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Efficient Cost Allocation

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Firms routinely allocate the costs of common corporate resources down to divisions. The main insight of this paper is that any efficient allocation rule must reflect the firm's underlying cost structure. We propose a new allocation rule (the polynomial rule), which achieves efficiency and approximate budget balance. Welfare losses due to linear allocation rules increase with firm size, so polynomial allocation rules dominate linear rules for larger firms.

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1. Introduction

The multiple divisions within a firm often share a variety of common resources, such as information technology, legal services, human resource management, executive time, etc. Managerial accounting textbooks (Horngren et al. 2005, Zimmerman 2006) and surveys of company practice (Fremgen and Liao 1981, Atkinson 1987, Ramadan 1989, Dean et al. 1991) document the widespread practice of common cost allocation to induce appropriate consumption of corporate resources. For example, if divisions are not allocated any corporate costs, they may have adverse incentives to overconsume such common resources. The objective of this paper is to examine cost allocation rules that solve this free-rider problem, i.e., induce efficient resource use by divisions acting simultaneously and independently.

We demonstrate that the main feature of any efficient allocation is that it must reflect the firm's underlying costs. While this point may seem obvious, the linear rules used in practice make allocations without regard to the shape of the firm's cost function, and this keeps such rules from achieving efficiency. The reason for this failure is straightforward: Charging for each unit of common resource used at the same constant rate (whether an actual average cost or a budgeted per-unit overhead rate) ignores the fact that the actual marginal cost of each unit of resource used may depend on the total amount of resources used. Consequently, under such cost allocation schemes, the price that a division pays for an additional unit of resource (i.e., the private cost to the division) differs from the actual marginal cost to the firm, which causes inefficient resource consumption decisions by the division.

The analysis here operates in environments that try to resemble real-world settings, with the aim of recommending cost allocations that will be practically useful to managers. We depart from formal mechanism design theory (such as Green and Laffont 1979) in that we assume that the private information of the divisional managers is too complex to be embedded in the firm's contracts. Therefore, the firm cannot perfectly obtain the manager's entire private information through a complex reporting game and through contracts that depend on announcements of private information. Private information is sufficiently complex, communication is sufficiently costly, and contracts are sufficiently incomplete that the Revelation Principle no longer applies. Despite this, our efficient mechanism operates similarly to the Groves (1973) scheme, in that it forces each division to fully internalize its externality on other divisions. In the Groves scheme, this is done by making each division's payment equal the sum of everybody else's payoffs and a term that does not depend on that division. In our mechanism, the same effect is achieved by making each division pay the total costs of the firm, plus a term that does not depend on that particular division's resource use.

The class of efficient cost allocations turns out to be large. However, the class of efficient cost allocations useful in practice can be reduced by imposing additional desirable properties on these allocations. In line with our main goal of capturing a more realistic firm environment, we require that cost allocation rules satisfy certain properties of actual allocation methods used in practice. Like the early cooperative cost allocation literature, we impose certain constraints on allocation rules and explore when these constraints can be satisfied. An allocation is *budget balancing* if the sum of the allocated costs equals total cost. Following Baldenius et al. (2007), an allocation satisfies *no-play-no-pay* (*NPNP*) when a division pays nothing if it consumes none of the resource. Linear allocation rules commonly used in practice satisfy both properties, though they are not efficient. Requiring these properties constrains the set of possible efficient allocation rules. For example, the firm could easily charge every division the full corporate cost. While this would achieve efficiency for each division, it would grossly break the budget. Is it possible to construct efficient cost allocation rules that have these additional desirable properties?

Almost. There exist allocations that are both efficient and *approximate* budget balancing. This allocation rule (called the polynomial allocation) induces efficient resource levels, but may exhibit a small budget imbalance. For firms with more divisions, this budget imbalance shrinks, eventually vanishing altogether. Numerical simulations show that for a firm with as few as four divisions, these imbalances are a small fraction of total cost. We give an explicit algorithm for calculating the polynomial allocation from the firm's cost function. First, fit a polynomial to the firm's cost function. Then use the coefficients of that polynomial to construct the allocation rule (specifically, use the coefficients to determine the transfers to different divisions). In fact, the firm can use this explicit algorithm even if it does not know its cost function exactly, but must estimate that function from internal cost data. This makes the polynomial allocation useful in practice, as it reduces the informational requirements of the allocation.

Even though linear rules are, in general, not efficient, they are widely used in practice. We conclude our analysis by exploring the welfare losses of linear rules. In particular, welfare losses increase with the number of divisions. Intuitively, linear rules are inefficient because they do not reflect the firm's underlying costs, and therefore do not adjust to changes in the firm's cost function. The linear rule is a blunt instrument to control managerial behavior compared to the efficient rule, which varies with the firm's underlying costs. An increase in the number of divisions aggravates the free-rider problem, and linear rules are less capable of resolving this problem compared to efficient rules.

Agency models of cost allocation take place in single-agent and multiple-agent settings. Single-agent settings consider a principal who must compensate and possibly allocate costs to an agent. Baiman and Noel (1985) show that allocating costs can assist in dynamic performance measurement. Magee (1988) shows that the agent's optimal contract can include a cost component based on activity levels to better control his unobservable effort levels. Demski (1981) argues that cost allocation is valuable if it provides additional information for contracting purposes.

Some papers consider multiple agents. Suh (1987) shows that the principal may want to include noncontrollable costs to discourage collusion. Rajan (1992) shows that cost allocation schemes can serve a coordination purpose when multiple agents have correlated private information. Baldenius et al. (2007) find that a cost allocation based on hurdle rates of divisional reports to a central office is an optimal mechanism in a multiple division, multiperiod setting. These last two papers both allow communication between the agents and the principal, and assume the principal can commit to a menu of contracts. We do not make these assumptions on communication and commitment here.

There has been a recent surge of interest in simple and robust mechanism design. A handful of papers seek to calculate the welfare losses from simple, common mechanisms used in practice (Rogerson 2003, McAfee 2002, Satterthwaite and Williams 2002). All three papers show that simple mechanisms fare quite well, despite small efficiency losses. Hansen and Magee (2003) show that linear allocation rules are robust in a model of a single decision maker who must allocate capacity to multiple products. Bergemann and Morris (2005) and Arya et al. (2009) consider mechanisms that are robust to small perturbations in the environment.

2. The Model

Consider a firm with n divisions and a central office. The firm has a decentralized structure: Each division acts as a profit center and therefore each divisional manager's goal is to maximize the profit of his or her division. Each division simultaneously selects a resource level k_i.[1] These resources are assets such as plants, machines, human capital, etc. Each division i has a production function $f_i(k_i)$, which is a strictly increasing and strictly concave function of division i's resource choice k_i. The strict concavity reflects diminishing marginal returns from resource use, and guarantees that the first order conditions from the firm's maximization problem are sufficient.

All divisions of the firm make use of a common, firm-wide resource, such as information technology, corporate human resources, executive time, etc.

[1] For simplicity, we assume k_i is one dimensional (i.e., just a real number). The analysis generalizes to multidimensional resources, though the interpretation of the paper becomes more complex.

The cost to the firm of the use of this common resource given each division's resource level is $C(k_1+\cdots+k_n)$, where C is strictly increasing, weakly convex, continuous, and twice continuously differentiable.[2] Let $k=(k_1,\ldots,k_n)$ be the resource vector, and let $k_{-i}=(k_1,\ldots,k_{i-1},k_{i+1},\ldots k_n)$ be the resource vector for all divisions other than i. Furthermore, let $K=k_1+\cdots+k_n$ be the total resource level and $K_{-i}=\sum_{j\neq i}k_j$ be the total resource level for all divisions except i. We assume that the feasible resource level set for each division i is bounded above by $\bar{k}_i$, so that $k\in\prod_{i=1}^{n}[0,\bar{k}_i]$ and $K\in[0,\bar{K}]$, where $\bar{K}=\sum_{i=1}^{n}\bar{k}_i$. The cost of the common resource is an increasing function of the sum of each division's *individual* resource k_i. Thus, k_i measures *individual* activity, whereas $C(K)$ measures collective use of the shared activity.[3] The additive nature of the total resource level K reflects the natural assumption that the costs of the common resource depend on the aggregate use of all individual parties.[4]

The firm's total profit is

$$\sum_{i=1}^{n}f_i(k_i)-C(K). \tag{1}$$

Let $k^*\equiv\{k_i^*\}_{i=1}^{n}$ denote the first-best efficient resource levels, i.e., the resource levels that maximize the firm's total profit. The first-order conditions for a maximum require that (for all i) $f_i'(k_i^*)=C'(K^*)$, where $K^*=\sum k_j^*$ is the efficient total resource level.[5] The production functions f_i are private information of the respective divisions, but resource level decisions k_i, current production levels $f_i(k_i)$, and costs $C(\cdot)$ are common knowledge.[6] This stands in contrast to many agency models where effort is unobservable, but utility functions are common knowledge.

Contracts within the firm are incomplete. So the firm cannot perfectly obtain the divisional managers' private information through a complex menu of contracts and incentive constraints. In this setting, the Revelation Principle does not apply. The private information of the divisions prevents the firm from implementing first-best resource levels through a forcing contract (i.e., a contract that pays each division a positive amount if it selects the first-best resource level, and zero otherwise). A forcing contract is impossible because the firm does not even know the first-best resource levels. The firm can, however, induce first-best resource levels through an appropriate cost allocation rule. Suppose that the firm charges $A_i(k)$ to division i, based on the resource levels of all divisions.[7] Let S_i be the *proportion* of common costs charged to division i, so $S_i=A_i/C$. Each division then maximizes

$$\Pi_i=f_i(k_i)-S_i(k_i,k_{-i})C(k_1+\cdots+k_n). \tag{2}$$

The agency problem here is the classic free-rider problem. Each division's resource consumption generates common costs for the firm, and thus imposes negative externalities on other divisions. The objective of the firm is to choose the allocation rule to induce the selection of efficient resource levels. The divisions are playing a simultaneous-move game, and each adopts the standard Nash assumption in (2) that all other divisions choose its resources at its equilibrium level. Thus, even though the common cost is an additive (and hence separable) function of the individual resources, the Nash assumption guarantees that each individual's optimal choice k_i depends on choices of other divisions k_{-i}.

2.1. Efficient Allocation Rules

Let $\tilde{k}_i$ denote the equilibrium resource level actually chosen by division i. These actual resource levels will be determined by the system of n first-order conditions from the individual divisions' optimization problems:

$$f_i'(\tilde{k}_i)=S_i(\tilde{k}_i,\tilde{k}_{-i})C'(\tilde{K})+C(\tilde{K})\frac{\partial S_i(\tilde{k}_i,\tilde{k}_{-i})}{\partial k_i}, \tag{3}$$

where $\tilde{k}_{-i}$ is the equilibrium resource level of all divisions other than i, and $\tilde{K}$ is the equilibrium total

[2] Convexity of the cost function is essential to guarantee the existence of a solution to the firm's optimization problem. Differentiability of the cost function is needed to generate analytical expressions defining this optimal solution. Twice differentiability is a technical condition needed for the approximations later in the paper. The model does generalize to a weak form of non differentiability at $K=0$ (the case with a positive fixed cost) as long as the optimal production level is positive. If the cost function were nondifferentiable at a finite number of points (such as lumpy costs), the analysis would still go through if the cost function were locally differentiable around the optimal solution. Details can be furnished upon request.

[3] We measure the resource level k_i in units (plants, machines, factories), whereas we measure the cost of the common resource $C(K)$ in dollars (cost of information technology, human resources, executive time, etc.).

[4] Making the common cost function a nonadditive (nonseparable) function of the individual resource levels (or, more generally, an arbitrary function of the vector k) is possible, but would significantly complicate the analysis without any clear theoretical gain.

[5] The assumptions on C and f_i guarantee that the second-order conditions for a maximum are met, and Lemma 1 in the appendix shows that the solution k^* is unique.

[6] That is, each division (and the central office of the firm) observes the current production of the other divisions, but it does not observe the other divisions' full production *functions*.

[7] This includes charging each division a capital charge rate for its resource level, in which case $A_i(k)=\mu_i k_i$ for some $\mu_i>0$. Of course, A_i can be much more general than this.

resource level. Thus in equilibrium, the marginal return to additional resource consumption equals the marginal cost. Observe that there are in fact two marginal costs of resource consumption. For every dollar's worth of resources, the division bears not only the direct marginal cost from use of the common resource, but also the marginal change in the allocation rule. This shows that cost allocations indeed have incentive effects.

Let $S \equiv \{S_i\}_{i=1}^n$ be a set of cost allocation rules.

Definition 1. S is *efficient* if, for any set of production functions, $\tilde{k}_i = k_i^*$ for all i.

A set of cost allocation rules S is efficient if each allocation rule S_i induces efficient resource levels for every division. Let S_i^* denote an efficient allocation rule and S^* the corresponding set of efficient allocation rules. Because the firm does not know the individual production functions, it can only ensure efficiency if it induces $\tilde{k}_i = k_i^*$ for *all* possible production functions. The differential equations given by the first-order conditions for the first-best and for the individual divisions' problems immediately yield a straightforward characterization of efficient allocation rules (all proofs are in the appendix).

Proposition 1. *S is efficient if and only if there exist transfers r_i: $\mathbb{R}^{n-1} \to \mathbb{R}$ such that, for all i and all $(k_1, \ldots, k_n)$,*

$$S_i^*(k_i, k_{-i}) = 1 - \frac{r_i(k_{-i})}{C(K)}. \tag{4}$$

The firm can implement efficiency (i.e., induce first-best resource levels) by setting an allocation rule with an appropriate transfer scheme $r_i(k_{-i})$, which constitutes a payment between division i and the central office. The intuition behind this result becomes apparent if we rewrite (4) by multiplying through by $C(K)$: $A(k_i, k_{-i}) = C(K) - r_i(k_{-i})$. Each division first pays the full cost of the firm ($C(K)$) and then receives a refund in the form of a transfer that depends only on the *other* divisions' resource choices ($r_i(k_{-i})$). The first step (paying the full common cost) makes each division's perceived cost move one-to-one with the firm's common cost (i.e., it equates each division's individual marginal cost of resource consumption to that of the firm), thus inducing the division to select the optimal resource level. The second step (the transfer) allows the firm to actually charge each division less than the total common cost without distorting the incentives of the division. This is because the transfer to each division does not depend on that division's decisions; the division cannot affect its own transfer by manipulating its resource level.

To see the logic in the proposition above, note that, under efficiency, the allocation to division i ($A_i(k_i, k_{-i})$) as a function of k_i must be a parallel shift of the total cost ($C(K) = C(k_i + K_{-i})$). This is because the division equates its marginal benefit $f_i'(k_i)$ to its *private* marginal cost, whereas efficiency requires that the same marginal benefit be equated to the firm's *overall* marginal cost $(\partial/\partial k_i)C(k_i + K_{-i}) = C'(K)$. Thus, if the division's decision is to coincide with the efficient decision, its private marginal cost must equal the overall marginal cost, i.e., $(\partial/\partial k_i)A_i(k_i, k_{-i}) = C'(K)$. Put differently, the functions $A_i(k_i, k_{-i})$ and $C(k_i+K_{-i})$ must have the same slope at every value of k_i, which means that one must be a parallel shift of the other. This term is the transfer $r_i(k_{-i})$. Note that, as far as efficiency is concerned, the transfer can be *any* function of k_{-i}: After receiving the payment $C(K)$ from each division, the firm can pay back as much or as little of it as it pleases, as long as the transfer given back to each division is independent of that division's own resource use.

2.2. Relation to the Groves Scheme

That the transfer for division i in Proposition 1 depends only on k_{-i} bears similarity to the Groves (1973) scheme in direct revelation mechanisms. However, the efficient rule S_i^* in Proposition 1 is not a Groves mechanism, because the game played here is not a direct revelation game. Nonetheless, the essential logic of the Groves scheme applies. Division i's transfer, independent of division i's actions, allows the mechanism designer (in this case, the firm) to adjust the total payment by division i without negatively affecting the division's incentives.

To see this concretely, let us revisit the direct revelation framework of the Groves scheme, adapting it for the particular allocation problem we study. In the Groves setup, it would be assumed that each division's production function f_i is its private information. Each division submits a report $\hat{f}_i$ to the firm, which then sets

$$\hat{k}(\hat{f}) = \arg\max_k \left[\sum_{i=1}^N \hat{f}_i(k_i) - C\left(\sum_{i=1}^N k_i\right)\right]$$

and decides on monetary transfers $t_i(\hat{f})$ to each division (note that the transfer to each division is a function of the entire vector of all divisions' reports, $\hat{f}$). Thus, given the report vector $\hat{f} = (\hat{f}_i, \hat{f}_{-i})$, division i's payoff is

$$u_i(\hat{f}_i, \hat{f}_{-i}) = f_i(\hat{k}_i(\hat{f}_i, \hat{f}_{-i})) + t_i(\hat{f}_i, \hat{f}_{-i}). \tag{5}$$

Groves (1973) shows that it is possible to set the transfers in such a way that it is a weakly dominant strategy for each division to report its true production function ($\hat{f}_i = f_i$ for all i), so that the resulting resource allocation $\hat{k}$ maximizes the true total surplus, that is, $\hat{k}(\hat{f}) = k^*(f)$, where

$$k^*(f) = \arg\max_k \left[\sum_{i=1}^N f_i(k_i) - C\left(\sum_{i=1}^N k_i \right) \right].$$

Transfers have this truthful-reporting-inducing property if and only if they are of the following form (Groves transfers):

$$t_i(\hat{f}) = \sum_{j \neq i} \hat{f}_j(\hat{k}(\hat{f})) - C\left(\sum_{i=1}^N \hat{k}_i(\hat{f}_i, \hat{f}_{-i}) \right) + h_i(\hat{f}_{-i}), \quad (6)$$

where h_i is an arbitrary function of $\hat{f}_{-i}$. If we combine (5) and (6), we see that the division's payoff is

$$U_i(\hat{f}_i, \hat{f}_{-i}) = f_i(\hat{k}(\hat{f}_i, \hat{f}_{-i})) + \sum_{j \neq i} \hat{f}_j(\hat{k}(\hat{f}_i, \hat{f}_{-i})) - C\left(\sum_{i=1}^N \hat{k}_i(\hat{f}_i, \hat{f}_{-i}) \right) + h_i(\hat{f}_{-i}). \quad (7)$$

The last term, $h_i(\hat{f}_{-i})$, does not depend on division i's report. Because the rest of the expression coincides with the objective function in the definition of $\hat{k}(\hat{f})$ when $\hat{f}_i = f_i$, the division can do no better than reporting truthfully, as this induces the firm to set $\hat{k}$ to maximize the part of i's payoff that i can influence.

In equilibrium, $\hat{f}_{-i} = f_{-i}$, so that division i's payoff (7) becomes exactly equal to the firm's total surplus plus the term $h_i(\hat{f}_{-i})$. Because the latter term cannot be influenced by division i, this division is effectively maximizing the total surplus of the firm: The externality that the division imposes on other divisions (through C) is fully internalized by the transfer.[8]

In our model, the divisions do not report their private information, but rather choose resource levels k_i directly. In addition, we assume that the information asymmetry pertains only to the contracting stage; all divisions know each other's production functions at the actual production stage. Yet the underlying logic is identical: To force division i to choose the action that is optimal for the entire firm, we set the divisional payments so as to force each division to maximize the entire firm's profits, thus internalizing the externalities that divisions impose on each other. Inserting the efficient transfers from Proposition 1 into each division's payoff function shows that each division's profit in our equilibrium is

$$\Pi_i(k_i, k_{-i}) = f_i(k_i) - C(K) - r_i(k_{-i}). \quad (8)$$

Furthermore, because $\sum_{j \neq i} f_j(k_j)$ is just a constant from division i's perspective, maximizing $\Pi_i(k_i, k_{-i})$ is equivalent to maximizing

$$\hat{\Pi}_i(k_i, k_{-i}) = \sum_{j=1}^N f_j(k_j) - C(K) - r_i(k_{-i}).$$

Because the last term is just another constant for division i, division i ends up maximizing the entire surplus of the firm, which results in an efficient choice.

The fundamental difference between our mechanism and that of Groves is in the information and contracting environment. In the Groves scheme, divisions can communicate their entire private information to the firm, which can then make optimal decisions based on that information. In our mechanism, it is not possible to write contracts on divisions' private information. Thus, a direct mechanism (one relying on individual divisions' reports of private information to the firm) is not possible. Instead, we assume that the firm must rely on the divisions to make resource choices on their own. The only information that the firm obtains is the actual resource choices made by the divisions, and the firm can make divisions' payments contingent only on those choices. The restricted communication between divisions and the firm in our model does result in some weakening of the result that the mechanism can achieve: In the Groves scheme, efficient allocation obtains as an equilibrium in weakly dominant strategies, whereas in our mechanism efficiency is simply a Nash equilibrium, but not a dominant strategy solution.

An allocation rule commonly used in practice is the linear rule $S_i^L(k_1, \ldots, k_n) = k_i/K$, where each division is allocated costs based on its relative resource level. The linear rule does not include the common cost function, and therefore it is not efficient. Nonetheless, the linear rule satisfies some convenient and intuitive properties. We now consider more general allocation rules that also satisfy these properties.

3. Budget Balance

In this section, we explore the implications of the additional requirement of budget balance, namely, the idea that the cost shares allocated should sum up to one.

Definition 2. S is *budget balancing* (*BB*) if, for all $(k_1, \ldots k_n)$,

$$\sum_{i=1}^n S_i(k_i, k_{-i}) = 1. \quad (9)$$

[8] The most commonly used type of Groves scheme, the Vickrey–Clarke–Groves (VCG) mechanism, in fact sets the terms $h_i(\hat{f}_{-i}) = \max_k [\sum_{j \neq i} \hat{f}_j(k_j) - C(\sum_{j \neq i}^N k_j)]$, so that $-t_i(f_i, f_{-i})$ becomes exactly equal to the externality that i imposes on the rest of the firm, i.e., to the difference between others' combined profit in the absence of division i and in its presence.

Budget balance simply requires the allocations to sum to one, or the sum of the allocated costs to exactly equal total costs.[9] The equality must hold at all values of $(k_1, \ldots, k_n)$, not just at the equilibrium, because the firm does not know the production functions and therefore does not know the equilibrium or efficient resource levels.

With this qualification, the requirement of budget balance is an intuitive and natural one, and typically satisfied by actual cost allocation rules used in practice (e.g., the linear rule). Textbook examples of cost allocations (such as Zimmerman 2006, Chap. 7) are also budget balancing: The identified common costs are fully distributed among cost objects (such as divisions of a firm) based on some allocation base (such as hours of resource use). Furthermore, budget balance also has normative appeal: It simplifies accounting and allows the firm to cover the full costs incurred without putting undue stress on the individual divisions' budgets.

3.1. Exact Budget Balance

When do efficient and budget balancing allocation rules exist in general? The following example shows that the search is not futile, even with strictly convex costs and zero fixed costs.

Example. Let $n = 3$ and let $C(K) = K^2$. Our goal is to create an efficient and budget balancing cost allocation.

Recall from Proposition 1 that efficiency requires the allocations to take the form

$$A_i(k_i, k_{-i}) = C(K) - r_i(k_{-i}). \tag{10}$$

Therefore, all three allocations together sum to

$$A_1(k_1, k_{-1}) + A_2(k_2, k_{-2}) + A_3(k_3, k_{-3}) = 3C(K) - r_1(k_{-1}) - r_2(k_{-2}) - r_3(k_{-3}). \tag{11}$$

Budget balance requires that this total allocated cost be equal to the total common cost, so

$$r_1(k_{-1}) + r_2(k_{-2}) + r_3(k_{-3}) = 2C(K). \tag{12}$$

Expanding this and plugging into the expression for the sum of the transfers above yields

$$r_1(k_{-1}) + r_2(k_{-2}) + r_3(k_{-3}) = 2k_1^2 + 2k_2^2 + 2k_3^2 + 4k_1k_2 + 4k_1k_3 + 4k_2k_3. \tag{13}$$

To obtain the individual transfers, we now just have to regroup the terms in the sum above, making sure that r_i does not contain any terms containing k_i for any i. One (symmetric) way to do this is to write

$$r_1(k_{-1}) = k_2^2 + k_3^2 + 4k_2k_3, \tag{14}$$

$$r_2(k_{-2}) = k_1^2 + k_3^2 + 4k_1k_3, \tag{15}$$

$$r_3(k_{-3}) = k_1^2 + k_2^2 + 4k_1k_2. \tag{16}$$

Letting $A_i(k_i, k_{-i}) = C(K) - r_i(k_{-i})$ for all i now yields our desired efficient, budget balancing solution.

Notice that in the three-division example above the third derivative of the cost function was zero. The following proposition shows that this is no coincidence.

Proposition 2. *An efficient and budget balancing allocation rule exists if and only if the nth derivative of C is identically* 0.

This completely characterizes the set of efficient and budget balancing allocation rules. The main insight is that every efficient rule must satisfy Proposition 1, and the allocations must sum to one. This reduces to the expression:

$$\frac{1}{n-1}\sum_{i=1}^{n} r_i(k_{-i}) = C(K). \tag{17}$$

In words, the average transfer must equal the total cost. Differentiating both sides of the equation above n times with respect to $k_1, \ldots, k_n$ shows that the nth derivative of C is 0. Moreover, any cost function whose nth derivative is 0 must be a polynomial of degree less than or equal to $n-1$. The proof of Proposition 2 shows that it is possible to construct a set of transfers based on the coefficients of that polynomial such that the equation above holds.

The result of Proposition 2 makes a theoretical link between the number of divisions in the firm and its cost function. As long as there are more divisions in the firm than the degree of the cost function, there will exist an efficient and budget balancing allocation rule. For example, suppose the cost function is the quadratic cost function $C(K) = \gamma K^2$. The third derivative of this function is 0, so any firm with three or more divisions can use the allocation rule constructed in the proof of Proposition 2. This allocation rule is efficient and budget balancing. One implication is that it is easier to achieve efficiency and budget balance in firms with many divisions. These firms allow for cost functions with large degrees, because the high number of divisions permits a class of increasingly fine polynomials.

3.2. Approximate Budget Balance with Known Cost Function

While exact budget balance is a desirable feature, it may often be sufficient to get "close enough." This

[9] Demski (1981) called allocations that sum to one "tidy." We use the term "budget balance," following the extensive literature on public decisions and cost-sharing (Groves 1973, Green and Laffont 1979, Moulin and Shenker 1992, Moulin 2005).

is analogous to the accounting notion of materiality, well documented in textbooks of financial and managerial accounting (Garrison et al. 2004, Stickney et al. 2009). What happens when the budget does not balance perfectly, but the amounts over and underbudget are immaterial? In this case, we can expand our analysis beyond polynomial cost functions to generate allocations that are still approximate budget balancing (i.e., they balance the budget almost, but not exactly). The central office can then determine the materiality threshold to which the accounting system must adhere. This allows us to expand our domain of analysis by considering a much wider range of cost functions.

Proposition 2 also begs the question of what happens if the cost function is not a polynomial, because some cost functions do not satisfy this property for any n (consider, for example, $C(K)=e^K$). However, recalling the basic result from real analysis that every continuous function can be arbitrarily well approximated by polynomials (Weierstrass theorem; see Royden 1988), it is possible to construct efficient rules that converge to budget balance, as long as the number of divisions is sufficiently large.

Definition 3. Let $\{\{S_i^n\}_{i=1}^n\}_{n=1}^\infty$ be a sequence of sets of allocation rules. The sequence *converges to budget balance in n* if

$$\lim_{n\to\infty}\max_{k}\left|\sum_{i=1}^{n} S_i^n(k_i,k_{-i})-1\right|=0. \quad (18)$$

Thus, if a sequence of sets of allocation rules converges to budget balance in n, it is approximate budget balancing in the sense discussed above: As the number of divisions increases, the maximum possible deviation from budget balance (over all possible equilibrium values of k) gets arbitrarily small. The reason we focus on the maximum possible deviation over all k is that the firm does not know the production functions of the individual divisions in advance, and hence also does not know the equilibrium values of k. Because any k is a possible equilibrium (as shown in Lemma 2 in the appendix), the firm must design the cost allocation rule to guarantee that the deviation from budget balance will be small, no matter what equilibrium k turns out to be.

Proposition 3. *There always exists a sequence of efficient allocation rules that converges to budget balance in n.*

We can achieve efficiency under many cost functions that are approximate budget balancing. As the materiality threshold becomes tighter, so does the hurdle for the number of divisions. Call the allocation rule constructed in Proposition 3 the "polynomial allocation," because the allocation itself is built from a Chebyshev polynomial approximation of the cost function. We know from Proposition 2 how to construct an efficient and budget balancing rule when the cost function is a polynomial. When the cost function is not a polynomial, we can *approximate* the function by a polynomial and construct the allocation rule from this approximated function, instead of the true one. By construction, the rule will be efficient. Furthermore, the allocations will sum to the approximated cost function. This will result in a budget imbalance. However, as the quality of the approximation improves, the approximated function (and hence also the sum of the allocations) will be closer and closer to the true cost function. Now, as the number of divisions grows, higher order polynomials can be used in the approximation (recall that we can use polynomials of order at most $n-1$). Consequently, the quality of the approximation improves, and the budget imbalance gradually vanishes.

As an example, consider the case $C(K)=e^K$ and let $\bar{K}=5$ (so that K is restricted to the interval $(0,5]$). To construct the polynomial allocation, an n-division firm begins by approximating the cost function using Chebyshev least squares approximation of degree $n-1$ over the interval $(0,\bar{K}]$. If the firm has five divisions, it will use the degree-4 approximation; in the $C(K)=e^K$ case, $\hat{C}(K)=1.66-5.14K+9.28K^2-3.93K^3+0.69K^4$. The first five polynomial approximations are shown in Figure 1. The approximated cost functions are quite close to the true C and the fit improves as the order of the polynomial increases. Once the cost function has been approximated by a polynomial, the firm uses the coefficients of the polynomial obtained to calculate the cost allocations according to the algorithm in the proof of Proposition 2. Finally, the allocation is given by $A_i=C(K)-r_i$.

Because, by construction, the allocated costs sum to the value of the estimated cost function, the budget

Figure 1 **The Quality of Fit of Polynomial Approximations**

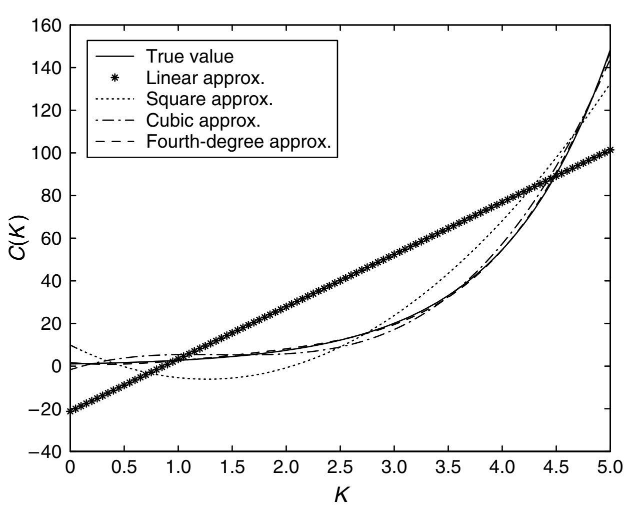
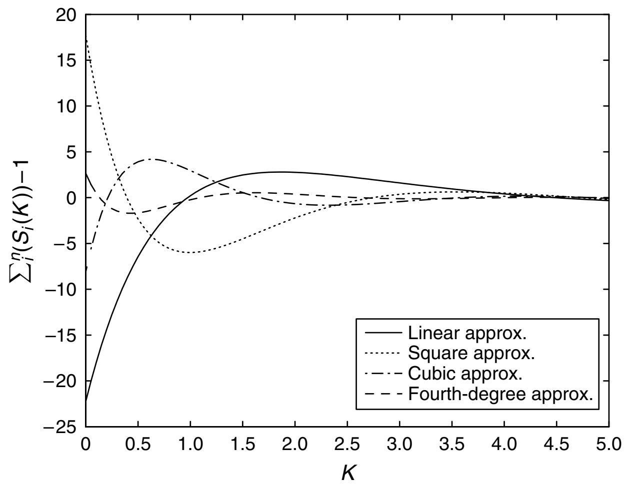


Figure 2 Deviations from Budget Balance for Polynomial Rules



imbalance at a given point equals the difference between the true and estimated cost functions at that point. Figure 2 uses this fact to show deviations from budget balance ($|\sum_{i=1}^n S_i^n(k_i, k_{-i}) - 1|$) as a function of K for the first few polynomial approximations. The convergence is very fast (note that the y-axis is on a logarithmic scale). Linear approximations allow for budget imbalances more than 20 times the total cost (however, as we saw before, these occur only when the equilibrium K is very low; the "typical" budget imbalance is much lower than this upper bound), whereas high-order approximations have essentially negligible imbalances. In particular, with 10 divisions (ninth-degree approximation), the budget imbalance is no more than 0.06% times the total cost, and with 20 divisions it is at most 8.4×10^{-13} times the total cost; that is, budget balance is achieved almost exactly.

The polynomial allocation constructed above is efficient for any size firm, and converges to budget balance as the number of divisions increase. Therefore, firms with more divisions that make use of this allocation rule eventually achieve budget balance. This feature of the polynomial allocation makes it superior to other efficient rules. The allocation rule constructed here, on the other hand, is not only efficient, but also achieves approximate budget balance, with the deviation from budget balance vanishing as the number of divisions increase. Furthermore, because the coefficients of the polynomial fitted to the cost function form the basis for the allocation, Proposition 3 further illustrates the message that the "right" allocation must reflect the firm's costs.

This analysis shows that budget balance and efficiency are not as incompatible as previously thought, especially in light of the impossibility results of the formal mechanism design literature (Green and Laffont 1979). We have always known that it is possible to achieve efficiency by breaking the budget, and that it is impossible to simultaneously achieve efficiency and budget balance for general cost functions. Yet the impossibility is not as severe as it seems. In particular, it is possible to construct allocation rules that are simultaneously efficient and converge to budget balance. Thus, if the firm is willing to relax its need for exact budget balance, and is willing to replace it with approximate budget balance, the polynomial allocation can do this and achieve efficiency as well. In some sense, this shows that the impossibility result is *discontinuous* because it is possible to approximate the cost function arbitrarily closely and still achieve efficiency and approximate budget balance.

3.3. Approximate Budget Balance with Estimated Cost Function

In the previous subsection, we assumed that the firm knows the entire cost function. However, this may be unlikely in practice. It is much more likely that the firm has observed the values of the cost function at a finite set of points and has to extrapolate the function elsewhere.

In this case, the firm can obtain a polynomial approximation of the cost function by running an ordinary least squares (OLS) regression of observed cost function values on observed resource levels. The firm can then use the polynomial allocation rule with polynomial coefficients from the OLS regression instead of those from the Chebyshev approximation (the latter is unavailable for an unknown cost function). However, it is not immediately obvious that the resulting rule will approach budget balance, even when both the number of divisions and the number of sample points are high; for each n, the OLS regression used suffers from omitted variable bias (attributable to truncating the series of powers of K at $n-1$), which usually does not vanish as sample size goes to infinity. Fortunately, the special structure of the regressor matrix, along with the convergence result from the previous subsection, guarantees that, in this particular case, the bias does disappear and the rule does become budget balancing as *both* the number of divisions and the sample size for each number of divisions increase. We now turn to stating this result more formally.

Let an n-division firm have data on cost function values at m points. The data consist of a vector $K^{n,m}$ of m observed total resource levels and a vector $y^{n,m}$ of the corresponding cost levels:

$$\tilde{K}^{n,m} = (\tilde{K}_1^{n,m}, \tilde{K}_2^{n,m}, \ldots, \tilde{K}_m^{n,m})', \tag{19}$$

$$\tilde{y}^{n,m} = (C(\tilde{K}_1^{n,m}), C(\tilde{K}_2^{n,m}), \ldots, C(\tilde{K}_m^{n,m}))'. \tag{20}$$

The firm constructs an allocation rule in two steps as follows:

1. Run an OLS regression of $\tilde{y}^{n,m}$ on the first $n-1$ powers of $\tilde{K}^{n,m}$ and a constant to estimate the cost function by $\tilde{C}$:

$$\tilde{C}(K) = \tilde{c}_0^{n,m} + \tilde{c}_1^{n,m} K + \cdots + \tilde{c}_{n-1}^{n,m} K^{n-1}, \tag{21}$$

where $\tilde{c}^{n,m} = (\tilde{c}_0^{n,m}, \tilde{c}_1^{n,m}, \ldots, \tilde{c}_{n-1}^{n,m})'$ is the vector of estimated coefficients from the OLS regression.

2. Construct the cost allocation rules $\{S_i^{n,m}\}_{i=1}^n$ as outlined in the proof of Proposition 3, with $\tilde{c}_i^{n,m}$ in place of $\hat{c}_i$.

Definition 4. Let $\{\{S_i^{n,m}\}_{i=1}^n\}_{n=1}^\infty$ be a sequence of sets of allocation rules. The sequence *converges to budget balance in n and m* if

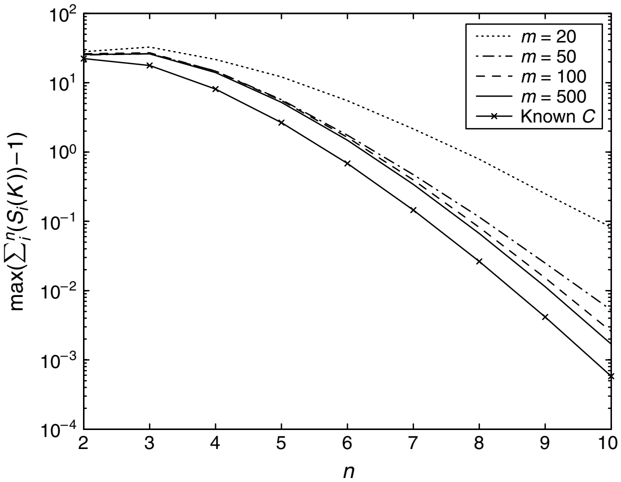
$$\lim_{n\to\infty} \lim_{m\to\infty} \max_k \left| \sum_{i=1}^n S_i^{n,m}(k_i, k_{-i}) - 1 \right| = 0. \tag{22}$$

That is, the sharing rules converge to budget balance in n and m if the deviation from budget balance becomes negligible when the firm estimates its cost functions from large samples and the number of divisions becomes large. The intuition for this definition is essentially the same as that for convergence to budget balance in n alone (from the case of known cost functions). The only difference is that in the case of estimated cost functions, one additional variable influences the quality of the approximation and hence also the maximum deviation from budget balance. This variable is sample size: The more observations of the cost function the firm has made in the past, the better it is able to estimate the cost function. An allocation rule converges to budget balance if the maximum possible deviation from budget balance goes to zero as the number of divisions and the number of cost function sample points grow.

Intuitively, this happens for two reasons. First, for any firm with a given number of divisions, the estimated cost function becomes closer and closer to the true cost function as the number of cost function observations (m) grows (this is the inner limit in the definition above). Second, just as in §3.2, the allocation rule guarantees that the maximum possible deviation from budget balance diminishes as the number of divisions (n) grows (this is the outer limit in the definition). Consequently, if a large firm with a large number of prior observations of the cost function implements cost allocation rules that converge to budget balance, it can be certain that the deviation from budget balance will be small, regardless of the shapes of the divisions' production functions.

It turns out that, as long as the firms' samples are sufficiently well dispersed over the range of feasible resource levels, the estimated polynomial allocation rule does indeed converge to budget balance.

Figure 3 **Maximum Deviation from Budget Balance as a Function of *n* and *m***



Proposition 4. *The estimated polynomial allocation rule converges to budget balance in n and m, if for each n and i,* $\lim_{m\to\infty}(\tilde{K}_i^{m,n} - (\bar{K}/m)i) = 0$.

Thus, the algorithm for the estimated polynomial rule differs from the simple polynomial rule only in the way the cost function is approximated by a polynomial: The simple polynomial rule uses the Chebyshev least squares approximation, whereas the estimated polynomial rule relies on OLS regression on a set of previously observed values. The simple polynomial rule requires exact knowledge of the cost function, whereas the estimated rule uses random samples of values of the function.

Let us return to our earlier example $C(K) = e^K$ for $K \in [0, 5]$. Suppose the firm has a sample of m observations of the values of the cost function. As described above, the firm runs the OLS regression on these observations to obtain a polynomial approximation of the cost function and then constructs the cost allocation, as described in the examples in the previous two sections. Figure 3 illustrates the speed of convergence to budget balance. Each line corresponds to a sample size. For each sample size the budget imbalance decreases as the number of divisions grows, and the budget imbalance for any given number of divisions is smaller as the sample size grows. With a sample size of 100, the deviation from budget balance is less than 10% (10^{-1}) for firms with more than seven divisions, less than 1% (10^{-2}) for firms with more than eight divisions, and less than one-hundredth of 1% (10^{-4}) for firms with twelve or more divisions.

4. No-Play-No-Pay Allocation Rules

In the previous section, we analyzed conditions under which cost allocation rules are budget balancing.

Following Baldenius et al. (2007), we now consider allocations that satisfy NPNP.

Definition 5. S satisfies NPNP if $S_i(0, k_{-i}) = 0$ for all i and all k_{-i}.

NPNP allocation rules neither reward nor penalize divisions for not consuming any of the common resource. Like budget balance, we take NPNP as an exogenous constraint on the class of feasible allocation rules. Combining efficiency (Proposition 1) and NPNP yields the following (unique) allocation rule:

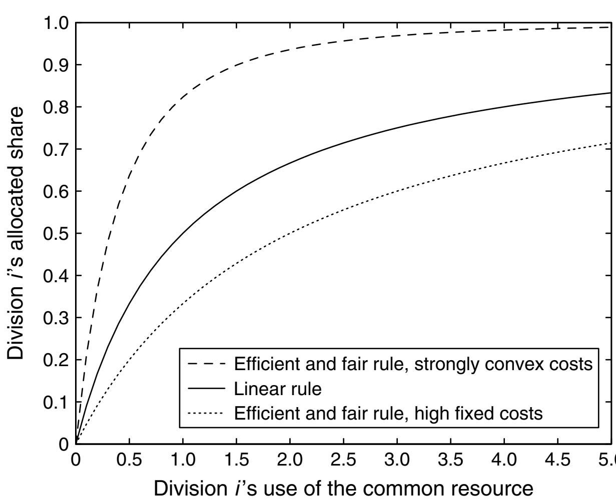
$$S_i(k_i, k_{-i}) = 1 - \frac{C(K_{-i})}{C(K)}. \tag{23}$$

NPNP implies that $r_i(k_{-i}) = C(K_{-i})$: Each division pays the additional costs that it incurs. The efficient allocation above that satisfies NPNP allocates to each division its relative incremental contribution to total cost. Specifically, this is its incremental contribution to total cost, $C(K) - C(K_{-i})$, divided by the total cost $C(K)$. Charging each division its relative incremental contribution to total cost induces each division to consume resources at the efficient level.

Recall that any efficient rule *must* include the common cost function of the firm. This is essential to obtain efficiency and is a key distinction between efficient allocation rules and linear allocation rules. Consider a particular example. Let $C(K) = F + K^q$, pick one division (labeled i), and fix the total resource use of all other divisions at $K_{-i} = 1$. Figure 4 graphs the share allocated to division i according to the linear rule and according to the efficient NPNP rule under two scenarios: highly convex C ($F = 0$, $q = 2.5$) and high fixed costs with linear variable costs ($F = 1$, $q = 1$).

Suppose that the cost function C is highly convex. Therefore, additional resources for any division are highly costly for the firm. The firm would like to discourage such resource use, and can do this by increasing the share of allocated costs. In particular, for any given resource level k_i, the firm will allocate more of the common cost under the efficient rule satisfying NPNP than under the linear rule. Essentially, the firm adjusts its allocation to respond to its highly convex cost function. Figure 4 shows that for the cost function $C(K) = F + K^q$ the efficient rule satisfying NPNP lies above the linear rule if costs are sufficiently convex (if q is sufficiently high).[10] The efficient rule essentially *accelerates* the cost allocation for any given resource level.

Figure 4 **Efficient vs. Linear Allocation Rules**



Alternatively, suppose that the firm's cost function has a high fixed component but a low variable component. In this case, the marginal effect of additional resource use by any division on the total resource level will be small. The firm would like to encourage additional resource use, and can do so by reducing the share of allocated cost for any given resource level. Therefore, the efficient allocation satisfying NPNP will lie below the linear allocation rule as shown in Figure 4. The efficient rule satisfying NPNP essentially *decelerates* the cost allocation for any given resource level, compared to the linear rule. Unlike the linear rule, the efficient rule satisfying NPNP varies as the firm's cost function varies. Efficiency forces the allocation to reflect the underlying costs; linear rules do not.

A natural question to ask is whether efficient, budget balancing, and NPNP-satisfactory allocation rules even exist. For example, linear rules are budget balancing and satisfy NPNP, but are not necessarily efficient. The polynomial rule constructed in the previous section is efficient and approximate budget balancing, but does not necessarily satisfy NPNP. The efficient allocation rule satisfying NPNP in (23) does not always balance the budget. Unfortunately, these examples are not a coincidence. For virtually any commonly used cost function (except for constant marginal cost with zero fixed cost) allocation rules that satisfy all three of the criteria above (efficiency, budget balance, and NPNP) do not exist.

Proposition 5. *An efficient and budget balancing allocation rule that satisfies NPNP exists if and only if* $C(K) \equiv \alpha K$, *for some* $\alpha \in \mathbb{R}_+$.

Any efficient allocation rule satisfying NPNP must satisfy (23) or, in words, must allocate to each division its relative incremental contribution to total cost. Yet budget balance constrains these allocations to one. Said differently, budget balance requires that the sum

[10] See Corollary 1 in the appendix for a derivation of this result.

of each division's incremental contribution to total cost exactly equals total cost:

$$\sum_{i=1}^{n}(C(K)-C(K_{-i}))=C(K)=C\left(\sum_{i=1}^{n}(K-K_{-i})\right). \quad (24)$$

The only cost function that satisfies this condition for all $(k_1,\ldots,k_n)$ is the constant-marginal-cost function with no fixed cost ($C(K)=\alpha K$). Intuitively, the linearity of the variable costs (constant marginal cost), along with the absence of fixed costs, allows us to exchange the C and $\sum$ in the equation above. Moreover, no other cost function permits this.

Green and Laffont (1979) show that it is impossible to find mechanisms that satisfy ex-post efficiency, budget balance, and implementation in dominant strategies. Their setting is different from here. We use a weaker equilibrium notion (Nash instead of dominant-strategy), a less complete class of contracts (those that do not vary with private information), a more restrictive set of mechanisms (allocation rules instead of general mechanisms), and an additional requirement that the allocations satisfy NPNP. Nonetheless, efficiency, budget balance, and NPNP are generally incompatible (except in the knife-edge case when costs are linear and with zero fixed costs).

To see this explicitly, recall the Groves scheme discussed in §2.2. There, the firm will set the transfers for division i equal to the externality imposed on all other agents, plus a term that does not vary with that division's choice. With these transfers, each division faces the same problem as the firm, and therefore maximizes total benefits and makes the efficient choice. Yet budget balance in the Groves scheme imposes that the transfers must sum to zero. Imposing $\sum_{i=1}^{N} t_i(\hat{f})=0$ on each of the individual transfers in the Groves scheme in (5) shows that this is clearly impossible. To guarantee efficiency, each transfer must internalize the externality imposed on others. Imposing the additional constraint that the transfers sum to zero obstructs this process. There are no more degrees of freedom to adjust the transfers, and so imposing budget balance will only shift the equilibrium away from efficiency.

The same logic applies here. Budget balance in this context means that the allocations must sum to total cost, so $\sum S_i=1$. NPNP makes the transfers $r_i(k_{-i})=C(K_{-i})$. Like the Groves scheme, Proposition 1 ensures that the efficient cost allocation forces each division to act in the interest of the firm as a whole by charging each division the full cost. Each division then has the proper incentives to constrain cost because a marginal increase in resource leads to a marginal increase in total common cost. Thus, in equilibrium, the division's marginal private benefit equals the marginal social cost. Yet budget balance forces the sharing rule to sum to one, and this breaks the equality between private benefit and social cost.

In sum, the common thread between the public decision literature and our study is that budget balance interferes with efficient incentives. The transfer in the Groves scheme matches social benefits with private benefits, just as our efficient allocation rule in Proposition 1 matches private costs with social costs. It is this equivalence that leads the individual divisions to choose privately what is optimal socially. Yet to do so, the mechanism designer must structure the transfer so that the individual parties act in the interests of all parties. Doing so breaks the budget. Forcing budget balance prevents each individual party from fully internalizing the costs and benefits from the planner's problem, and this is what causes the inefficiency.

5. Inefficiency of Linear Rules

In practice, linear rules are often used even though they may not be efficient. This section considers the relationship between linear and efficient allocation rules. This will give insight into the welfare losses from using linear rules compared to the efficient rule. In general, the use of linear rules leads to overconsumption of resources relative to first-best. The superscript "L" designates the linear rule and the superscript "$*$" designates the efficient rule.

Proposition 6. *Let $C(0)=0$. If $f_i=f$ for all i, then $k_i^L>k_i^*$.*

In the case of symmetric production, at the symmetric equilibrium, the resource levels under a linear allocation rule will be larger for each division than the efficient symmetric value. Consider

$$f_i(k_i)=A(k_i)^p \quad \text{for each } i \quad \text{and} \quad C(K)=BK^q, \quad (25)$$

where $0<p<1<q<\infty$. Let $\Delta\equiv 1-\pi^L/\pi^*$ be the measure of welfare loss; a larger Δ corresponds to a greater welfare loss.

Proposition 7. *The welfare loss Δ increases in n, increases in q for sufficiently high values of n or p and for sufficiently low values of q, and decreases in q for sufficiently high values of q.*

Most importantly, the welfare loss increases with the number of divisions. Larger firms with more divisions suffer more from using linear rules than smaller firms with fewer divisions. Combined with Proposition 3, this gives a general prediction on efficient cost allocation and the size of firms. If n is small, then Proposition 7 shows that the welfare loss from using the linear rule is small, and so small firms can use linear rules with few efficiency losses. But if n is big, not only do the welfare losses from the linear rules increase, but the benefits of the polynomial allocation also increase, because the polynomial allocation converges to budget balance for large n. Therefore, small firms can use linear rules, and large firms should use nonlinear rules (such as the polynomial allocation). This recommendation from the model is consistent with the normative flavor of much of the cost allocation literature.

6. Conclusion

This study proposes a new cost allocation—the polynomial allocation—that achieves efficiency and converges to budget balance, as the number of divisions in the firm increases. The main message is that any efficient allocation must reflect the firm's underlying costs. As the cost of a common resource increases, the firm would like to discourage use of the common resource (to mitigate the negative externalities of resource consumption) and therefore will accelerate the allocation, relative to the linear rule used in practice. Thus, the efficient rule will vary with the firm's cost structure. Future research in this area should proceed in a similar manner by examining what firms actually do and making explicit, concrete recommendations for changes to practice.

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Appendix

Lemma 1. *Given C and the set of production functions $(f_1, \ldots, f_n)$, there is a single vector of resource levels $(k_1^*, \ldots, k_n^*)$ satisfying the first-best conditions $f_i'(k_i^*) = C'(\sum_i k_i^*) = 0$ for all i.*

Proof of Lemma 1. Suppose that $k^a = (k_1^a, \ldots, k_n^a)$ and $k^b = (k_1^b, \ldots, k_n^b)$ both satisfy $f_i'(k_i^j) = C'(K^j)$ for $j \in \{a, b\}$, where $K^j \equiv \sum_{i=1}^n k_i^j$. First, suppose $K^a \neq K^b$. Without loss of generality, $K^a < K^b$. Then

$$f_i'(k_i^a) = C'(K^a) < C'(K^b) = f_i'(k_i^b). \tag{26}$$

This implies that $k_i^a > k_i^b$ for each i, since $f_i'' < 0$. So $K^a = \sum k_i^a > \sum k_i^b = K^b$, which is a contradiction.

Therefore, $K^a = K^b = K^*$. Because $f_i'(k_i^j) = C'(K^*)$, it must also hold that $k_i^a = k_i^b = k_i^*$ for each i. □

Lemma 2. *For any vector $(\hat{k}_1, \ldots, \hat{k}_n)$ there is a vector of production functions $(\hat{f}_1, \ldots, \hat{f}_n)$ that leads to optimal resource levels $k_i^* = \hat{k}_i$ for each i.*

Proof of Lemma 2. Let the function $f^\alpha(x) \equiv \alpha x^{1/2}$. Then $f^{\alpha\prime}(x) = \alpha/(2x^{1/2})$. So for each i, let

$$\hat{f}_i = f^{\alpha_i} \quad \text{for } \alpha_i = 2\hat{k}_i^{1/2} C'\left(\sum_{i=1}^n \hat{k}_i\right). \tag{27}$$

These production functions generate our desired optimal resource levels. □

Proof of Proposition 1. We will prove the equivalent statement that a set of cost allocations induces $k_i^* = \tilde{k}_i$ for all i and all $\{f_i\}_{i=1}^n$ if and only if there exist $r_i: \mathbb{R}^{n-1} \to \mathbb{R}$ such that, for all i and all $(k_1, \ldots, k_n)$, $A_i(k_i, k_{-i}) = C(\sum_{i=1}^n k_i) - r_i(k_{-i})$.

Recall that, for a given set of production functions, k_i^* and $\tilde{k}_i$ are defined by the first-order conditions for the firm's and the individual division's optimization problems, respectively:

$$f_i'(k_i^*) = C'\left(\sum_{i=1}^n k_i^*\right) \quad \text{and} \quad f_i'(\tilde{k}_i) = \frac{\partial A_i}{\partial k_i}(\tilde{k}_i, \tilde{k}_{-i}). \tag{28}$$

Therefore, $k_i^* = \tilde{k}_i$ for all i if and only if

$$C'\left(\sum_{i=1}^n k_i^*\right) = \frac{\partial A_i}{\partial k_i}(k_i^*, k_{-i}^*). \tag{29}$$

This completes the proof of the "if" part of Proposition 1: If $A_i(k_i, k_{-i}) = C(\sum_{i=1}^n k_i) - r_i(k_{-i})$, then $C'(\sum_{i=1}^n k_i) = (\partial A_i/\partial k_i)(k_i, k_{-i})$ at all $(k_1, \ldots, k_n)$ and thus also at $(k_1^*, \ldots, k_n^*)$.

The "only if" part requires some more work because, for a given set of production functions, the relationship $C' = \partial A_i/\partial k_i$ must hold only at one point, namely, at the corresponding first-best resource vector. This precludes us from determining a global relationship between A and C directly. However, the fact that we must ensure $k_i^* = \tilde{k}_i$ for *arbitrary* production functions guarantees that the relationship will hold at every point $(k_1, \ldots, k_n)$, as Lemma 2 shows.

Because every vector $(k_1, \ldots, k_n)$ is first-best for some set of production functions, the relationship

$$C'\left(\sum_{i=1}^n k_i\right) = \frac{\partial A_i}{\partial k_i}(k_i, k_{-i}) \tag{30}$$

must hold at all $(k_1, \ldots, k_n)$. Holding k_{-i} fixed and integrating with respect to k_i, we readily obtain

$$A_i(k_i, k_{-i}) = C\left(\sum_{i=1}^n k_i\right) - r_i(k_{-i}). \quad \square \tag{31}$$

Corollary 1. *Let $C(K) = F + K^q$. Let S_i^L and S_i^* denote the "linear" and "NPNP-satisfactory and efficient" allocation rules, respectively. For any i and k_{-i},*
1. $S_i^*(k_i, k_{-i}) > S_i^L(k_i, k_{-i})$ for sufficiently high m;
2. $S_i^*(k_i, k_{-i}) < S_i^L(k_i, k_{-i})$ for sufficiently high F.

Proof of Corollary 1. Observe $S_i^L(k_i, k_{-i}) = k_i/K$ and $S_i^*(k_i, k_{-i}) = 1 - C(K_{-i})/C(K)$. So $S_i^*(k_i, k_{-i}) > S_i^L(k_i, k_{-i})$ if and only if

$$\frac{K_{-i}}{K} > \frac{C(K_{-i})}{C(K)} = \frac{F + (K_{-i})^q}{F + K^q} \equiv D. \tag{32}$$

Observe that

$$D = \frac{1}{1+K^q/F} + \frac{(K_{-i}/K)^q}{F/K^q+1} \to 0 \quad \text{as } q \to \infty. \tag{33}$$

Both terms vanish as $q \to \infty$ since $K_{-i} < K$. This proves part one.

Observe that

$$D = \frac{1+K_{-i}^q/F}{1+K^q/F} \to 1 \quad \text{as } F \to \infty, \tag{34}$$

since $K_{-i}/K < 1$. This proves part two. $\square$

Proof of Proposition 2. $\Longrightarrow$: Suppose an efficient and budget balancing allocation rule does exist. By efficiency, there are functions $(r_1, \ldots, r_n)$ satisfying

$$S_i(k_i, k_{-i}) = 1 - \frac{r_i(k_{-i})}{C(\sum_{i=1}^n k_i)}. \tag{35}$$

Summing over all divisions and applying the BB condition yields

$$\sum_{i=1}^n r_i(k_{-i}) = (n-1)C\left(\sum_{i=1}^n k_i\right). \tag{36}$$

Apply $\partial^n/(\partial k_1 \cdots \partial k_n)$ to both sides to get

$$0 = (n-1)C^{(n)}\left(\sum_{i=1}^n k_i\right). \tag{37}$$

$\Longleftarrow$: If $C^{(n)}$ is identically 0, then C must be a polynomial of degree less than or equal to $n-1$:

$$C(K) = a_{n-1}K^{n-1} + \cdots + a_1 K + a_0. \tag{38}$$

We will now define some helpful terminology. First define the sets

$$P^j \equiv \left\{ p = (p_1, \ldots, p_n) \,\middle|\, p_l \text{ a nonnegative integer}, \left(\sum_{l=1}^n p_l\right) = j \right\}, \tag{39}$$

$$P_i^j \equiv \{p \in P^j \mid p_i = 0\} \tag{40}$$

for $j = 1, \ldots, n-1$; $i = 1, \ldots, n$. Next, for $p \in P^j$, let $G(p)$ be the number of nonzero coordinates of p: $G(p) \equiv |\{l \mid p_l \neq 0\}|$. Note that $G(p)$ is at most j. Finally, define $\binom{j}{p} \equiv j!/(p_1! \cdots p_n!)$.

By the multinomial expansion theorem, it holds that

$$(k_1 + \cdots + k_n)^j = \sum_{p \in P^j} \binom{j}{p} k_1^{p_1} \ldots k_n^{p_n}. \tag{41}$$

Now we will define a series of β_i^j for $j = 0, \ldots, n-1$; $i = 1, \ldots, n$:

$$\beta_i^0 = a_0 \frac{n-1}{n}, \tag{42}$$

$$\beta_i^j = a_j \sum_{p \in P_i^j} \frac{n-1}{n-G(p)} \binom{j}{p} k_1^{p_1} \cdots k_n^{p_n} \quad \text{for } j = 1, \ldots, n-1. \tag{43}$$

Observe that for a given vector $p \in P^j$, p is in P_i^j for $n - G(p)$ values of i. Therefore,

$$\sum_{i=1}^n \beta_i^j = a_j(n-1)(k_1 + \cdots + k_n)^j = a_j(n-1)K^j. \tag{44}$$

By construction, β_i^j is independent of k_i for each j. So now let

$$r_i(k_{-i}) = \sum_{j=0}^{n-1} \beta_i^j \tag{45}$$

and define

$$S_i(k_i, k_{-i}) = 1 - \frac{r_i(k_{-i})}{C(K)}. \tag{46}$$

By Proposition 1, this rule is efficient. Furthermore, it satisfies budget balance, because

$$\begin{aligned}\sum_{i=1}^n r_i(k_{-i}) &= \sum_{i=1}^n \sum_{j=0}^{n-1} \beta_i^j = \sum_{j=0}^{n-1} \sum_{i=1}^n \beta_i^j \\ &= \sum_{j=0}^{n-1} a_j(n-1)K^j = (n-1)C(K). \quad \square\end{aligned} \tag{47}$$

Proof of Proposition 3. For each $n \geq 2$, let $\hat{C}_{n-1}$ be the Chebyshev least squares approximation of C over the interval $(0, \bar{K}]$. Note that, by definition, $\hat{C}_{n-1}$ is a polynomial of degree $n-1$,

$$\hat{C}_{n-1}(K) = \hat{c}_{n-1}K^{n-1} + \cdots + \hat{c}_1 K + \hat{c}_0. \tag{48}$$

Construct the sets of allocation rules as in the proof of Proposition 2. That is, let

$$\beta_i^0 = \hat{c}_0 \frac{n-1}{n}, \tag{49}$$

$$\beta_i^j = \hat{c}_j \sum_{p \in P_i^j} \frac{n-1}{n-G(p)} \binom{j}{p} k_1^{p_1} \cdots k_n^{p_n} \qquad \text{for } j = 1, \ldots, n-1, \tag{50}$$

where P_i^j and $G(p)$ are as defined in the proof of Proposition 2, and let

$$S_i^n(k_i, k_{-i}) = 1 - \frac{r_i^n(k_{-i})}{C(K)}, \tag{51}$$

where

$$r_i^n(k_{-i}) = \sum_{j=0}^{n-1} \beta_i^j. \tag{52}$$

As before, the rule is efficient by Proposition 1, and

$$\begin{aligned}\sum_{i=1}^n r_i^n(k_{-i}) &= \sum_{i=1}^n \sum_{j=0}^{n-1} \beta_i^j = \sum_{j=0}^{n-1} \sum_{i=1}^n \beta_i^j \\ &= \sum_{j=0}^{n-1} \hat{c}_j(n-1)K^j = (n-1)\hat{C}_{n-1}(K).\end{aligned} \tag{53}$$

Therefore, for a given $k \in \prod_{i=1}^n [0, \bar{k}_i]$,

$$\left| \sum_{i=1}^n S_i^n(k_i, k_{-i}) - 1 \right| = \frac{n-1}{C(K)} |C(K) - \hat{C}_{n-1}(K)|. \tag{54}$$

Since C is C^2, the standard results on the convergence of Chebyshev approximations apply (see, e.g., Judd 1998, pp. 210–215). In particular, there exists a $B < \infty$ such that

$|C(K) - \hat{C}_{n-1}(K)| \le B(\ln(n)/n^2)$ for all $K \in (0, \bar{K}]$. Therefore, for each K,

$$\left|\sum_{i=1}^{n} S_i^n(k_i, k_{-i}) - 1\right| \le \frac{B}{C(K)} \frac{\ln(n)}{n} \to 0, \tag{55}$$

as $n \to \infty$, because $C(K) > 0$ for $K > 0$. □

Proof of Proposition 4. In what follows, we will omit the superscript m, n for visual clarity. Thus, we will use $\tilde{K}$, $\tilde{y}$, and $\tilde{c}$ to denote $\tilde{K}^{m,n}$, $\tilde{y}^{m,n}$, and $\tilde{c}^{m,n}$, respectively.

Let

$$\tilde{\mathbf{K}} = \begin{pmatrix} 1 & \tilde{K}_1 & (\tilde{K}_1)^2 & \cdots & (\tilde{K}_1)^{n-1} \\ 1 & \tilde{K}_2 & (\tilde{K}_2)^2 & \cdots & (\tilde{K}_2)^{n-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & \tilde{K}_m & (\tilde{K}_m)^2 & \cdots & (\tilde{K}_m)^{n-1} \end{pmatrix}. \tag{56}$$

Estimating the polynomial coefficients by OLS, we obtain

$$\tilde{c} = (\tilde{\mathbf{K}}'\tilde{\mathbf{K}})^{-1}\tilde{\mathbf{K}}'\tilde{y}. \tag{57}$$

Recall that we construct the estimated polynomial sharing rule by using $\tilde{c}$ instead of $\hat{c}$ in the algorithm outlined in the proof of Proposition 3. That proof then shows that the budget imbalance arising from using this rule when total resource level is K is

$$\frac{n-1}{C(K)}|C(K) - \tilde{C}(K)|, \tag{58}$$

where $\tilde{C}(K) = \tilde{c}_0 + \tilde{c}_1 K + \cdots + \tilde{c}_{n-1}K^{n-1}$. Our goal is therefore to show that (for any K)

$$\lim_{n\to\infty}\lim_{m\to\infty} \frac{n-1}{C(K)}|C(K) - \tilde{C}(K)| = 0. \tag{59}$$

Now, let $\hat{C}$ be the $(n-1)$st-degree Chebyshev least squares approximation of C, and let δ be the residual, $\delta(K) = C(K) - \hat{C}(K)$. Recall that $\delta(K) = O((\ln n)/n^2)$ (where $O(\cdot)$ denotes the asymptotic operator "of order no more than"). We can now write

$$\begin{aligned} \tilde{y}_i &= C(\tilde{K}_i) = \hat{C}(\tilde{K}_i) + \delta(\tilde{K}_i) \\ &= (1 \quad \tilde{K}_i \quad (\tilde{K}_i)^2 \quad \ldots \quad (\tilde{K}_i)^{n-1})\hat{c} + \delta(\tilde{K}_i), \end{aligned} \tag{60}$$

so that $\tilde{y} = \tilde{\mathbf{K}}\hat{c} + \Delta$, where $\Delta = (\delta(\tilde{K}_1) \;\; \delta(\tilde{K}_2) \;\; \ldots \;\; \delta(\tilde{K}_m))'$. Now,

$$\tilde{C}(K) = (1 \quad K \quad K^2 \quad \cdots \quad K^{n-1})\tilde{c} \tag{61}$$

$$= (1 \quad K \quad K^2 \quad \cdots \quad K^{n-1})(\tilde{\mathbf{K}}'\tilde{\mathbf{K}})^{-1}\tilde{\mathbf{K}}'(\tilde{\mathbf{K}}\hat{c} + \Delta) \tag{62}$$

$$= \hat{C}(K) + (1 \quad K \quad K^2 \quad \cdots \quad K^{n-1})((\tilde{\mathbf{K}}'\tilde{\mathbf{K}})^{-1}\tilde{\mathbf{K}}'\Delta). \tag{63}$$

Now, given our assumption that $\lim_{m\to\infty}(K_i - (\bar{K}/m)i) = 0$, we know that

$$\begin{aligned} \lim_{m\to\infty} \tilde{\mathbf{K}} &= \lim_{m\to\infty} \begin{pmatrix} 1 & \bar{K}/m & (\bar{K}/m)^2 & \cdots & (\bar{K}/m)^{n-1} \\ 1 & 2(\bar{K}/m) & (2(\bar{K}/m))^2 & \cdots & ((2\bar{K}/m))^{n-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & \bar{K} & (\bar{K})^2 & \cdots & (\bar{K})^{n-1} \end{pmatrix} \\ &= \lim_{m\to\infty} VD, \end{aligned} \tag{64}$$

where V is the rectangular (n, m) Vandermonde matrix on integer nodes ($V_{ij} = i^{j-1}$) and $D = \operatorname{diag}(1 \;\; \bar{K}/m \;\; \ldots \;\; (\bar{K}/m)^{n-1}$. Therefore, for each n,

$$\begin{aligned} &\lim_{m\to\infty}(1 \quad K \quad K^2 \quad \cdots \quad K^{n-1})(\tilde{\mathbf{K}}'\tilde{\mathbf{K}})^{-1}\tilde{\mathbf{K}}'\Delta) \\ &\quad = \lim_{m\to\infty}(1 \quad K \quad K^2 \quad \cdots \quad K^{n-1})D^{-1}V^{+}\Delta, \end{aligned} \tag{65}$$

where V^+ is the Moore–Penrose pseudoinverse of V, $V^+ = (V'V)^{-1}V'$.

Using the factorization in Eisinberg et al. (2001), we can write $V^+ = \bar{S}\mathcal{M}$, where $\bar{S}$ is an $n \times n$ upper-triangular matrix whose entries do not depend on m and $\mathcal{M}$ is given by

$$\mathcal{M}_{i,j} = \sum_{t=1}^{n}\sum_{k=1}^{n}(-1)^t\binom{j-1}{t-1}\frac{\binom{i+k-2}{i-1}\binom{m-i}{m-k}\binom{t+k-2}{t-1}\binom{m-t}{m-k}}{\binom{2k-2}{k-1}\binom{m+k-1}{2k-1}}. \tag{66}$$

Now, $\binom{j-1}{t-1} \le \binom{m-1}{t-1} = \Theta(m^{t-1})$, where Θ denotes the asymptotic operator "of the same order as." Similarly, $\binom{m-i}{m-k} = \Theta(m^{k-i})$, $\binom{m-t}{m-k} = \Theta(m^{k-t})$, and $\binom{2k-2}{k-1}\binom{m+k-1}{2k-1} = \Theta(m^{2k-1})$. Consequently, each term in the finite summation for $\mathcal{M}_{ij}$ (and therefore $\mathcal{M}_{ij}$ itself) is

$$O(m^{t-1+k-i+k-t-(2k-1)}) = O(m^{-i}), \tag{67}$$

where O denotes the asymptotic operator "of order no more than." Finally, the (i, j) element of $D^{-1}V^+ = D^{-1}\bar{S}\mathcal{M}$ is

$$(D^{-1}V^+)_{i,j} = \sum_{l=1}^{n}\left(\frac{m}{\bar{K}}\right)^{i-1}\bar{S}_{i,l}\mathcal{M}_{l,j}. \tag{68}$$

Since $\mathcal{M}_{l,j} = O(m^{-l})$, all terms are $O(m^{i-l-1})$. Noting that $\bar{S}_{i,l} = 0$ when $i > l$ we see that $i - l - 1 \le -1$ for all nonzero terms, so that each element of $(D^{-1}V^+)_{i,j}$ is decreasing at a rate at least m^{-1}. Hence, for any n,

$$\lim_{m\to\infty}(1 \quad K \quad K^2 \quad \cdots \quad K^{n-1})D^{-1}V^+\Delta = \mathbf{0}, \tag{69}$$

so that $\lim_{m\to\infty}\tilde{C}(K) = \hat{C}(K)$. Now,

$$\begin{aligned} \lim_{m\to\infty}\frac{n-1}{C(K)}|C(K) - \tilde{C}(K)| &= \frac{n-1}{C(K)}|C(K) - \hat{C}(K)| \\ &= \frac{n-1}{C(K)}|\delta(K)| = O\left(\frac{\ln(n)}{n}\right), \end{aligned} \tag{70}$$

so that

$$\lim_{n\to\infty}\lim_{m\to\infty}\frac{n-1}{C(K)}|C(K) - \tilde{C}(K)| = 0. \quad \square \tag{71}$$

Proof of Proposition 5. A set of allocation rules is efficient, budget balancing, and satisfies NPNP if and only if the following three conditions hold for all i and all $(k_1, \ldots, k_n)$:

$$S_i(k_i, k_{-i}) = 1 - \frac{r_i(k_{-i})}{C(\sum_{i=1}^{n} k_i)}, \tag{72}$$

$$\sum_{i=1}^{n} S_i(k_i, k_{-i}) = 1, \tag{73}$$

$$S_i(0, k_{-i}) = 0. \tag{74}$$

The first and third conditions (i.e., efficiency and NPNP) together are equivalent to

$$S_i(k_i, k_{-i}) = 1 - \frac{C(\sum_{j\neq i} k_j)}{C(\sum_{i=1}^n k_i)}. \tag{75}$$

Combining this with budget balance yields

$$n - \frac{\sum_{i=1}^n C(\sum_{j\neq i} k_j)}{C(\sum_{i=1}^n k_i)} = 1, \tag{76}$$

that is,

$$C\left(\sum_{i=1}^n k_i\right) = \frac{1}{n-1}\sum_{i=1}^n C\left(\sum_{j\neq i} k_j\right). \tag{77}$$

We therefore see that a set of efficient and budget balancing allocation rules that satisfy NPNP exists if and only if (77) holds for all $k \in \prod_{i=1}^n [0, \bar{k}_i]$.

Now, pick any $k_1 \in \mathbb{R}$ and take $k = (k_1, 0, \ldots, 0)$. By (77), if an efficient budget balancing allocation rule that satisfies NPNP exists, then $C(k_1) = (1/(n-1))(C(0) + (n-1)C(k_1))$, that is, $C(0) = 0$.

Next, suppose that C is strictly convex and there exists an efficient, budget balancing allocation rule that satisfies NPNP. Let $K \equiv \sum_{i=1}^n k_i$ and let $K_{-i} \equiv \sum_{j\neq i} k_j$. Now, observe that

$$C(K_{-i}) = C\left(\frac{K_{-i}}{K}K + \frac{k_i}{K}\cdot 0\right) < \frac{K_{-i}}{K}C(K) + \frac{k_i}{K}C(0)$$
$$= \frac{K_{-i}}{K}C(K) \tag{78}$$

for all i, where the inequality follows by the convexity of C and $C(0) = 0$ as shown above. Therefore,

$$\sum_{i=1}^n C(K_{-i}) < \sum_{i=1}^n \frac{K_{-i}}{K}C(K) = \frac{C(K)}{K}\sum_{i=1}^n K_{-i} \tag{79}$$
$$= \frac{C(K)(n-1)K}{K}$$
$$= (n-1)C(K), \tag{80}$$

which contradicts (77).

Therefore, we see that if an efficient, budget balancing allocation rule that satisfies NPNP exists, C must not be strictly convex and must have $C(0) = 0$. That is, the only candidate functions are the linear cost functions $C(K) = \alpha K$. This family of functions satisfies (77):

$$\frac{1}{n-1}\sum_{i=1}^n C(K_{-i}) = \frac{1}{n-1}\left(\alpha\sum_{i=1}^n\sum_{j\neq i} k_j\right)$$
$$= \alpha K = C(K). \quad \square \tag{81}$$

Proof of Proposition 6. Fix K_{-i}. Let k_i^{Lc} be the resource levels induced by the linear cost allocation rule for division i, conditional on K_{-i}, and let k_i^{*c} be the conditional efficient resource level. Given K_{-i}, we see from (94) that the conditional efficient k_i^{*c} will satisfy

$$f_i'(k_i^{*c}) = C'(K_{-i} + k_i^{*c}). \tag{82}$$

The linear rule will have

$$S_i^L(k) = \frac{k}{K_{-i}+k}, \quad S_i^{L\prime}(k) = \frac{K_{-i}}{(K_{-i}+k)^2}, \tag{83}$$

and the linear k_i^{Lc} will satisfy

$$f_i'(k_i^{Lc}) = \frac{K_{-i}}{(K_{-i}+k_i^{Lc})^2}C(K_{-i}+k_i^{Lc})$$
$$+ \frac{k_i^{Lc}}{K_{-i}+k_i^{Lc}}C'(K_{-i}+k_i^{Lc}). \tag{84}$$

And, because C is convex,

$$C(x) < xC'(x) + C(0). \tag{85}$$

We have assumed $C(0) = 0$, so plug this back in with $x = K_{-i} + k_i^{Lc}$:

$$f_i'(k_i^{Lc}) < \frac{K_{-i}(K_{-i}+k_i^{Lc})}{(K_{-i}+k_i^{Lc})^2}C'(K_{-i}+k_i^{Lc}) + \frac{k_i^{Lc}}{K_{-i}+k_i^{Lc}}C'(K_{-i}+k_i^{Lc})$$
$$= C'(K_{-i}+k_i^{Lc}). \tag{86}$$

Because $f_i'(k)$ is decreasing in k and $C'(K_{-i}+k)$ is increasing in k, we have that

$$f_i'(k_i^{Lc}) < C'(K_{-i}+k_i^{Lc}) \quad \text{and} \tag{87}$$
$$f_i'(k_i^{*c}) = C'(K_{-i}+k_i^{*c}) \tag{88}$$

if and only if $k_i^{Lc} > k_i^{*c}$. If production is symmetric, $k_i^* = k^*$ for each i. Let $k_i^{*c}(K_{-i})$ be the optimal conditional resource levels for a given K_{-i}. $k_i^{*c}(K_{-i})$ satisfies $f'(k_i^{*c}(K_{-i})) - C'(k_i^{*c}(K_{-i}) + K_{-i}) = 0$, so we can implicitly differentiate and see that

$$\frac{dk_i^{*c}(K_{-i})}{dK_{-i}} = \frac{C''}{f'' - C''} < 0. \tag{89}$$

At any symmetric equilibrium, $K_{-i} = (n-1)k$ and $K = nk$. Now $k_i^{Lc}(K_{-i}) > k_i^{*c}(K_{-i})$ for any K_{-i}. Also, for $K_{-i} \le K_{-i}^*$, the above differentiation shows that $k_i^{*c}(K_{-i}) \ge k_i^{*c}(K_{-i}^*) = k^*$. So assume that $K^L \le K^*$. Then, $K_{-i}^L \le K_{-i}^*$ for all i, and therefore

$$k_i^L = k_i^{Lc}(K_{-i}^L) > k_i^{*c}(K_{-i}^L) \ge k^*. \tag{90}$$

Yet this implies that $K^L > K^*$, which is a contradiction. So $K^L > K^*$ and $k^L > k^*$. $\square$

Lemma 3. *If $f_i(k_i) = A(k_i)^p$ for each i and $C(K) = BK^q$ for $0 < p < 1 < q < \infty$, then*

$$k^* = \left(\frac{Ap}{Bq}n^{1-q}\right)^{1/(q-p)}, \tag{91}$$

$$k^L = \left(\frac{Apn^{2-q}}{B(n-1+q)}\right)^{1/(q-p)}, \tag{92}$$

$$\frac{\pi^L}{\pi^*} = n^{p/(q-p)}\left(\frac{q}{n-1+q}\right)^{p/(q-p)}\left(\frac{1-(np)/(n-1+q)}{1-p/q}\right). \tag{93}$$

Proof. From the firm's objective function, we see that k_i^* is defined by the first-order condition:

$$f_i'(k_i^*) - C'(K^*) = 0. \tag{94}$$

Division i maximizes $f_i(k_i) - S_i(k_1, \ldots, k_n)C(K)$. Let $\partial_i S_i$ denote the derivative of $S_i(k_1, \ldots, k_n)$ with respect to k_i. With an arbitrary allocation rule S, we get the first order condition for division i's problem,

$$f_i'(\tilde{k}_i) - \partial_i S_i(\tilde{k}_1, \ldots, \tilde{k}_n)C(\tilde{K}) - S_i(\tilde{k}_1, \ldots, \tilde{k}_n)C'(\tilde{K}) = 0. \tag{95}$$

In the respective symmetric equilibria, $k_i^* = k_j^*$ and $k_i^L = k_j^L$, so we can drop the subscripts. The optimal production k^* solves (94), and k^L solves (95), so straightforward calculation gives k^* and k^L in the statement of the lemma.

Now, $\pi(k) = nAk^p - B(nk)^q$ for a symmetric output level k, so straightforward calculation of these terms gives π^L/π^* in the statement of the lemma. □

Proof of Proposition 7. Note $\Delta = 1 - \pi^L/\pi^*$, where π^L/π^* is calculated in Lemma 3. Without calculations, it is clear that $\partial\Delta/\partial A = \partial\Delta/\partial B = 0$. Now

$$\frac{\partial\Delta}{\partial n} = \frac{(n-1)n^{p/(q-p)-1}p(q-1)^2 q(q/(n+q-1))^{p/(q-p)}}{(q-p)^2(n+q-1)^2} > 0, \tag{96}$$

$$\frac{\partial\Delta}{\partial p} = \frac{1}{(p-q)^3}\left(n^{p/(q-p)}\left(\frac{q}{n-1+q}\right)^{q/(q-p)}\right. \tag{97}$$

$$\left.\cdot\left((n-1)(p-q)(q-1)+q(n-np+q-1)\log\left(\frac{nq}{n-1+q}\right)\right)\right). \tag{98}$$

So $\partial\Delta/\partial p$ has the opposite sign of $(n-1)(p-q)(q-1) + q(n-np+q-1)\log(nq/(n-1+q))$. Plugging in the inequality $\log(x) \le x-1$,

$$(n-1)(p-q)(q-1)+q(n-np+q-1)\log\left(\frac{nq}{n-1+q}\right) \tag{99}$$

$$\le (n-1)(p-q)(q-1)+q(n-np+q-1)\cdot\left(\frac{nq}{n-1+q}-1\right) \tag{100}$$

$$= -\frac{(n-1)^2p(q-1)^2}{n-1+q} \tag{101}$$

$$< 0. \tag{102}$$

Thus, $\partial\Delta/\partial p > 0$. Now

$$\frac{\partial\Delta}{\partial q} = \frac{1}{(p-q)^3(n-1+q)^2}\left(n^{p/(q-p)}pq\left(\frac{q}{n-1+q}\right)^{p/(q-p)}\right. \tag{103}$$

$$\left.\cdot\left((n-1)(q-p)(q-1)-(q-1+n-np)\cdot(n-1+q)\log\left(\frac{nq}{n-1+q}\right)\right)\right). \tag{104}$$

So $\partial\Delta/\partial q$ has the opposite sign of

$$(n-1)(q-p)(q-1)-(q-1+n-np)(n-1+q)\log\left(\frac{nq}{n-1+q}\right).$$

In other words, $\partial\Delta/\partial q$ has the sign of

$$\log\left(\frac{nq}{n-1+q}\right) - \frac{(n-1)(q-p)(q-1)}{(n-1+q)(n-np+q-1)}. \tag{105}$$

As $n \to \infty$, this goes to $\log(q) - 0 = \log(q) > 0$. So for n large enough holding other parameters fixed, $\partial\Delta/\partial q > 0$.

As $q \to \infty$, this goes to $\log(n) - (n-1)$ which is strictly less than 0 for $n \ge 2$. So for q large enough, $\partial\Delta/\partial q < 0$.

As $q \to 1$, we can find the sign by dividing through by the fraction and applying L'Hôpital's rule to the limit. This has the sign of $n-1 > 0$, so for q small enough, $\partial\Delta/\partial q > 0$.

As $p \to 1$, the expression goes to $\log(nq/(n-1+q)) - ((n-1)(q-1))/(n-1+q)$. Applying the inequality $\log(x) \le x-1$, we get $\log(nq/(n-1+q)) \le ((n-1)(q-1))/(n-1+q)$. So $\partial\Delta/\partial p < 0$ for p large enough, when other parameters are held constant. □

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