MAE 4730 - FINAL PROJECT

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1. Problem Statement

The purpose of this project is to simulate and animate the the motions of a triple pendulum and a 4-bar linkage. For the triple-pendulum, the equations of motion must be found through three different methods. In both problems, numerical solutions must be checked though as many ways as possible, such as through verifying energy conservation and limiting cases where simple pendulum motion is expected.

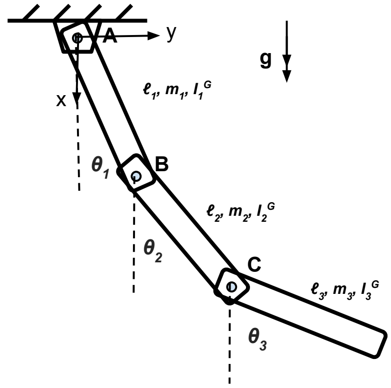

Figure 1. Setup for Triple Pendulum Problem

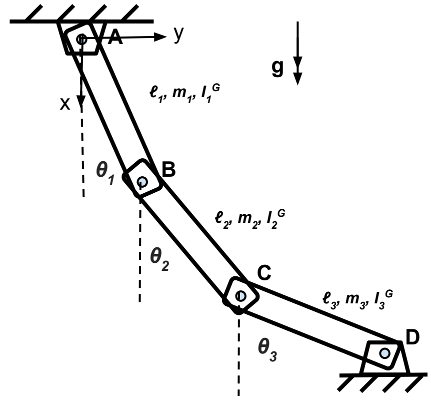

Figure 2. Setup for 4-Bar Linkage Problem.

Date: December 17, 2014 @ 2:00 pm.

2. Assumptions

The assumptions made for both problems are as follows:

(1) No friction or other non-conservative forces are present.
(2) Gravity acts in the downward direction.
(3) Links have uniform shape. In particular, when solving the numerical solutions, they are assumed rods with pins attached the end(s).
(4) Links do not change in length. Links stay connected to pin at joints.
(5) Pin connecting first link to wall (pin A in Figure 1), as well as the wall, is stationary.

3. Triple Pendulum Equations of Motion

The physical system we are modeling is shown in Figure 1. The motion of the triple pendulum will be solved for through the following three approaches: angular momentum balances (AMB), differential algebraic equations (DAE), and Lagrange equations

3.a. **AMB Approach.** First, we draw the three free body diagrams (FBDs) shown in Figure 3 for the triple pendulum system.

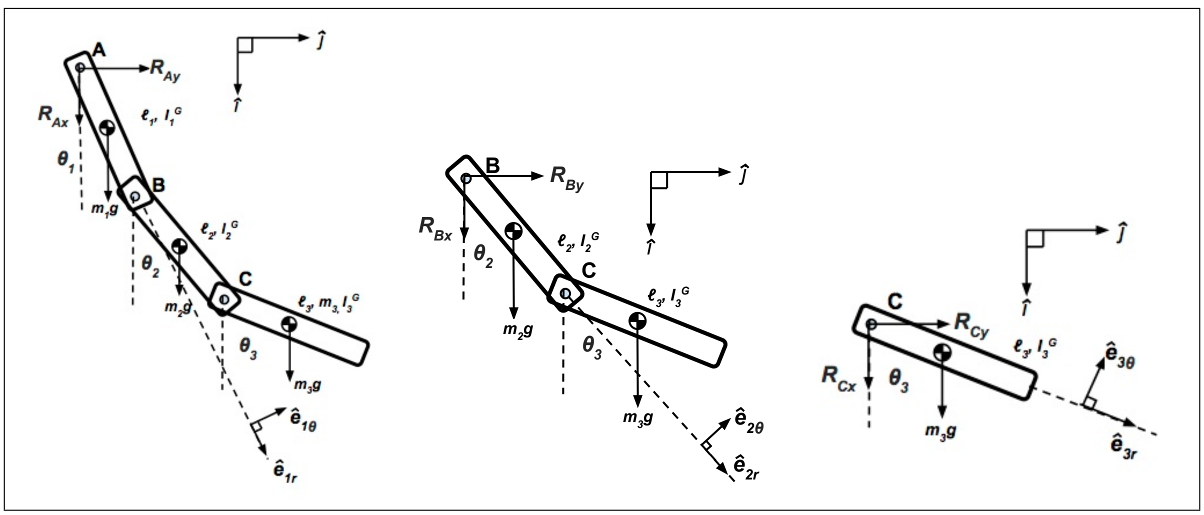

Figure 3. Three FBDs for AMB approach of triple pendulum problem

In the FBDs shown in Figure 3, the body coordinates of each link $i \in \{1, 2, 3\}$ are represented as follows:

$$\hat{e}_{ir} = \cos\theta_i\hat{\imath} + \sin\theta_i\hat{\jmath}, \hat{e}_{i\theta} = -\sin\theta_i\hat{\imath} + \cos\theta_i\hat{\jmath} \tag{1}$$

For each FBD, we then determine the position vector from each pin to the pins and center of gravity that follow down the link in terms of θ's and the body coordinates.

$$\vec{r}_{G_1/A} = \frac{l_1}{2}\hat{e}_{1r} \tag{2}$$

$$\vec{r}_{B/A} = l_1\hat{e}_{1r} \tag{3}$$

$$\vec{r}_{G_2/A} = \vec{r}_{B/A} + \frac{l_2}{2}\hat{e}_{2r} \tag{4}$$

$$\vec{r}_{C/A} = \vec{r}_{B/A} + l_2\hat{e}_{2r} \tag{5}$$

$$\vec{r}_{G_3/A} = \vec{r}_{C/A} + \frac{l_3}{2}\hat{e}_{3r} \tag{6}$$

$$\vec{r}_{G_2/B} = \frac{l_2}{2}\hat{e}_{2r} \tag{7}$$

$$\vec{r}_{C/B} = l_2\hat{e}_{2r} \tag{8}$$

$$\vec{r}_{G_3/B} = \vec{r}_{C/B} + \frac{l_3}{2}\hat{e}_{3r} \tag{9}$$

$$\vec{r}_{G_3/C} = \frac{l_3}{2}\hat{e}_{3r} \tag{10}$$

Similarly, we solve for the accelerations of each pin and center of gravity down the pendulum in the intertial frame of each FBD; for a combination of points $p \in A, B, C, G_1, G_2, G_3$ and links $i \in 1, 2, 3$, we use the 5 term acceleration formula provided below.

$$\vec{a}_{G_1/\mathcal{F}} = \vec{a}_{A/\mathcal{F}} + \vec{a}_{G_1/\mathcal{B}} + \vec{\omega}_1 \times \left(\vec{\omega}_1 \times \vec{r}_{G_1/A}\right) + \dot{\vec{\omega}}_1 \times \vec{r}_{G_1/A} + 2\vec{\omega}_1\dot{\vec{r}}_{G_1/A} \tag{11}$$

Assuming that the rods do not change in length, each acceleration in the inertial frame is computed below.

$$\vec{a}_{G_1} = \frac{l_1}{2}\big(- \dot{\theta}_1^2\hat{e}_{1r} + \ddot{\theta}_1\hat{e}_{1\theta}\big) \tag{12}$$

$$\vec{a}_B = l_1\big(- \dot{\theta}_1^2\hat{e}_{1r} + \ddot{\theta}_1\hat{e}_{1\theta}\big) \tag{13}$$

$$\vec{a}_{G_2} = \vec{a}_B + \frac{l_2}{2}\big(- \dot{\theta}_2^2\hat{e}_{2r} + \ddot{\theta}_2\hat{e}_{1\theta}\big) \tag{14}$$

$$\vec{a}_C = \vec{a}_B + l_2\big(- \dot{\theta}_2^2\hat{e}_{2r} + \ddot{\theta}_2\hat{e}_{2\theta}\big) \tag{15}$$

$$\vec{a}_{G_3} = \vec{a}_C + \frac{l_3}{2}\big(- \dot{\theta}_3^2\hat{e}_{3r} + \ddot{\theta}_3\hat{e}_{3\theta}\big) \tag{16}$$

Finally, we set the sum of moments about A, B, and C in each FBD diagram, respectively, equal to the change in angular momentum about that point; for $p \in \{A, B, C\}$, where G_i represents the center of mass for link i, we solve the following.

$$\sum M_{/p} = \dot{\vec{H}}_{/p} \tag{17}$$

Applying Equation 17 to each FBD, we arrive at the following equations.

$$\vec{r}_{G_1/A} \times m_1 g\hat{\imath} + \vec{r}_{G_2/A} \times m_2 g\hat{\imath} + \vec{r}_{G_3/A} \times m_3 g\hat{\imath} = \vec{r}_{G_1/A} \times m_1 \vec{a}_{G_1} + \vec{r}_{G_2/A} \times m_2 \vec{a}_{G_2} + \vec{r}_{G_3/A} \times m_3 \vec{a}_{G_3} \\ I^{G_1}\ddot{\theta}_1\hat{k} + I^{G_2}\ddot{\theta}_2\hat{k} + I^{G_3}\ddot{\theta}_3\hat{k} \tag{18}$$

$$\vec{r}_{G_2/B} \times m_2 g\hat{\imath} + \vec{r}_{G_3/B} \times m_3 g\hat{\imath} = \vec{r}_{G_2/B} \times m_2 \vec{a}_{G_2} + \vec{r}_{G_3/B} \times m_3 \vec{a}_{G_3} \\ I^{G_2}\ddot{\theta}_2\hat{k} + I^{G_3}\ddot{\theta}_3\hat{k} \tag{19}$$

$$\vec{r}_{G_3/C} \times m_3 g\hat{\imath} = \vec{r}_{G_3/C} \times m_3 \vec{a}_{G_3} + I^{G_3}\ddot{\theta}_3\hat{k} \tag{20}$$

By substituting in the accelerations from Equations 12 - 16, the positions from Equations 2 - 10, and, in turn, the body-coordinate unit vectors from Equation 1, we can solve the equations of motion stated above for the angles θ_1, θ_2, and θ_3. Because this substitution requires a tremendous amount of algebra, it was done symbolically using MATLAB to solve for $\ddot{\theta}_1$, $\ddot{\theta}_2$, and $\ddot{\theta}_3$ (refer to the function `MakeTriplePendSolverFile` in the Appendix).

3.b. **DAE Approach.** For the DAE approach, we first choose the maximal coordinates θ_1, θ_2, θ_3, x_{G_1}, x_{G_2}, x_{G_3}, y_{G_1}, y_{G_2}, and y_{G_3}, where x_{G_i} and y_{G_i} mark the center of gravity position of link i based on the coordinate system shown in Figure 1. To solve for these values, the three FBDs shown in Figure 4 are used.

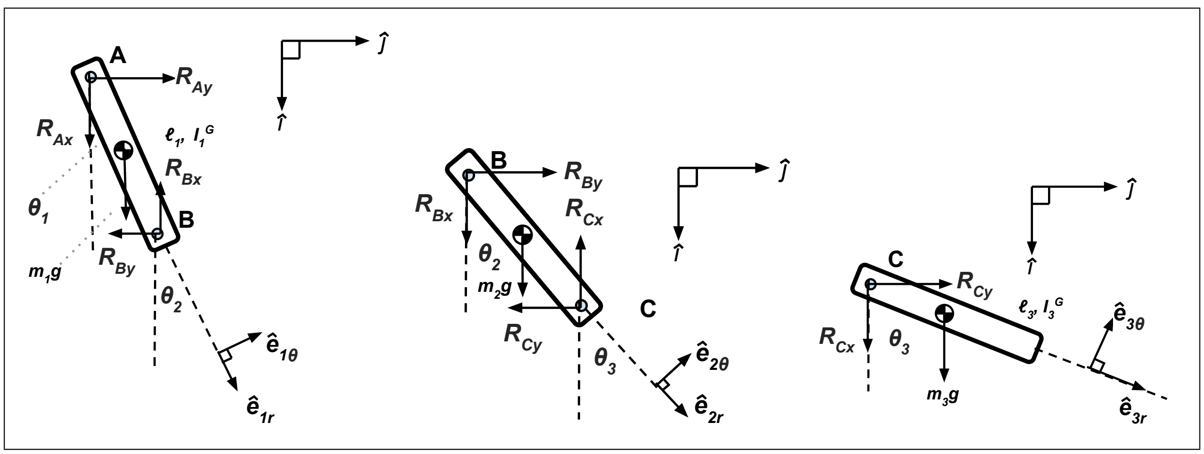

FIGURE 4. Three FBDs for DAE approach of triple pendulum problem

First, the linear momentum balance of external forces $\{F_1, F_2, ..., F_n\}$,

$$\sum_{j=1}^{n} F_j = m_i a_{G_i} \tag{21}$$

is performed for each link $i \in \{1, 2, 3\}$ in the three FBDs provided above, resulting in the following equations.

$$(m_1 g + R_{Ax} - R_{Bx})\hat{\imath} + (R_{Ay} - R_{By})\hat{\jmath} = m_1(\ddot{x}_{G_1}\hat{\imath} + \ddot{y}_{G_1}\hat{\jmath}) \tag{22}$$
$$(m_2 g + R_{Bx} - R_{Cx})\hat{\imath} + (R_{By} - R_{Cy})\hat{\jmath} = m_2(\ddot{x}_{G_2}\hat{\imath} + \ddot{y}_{G_2}\hat{\jmath}) \tag{23}$$
$$(m_3 g + R_{Cx})\hat{\imath} + (R_{Cy})\hat{\jmath} = m_3(\ddot{x}_{G_3}\hat{\imath} + \ddot{y}_{G_3}\hat{\jmath}) \tag{24}$$

By dotting Equations 22 - 24 with $\hat{\imath}$, we obtain the following set of equations.

$$m_1\ddot{x}_{G_1} - R_{Ax} + R_{Bx} = m_1 g \tag{25}$$
$$m_2\ddot{x}_{G_2} - R_{Bx} + R_{Cx} = m_2 g \tag{26}$$
$$m_3\ddot{x}_{G_3} - R_{Cx} = m_3 g \tag{27}$$

Similarly, we obtain a second set of equations by dotting Equations 22 - 24 with with $\hat{\jmath}$.

$$m_1\ddot{y}_{G_1} - R_{Ay} + R_{By} = 0 \tag{28}$$
$$m_2\ddot{y}_{G_2} - R_{By} + R_{Cy} = 0 \tag{29}$$
$$m_3\ddot{y}_{G_3} - R_{Cy} = 0 \tag{30}$$

Then, we perform an angular momentum balance about each rod's center of mass, which results in the following three equations.

$$\vec{r}_{A/G_1} \times R_{Ax}\hat{\imath} + \vec{r}_{A/G_1} \times R_{Ay}\hat{\jmath} + \vec{r}_{B/G_1} \times -R_{Bx}\hat{\imath} + \vec{r}_{B/G_1} \times -R_{By}\hat{\jmath} = I^{G_1}\ddot{\theta}_1\hat{k} \tag{31}$$
$$\vec{r}_{B/G_2} \times R_{Bx}\hat{\imath} + \vec{r}_{B/G_2} \times R_{By}\hat{\jmath} + \vec{r}_{C/G_2} \times -R_{Cx}\hat{\imath} + \vec{r}_{C/G_2} \times -R_{Cy}\hat{\jmath} = I^{G_2}\ddot{\theta}_2\hat{k} \tag{32}$$
$$\vec{r}_{C/G_3} \times R_{Cx}\hat{\imath} + \vec{r}_{C/G_3} \times R_{Cy}\hat{\jmath} = I^{G_3}\ddot{\theta}_3\hat{k} \tag{33}$$

By substituting in $\vec{r}_{A/G_1} = -\vec{r}_{B/G_1} = -\frac{l_1}{2}\hat{e}_{1r}$, $\vec{r}_{B/G_2} = -\vec{r}_{C/G_2} = -\frac{l_2}{2}\hat{e}_{2r}$, and $\vec{r}_{C/G_3} = -\frac{l_3}{2}\hat{e}_{3r}$ into 31 - 33, and dotting the result by $\hat{k}$, we arrive at three new equations.

$$-I^{G_1}\ddot{\theta}_1 + \left(\frac{l_1}{2}\sin\theta_1\right)R_{Ax} - \left(\frac{l_1}{2}\cos\theta_1\right)R_{Ay} + \left(\frac{l_1}{2}\sin\theta_1\right)R_{Bx} - \left(\frac{l_1}{2}\cos\theta_1\right)R_{By} = 0 \tag{34}$$
$$-I^{G_2}\ddot{\theta}_2 + \left(\frac{l_2}{2}\sin\theta_2\right)R_{Bx} - \left(\frac{l_2}{2}\cos\theta_2\right)R_{By} + \left(\frac{l_2}{2}\sin\theta_2\right)R_{Cx} - \left(\frac{l_2}{2}\cos\theta_2\right)R_{Cy} = 0 \tag{35}$$
$$-I^{G_3}\ddot{\theta}_3 + \left(\frac{l_3}{2}\sin\theta_3\right)R_{Cx} - \left(\frac{l_3}{2}\cos\theta_3\right)R_{Cy} = 0 \tag{36}$$

Next, using the constraints on each pin $p \in \{A, B, C\}$, we identify equal representations of each acceleration $\vec{a}_p$ using the 5-term acceleration formula provided in Equation 11. For pin A, we solve for the following equation.

$$\begin{aligned}\vec{a}_A &= \vec{a}_A \\ \vec{0} &= \vec{a}_{G_1} + \vec{a}_{A/G_1} \\ \vec{0} &= \ddot{x}_{G_1}\hat{\imath} + \ddot{y}_{G_1}\hat{\jmath} + \ddot{\theta}_1\hat{k}\times\vec{r}_{A/G_1} + \dot{\theta}_1\hat{k}\times\big(\dot{\theta}_1\hat{k}\times\vec{r}_{A/G_1}\big)\end{aligned} \tag{37}$$

Noting that $\vec{r}_{A/G_1} = -\frac{l_1}{2}\hat{e_1}_r = -\frac{l_1}{2}\big(\cos\theta_1\hat{\imath} + \sin\theta_1\hat{\jmath}\big)$, Equation 37 can be simplified to the following equations after taking the $\hat{\imath}$ and $\hat{\jmath}$ components, respectively.

$$\ddot{x}_{G_1} + \big(\frac{l_1}{2}\sin\theta_1\big)\ddot{\theta_1} = -\big(\frac{l_1}{2}\cos\theta_1\big)\dot{\theta}_1^2 \tag{38}$$

$$\ddot{y}_{G_1} - \big(\frac{l_1}{2}\cos\theta_1\big)\ddot{\theta_1} = -\big(\frac{l_1}{2}\sin\theta_1\big)\dot{\theta}_1^2 \tag{39}$$

Similarly, we can obtain four more second-order differential equations by looking at hinges A and B, as follows.

$$\begin{aligned}\vec{a}_B &= \vec{a}_B \\ \vec{a}_{G_1} + \vec{a}_{B/G_1} &= \vec{a}_{G_2} + \vec{a}_{B/G_2}\end{aligned}$$

$$\Big\{\ddot{x}_{G_1}\hat{\imath} + \ddot{y}_{G_1}\hat{\jmath} + \ddot{\theta}_1\hat{k}\times\vec{r}_{B/G_1} + \dot{\theta}_1\hat{k}\times\big(\dot{\theta}_1\hat{k}\times\vec{r}_{B/G_1}\big) = \ddot{x}_{G_2}\hat{\imath} + \ddot{y}_{G_2}\hat{\jmath} + \ddot{\theta}_2\hat{k}\times\vec{r}_{B/G_2} + \dot{\theta}_2\hat{k}\times\big(\dot{\theta}_2\hat{k}\times\vec{r}_{B/G_2}\big)\Big\}$$

$$\Big\{\Big\}\cdot\hat{\imath} \Rightarrow \ddot{x}_{G_2} - \ddot{x}_{G_1} + \big(\frac{l_1}{2}\sin\theta_1\big)\ddot{\theta_1} + \big(\frac{l_2}{2}\sin\theta_2\big)\ddot{\theta_2} = -\big(\frac{l_1}{2}\cos\theta_1\big)\dot{\theta}_1^2 - \big(\frac{l_2}{2}\cos\theta_2\big)\dot{\theta}_2^2 \tag{40}$$

$$\Big\{\Big\}\cdot\hat{\jmath} \Rightarrow \ddot{y}_{G_2} - \ddot{y}_{G_1} - \big(\frac{l_1}{2}\cos\theta_1\big)\ddot{\theta_1} - \big(\frac{l_2}{2}\cos\theta_2\big)\ddot{\theta_2} = -\big(\frac{l_1}{2}\sin\theta_1\big)\dot{\theta}_1^2 - \big(\frac{l_2}{2}\sin\theta_2\big)\dot{\theta}_2^2 \tag{41}$$

$$\begin{aligned}\vec{a}_C &= \vec{a}_C \\ \vec{a}_{G_2} + \vec{a}_{C/G_2} &= \vec{a}_{G_3} + \vec{a}_{C/G_3}\end{aligned}$$

$$\Big\{\ddot{x}_{G_2}\hat{\imath} + \ddot{y}_{G_2}\hat{\jmath} + \ddot{\theta}_2\hat{k}\times\vec{r}_{C/G_2} + \dot{\theta}_2\hat{k}\times\big(\dot{\theta}_2\hat{k}\times\vec{r}_{C/G_2}\big) = \ddot{x}_{G_3}\hat{\imath} + \ddot{y}_{G_3}\hat{\jmath} + \ddot{\theta}_3\hat{k}\times\vec{r}_{C/G_3} + \dot{\theta}_3\hat{k}\times\big(\dot{\theta}_3\hat{k}\times\vec{r}_{C/G_3}\big)\Big\}$$

$$\Big\{\Big\}\cdot\hat{\imath} \Rightarrow \ddot{x}_{G_3} - \ddot{x}_{G_2} + \big(\frac{l_2}{2}\sin\theta_2\big)\ddot{\theta_2} + \big(\frac{l_3}{2}\sin\theta_3\big)\ddot{\theta_3} = -\big(\frac{l_2}{2}\cos\theta_2\big)\dot{\theta}_2^2 - \big(\frac{l_3}{2}\cos\theta_3\big)\dot{\theta}_3^2 \tag{42}$$

$$\Big\{\Big\}\cdot\hat{\jmath} \Rightarrow \ddot{y}_{G_3} - \ddot{y}_{G_2} - \big(\frac{l_2}{2}\cos\theta_2\big)\ddot{\theta_2} - \big(\frac{l_3}{2}\cos\theta_3\big)\ddot{\theta_3} = -\big(\frac{l_2}{2}\sin\theta_2\big)\dot{\theta}_2^2 - \big(\frac{l_3}{2}\sin\theta_3\big)\dot{\theta}_3^2 \tag{43}$$

Finally, we combine Equations 25 - 30, 34 - 36, and 38 - 43 into Matrix form, shown on the next page and represented as follows.

$$\begin{aligned} \mathbf{A}\vec{z} &= \vec{b} \\ \vec{z} &= \mathbf{A}^{-1}\vec{b} \end{aligned} \tag{44}$$

By solving Equation 44, the equations of motion can be extracted for the three bars. This equation is solved in the MATLAB `ODE45` right-hand-side function through the command `z = A \ b` (refer to the Appendix).

$$\mathbf{A} = \begin{pmatrix}
m_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\
0 & m_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\
0 & 0 & m_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & m_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\
0 & 0 & 0 & 0 & m_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\
0 & 0 & 0 & 0 & 0 & m_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & -I^{G_1} & 0 & 0 & \left(\frac{l_1}{2}\sin\theta_1\right) & \left(\frac{l_1}{2}\sin\theta_1\right) & 0 & -\left(\frac{l_1}{2}\cos\theta_1\right) & \left(-\frac{l_1}{2}\cos\theta_1\right) & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -I^{G_2} & 0 & 0 & \left(\frac{l_2}{2}\sin\theta_2\right) & \left(\frac{l_2}{2}\sin\theta_2\right) & 0 & -\left(\frac{l_2}{2}\cos\theta_2\right) & -\left(\frac{l_2}{2}\cos\theta_2\right) \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -I^{G_3} & 0 & 0 & \left(\frac{l_3}{2}\sin\theta_3\right) & 0 & 0 & \left(-\frac{l_3}{2}\cos\theta_3\right) \\
1 & 0 & 0 & 0 & 0 & 0 & \left(\frac{l_1}{2}\sin\theta_1\right) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 1 & 0 & 0 & 0 & 0 & \left(\frac{l_1}{2}\sin\theta_1\right) & \left(\frac{l_2}{2}\sin\theta_2\right) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 1 & 0 & 0 & 0 & 0 & \left(\frac{l_2}{2}\sin\theta_2\right) & \left(\frac{l_3}{2}\sin\theta_3\right) & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & -\left(\frac{l_1}{2}\cos\theta_1\right) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 1 & 0 & -\left(\frac{l_1}{2}\cos\theta_1\right) & -\left(\frac{l_2}{2}\cos\theta_2\right) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 1 & 0 & -\left(\frac{l_2}{2}\cos\theta_2\right) & -\left(\frac{l_3}{2}\cos\theta_3\right) & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

$$\vec{z} = \begin{bmatrix} \ddot{x}_{G_1} \\ \ddot{x}_{G_2} \\ \ddot{x}_{G_3} \\ \ddot{y}_{G_1} \\ \ddot{y}_{G_2} \\ \ddot{y}_{G_3} \\ \ddot{\theta}_1 \\ \ddot{\theta}_2 \\ \ddot{\theta}_3 \\ R_{Ax} \\ R_{Bx} \\ R_{Cx} \\ R_{Ay} \\ R_{By} \\ R_{Cy} \end{bmatrix}$$

$$\vec{b} = \begin{bmatrix} m_1 g \\ m_2 g \\ m_3 g \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -\left(\frac{l_1}{2}\cos\theta_1\right)\dot{\theta}_1^2 \\ -\left(\frac{l_1}{2}\cos\theta_1\right)\dot{\theta}_1^2 - \left(\frac{l_2}{2}\cos\theta_2\right)\dot{\theta}_2^2 \\ -\left(\frac{l_2}{2}\cos\theta_2\right)\dot{\theta}_2^2 - \left(\frac{l_3}{2}\cos\theta_3\right)\dot{\theta}_3^2 \\ -\left(\frac{l_1}{2}\sin\theta_1\right)\dot{\theta}_1^2 \\ -\left(\frac{l_1}{2}\sin\theta_1\right)\dot{\theta}_1^2 - \left(\frac{l_2}{2}\sin\theta_2\right)\dot{\theta}_2^2 \\ -\left(\frac{l_2}{2}\sin\theta_2\right)\dot{\theta}_2^2 - \left(\frac{l_3}{2}\sin\theta_3\right)\dot{\theta}_3^2 \end{bmatrix}$$

3.c. **Lagrange Equations Approach.** The third approach used to solve the triple pendulum problem involves solving the lagrange equation,

$$\frac{d}{dt}\Big(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\Big) - \frac{\partial \mathcal{L}}{\partial q_i} = Q_i, \tag{45}$$

where q_i is the generalized coordinate, Q_i are the generalized forces associated with each q_i, and the Lagrangian $\mathcal{L}$ is calculated as the kinetic subtracted by the potential energy of the system,

$$\mathcal{L} = T - U \tag{46}$$

For this problem, we will use the general coordinates θ_1, θ_2, and θ_3 illustrated in Figure 1, and will take our reference point for the potential energy to be the stationary point A. Thus, the potential energy of the system is defined as follows, where h_{G_i} corresponds to the vertical height of the center of mass of link i relative to point A.

$$\begin{aligned} U &= m_1 g h_{G_1} + m_2 g h_{G_2} + m_3 g h_{G_3} \\ &= m_1 g\big(\vec{r}_{G_1/A} \cdot \hat{\jmath}\big) + m_2 g\big(\vec{r}_{B/A} \cdot \hat{\jmath} + \vec{r}_{G_2/B} \cdot \hat{\jmath}\big) + m_3 g\big(\vec{r}_{B/A} \cdot \hat{\jmath} + \vec{r}_{C/B} \cdot \hat{\jmath} + \vec{r}_{G_3/C} \cdot \hat{\jmath}\big) \\ &= m_1 g\big(-\frac{l_1}{2}\cos\theta_1\big) + m_2 g\big(-l_1\cos\theta_1 - \frac{l_2}{2}\cos\theta_2\big) + m_3 g\big(-l_1\cos\theta_1 - l_2\cos\theta_2 + -\frac{l_3}{2}\cos\theta_3\big) \\ &= -g l_1 \cos\theta_1\big(\frac{m_1}{2} + m_2 + m_3\big) - g l_2\cos\theta_2\big(\frac{m_2}{2} + m_3\big) - g l_3 \cos\theta_3\big(\frac{m_3}{2}\big) \end{aligned} \tag{47}$$

Next, the linear kinetic energy T is calculated using the squared velocities $V_{G_i}^2$ of each link i, which are calculated as follows, assuming that point A is stationary and that the rod lengths are fixed.

$$
\begin{aligned}
V_{G_1}^2 &= ||\frac{d}{dt}\vec{r}_{G_1/A}||^2 \\
&= ||\frac{l_1}{2}\dot{\theta}_1\hat{e}_{1\theta}||^2 \\
&= \big(\frac{l_1}{2}\dot{\theta}_1\big)^2 \qquad (48)
\end{aligned}
$$

$$
\begin{aligned}
V_{G_2}^2 &= ||\frac{d}{dt}\big(\vec{r}_{B/A} + \vec{r}_{G_2/B}\big)||^2 \\
&= ||\frac{d}{dt}\big(l_1\hat{e}_{1r} + \frac{l_2}{2}\hat{e}_{2r}\big)||^2 \\
&= ||\frac{d}{dt}\big(l_1\dot{\theta}_1\hat{e}_{1\theta} + \frac{l_2}{2}\dot{\theta}_2\hat{e}_{2\theta}||^2 \\
&= \big(-l_1\dot{\theta}_1\sin\theta_1 - \frac{l_2}{2}\dot{\theta}_2\sin\theta_2\big)^2 + \big(l_1\dot{\theta}_1\cos\theta_1 + \frac{l_2}{2}\dot{\theta}_2\cos\theta_2\big)^2 \qquad (49)
\end{aligned}
$$

$$
\begin{aligned}
V_{G_3}^2 &= ||\frac{d}{dt}\big(\vec{r}_{B/A} + \vec{r}_{C/B} + \vec{r}_{G_3/C}\big)||^2 \\
&= ||\frac{d}{dt}\big(l_1\hat{e}_{1r} + l_2\hat{e}_{2r} + \frac{l_3}{2}\hat{e}_{3r}\big)||^2 \\
&= ||\big(l_1\dot{\theta}_1\hat{e}_{1\theta} + l_2\dot{\theta}_2\hat{e}_{2\theta} + \frac{l_3}{2}\dot{\theta}_3\hat{e}_{3\theta}||^2 \\
&= \big(-l_1\dot{\theta}_1\sin\theta_1 - l_2\dot{\theta}_2\sin\theta_2 - \frac{l_3}{2}\dot{\theta}_3\sin\theta_3\big)^2 + \big(l_1\dot{\theta}_1\cos\theta_1 + l_2\dot{\theta}_2\cos\theta_2 + \frac{l_3}{2}\dot{\theta}_3\cos\theta_3\big)^2 \qquad (50)
\end{aligned}
$$

Using the velocities derived in Equations 48 - 50, and taking into account rotational kinetic energy, we define the total kinetic energy T of the system as follows.

$$
\begin{aligned}
T &= \frac{1}{2}I^{G_1}\dot{\theta}_1^2 + \frac{1}{2}I^{G_2}\dot{\theta}_1^2 + \frac{1}{2}I^{G_3} + \frac{1}{2}m_1V_{G_1}^2 + \frac{1}{2}m_2V_{G_2}^2 + \frac{1}{2}m_3V_{G_3}^2 \\
&= \frac{1}{2}I^{G_1}\dot{\theta}_1^2 + \frac{1}{2}I^{G_2}\dot{\theta}_1^2 + \frac{1}{2}I^{G_3} + \frac{1}{2}m_1\big(\frac{l_1}{2}\dot{\theta}_1\big)^2 \\
&\quad + \frac{1}{2}m_2\Big[\big(-l_1\dot{\theta}_1\sin\theta_1 - \frac{l_2}{2}\dot{\theta}_2\sin\theta_2\big)^2 + \big(l_1\dot{\theta}_1\cos\theta_1 + \frac{l_2}{2}\dot{\theta}_2\cos\theta_2\big)^2\Big] \\
&\quad + \frac{1}{2}m_3\Big[\big(-l_1\dot{\theta}_1\sin\theta_1 - l_2\dot{\theta}_2\sin\theta_2 - \frac{l_3}{2}\dot{\theta}_3\sin\theta_3\big)^2 + \big(l_1\dot{\theta}_1\cos\theta_1 + l_2\dot{\theta}_2\cos\theta_2 + \frac{l_3}{2}\dot{\theta}_3\cos\theta_3\big)^2\Big] \qquad (51)
\end{aligned}
$$

Therefore, the Lagrangian $\mathcal{L}$ is represented as

$$
\begin{aligned}
\mathcal{L} &= T - U \\
&= \frac{1}{2}I^{G_1}\dot{\theta}_1^2 + \frac{1}{2}I^{G_2}\dot{\theta}_1^2 + \frac{1}{2}I^{G_3} + \frac{1}{2}m_1\big(\frac{l_1}{2}\dot{\theta}_1\big)^2 \\
&\quad + \frac{1}{2}m_2\Big[\big(-l_1\dot{\theta}_1\sin\theta_1 - \frac{l_2}{2}\dot{\theta}_2\sin\theta_2\big)^2 + \big(l_1\dot{\theta}_1\cos\theta_1 + \frac{l_2}{2}\dot{\theta}_2\cos\theta_2\big)^2\Big] \\
&\quad + \frac{1}{2}m_3\Big[\big(-l_1\dot{\theta}_1\sin\theta_1 - l_2\dot{\theta}_2\sin\theta_2 - \frac{l_3}{2}\dot{\theta}_3\sin\theta_3\big)^2 + \big(l_1\dot{\theta}_1\cos\theta_1 + l_2\dot{\theta}_2\cos\theta_2 + \frac{l_3}{2}\dot{\theta}_3\cos\theta_3\big)^2\Big] \\
&\quad - \Big[-gl_1\cos\theta_1\big(\frac{m_1}{2} + m_2 + m_3\big) - gl_2\cos\theta_2\big(\frac{m_2}{2} + m_3\big) - gl_3\cos\theta_3\big(\frac{m_3}{2}\big)\Big]
\end{aligned}
\tag{52}
$$

Because we are assuming that there are no applied forces present (i.e., all forces are conservative), we set all generalized forces $Q_i = 0$ for each generalized coordinate q_i. Thus, the lagrange equation becomes

$$
\frac{d}{dt}\Big(\frac{\partial\mathcal{L}}{\partial\dot{q}_i}\Big) = \frac{\partial\mathcal{L}}{\partial q_i}
\tag{53}
$$

For $q_i = \theta_1$, we solve for $\frac{d}{dt}\Big(\frac{\partial\mathcal{L}}{\partial\dot{q}_i}\Big)$ as follows.

$$
\begin{aligned}
\frac{d}{dt}\Big(\frac{\partial \mathcal{L}}{\partial \dot\theta_1}\Big) &= \frac{d}{dt}\Big(I^{G_1}\dot\theta_1 + \frac{m_1 l_1^2}{4}\dot\theta_1 \\
&\quad + m_2 l_1\Big[\sin\theta_1\big(l_1\dot\theta_1\sin\theta_1 + \frac{l_2}{2}\dot\theta_2\sin\theta_2\big) + \cos\theta_1\big(l_1\dot\theta_1\cos\theta_1 + \frac{l_2}{2}\dot\theta_2\cos\theta_2\big)\Big] \\
&\quad + m_3 l_1\Big[\sin\theta_1\big(l_1\dot\theta_1\sin\theta_1 + l_2\dot\theta_2\sin\theta_2 + \frac{l_3}{2}\dot\theta_3\sin\theta_3\big) + \cos\theta_1\big(l_1\dot\theta_1\cos\theta_1 + l_2\dot\theta_2\cos\theta_2 + \frac{l_3}{2}\dot\theta_3\cos\theta_3\big)\Big]\Big) \\
&= \big(I^{G_1} + \frac{m_1 l_1^2}{4}\big)\ddot\theta_1 \\
&\quad + m_2 l_1\Big[\dot\theta_1\cos\theta_1\big(l_1\dot\theta_1\sin\theta_1 + \frac{l_2}{2}\dot\theta_2\sin\theta_2\big) \\
&\qquad + \sin\theta_1\big(l_1\ddot\theta_1\sin\theta_1 + l_1\dot\theta_1^2\cos\theta_1 + \frac{l_2}{2}\ddot\theta_2\sin\theta_2 + \frac{l_2}{2}\dot\theta_2^2\cos\theta_2\big) \\
&\qquad - \dot\theta_1\sin\theta_1\big(l_1\dot\theta_1\cos\theta_1 + \frac{l_2}{2}\dot\theta_2\cos\theta_2\big) \\
&\qquad + \cos\theta_1\big(l_1\ddot\theta_1\cos\theta_1 - l_1\dot\theta_1^2\sin\theta_1 + \frac{l_2}{2}\ddot\theta_2\cos\theta_2 - \frac{l_2}{2}\dot\theta_2^2\sin\theta_2\big)\Big] \\
&\quad + m_3 l_1\Big[\dot\theta_1\cos\theta_1\big(l_1\dot\theta_1\sin\theta_1 + l_2\dot\theta_2\sin\theta_2 + \frac{l_3}{2}\dot\theta_3\sin\theta_3\big) \\
&\qquad + \sin\theta_1\big(l_1\ddot\theta_1\sin\theta_1 + l_1\dot\theta_1^2\cos\theta_1 + l_2\ddot\theta_2\sin\theta_2 + l_2\dot\theta_2^2\cos\theta_2 + \frac{l_3}{2}\ddot\theta_3\sin\theta_3 + \frac{l_3}{2}\dot\theta_3^2\cos\theta_3\big) \\
&\qquad - \dot\theta_1\sin\theta_1\big(l_1\dot\theta_1\cos\theta_1 + l_2\dot\theta_2\cos\theta_2 + \frac{l_3}{2}\dot\theta_3\cos\theta_3\big) \\
&\qquad + \cos\theta_1\big(l_1\ddot\theta_1\cos\theta_1 - l_1\dot\theta_1^2\sin\theta_1 + l_2\ddot\theta_2\cos\theta_2 - l_2\dot\theta_2^2\sin\theta_2 + \frac{l_3}{2}\ddot\theta_3\cos\theta_3 - \frac{l_3}{2}\dot\theta_3^2\sin\theta_3\big)\Big] \\
&= \ddot\theta_1\Big[I^{G_1} + \frac{m_1 l_1^2}{4} + m_2 l_1^2\big(\sin^2\theta_1 + \cos^2\theta_1\big) + m_3 l_1^2\big(\sin^2\theta_1 + \cos^2\theta_1\big)\Big] \\
&\quad + \ddot\theta_2\Big[\frac{m_2 l_1 l_2}{2}\big(\sin\theta_1\sin\theta_2 + \cos\theta_1\cos\theta_2\big) + m_3 l_1 l_2\big(\sin\theta_1\sin\theta_2 + \cos\theta_1\cos\theta_2\big)\Big] \\
&\quad + \ddot\theta_3\Big[\frac{m_3 l_1 l_3}{2}\big(\sin\theta_1\sin\theta_3 + \cos\theta_1\cos\theta_3\big)\Big] \\
&\quad + m_2 l_1\Big[\cos\theta_1\big(\frac{l_2}{2}\dot\theta_1\dot\theta_2\sin\theta_2 - \frac{l_2}{2}\dot\theta_2^2\sin\theta_2\big) - \sin\theta_1\big(\frac{l_2}{2}\dot\theta_1\dot\theta_2\cos\theta_2 - \frac{l_2}{2}\dot\theta_2^2\cos\theta_2\big)\Big] \\
&\quad + m_3 l_1\Big[\cos\theta_1\big(l_2\dot\theta_1\dot\theta_2\sin\theta_2 - l_2\dot\theta_2^2\sin\theta_2 + \frac{l_3}{2}\dot\theta_1\dot\theta_3\sin\theta_3 - \frac{l_3}{2}\dot\theta_3^2\sin\theta_3\big) \\
&\qquad - \sin\theta_1\big(l_2\dot\theta_1\dot\theta_2\cos\theta_2 - l_2\dot\theta_2^2\cos\theta_2 + \frac{l_3}{2}\dot\theta_1\dot\theta_3\cos\theta_3 - \frac{l_3}{2}\dot\theta_2^2\cos\theta_2\big)\Big] \\
&= \ddot\theta_1\Big[I^{G_1} + l_1^2\big(\frac{m_1}{4} + m_2 + m_3\big)\Big] + \ddot\theta_2 l_1 l_2\big(\frac{m_2}{2} + m_3\big)\cos\left(\theta_1 - \theta_2\right) + \ddot\theta_3 l_1 l_3\big(\frac{m_3}{2}\big)\cos\left(\theta_1 - \theta_3\right) \\
&\quad + l_1\cos\theta_1\Big[\big(\frac{m_2}{2} + m_3\big)\big(l_2\dot\theta_1\dot\theta_2\sin\theta_2 - l_2\dot\theta_2^2\sin\theta_2\big) + \frac{m_3}{2}\big(l_3\dot\theta_1\dot\theta_3\sin\theta_3 - l_3{\dot\theta_3}^2\sin\theta_3\big)\Big] \\
&\quad - l_1\sin\theta_1\Big[\big(\frac{m_2}{2} + m_3\big)\big(l_2\dot\theta_1\dot\theta_2\cos\theta_2 - l_2\dot\theta_2^2\cos\theta_2\big) + \frac{m_3}{2}\big(l_3\dot\theta_1\dot\theta_3\cos\theta_3 - l_3\dot\theta_2^2\cos\theta_2\big)\Big] \\
&= \ddot\theta_1\Big[I^{G_1} + l_1^2\big(\frac{m_1}{4} + m_2 + m_3\big)\Big] + \ddot\theta_2 l_1 l_2\big(\frac{m_2}{2} + m_3\big)\cos\left(\theta_1 - \theta_2\right) + \ddot\theta_3 l_1 l_3\big(\frac{m_3}{2}\big)\cos\left(\theta_1 - \theta_3\right) \\
&\quad + l_1\dot\theta_1\Big[\big(\frac{m_2}{2} + m_3\big)\big(l_2\dot\theta_2\big)\sin\left(\theta_2 - \theta_1\right) + \frac{m_3}{2}\big(l_3\dot\theta_3\big)\sin\left(\theta_3 - \theta_1\right)\Big] \\
&\quad - l_1\Big[\big(\frac{m_2}{2} + m_3\big)\big(l_2\dot\theta_2^2\big)\sin\left(\theta_2 - \theta_1\right) + \frac{m_3}{2}\big(l_3\dot\theta_3^2\big)\sin\left(\theta_3 - \theta_1\right)\Big]
\end{aligned}
$$

(54)

Next, we solve for the right-hand-side of Equation 53.

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \theta_1} &= m_2 l_1 \dot{\theta}_1 \Big[\cos\theta_1 \big(l_1 \dot{\theta}_1 \sin\theta_1 + \frac{l_2}{2}\dot{\theta}_2 \sin\theta_2 \big) \\
&\qquad\qquad + \sin\theta_1 \big(- l_1 \dot{\theta}_1 \cos\theta_1 - \frac{l_2}{2}\dot{\theta}_2 \cos\theta_2 \big) \Big] \\
&\quad + m_3 l_1 \dot{\theta}_1 \Big[\cos\theta_1 \big(l_1 \dot{\theta}_1 \sin\theta_1 + l_2 \dot{\theta}_2 sin\theta_2 + \frac{l_3}{2}\dot{\theta}_3 \sin\theta_3 \big) \\
&\qquad\qquad + \sin\theta_1 \big(- l_1 \dot{\theta}_1 \cos\theta_1 - l_2 \dot{\theta}_2 \cos\theta_2 - \frac{l_3}{2}\dot{\theta}_3 \cos\theta_3 \big) \Big] \\
&\quad - g l_1 \sin\theta_1 \big(\frac{m_1}{2} + m_2 + m_3 \big) \\
&= l_1 \dot{\theta}_1 \cos\theta_1 \Big[\big(\frac{m_2}{2} + m_3 \big) \big(l_2 \dot{\theta}_2 \sin\theta_2 \big) + \frac{m_3}{2} \big(l_3 \dot{\theta}_3 \sin\theta_3 \big) \Big] \\
&\quad - l_1 \dot{\theta}_1 \sin\theta_1 \Big[\big(\frac{m_2}{2} + m_3 \big) \big(l_2 \dot{\theta}_2 \cos\theta_2 \big) + \frac{m_3}{2} \big(l_3 \dot{\theta}_3 \cos\theta_3 \big) \Big] \\
&\quad - g l_1 \sin\theta_1 \big(\frac{m_1}{2} + m_2 + m_3 \big) \\
&= l_1 \dot{\theta}_1 \Big[\big(\frac{m_2}{2} + m_3 \big) \big(l_2 \dot{\theta}_2 \big) \sin(\theta_2 - \theta_1) + \frac{m_3}{2} \big(l_3 \dot{\theta}_3 \big) \sin(\theta_3 - \theta_1) \Big] \\
&\quad - g l_1 \sin\theta_1 \big(\frac{m_1}{2} + m_2 + m_3 \big)
\end{aligned} \tag{55}$$

By plugging Equations 54 and 55 into the lagrange equation (Equation 53), we arrive at the following expression, in terms of all three minimal coordinates.

$$\begin{aligned}
&\ddot{\theta}_1 \Big[I^{G_1} + l_1^2 \big(\frac{m_1}{4} + m_2 + m_3 \big) \Big] + \ddot{\theta}_2 l_1 l_2 \big(\frac{m_2}{2} + m_3 \big) \cos(\theta_1 - \theta_2) + \ddot{\theta}_3 l_1 l_3 \big(\frac{m_3}{2} \big) \cos(\theta_1 - \theta_3) \\
&\qquad + l_1 \dot{\theta}_1 \Big[\big(\frac{m_2}{2} + m_3 \big) \big(l_2 \dot{\theta}_2 \big) \sin(\theta_2 - \theta_1) + \frac{m_3}{2} \big(l_3 \dot{\theta}_3 \big) \sin(\theta_3 - \theta_1) \Big] \\
&\quad - l_1 \Big[\big(\frac{m_2}{2} + m_3 \big) \big(l_2 \dot{\theta}_2^2 \big) \sin(\theta_2 - \theta_1) + \frac{m_3}{2} \big(l_3 \dot{\theta}_3^2 \big) \sin(\theta_3 - \theta_1) \Big] \\
&\quad = l_1 \dot{\theta}_1 \Big[\big(\frac{m_2}{2} + m_3 \big) \big(l_2 \dot{\theta}_2 \big) \sin(\theta_2 - \theta_1) + \frac{m_3}{2} \big(l_3 \dot{\theta}_3 \big) \sin(\theta_3 - \theta_1) \Big] \\
&\qquad - g l_1 \sin\theta_1 \big(\frac{m_1}{2} + m_2 + m_3 \big) \\
&\Rightarrow \ddot{\theta}_1 \Big[I^{G_1} + l_1^2 \big(\frac{m_1}{4} + m_2 + m_3 \big) \Big] + \ddot{\theta}_2 l_1 l_2 \big(\frac{m_2}{2} + m_3 \big) \cos(\theta_1 - \theta_2) + \ddot{\theta}_3 l_1 l_3 \big(\frac{m_3}{2} \big) \cos(\theta_1 - \theta_3) \\
&\quad = l_1 \Big[\big(\frac{m_2}{2} + m_3 \big) \big(l_2 \dot{\theta}_2^2 \big) \sin(\theta_2 - \theta_1) + \frac{m_3}{2} \big(l_3 \dot{\theta}_3^2 \big) \sin(\theta_3 - \theta_1) \Big] - g l_1 \sin\theta_1 \big(\frac{m_1}{2} + m_2 + m_3 \big)
\end{aligned} \tag{56}$$

Similarly, for the minimal coordinate $q_i = \theta_2$, we solve for $\frac{d}{dt}\Big(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\Big)$ as follows.

$$
\begin{aligned}
\frac{d}{dt}\Big(\frac{\partial \mathcal{L}}{\partial \dot\theta_2}\Big) &= \frac{d}{dt}\Big(I^{G_2}\dot\theta_2 + \frac{m_2 l_2}{2}\Big[\sin\theta_2\big(l_1\dot\theta_1\sin\theta_1 + \frac{l_2}{2}\dot\theta_2\sin\theta_2\big) + \cos\theta_2\big(l_1\dot\theta_1\cos\theta_1 + \frac{l_2}{2}\dot\theta_2\cos\theta_2\big)\Big] \\
&\quad + m_3 l_2\Big[\sin\theta_2\big(l_1\dot\theta_1\sin\theta_1 + l_2\dot\theta_2\sin\theta_2 + \frac{l_3}{2}\dot\theta_3\sin\theta_3\big) + \cos\theta_2\big(l_1\dot\theta_1\cos\theta_1 + l_2\dot\theta_2\cos\theta_2 + \frac{l_3}{2}\dot\theta_3\cos\theta_3\big)\Big]\Big) \\
&= I^{G_2}\ddot\theta_2 + \frac{m_2 l_2}{2}\Big[\dot\theta_2\cos\theta_2\big(l_1\dot\theta_1\sin\theta_1 + \frac{l_2}{2}\dot\theta_2\sin\theta_2\big) \\
&\qquad + \sin\theta_2\big(l_1\ddot\theta_1\sin\theta_1 + l_1\dot\theta_1^2\cos\theta_1 + \frac{l_2}{2}\ddot\theta_2\sin\theta_2 + \frac{l_2}{2}\dot\theta_2^2\cos\theta_2\big) \\
&\qquad - \dot\theta_2\sin\theta_2\big(l_1\dot\theta_1\cos\theta_1 + \frac{l_2}{2}\dot\theta_2\cos\theta_2\big) \\
&\qquad + \cos\theta_2\big(l_1\ddot\theta_1\cos\theta_1 - l_1\dot\theta_1^2\sin\theta_1 + \frac{l_2}{2}\ddot\theta_2\cos\theta_2 - \frac{l_2}{2}\dot\theta_2^2\sin\theta_2\big)\Big] \\
&\quad + m_3 l_2\Big[\dot\theta_2\cos\theta_2\big(l_1\dot\theta_1\sin\theta_1 + l_2\dot\theta_2\cos\theta_2 + \frac{l_3}{2}\dot\theta_3\cos\theta_3\big) \\
&\qquad + \sin\theta_2\big(l_1\ddot\theta_1\sin\theta_1 + l_1\dot\theta_1^2\cos\theta_1 + l_2\ddot\theta_2\sin\theta_2 + l_2\dot\theta_2^2\cos\theta_2 + \frac{l_3}{2}\ddot\theta_3\sin\theta_3 + \frac{l_3}{2}\dot\theta_3^2\cos\theta_3\big) \\
&\qquad - \dot\theta_2\sin\theta_2\big(l_1\dot\theta_1\cos\theta_1 + l_2\dot\theta_2\cos\theta_2 + \frac{l_3}{2}\dot\theta_3\cos\theta_3\big) \\
&\qquad + \cos\theta_2\big(l_1\ddot\theta_1\cos\theta_1 - l_1\dot\theta_1^2\sin\theta_1 + l_2\ddot\theta_2\cos\theta_2 - l_2\dot\theta_2^2\sin\theta_2 + \frac{l_3}{2}\ddot\theta_3\cos\theta_3 - \frac{l_3}{2}\dot\theta_3^2\sin\theta_3\big)\Big] \\
&= \ddot\theta_1 l_1 l_2\big(\frac{m_2}{2} + m_3\big)\cos\big(\theta_2 - \theta_1\big) + \ddot\theta_2\Big[I^{G_2} + l_2^2\big(\frac{m_2}{4} + m_3\big)\Big] + \ddot\theta_3 l_2 l_3\big(\frac{m_3}{2}\big)\cos\big(\theta_2 - \theta_3\big) \\
&\quad + l_2\dot\theta_2\Big[\big(\frac{m_2}{2} + m_3\big)\big(l_1\dot\theta_1\big)\sin\left(\theta_1 - \theta_2\right) + \frac{m_3}{2}\big(l_3\dot\theta_3\big)\sin\left(\theta_3 - \theta_2\right)\Big] \\
&\quad - l_2\Big[\big(\frac{m_2}{2} + m_3\big)\big(l_1\dot\theta_1^2\big)\sin\left(\theta_1 - \theta_2\right) + \frac{m_3}{2}\big(l_3\dot\theta_3^2\big)\sin\left(\theta_3 - \theta_2\right)\Big]
\end{aligned}
\tag{57}
$$

Then, solving for the right-hand-side of Equation 53,

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \theta_2} = {} & m_2\dot{\theta}_2\frac{l_2}{2}\Big[\cos\theta_2\big(l_1\dot{\theta}_1\sin\theta_1 + \frac{l_2}{2}\dot{\theta}_2\sin\theta_2\big) \\
& \qquad + \sin\theta_2\big(-l_1\dot{\theta}_1\cos\theta_1 - \frac{l_2}{2}\dot{\theta}_2\cos\theta_2\big)\Big] \\
& + m_3 l_2\dot{\theta}_2\Big[\cos\theta_2\big(l_1\dot{\theta}_1\sin\theta_1 + l_2\dot{\theta}_2 sin\theta_2 + \frac{l_3}{2}\dot{\theta}_3\sin\theta_3\big) \\
& \qquad + \sin\theta_2\big(-l_1\dot{\theta}_1\cos\theta_1 - l_2\dot{\theta}_2\cos\theta_2 - \frac{l_3}{2}\dot{\theta}_3\cos\theta_3\big)\Big] \\
& - gl_2\sin\theta_2\big(\frac{m_2}{2} + m_3\big) \\
= {} & l_2\dot{\theta}_2\cos\theta_2\Big[\big(\frac{m_2}{2} + m_3\big)\big(l_1\dot{\theta}_1\sin\theta_1\big) + \frac{m_3}{2}\big(l_3\dot{\theta}_3\sin\theta_3\big)\Big] \\
& - l_2\dot{\theta}_2\sin\theta_2\Big[\big(\frac{m_2}{2} + m_3\big)\big(l_1\dot{\theta}_1\cos\theta_1\big) + \frac{m_3}{2}\big(l_3\dot{\theta}_3\cos\theta_3\big)\Big] \\
& - gl_2\sin\theta_2\big(\frac{m_2}{2} + m_3\big) \\
= {} & l_2\dot{\theta}_2\Big[\big(\frac{m_2}{2} + m_3\big)\big(l_1\dot{\theta}_1\big)\sin(\theta_1 - \theta_2) + \frac{m_3}{2}\big(l_3\dot{\theta}_3\big)\sin(\theta_3 - \theta_2)\Big] \\
& - gl_2\sin\theta_2\big(\frac{m_2}{2} + m_3\big)
\end{aligned}
\tag{58}
$$

By plugging Equations 57 and 58 into the lagrange equation, we arrive at yet another expression in terms of all three minimal coordinates.

$$
\begin{aligned}
& \ddot{\theta}_1 l_1 l_2\big(\frac{m_2}{2} + m_3\big)\cos\big(\theta_2 - \theta_1\big) + \ddot{\theta}_2\Big[I^{G_2} + l_2^2\big(\frac{m_2}{4} + m_3\big)\Big] + \ddot{\theta}_3 l_2 l_3\big(\frac{m_3}{2}\big)\cos\big(\theta_2 - \theta_3\big) \\
& \quad + l_2\dot{\theta}_2\Big[\big(\frac{m_2}{2} + m_3\big)\big(l_1\dot{\theta}_1\big)\sin(\theta_1 - \theta_2) + \frac{m_3}{2}\big(l_3\dot{\theta}_3\big)\sin(\theta_3 - \theta_2)\Big] \\
& \quad - l_2\Big[\big(\frac{m_2}{2} + m_3\big)\big(l_1\dot{\theta}_1^2\big)\sin(\theta_2 - \theta_1) + \frac{m_3}{2}\big(l_3\dot{\theta}_3^2\big)\sin(\theta_3 - \theta_1)\Big] \\
& \quad = l_2\dot{\theta}_2\Big[\big(\frac{m_2}{2} + m_3\big)\big(l_2\dot{\theta}_2\big)\sin(\theta_2 - \theta_1) + \frac{m_3}{2}\big(l_3\dot{\theta}_3\big)\sin(\theta_3 - \theta_1)\Big] \\
& \quad - gl_2\sin\theta_2\big(\frac{m_2}{2} + m_3\big) \\
& \Rightarrow \ddot{\theta}_1 l_1 l_2\big(\frac{m_2}{2} + m_3\big)\cos\big(\theta_2 - \theta_1\big) + \ddot{\theta}_2\Big[I^{G_2} + l_2^2\big(\frac{m_2}{4} + m_3\big)\Big] + \ddot{\theta}_3 l_2 l_3\big(\frac{m_3}{2}\big)\cos\big(\theta_2 - \theta_3\big) \\
& \quad = l_2\Big[\big(\frac{m_2}{2} + m_3\big)\big(l_1\dot{\theta}_1^2\big)\sin(\theta_1 - \theta_2) + \frac{m_3}{2}\big(l_3\dot{\theta}_3^2\big)\sin(\theta_3 - \theta_2)\Big] - gl_2\sin\theta_2\big(\frac{m_2}{2} + m_3\big)
\end{aligned}
\tag{59}
$$

Last, to solve for the minimal coordinate $q_i = \theta_3$, we solve for $\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right)$,

$$
\begin{aligned}
\frac{d}{dt}\Big(\frac{\partial\mathcal{L}}{\partial\dot{\theta}_3}\Big) &= \frac{d}{dt}\Big(I^{G_3}\dot{\theta}_3 + \frac{m_3 l_3}{2}\Big[\sin\theta_3\big(l_1\dot{\theta}_1\sin\theta_1 + l_2\dot{\theta}_2\sin\theta_2 + \frac{l_3}{2}\dot{\theta}_3\sin\theta_3\big) + \cos\theta_3\big(l_1\dot{\theta}_1\cos\theta_1 + l_2\dot{\theta}_2\cos\theta_2 + \frac{l_3}{2}\dot{\theta}_3 \\
&= I^{G_3} + \ddot{\theta}_3 + \frac{m_3 l_3}{2}\Big[\dot{\theta}_3\cos\theta_3\big(l_1\dot{\theta}_1\sin\theta_1 + l_2\dot{\theta}_2\cos\theta_2 + \frac{l_3}{2}\dot{\theta}_3\cos\theta_3\big) \\
&\qquad + \sin\theta_3\big(l_1\ddot{\theta}_1\sin\theta_1 + l_1\dot{\theta}_1^2\cos\theta_1 + l_2\ddot{\theta}_2\sin\theta_2 + l_2\dot{\theta}_2^2\cos\theta_2 + \frac{l_3}{2}\ddot{\theta}_3\sin\theta_3 + \frac{l_3}{2}\dot{\theta}_3^2\cos\theta_3\big) \\
&\qquad - \dot{\theta}_3\sin\theta_3\big(l_1\dot{\theta}_1\cos\theta_1 + l_2\dot{\theta}_2\cos\theta_2 + \frac{l_3}{2}\dot{\theta}_3\cos\theta_3\big) \\
&\qquad + \cos\theta_3\big(l_1\ddot{\theta}_1\cos\theta_1 - l_1\dot{\theta}_1^2\sin\theta_1 + l_2\ddot{\theta}_2\cos\theta_2 - l_2\dot{\theta}_2^2\sin\theta_2 + \frac{l_3}{2}\ddot{\theta}_3\cos\theta_3 - \frac{l_3}{2}\dot{\theta}_3^2\sin\theta_3\big)\Big] \\
&= \ddot{\theta}_1 l_1 l_3\big(\frac{m_3}{2}\big)\cos\big(\theta_3 - \theta_1\big) + \ddot{\theta}_2 l_2 l_3\big(\frac{m_3}{2}\big)\cos\big(\theta_3 - \theta_2\big) + \ddot{\theta}_3\Big[I^{G_3} + l_3^2\big(\frac{m_3}{4}\big)\Big] \\
&\quad + l_3\dot{\theta}_3\frac{m_3}{2}\Big[\big(l_1\dot{\theta}_1\big)\sin\left(\theta_1 - \theta_3\right) + \big(l_2\dot{\theta}_2\big)\sin\left(\theta_2 - \theta_3\right)\Big] \\
&\quad - l_3\frac{m_3}{2}\Big[\big(l_1\dot{\theta}_1^2\big)\sin\left(\theta_1 - \theta_3\right) + \big(l_2\dot{\theta}_2^2\big)\sin\left(\theta_2 - \theta_3\right)\Big]
\end{aligned}
\tag{60}
$$

and for $\frac{\partial\mathcal{L}}{\partial_i}$,

$$
\begin{aligned}
\frac{\partial\mathcal{L}}{\partial\theta_3} &= \frac{m_3 l_3\dot{\theta}_3}{2}\Big[\cos\theta_3\big(l_1\dot{\theta}_1\sin\theta_1 + l_2\dot{\theta}_2 sin\theta_2 + \frac{l_3}{2}\dot{\theta}_3\sin\theta_3\big) \\
&\qquad + \sin\theta_3\big(- l_1\dot{\theta}_1\cos\theta_1 - l_2\dot{\theta}_2\cos\theta_2 - \frac{l_3}{2}\dot{\theta}_3\cos\theta_3\big)\Big] \\
&\quad - g l_3\sin\theta_3\big(\frac{m_3}{2}\big) \\
&= l_3\dot{\theta}_3\frac{m_3}{2}\Big[\big(l_1\dot{\theta}_1\big)\sin\left(\theta_1 - \theta_3\right) + \big(l_2\dot{\theta}_2\big)\sin\left(\theta_2 - \theta_3\right)\Big] \\
&\quad - g l_3\sin\theta_3\big(\frac{m_3}{2}\big)
\end{aligned}
\tag{61}
$$

Using Equations 60 - 61, we arrive at the third expression in terms of all three minimal coordinates.

$$\begin{aligned}
&\ddot{\theta}_1 l_1 l_3\big(\frac{m_3}{2}\big)\cos\big(\theta_3-\theta_1\big)+\ddot{\theta}_2 l_2 l_3\big(\frac{m_3}{2}\big)\cos\big(\theta_3-\theta_2\big)+\ddot{\theta}_3\Big[I^{G_3}+l_3^2\big(\frac{m_3}{4}\big)\Big]\\
&\quad+l_3\dot{\theta}_3\frac{m_3}{2}\Big[\big(l_1\dot{\theta}_1\big)\sin\left(\theta_1-\theta_3\right)+\big(l_2\dot{\theta}_2\big)\sin\left(\theta_2-\theta_3\right)\Big]\\
&\quad-l_3\frac{m_3}{2}\Big[\big(l_1\dot{\theta}_1^2\big)\sin\left(\theta_1-\theta_3\right)+\big(l_2\dot{\theta}_2^2\big)\sin\left(\theta_2-\theta_3\right)\Big]\\
&\quad=l_3\dot{\theta}_3\frac{m_3}{2}\Big[\big(l_1\dot{\theta}_1\big)\sin\left(\theta_1-\theta_3\right)+\big(l_2\dot{\theta}_2\big)\sin\left(\theta_2-\theta_3\right)\Big]\\
&\quad-gl_3\sin\theta_3\big(\frac{m_3}{2}\big)\\
&\Rightarrow\ddot{\theta}_1 l_1 l_3\big(\frac{m_3}{2}\big)\cos\big(\theta_3-\theta_1\big)+\ddot{\theta}_2 l_2 l_3\big(\frac{m_3}{2}\big)\cos\big(\theta_3-\theta_2\big)+\ddot{\theta}_3\Big[I^{G_3}+l_3^2\big(\frac{m_3}{4}\big)\Big]\\
&\quad=l_3\frac{m_3}{2}\Big[\big(l_1\dot{\theta}_1^2\big)\sin\left(\theta_1-\theta_3\right)+\big(l_2\dot{\theta}_2^2\big)\sin\left(\theta_2-\theta_3\right)\Big]-gl_3\sin\theta_3\big(\frac{m_3}{2}\big)
\end{aligned}\tag{62}$$

By combining the three solutions to the lagrange equations (Equations 56, 59, and 62) in matrix form, we can solve for the equations of motion of each minimal coordinate used, as follows.

$$\mathbf{A}=\begin{bmatrix} I^{G_1}+l_1^2\big(\frac{m_1}{4}+m_2+m_3\big) & l_1l_2\big(\frac{m_2}{2}+m_3\big)\cos\big(\theta_1-\theta_2\big) & l_1l_3\big(\frac{m_3}{2}\cos\big(\theta_1-\theta_3\big)\big)\\ l_1l_2\big(\frac{m_2}{2}+m_3\big)\cos\big(\theta_2-\theta_1\big) & I^{G_2}+l_2^2\big(\frac{m_2}{4}+m_3\big) & l_2l_3\big(\frac{m_3}{2}\big)\cos\big(\theta_2-\theta_3\big)\\ l_1l_3\big(\frac{m_3}{2}\big)\cos\big(\theta_3-\theta_1\big) & l_2l_3\big(\frac{m_3}{2}\big)\cos\big(\theta_3-\theta_2\big) & I^{G_3}+l_3^2\big(\frac{m_3}{4}\big)\end{bmatrix}$$

$$\vec{z}=\begin{bmatrix}\ddot{\theta}_1\\ \ddot{\theta}_2\\ \ddot{\theta}_3\end{bmatrix}$$

$$\vec{b}=\begin{bmatrix} l_1\Big[\big(\frac{m_2}{2}+m_3\big)\big(l_2\dot{\theta}_2^2\big)\sin\left(\theta_2-\theta_1\right)+\frac{m_3}{2}\big(l_3\dot{\theta}_3^2\big)\sin\left(\theta_3-\theta_1\right)\Big]-gl_1\sin\theta_1\big(\frac{m_1}{2}+m_2+m_3\big)\\ l_2\Big[\big(\frac{m_2}{2}+m_3\big)\big(l_1\dot{\theta}_1^2\big)\sin\left(\theta_1-\theta_2\right)+\frac{m_3}{2}\big(l_3\dot{\theta}_3^2\big)\sin\left(\theta_3-\theta_2\right)\Big]-gl_2\sin\theta_2\big(\frac{m_2}{2}+m_3\big)\\ l_3\frac{m_3}{2}\Big[\big(l_1\dot{\theta}_1^2\big)\sin\left(\theta_1-\theta_3\right)+\big(l_2\dot{\theta}_2^2\big)\sin\left(\theta_2-\theta_3\right)\Big]-gl_3\sin\theta_3\big(\frac{m_3}{2}\big)\end{bmatrix}$$

$$\mathbf{A}\vec{z}=\vec{b}$$

$$\vec{z}=\mathbf{A}^{-1}\vec{b}\tag{63}$$

4. Analysis of Triple Pendulum Solutions

The set of numerical solutions of the triple pendulum system using the AMB, DAE, and Lagrange approach were obtained through the following MATLAB `ODE45` solver functions, respectively: `AMB_rhs` (symbolically derived from `MakeTriplePendSolverFile`), `DAE_rhs`, and `Lagrange_rhs`. The following constants and initial conditions were used in a 10-second simulation with the absolute and relative tolerances of the `ODE45` solver set to 1E-10:

$$
\begin{aligned}
l_1 &= l_2 = l_3 = 1 \text{ m} \\
m_1 &= m_2 = m_3 = 1 \text{ kg} \\
I_i &= \frac{1}{12} m_i l_i^2 = \frac{1}{12} \text{ kgm}^2 \\
g &= 9.8 \text{ m s}^{-2} \\
\theta_1 &= \frac{\pi}{2} \text{ rad}, \theta_2 = \frac{3\pi}{4} \text{ rad}, \theta_3 = \frac{5\pi}{4} \text{ rad} \\
\dot{\theta}_1 &= \dot{\theta}_2 = \dot{\theta}_3 = 0 \text{ rad/s}
\end{aligned}
$$

Using these numerical solutions, the end positions of links 1, 2, and 3 (refer to Figure 1) for each approach are plotted in Figures 5 - 7, shown below.

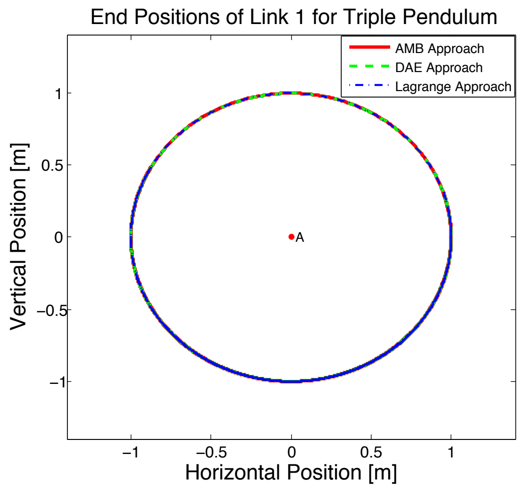

FIGURE 5. Positions of end of link 1 (i.e., pin B) for parameters stated above.

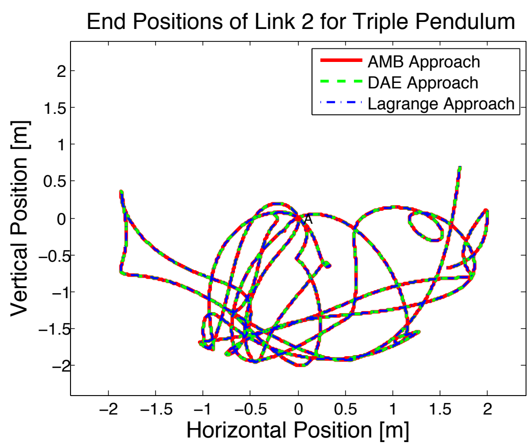

FIGURE 6. Positions of end of link 2 (i.e., pin C) for parameters stated above.

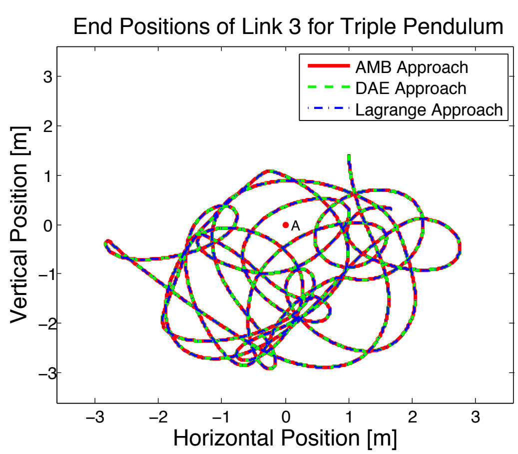

FIGURE 7. Positions of end of link 3 (i.e., pin D) for parameters stated above.

To better portray the motion of the triple pendulum, Figure 8 illustrates the end position of the third link, where the color of the plotted trajectory blends from red to blue from the start to end of the simulation, respectively, through constant time steps.

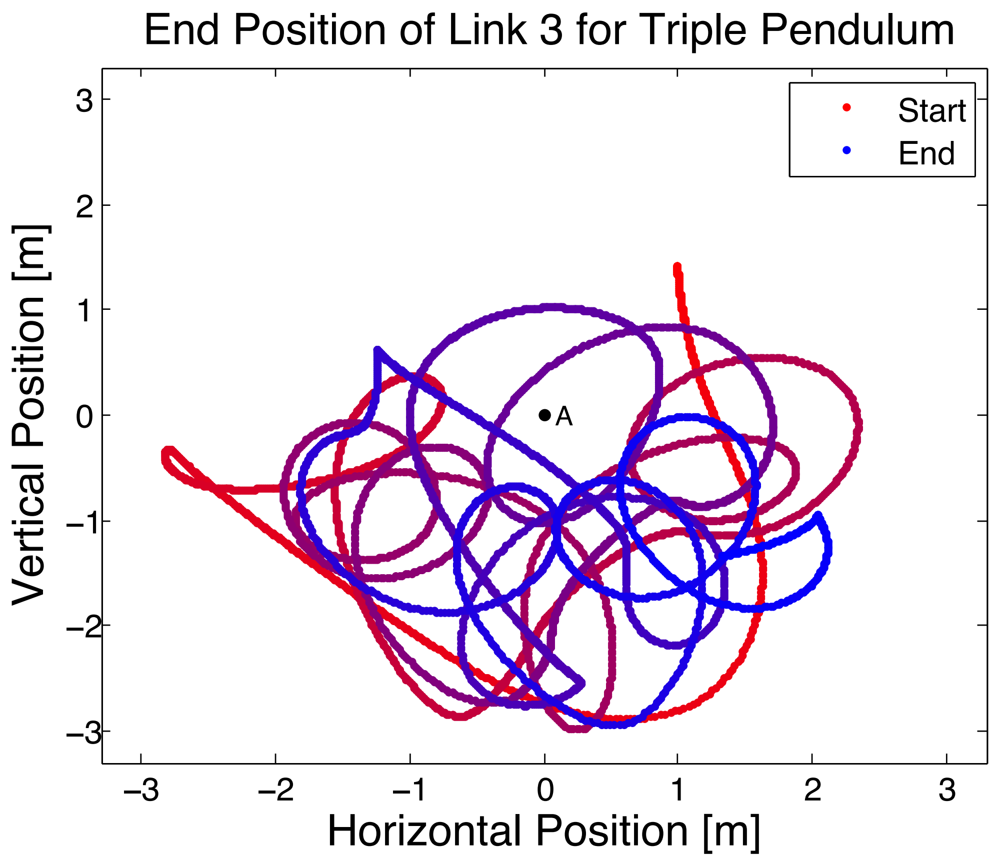

FIGURE 8. Trajectory of end position of link 3, blending from red (start) to blue (stop).

Because Figures 5 - 7 illustrate that the three numerical solutions are almost identical in yielding the links' position, the approaches are valid and have been implemented successfully. To further illustrate the similarity between the three solutions, Figure 9 illustrates the positions of the ends of each link for the AMB and Lagrange approaches, while Figure 10 the angles, relative to those for DAE solution at all equivalent times.

In addition, the total energies (kinetic + potential) of each link are plotted in Figures 11 - 13 with respect to their total energy calculated at the start of the simulation. Because all calculated energies are less than 2E-9 J (i.e., essentially zero), total energy is conserved, which further verifies the accuracies of all three solutions. The slight variation is simply due to numerical error, and could be decreased using lower `ODE45` tolerances.

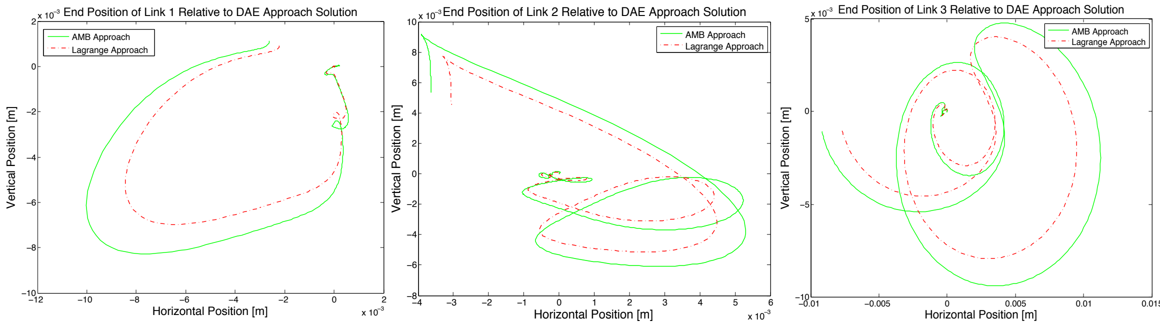

FIGURE 9. End positions of each link found using the AMB and Lagrange approaches relative to those found using the DAE method, using a constant time step of 1E-5 s. Note that the largest variation is between the AMB and DAE solutions towards the end of the simulation, of about 0.008 m. Also note that the ODE45 solver is slightly less accurate when using a time span determined by constant time steps rather than tolerances.

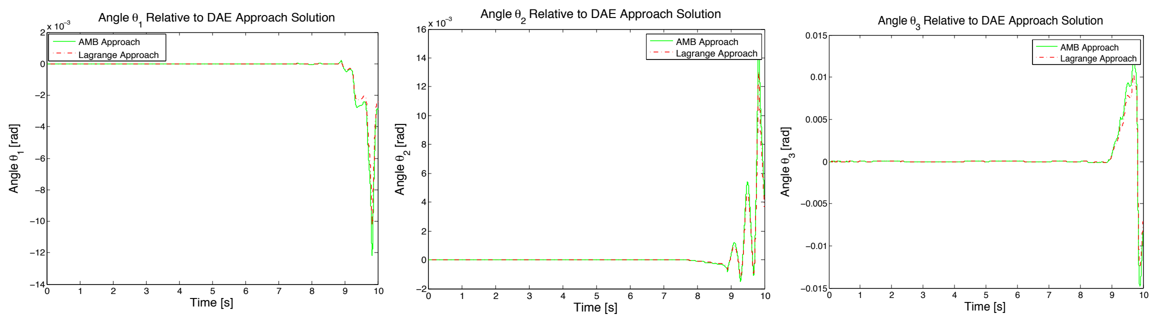

FIGURE 10. Angles of each link found using the AMB and Lagrange approaches relative to those found using the DAE method, using a constant time step of 1E-5 s. Note that the angles begin to deviate towards the end of the 10-second simulation.

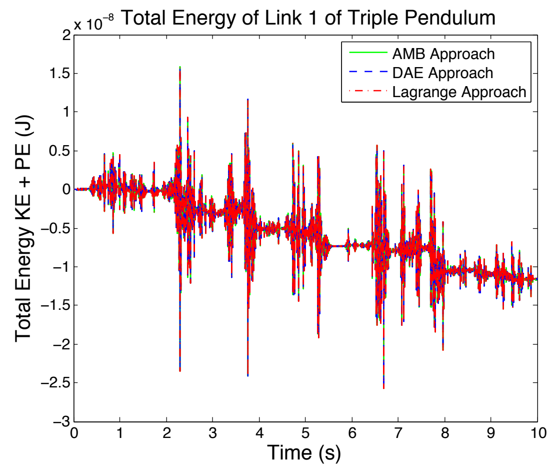

FIGURE 11. Total energy of Link 1 relative to start of simulation.

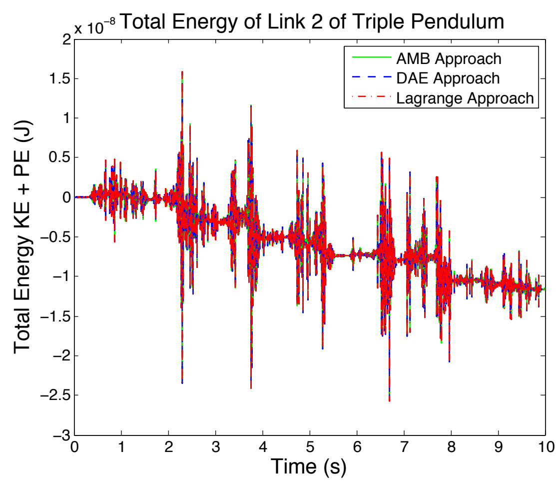

FIGURE 12. Total energy of Link 2 relative to start of simulation.

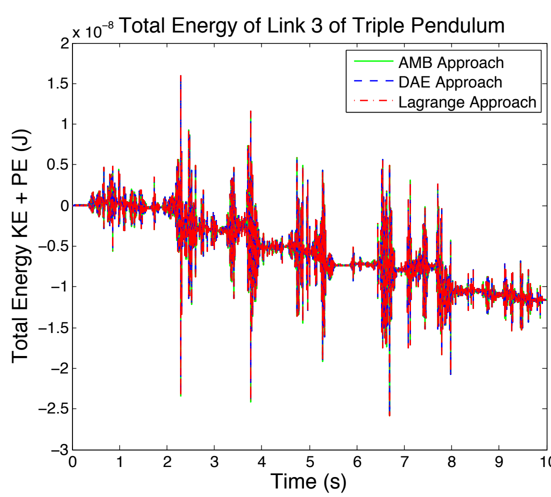

FIGURE 13. Total energy of Link 3 relative to start of simulation.

As a last verification, the solution of the triple pendulum was compared to the known solution of a single pendulum (solved using `SinglePendulum_rhs` in the function `ValidateSolutions`) by making the masses of the second two links negligible (1E-10 kg). The solutions found using each approach followed those of the single pendulum for three different starting angles of the first link, as illustrated below in Figure 14.

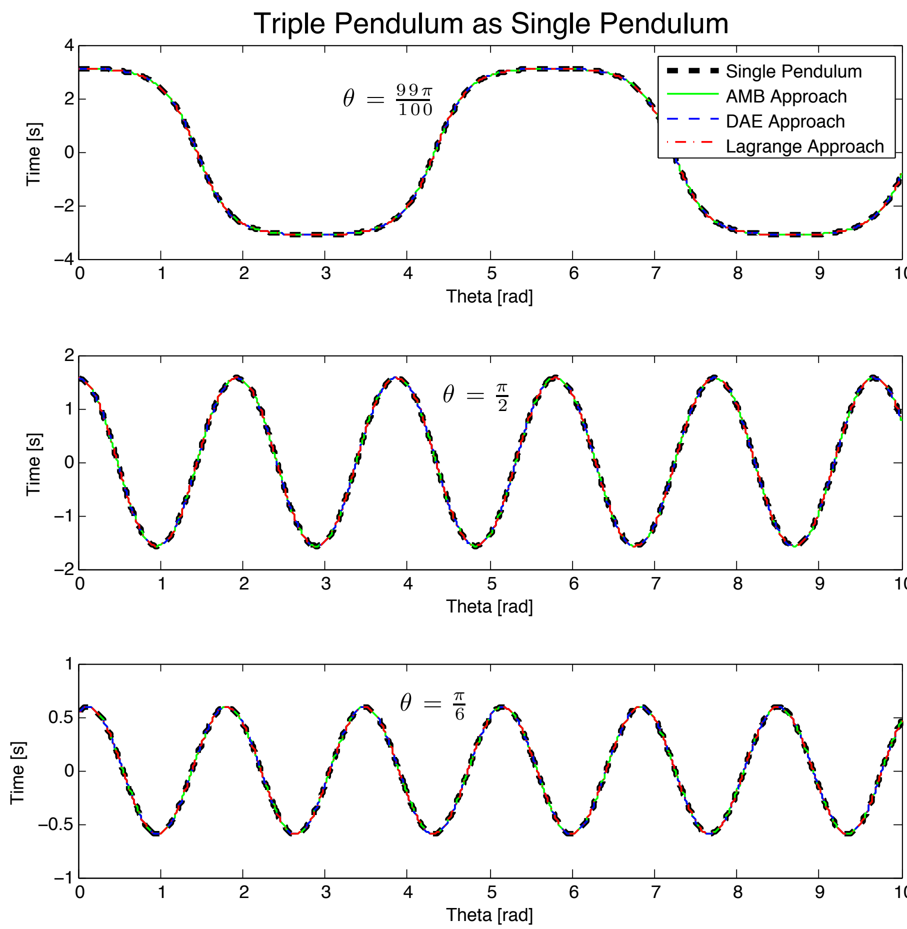

FIGURE 14. Simplification of triple pendulum to a single pendulum by making masses of second two links negligable. Note that the first start angle is slightly less than π to ensure that the pendulum tips the same way for each solution, and so that oscillations would occur in the 10-second simulation.

5. 4-Bar Linkage Equations of Motion

For the second part of the project, we derive the equations of motion for a 4-Bar Linkage, as modeled in Figure 2. To solve for the equations of motion, we will use the DAE approach since same FBDs created for the DAE solution to the triple pendulum problem (refer to Figure 4) are identical except for the additional reaction force at pin D for the third FBD, as labeled in Figure 15.

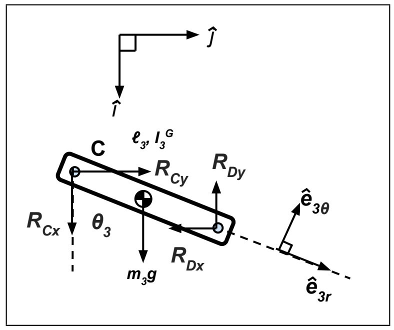

FIGURE 15. Third FBD used in DAE solution to 4-Bar linkage problem. Note that the first two are identical to those used in the triple pendulum solution.

To account for the additional reaction force at pin D (i.e., R_{Dx} and R_{Dy}), we must modify the linear momentum balance solutions found for Link 3 in the triple pendulum DAE approach (Equations 27 and 30) as follows.

$$m_3\ddot{x}_{G_3} - R_{Cx} + R_{Dx} = m_3 g \tag{64}$$

$$m_3\ddot{y}_{G_3} - R_{Cy} + R_{Dy} = 0 \tag{65}$$

Next, we must alter the angular momentum balance derived in Equation 36.

$$-I^{G_3}\ddot{\theta}_3 + \big(\frac{l_3}{2}\sin\theta_3\big)R_{Cx} + \big(\frac{l_3}{2}\sin\theta_3\big)R_{Dx} - \big(\frac{l_3}{2}\cos\theta_3\big)R_{Cy} - \big(\frac{l_3}{2}\cos\theta_3\big)R_{Dy} = 0 \tag{66}$$

Last, we derive two more equations by solving for the constraint, or by setting the acceleration at pin D equal to zero, and solving for the î and ĵ components. Similar to Equations 38 and 39, we arrive at the following expressions.

$$\ddot{x}_{G_1} + \big(\frac{l_1}{2}\sin\theta_1\big)\ddot{\theta_1} = -\big(\frac{l_1}{2}\cos\theta_1\big)\dot{\theta}_1^2 \tag{67}$$

$$\ddot{y}_{G_1} - \big(\frac{l_1}{2}\cos\theta_1\big)\ddot{\theta_1} = -\big(\frac{l_1}{2}\sin\theta_1\big)\dot{\theta}_1^2 \tag{68}$$

To solve the system of equations for ultimately $\ddot{\theta}_1$, $\ddot{\theta}_2$, and $\ddot{\theta}_3$, we combine these equations into matrix form, as shown on the next two pages, and, in the `ODE45` solver function, solve the matrix equation

$$\begin{aligned} \mathbf{A}\vec{z} &= \vec{b} \\ \vec{z} &= \mathbf{A}^{-1}\vec{b} \end{aligned} \tag{69}$$

$$
\mathbf{A} = \begin{pmatrix}
m_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & m_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & m_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & m_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & m_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & m_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & -I^{G_1} & 0 & 0 & \left(\frac{l_1}{2}\sin\theta_1\right) & \left(\frac{l_1}{2}\sin\theta_1\right) & 0 & 0 & -\left(\frac{l_1}{2}\cos\theta_1\right) & \left(-\frac{l_1}{2}\cos\theta_1\right) & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -I^{G_2} & 0 & 0 & \left(\frac{l_2}{2}\sin\theta_2\right) & \left(\frac{l_2}{2}\sin\theta_2\right) & 0 & 0 & -\left(\frac{l_2}{2}\cos\theta_2\right) & -\left(\frac{l_2}{2}\cos\theta_2\right) & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -I^{G_3} & 0 & 0 & \left(\frac{l_3}{2}\sin\theta_3\right) & \left(\frac{l_3}{2}\sin\theta_3\right) & 0 & 0 & \left(-\frac{l_3}{2}\cos\theta_3\right) & \left(-\frac{l_3}{2}\cos\theta_3\right) \\
1 & 0 & 0 & 0 & 0 & 0 & \left(\frac{l_1}{2}\sin\theta_1\right) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 1 & 0 & 0 & 0 & 0 & \left(\frac{l_1}{2}\sin\theta_1\right) & \left(\frac{l_2}{2}\sin\theta_2\right) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 1 & 0 & 0 & 0 & 0 & \left(\frac{l_2}{2}\sin\theta_2\right) & \left(\frac{l_3}{2}\sin\theta_3\right) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & \left(\frac{l_3}{2}\sin\theta_3\right) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & -\left(\frac{l_1}{2}\cos\theta_1\right) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & \\
0 & 0 & 0 & -1 & 1 & 0 & -\left(\frac{l_1}{2}\cos\theta_1\right) & -\left(\frac{l_2}{2}\cos\theta_2\right) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & -1 & 1 & 0 & -\left(\frac{l_2}{2}\cos\theta_2\right) & -\left(\frac{l_3}{2}\cos\theta_3\right) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -\left(\frac{l_3}{2}\cos\theta_3\right) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

$$\vec{z} = \begin{bmatrix} \ddot{x}_{G_1} \\ \ddot{x}_{G_2} \\ \ddot{x}_{G_3} \\ \ddot{y}_{G_1} \\ \ddot{y}_{G_2} \\ \ddot{y}_{G_3} \\ \ddot{\theta}_1 \\ \ddot{\theta}_2 \\ \ddot{\theta}_3 \\ R_{Ax} \\ R_{Bx} \\ R_{Cx} \\ R_{Dx} \\ R_{Ay} \\ R_{By} \\ R_{Cy} \\ R_{Dy} \end{bmatrix}$$

$$\vec{b} = \begin{bmatrix} m_1 g \\ m_2 g \\ m_3 g \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -\left(\frac{l_1}{2}\cos\theta_1\right)\dot{\theta}_1^2 \\ -\left(\frac{l_1}{2}\cos\theta_1\right)\dot{\theta}_1^2 - \left(\frac{l_2}{2}\cos\theta_2\right)\dot{\theta}_2^2 \\ -\left(\frac{l_2}{2}\cos\theta_2\right)\dot{\theta}_2^2 - \left(\frac{l_3}{2}\cos\theta_3\right)\dot{\theta}_3^2 \\ -\left(\frac{l_3}{2}\cos\theta_3\right)\dot{\theta}_3^2 \\ -\left(\frac{l_1}{2}\sin\theta_1\right)\dot{\theta}_1^2 \\ -\left(\frac{l_1}{2}\sin\theta_1\right)\dot{\theta}_1^2 - \left(\frac{l_2}{2}\sin\theta_2\right)\dot{\theta}_2^2 \\ -\left(\frac{l_2}{2}\sin\theta_2\right)\dot{\theta}_2^2 - \left(\frac{l_3}{2}\sin\theta_3\right)\dot{\theta}_3^2 \\ -\left(\frac{l_3}{2}\sin\theta_3\right)\dot{\theta}_3^2 \end{bmatrix}$$

6. Analysis of Triple Pendulum Solutions

The numerical solution for the 4-bar linkage was found using the same constants, initial conditions, and simulation parameters as those stated in Section 4. The end positions of the first, second, and third link found from the numerical solution is shown below in Figure 16.

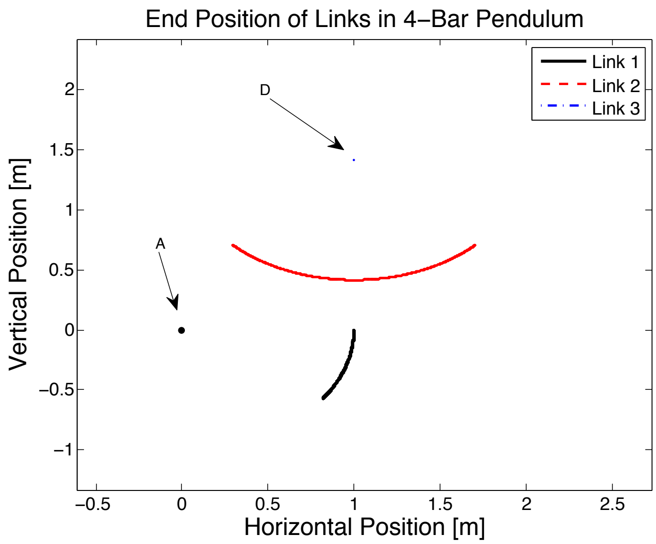

Figure 16. End positions of links 1, 2, and 3 of 4-bar linkage. Note that the position of pin D is constant since it is fixed.

To help visualize this motion, an instant of the animation for this system is shown in Figure 16.

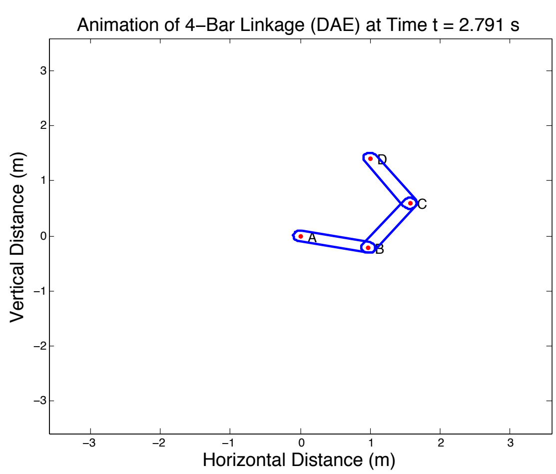

Figure 17. Instant from 4-Bar linkage animation, created from simulation using the same parameters described in Section 4.

To validate the numerical solution of the 4-bar linkage, the end positions of the third link was calculated with respect to its initial position. Since pin D is set to be fixed, there should be no variation in position; as shown in Figures 18, there was in fact negligable drift.

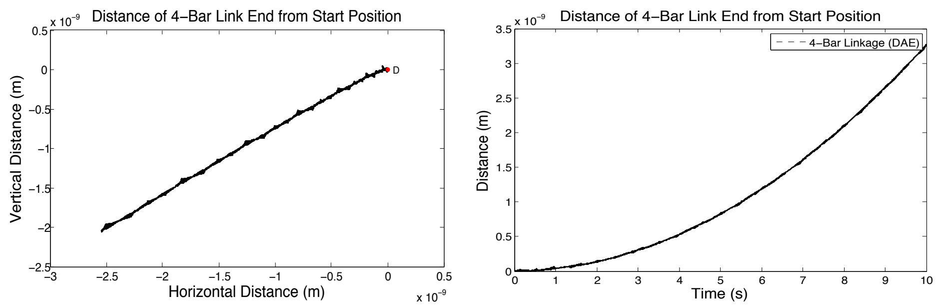

FIGURE 18. Drift in end position of third link in 4-Bar Linkage, plotted in the $\hat{\imath}$ and $\hat{\jmath}$ coordinate system and as a magnitude against time. Note that the distance scales of both plots are miniscule, indicating negligable drift.

It was observed, however, that if an initial angular rate is given to any of the links, the constraint is violated in that pin D is shown to move at a constant velocity. This is not an error in the implementation of the 4-bar linkage solver, but rather a shortcoming of the DAE approach, which only defines the acceleration at pin D to be zero; in other words, the change in velocity of point D will be zero, but given an initial velocity, which occurs for nonzero initial angular rates, its velocity will remain constant, as observed through animation.

In addition, similar to the triple pendulum analysis, the kinetic and potential energy of the 4-bar linkage was calculated for each link with respect to its total energy at the start of the simulation. As illustrated in Figure 19, the energy is essentially zero throughout the simulation. The slight decrease in energy is again associated with numerical error, and can be related to the fact that pin D is observed to move slightly downward in Figure 18, in turn decreasing the potential energy.

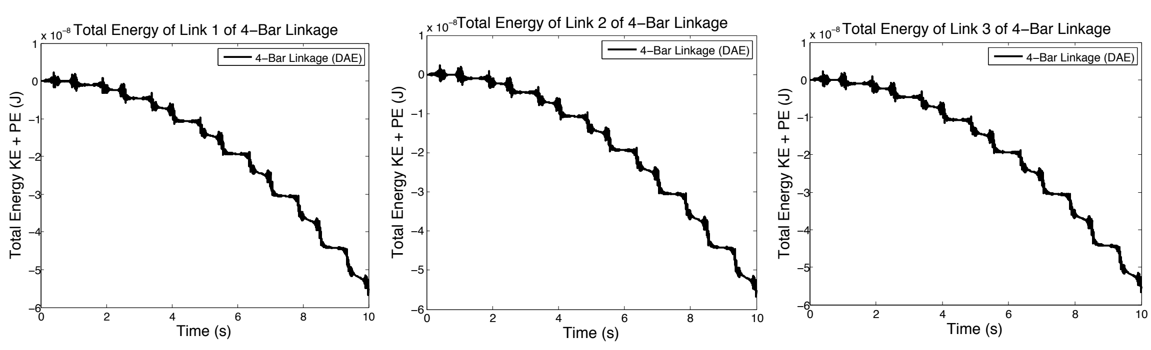

FIGURE 19. Total energy (kinetic + potential) of each link in the 4-bar linkage.

Further, as in the triple pendulum analysis, the motion of the 4-Bar Linkage was compared to a simplified case - a triangular 4-bar linkage, where pins A and D are fixed at the same position and all links have equivalent mass and length, as illustrated in Figure 20. In this case, the 4-bar linkage can be modeled as a single pendulum with the same moment of inertia, center-of-mass position, and total mass.

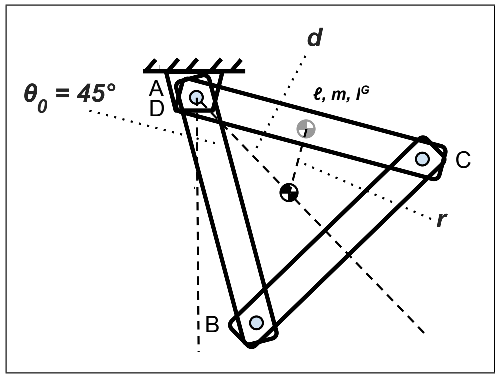

FIGURE 20. 4-Bar Linkage in triangular formation to solve for simplified case of a single pendulum with equivalent physical peroperties.

Through trigonometry, the distance d marked in Figure 20 from pin A/D to the center of mass of the triangle is found to be $\frac{l}{\sqrt{3}}$. Next, the total mass of the triangle is clearly $3m$. Third, using the parallel axis theorem for each link, with the distance r found to be $r = \frac{l}{2\sqrt{3}}$, the moment of inertia of the triangular 4-bar linkage can be calculated as follows.

$$\begin{aligned} I^G &= 3\Big(\frac{1}{12}ml^2 + mr^2\Big) \\ &= 3\Big(\frac{1}{12}ml^2 + \frac{1}{12}ml^2\Big) \\ &= \frac{1}{2}ml^2 \end{aligned} \tag{70}$$

With a start angle of $\theta = 45^o$, as illustrated in Figure 20, effectively identical numerical solutions were found between the 4-bar linkage and the equivalent single pendulum, as illustrated in Figure 21.

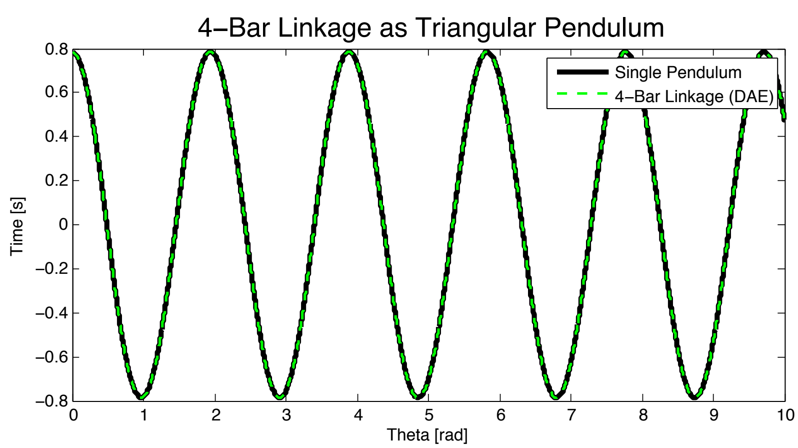

FIGURE 21. Comparison of 4-bar linkage triangle to equivalent single pendulum, at an initial angle of $\theta = 45^o$.

To verify that the 4-bar linkage triangle was set up correctly in the simulation, the animation was ran, of which a snapshot is provided below in Figure 22.

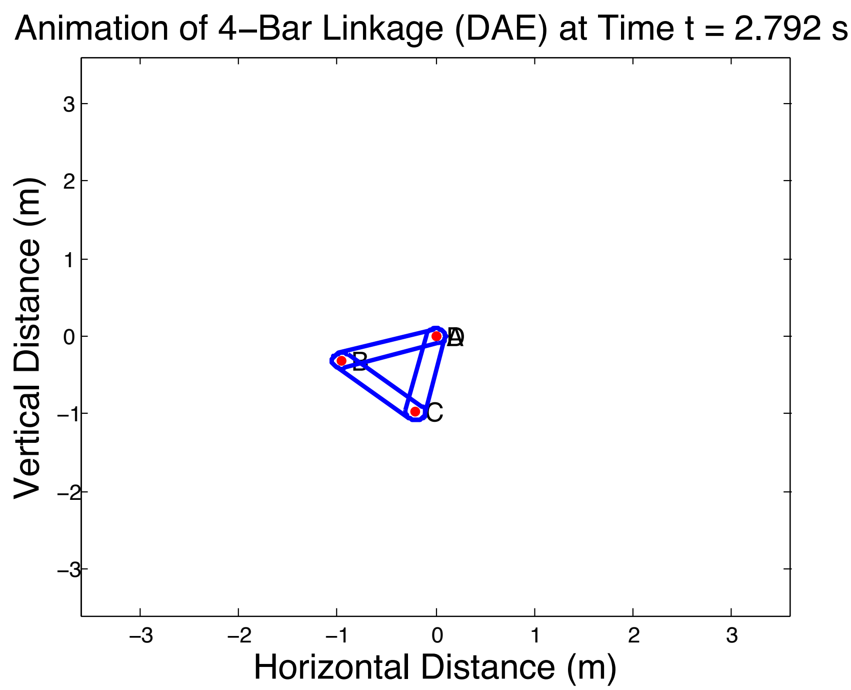

Figure 22. Snapshot of 4-bar linkage triangle animation.

7. Extras: Graphical User Interface and N-Link Solver/Animator

To make changing system constants and initial conditions easy and user-friendly, a MATLAB GUI was developed, as shown in Figure 23. This GUI also allows simulation parameters to be changed and can be used to select which approaches are solved, animated, and/or saved as movies from their animations.

In addition, the function `SolveAndAnimateNLinkPendulum` was developed to find the numerical solution for and animate the motion of an n-link pendulum. The user simply inputs the initial angles of each link, and can optionally input their initial angular rates, the duration of the simulation, gravity, and the masses, lengths, moments of inertia of each link. Given the number of links n determined by the length of the inputted initial angles array, the function `MakeNLinkPendulumSolver` writes the ODE45 solver function for the pendulum by printing the $\mathbf{A}$ matrix and $\vec{b}$ vector solved through the DAE approach, as explained in Section 3.b. A snapshot of a 6-link and 10-link pendulum animation is provided below in Figures 24 and 25.

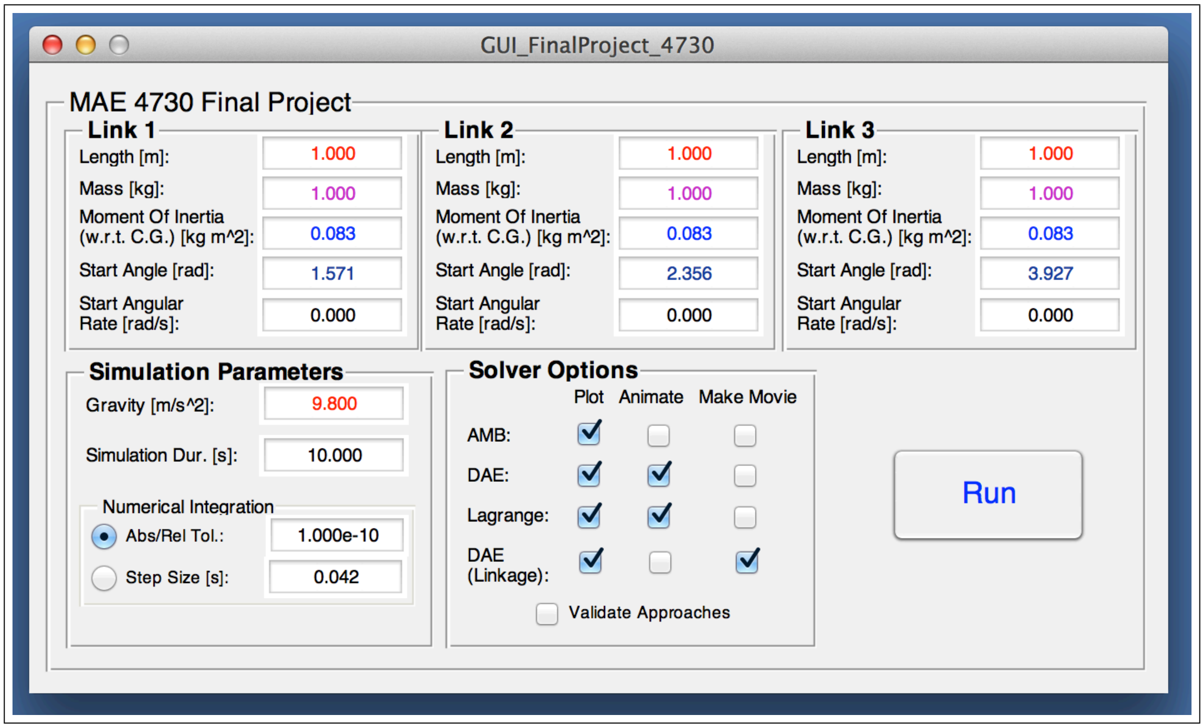

FIGURE 23. Graphical user interface to control parameters and outputs of triple pendulum and 4-bar linkage solvers

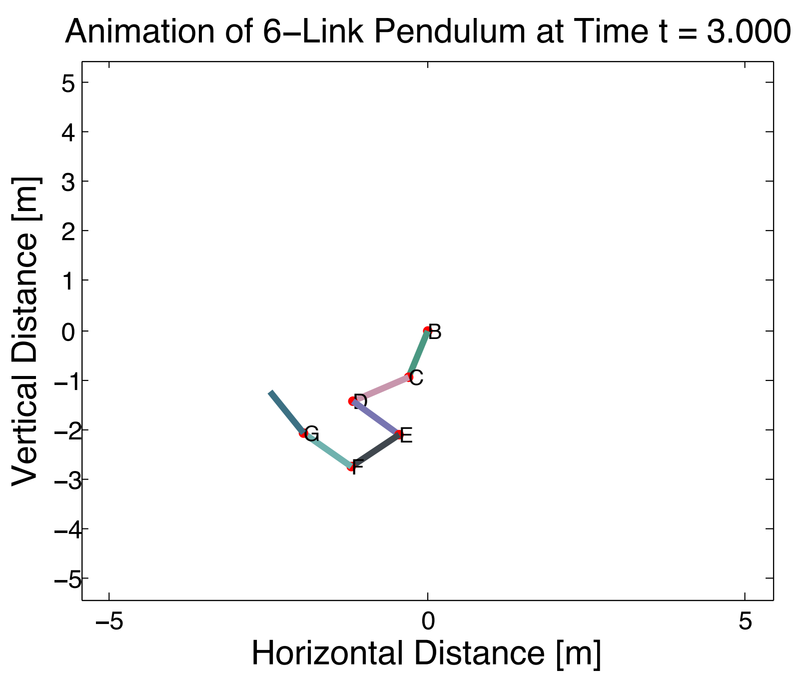

FIGURE 24. Snapshot of 6-Link Pendulum animation

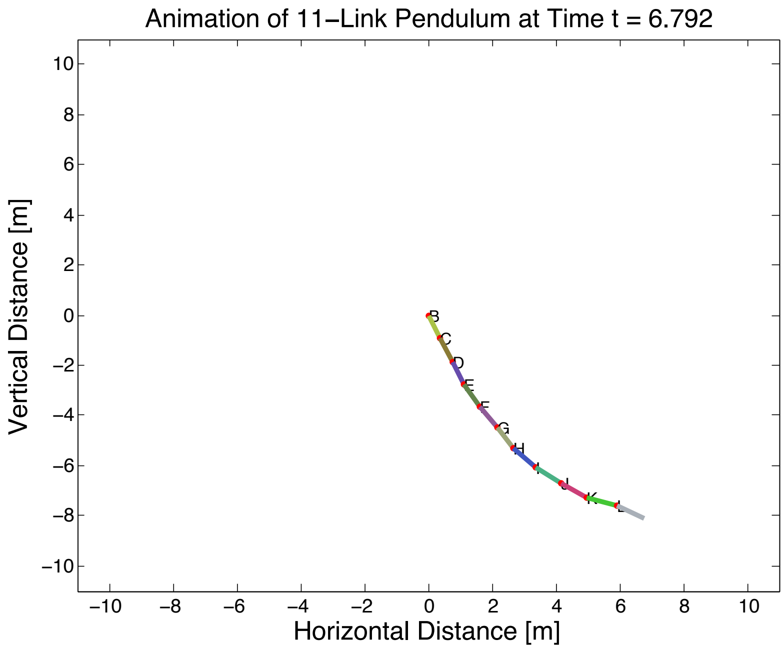

FIGURE 25. Snapshot of 10-Link Pendulum animation