Symbols: $\forall$ = "for all" $\in$ = "belongs to"
$\exists$ = "there exists"

Def:

Let E be a set of real numbers
- A number M is called an upper bound for E if $\boxed{x \leq M, \ \forall x \in E}$
- A number m is called a lower bound for E if $\boxed{m \leq x, \ \forall x \in E}$

- We say a set is bounded if it has a lower and an upper bound.
- If there is no lower (upper) bound, we say E is "unbounded below" ("above")

Example: $1, 1.5, 3, 10^4$ are all upper bounds for $E = [0,1]$
$0, -1, -6, -100$ are all lower bounds for $E = [0,1]$

Def:

Let E be a set of real numbers.
- A number M is the maximum of the set E, denoted $M = \max(E)$ if:
  $\boxed{M \in E}$ AND $\boxed{M \text{ is an upper bound for } E}$
- A number m is the minimum of the set E, denoted $m = \min(E)$ if:
  $\boxed{m \in E}$ AND $\boxed{m \text{ is a lower bound for } E}$

Example: $E = [0,1]$ $\min(E) = 0$, $\max(E) = 1$

What if $E = (0,1)$?

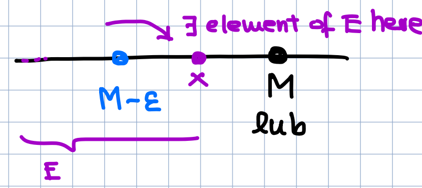
Can 0 be the minimum? No, because $0 \notin (0,1)$ and a min/max must live in the set!

But intuitively, 0 does feel like a "best" or "obvious" minimum, except we cannot call it a minimum. There is a term for this sort of situation, 0 will be a "greatest lower bound" or infimum.

Def.: Let E be a set of real numbers that is bounded above & non-empty. (LUB)

The supremum ("least upper bound") of E is a number M such that:

- M is an upper bound for E
- M is the least upper bound, i.e.

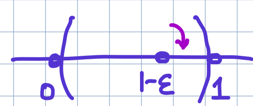


$$\forall \varepsilon>0, \ \exists \ x\in E \text{ s.t. } M-\varepsilon < x$$

(for all $\varepsilon>0$, there exists an element $x\in E$ s.t. $M-\varepsilon<x$)

If $E\neq\emptyset$ is unbounded above, we say simply $\sup(E)=\infty$.

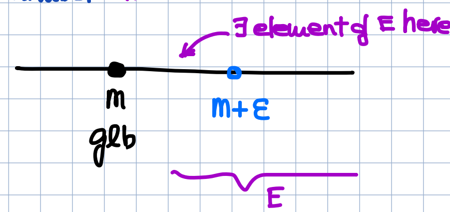
Example: $1=\sup(0,1)$



$\sup(1,\infty)=\infty$

Def.: Let E be a non-empty set of real numbers. If E is bounded below, (GLB)
the infimum ("greatest lower bound") of E is the number m such that:

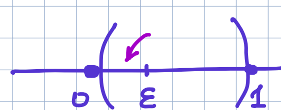
- m is a lower bound for E
- m is the greatest lower bound, i.e.



$$\forall \varepsilon>0, \ \exists \ x\in E \text{ s.t. } x< m+\varepsilon$$

If $E\neq\emptyset$ is unbounded below, we say $\inf(E)=-\infty$

Example: $0=\inf(0,1)$



$\inf \emptyset := \infty$

$\sup \emptyset := -\infty$

Example: Find the inf & sup of

$$E=\left\{\frac{1}{n} : n\in\mathbb{N}\right\}$$

$$E=\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots \right\}$$

Observe: $\lim_{n\to\infty} \frac{1}{n}=0$

Is this sequence increasing or decreasing?

$$\frac{1}{n} \ \boxed{>} \ \frac{1}{n+1}$$

$$n+1 \ \boxed{>} \ n \qquad \text{decreasing}$$

The sequence being decreasing immediately tells us that $a_1 = 1 \geq a_n,\ \forall n \in \mathbb{N}$

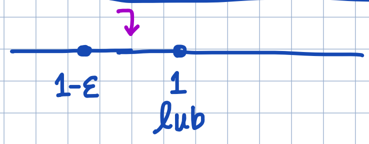
so $a_1 = 1$ is an upper bound for E, but moreover $1 \in E$, so really

$$\max(E) = \sup(E) = 1.$$

"need to show"

Side note: Why are they equal (i.e. why if a max exists, then it is the supremum?)

NTS: $\boxed{\forall \varepsilon > 0,\ \exists x \in E \text{ s.t. } x > 1 - \varepsilon}$ This is trivially true for $x = 1 \in E$.



What about inf? Since the sequence decreases to 0, it seems the inf should be 0.

Since no $\frac{1}{n}$ can ever equal 0, this would have to be an inf & not a min.

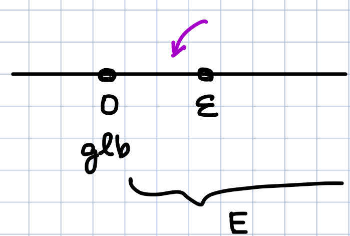
Now that we formulated a reasonable "guess", we have to actually prove that $\inf E = 0$

Claim: $\inf(E) = 0$

What would 0 have to do to be $\inf(E)$?

First of all, it needs to be a lower bound. Is it? Yes: $0 < \frac{1}{n},\ \forall n \in \mathbb{N}$.

Second, we would need to show:



$\boxed{\forall \varepsilon > 0,\ \exists x \in E \text{ s.t. } x < \varepsilon}$

$\boxed{\forall \varepsilon > 0,\ \exists n \in \mathbb{N} \text{ s.t. } \frac{1}{n} < \varepsilon}$ or $\frac{1}{\varepsilon} < n$

This seems obvious (and it is), but it must be justified, by something called the Archimedean Property of $\mathbb{R}$

This brings us to why we first need to set some ground rules about $\mathbb{R}$.

Properties of the Real Numbers

$\mathbb{N}$ = the natural numbers   1, 2, 3, ...

Two operations: addition & multiplication

Order relation: $m < n$

$\mathbb{Z}$ = the integers ... −3, −2, −1, 0, 1, 2, 3, ...

"Three" operations: addition, subtraction, multiplication

Order relation

$\mathbb{Q}$ = the rational numbers: $\mathbb{Q} = \left\{ \frac{m}{n} : m, n \in \mathbb{N} \right\}$

"Four" operations: addition, subtraction, multiplication, division

Order relation: $\frac{m}{n} < \frac{a}{b}$

$\mathbb{R}$ = the real numbers $\mathbb{R}$??   For analysis, we will define $\mathbb{R}$ to be a complete, ordered field.

(i.e. we will state clearly the rules)

I. Algebraic Structure

Many of the algebraic manipulations of real numbers you are used to performing (such as factoring) can be reduced to a small set of rules, and all other rules are just a consequence of those. There are many instances of objects that are not real numbers, but some other mathematical constructions, but that also follow the same algebraic rules. So this structure was given a name, field.

The field axioms: Assume $\mathbb{R}$ has two operations: "+" and "·"
(addition & multiplication) which satisfy the following:

(A1) $\forall a,b \in \mathbb{R}$, $\exists$ number $(a+b) \in \mathbb{R}$ and $a+b=b+a$. (commutativity of addition)

(A2) $\forall a,b,c \in \mathbb{R}$, there holds $(a+b)+c = a+(b+c)$ (associativity of addition)

(A3) $\exists$ unique number $0 \in \mathbb{R}$ s.t. $a+0 = 0+a = a$ (identity for addition)

(A4) $\forall a \in \mathbb{R}$, $\exists$ corresponding number denoted $-a \in \mathbb{R}$ such that $a+(-a)=0$ (inverse for addition)

These are all required for multiplication as well:

(M1) $\forall a,b \in \mathbb{R}$, $\exists$ number $ab \in \mathbb{R}$ and $ab = ba$

(M2) $\forall a,b,c \in \mathbb{R}$: $(ab)c = a(bc)$

(M3) $\exists$ unique number $1 \in \mathbb{R}$ s.t. $a \cdot 1 = 1 \cdot a = a$

(M4) $\forall a \in \mathbb{R}, a \neq 0$: $\exists$ corresponding number denoted $a^{-1} \in \mathbb{R}$ such that $aa^{-1} = 1$

Finally, addition & multiplication must interact in a certain way:

(AM1): $\forall a,b,c \in \mathbb{R}$: $(a+b)c = ac + bc$ (distributivity)

$$\underset{c(a+b)}{\underbrace{(a+b)c}_{\parallel}} = \overset{ca}{\overset{\parallel}{ac}} + \overset{cb}{\overset{\parallel}{bc}}$$

by commutativity of multiplication

(II) Order Structure

$\mathbb{R}$ is actually an <u>ordered</u> field:

(O1) $\forall a,b \in \mathbb{R}$, exactly <u>one</u> of the following statements is true: $a=b$, $a<b$, or $a>b$.

(O2) $\forall a,b,c \in \mathbb{R}$: If $a<b$ and $b<c$ are true, then $a<c$ is true.

(O3) $\forall a,b \in \mathbb{R}$: If $a<b$ is true, then $a+c<b+c$ is also true for <u>all</u> $c \in \mathbb{R}$.

(O4) $\forall a,b \in \mathbb{R}$: If $a<b$ is true, then $ac<bc$ is also true for <u>all</u> $c>0$.

III The Axiom of Completeness

Q: Can you think of a non-empty set, bounded above, which does not have a least upper bound? (it's OK if you can't!)

Completeness Axiom. A nonempty set of real numbers that is bounded above has a least upper bound.

So: $\mathbb{R}$ is a complete ordered field.

THEOREM: Archimedean Property of $\mathbb{R}$

The set of natural numbers $\mathbb{N}$ has no upper bound.

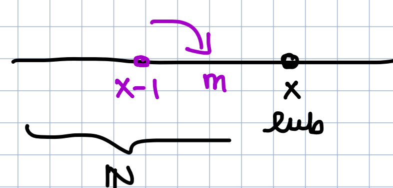
Proof (by contradiction)

Assume $\mathbb{N}$ has an upper bound.

By the Completeness Axiom, it must have a least upper bound, call it $x := \sup \mathbb{N}$ $(\in \mathbb{R})$.

Then

$n \leq x$ for all $n \in \mathbb{N}$



But there must be $m \in \mathbb{N}$ such that

$m > x - 1$

Then

$m + 1 > x$

but $m+1$ is also in $\mathbb{N}$ => contradiction! ∎

So, back to

$$\boxed{\forall \varepsilon > 0, \ \exists n \in \mathbb{N} \text{ s.t. } \frac{1}{n} < \varepsilon}$$

we can finish the proof: by the Archimedean property of $\mathbb{R}$, forall $\varepsilon > 0$, there is $n \in \mathbb{N}$ s.t. $\frac{1}{\varepsilon} < n$.

∎