The Root-Locus Method

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The Characteristic Equation (CE) for a Linear System

At the heart of any automatic control system's design is what that system's natural dynamic behavior will be—how stable will it be and how quickly and well damped will its natural motions be following any disturbance.

Suppose we have a set of linear differential equations that completely describe the dynamics of a physical system. To determine the character of the natural dynamic behavior of which the system is capable, we represent that the behavior for each variable (y) in the equations will be of the form $y = Ye^{st}$. This assumption always works because, when you differentiate each one, you get back the original form multiplied by s:

[1] $dy/dt = sYe^{st}$

With this substitution, each differential equation is converted to an algebraic one. Then by combining the resulting algebraic equations to eliminate all the capital letters (Y's), you get one simple algebraic equation in s, of the form

[2] $s^5 + A_4 s^4 + A_3 s^3 + A_2 s^2 + A_1 s + A_0 = 0$

This is the system's CE. The next step is to factor it:

[3] $(s + \alpha)(s + \beta)(s + \sigma - j\omega)(s + \sigma + j\omega)(s + \gamma) = 0$

When the roots α, ß, and so on are each put into the original assumed motion for y, they connote quantitatively the character of one of the natural motions the system can have.
For example:

[4] $y = Y_1 e^{\alpha t}$

Any of the Greek letters might be zero, connoting that a root has the value 0. For the complex pair of roots, the value of ω indicates the frequency of a sinusoidal motion and the value of σ, the motion's rate of damping.

Factoring the Characteristic Equation

This is the crux of the matter: determining each root of the CE. For a control system, this equation can be so written that the part of the equation that contains the control variables (which are at the disposal of the designer) is separated from the rest of the CE. For example, the CE could turn out to be of the form:

[5] $s (s+b) (s^2 + Bs + C) + K (s + a) = 0$

where K and a are the control variables that can be selected. When multiplied out, [5] would give back the original characteristic equation, [2].

This equation can be written in the powerful Evans form:

[6] $[s (s+b) (s^2 + B s + C)] / (s + a) = - K$

Next, the values of s that make this equation correct for each value of K are in fact roots of the system's Characteristic

Equation for that K. Plotted in the complex plane (i.e., the "s-plane": the y-axis is the imaginary part of s and the x-axis is the real part of s), these values of s form the locus of roots.

The Root-Locus Plot

The way the plot is constructed begins with finding the roots for the value K = 0. Then, in principle, one must “search” for all the other points in the s-plane that satisfy Equation [6]. What makes Evans’s reformulation of [5] into [6] so powerful?

Evans noted that to determine whether a given location on the s-plane is on the locus of roots of a characteristic equation you simply need to determine whether or not the sum of the angles of the components of the factored characteristic equation of the system is 180 degrees there. Following Evans's incisive instinct, the search for roots is reduced to two simple parts. Part 1 is finding the loci of all the locations that simply meet the total phase angle 180 degrees that corresponds to the sign in [6]. For a trial point, the angles of the vectors are summed. If the total is 180 degrees, the point will satisfy [6]. The locus of all these points in the s-plane is the “root locus” of Equation [6].

Part 2 is establishing the magnitude of K at important points of the locus. For example, we might determine the value of K at which the roots pass from a stable condition to an unstable one (or vice versa)—that is, the value of K when the roots cross the imaginary axis in the s-plane and the real part of s transitions from a negative, decaying exponential to a positive, growing exponential. Stability requires all roots to lie in the left-half-plane. Another important value of K on the root locus occurs where a desired level of damping is achieved; this is defined as a line in the s-plane crossing the real axis at specified value for re(S).

The Spirule: For Plotting the Locus of Roots Quickly and Accurately

As noted, all of the above can be done in seconds today on a common powerful laptop computer. But in 1948 there were no digital computers. So, Evans -invented a plastic device for making the above computations, in minutes, without electricity! (Author's note: The Spirule's primary inventor was Jeff Schmidt, as described in Chapter 7 and Appendix 5.)

To make a Spirule, start with a circular plastic disk of diameter 4 1/2 inches. Using radial lines, mark directions 0°, 90°, 180°, and 270°. Next, make a tick mark for each degree on the circumference and label every 10 degrees with its value.

Now drill a hole in the center of the disk for a small circular clamp with a hole in its center. Clamp to the protractor and atop it a straight-edge ruler ("arm") that extends out 9 1/4 inches from the center of the hole. Give the clamp just enough friction to permit setting the arm at any angle and having it hold well. Finally, give the bottom edge of the clamp a sharp edge so that when you push down on it, it holds fast to the paper.

Part 1. The Locus of Roots

You are now ready to carry out Part 1 of the process for plotting the roots of any CE. Begin by putting the CE in the form of Equation [6] and plotting the roots for K = 0 as Xs on the s-plane. Next, choose a place where, for some value of K, a root is likely to be found. This is trial and error. However, it can be fast because you can use quick rules Evans developed to sketch where loci are likely to go before refining their location with a

Spirule. Each trial takes seconds. Thus, to check whether a given spot in the s-plane yields a 180° product of the vectors, simply (i) draw a straight line horizontally to the left from the spot, (ii) set the straight edge to 0° and place the center of the Spirule's hole over that spot, and (iii) add up the angles between the horizontal and each X by swinging the straight edge, holding the disk down while going from horizontal to X, and then letting the disk stick to the arm when swinging back. It's really a quick process to find the angle at the point being measured. Mark the value of the product on the plot. Then try another point until you find a 180° point.

Part 2. The Magnitude of K at any Point on the Locus of Roots

With the Spirule's center sitting on any point on a locus, it is entirely straightforward to obtain the value of K by measuring the lengths of the vectors (using the scale on the straight edge) and then multiplying the lengths together, as Equation [6] indicates. In 1948, the multiplications could be done by using a slide rule or by entering the lengths into a Marchant calculator and punching "multiply." Time-consuming! Evans invented a more elegant way. You use the logarithmic spiral on the arm of the Spirule. You have a slide rule, but you don't need to set any numbers into it. Instead, you set in the length of each vector graphically. To insert the length of a vector, put the center hole on its point and the reference line (R) on its tail, and then—holding the disk fixed—rotate the Spirule until its curve is on the vector's tail. After you've done this for all the vectors you want to multiply together, look at the output arrow and read their product from its scale. You now have the locus of roots plotted with the values of K at the points of most interest. The first step in designing a good control system is finished: It will be stable, with K selected to provide a good margin of stability despite possible changes in physical parameters. Its natural motions will be acceptably quick, and its oscillations will be acceptably well damped.