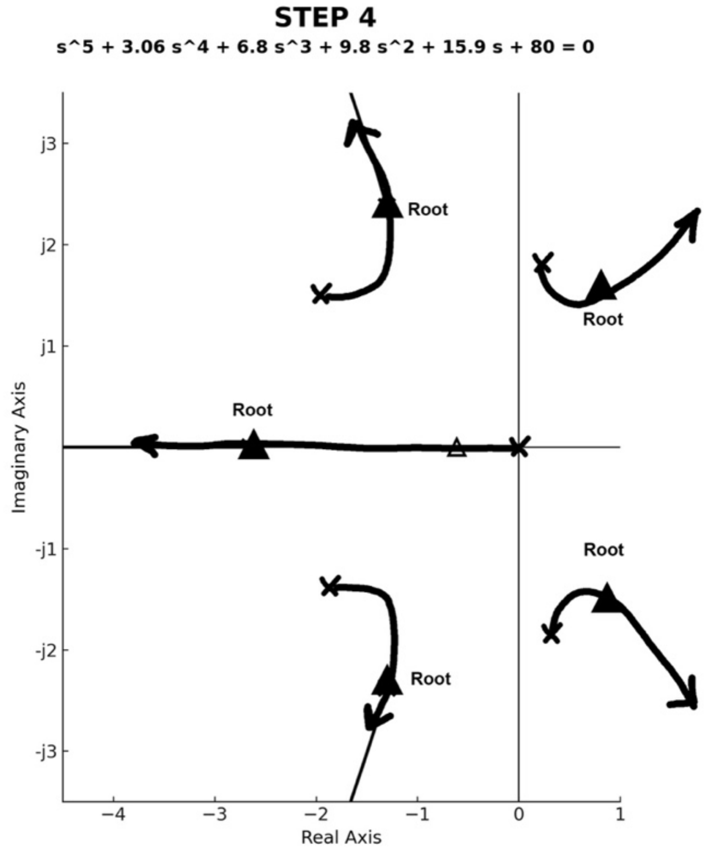
Factoring the fourth bracket equation is shown in the figure labeled Step 4. This is a fifth order root locus plot. The resulting roots are (to about two and one-half significant figures):

$$s \approx -2.66,\ \ s \approx -1.3 + j\,2.5,\ \ s \approx -1.3 - j\,2.5,\ \ s \approx 1.1 + j\,1.7,$$
$$s \approx 1.1 - j\,1.7$$

As a check, we simply multiply these factors out to obtain.

$$s^5 + 3.06s^4 + 6.8s^3 + 9.8s^2 + 15.9s + 80$$

This concludes the demonstration of the pinball method, in which each bracket is factored in turn, root loci plotted at each stage, and factors multiplied to verify the polynomial. While originally done with the spirule, the method translates directly to modern computational tools.


STEP 4
s^5 + 3.06 s^4 + 6.8 s^3 + 9.8 s^2 + 15.9 s + 80 = 0
Root
Root
Root
Root
Root
j3
j2
j1
-j1
-j2
-j3
Imaginary Axis
−4
−3
−2
−1
0
1
Real Axis