Pg. #
Date 21/3/23

Important Topics :-

Unit - 1 :- Differencial Equations - I :-

1. Differential Equation of first order & first degree-
* Leibnitz's linear.
* Bernoulli's.
* Exact.

2. Differential Equations of first order & higher degree.

3. Higher order differential Equations with Constant coefficients.

4. Homogeneous linear differential Equation.

5. Simultaneous differential equation.

Unit-1 => ordinary Differencial equation.

Basic's :-

1) An Equation which can be differenciated as per the requirement, represented in the form of $\frac{d}{dx}$ / $\frac{d}{dy}$ / $\frac{d}{dz}$ etc

is a differencial Equation.

* what do you mean by differenciation?

Ans. Differenciation is defined as the rate of change of the given quantity [in lamen terms].

3) Also note that; an equation which envolves differencial coefficient is known as a differencial Equation.

* Differencial equation of first order :-

$$M + N\left(\frac{dy}{dx}\right) = 0 \quad \rightarrow 1)$$

or;

$$M(dx) + N(dy) = 0 \quad \rightarrow 2)$$

5) To solve a first order differencial Equation we have the following methods :-

i) Leibnitz's linear differencial Equation
ii) Bernoulli's Equation.
iii) Exact differencial Equation

i) Lebnitz's Linear Differencial Equation :-

a) "A linear equation is called Lebnitz's linear differencial equation when the dependent variable y and all it's differencial coefficient occour in the first degree only & are not multiplyied together."

b) An equation of the form;

$$\frac{dy}{dx} + Py = Q \quad \ldots\ldots (i).$$

where; P & Q are functions of x;
y is the dependent variable.

c) Working rule :-

[i] first write the given differencial Equation.

[ii] write the differencial equation in the form:-

$$\frac{dy}{dx} + Py = Q \rightarrow 1)$$

or;

$$\frac{dx}{dy} + Px = Q \rightarrow 2)$$

(iii) Obtain the integrating factor $e^{\int P dx}$ or $e^{\int P dy}$.

(iv) The solution of the differencial equation is either;

$$y\,(I.F.) = \int \{Q \cdot (I.F.)\}\,dx + C \rightarrow 1)$$

or;

$$x\,(I.F.) = \int \{Q\,(I.F.)\}\,dy + C \rightarrow 2)$$

Example 8:- Determine defferencial Equation of the family of Curves; [R.G.P.V] ⇒ [June/July-2006]

$$y = A\cos x^2 + B\sin x^2.$$

Solⁿ ⇒ Here $y = A\cos x^2 + B\sin x^2. \rightarrow 1)$

Differenciating Equation(1) with respect to x, we get;

$$\frac{dy}{dx} = A(-\sin x^2)\cdot 2x + B(\cos x^2)\cdot 2x$$

or; $$\frac{dy}{dx} = 2x[-A\sin x^2 + B\cos x^2]$$

Again; $$\frac{d^2y}{dx^2} = 2x[-A\cos x^2\cdot(2x) + B(-\sin x^2)\cdot 2x] + 2(-A\sin x^2 + B\cos x^2)$$

or; $$\frac{d^2y}{dx^2} = -4x^2(A\cos x^2 + B\sin x^2) + 2(-A\sin x^2 + B\cos x^2)$$

or $$\frac{d^2y}{dx^2} = -4x^2y + \frac{1}{y}\frac{dy}{dx}$$

or $$x\frac{d^2y}{dx^2} = -4x^3y + \frac{dy}{dx}$$

$$\boxed{x\frac{d^2y}{dx^2} - \frac{dy}{dx} + 4x^3y = 0} \Rightarrow \text{Ans.}$$

Numerical problems :-

Prob → 1. - 3.

- Solve the following.

(i) $\frac{dy}{dx} + y = 1.$

Solⁿ ⇒ STEP - 1 :- Write the given differencial Equation -

$$\frac{dy}{dx} + y = 1 \rightarrow 1>.$$

STEP - 2 :- Compare the given differential Equation With standarad differential Equation -

$$\frac{dy}{dx} + Py = Q. \rightarrow 2>$$

we have; $P = 1; Q = 1.$

STEP3 :- Determine integrating Factor.

Trial Solution ← $\left(I.F. = e^{\int P\,dy} = e^{\int 1\,dy} = e^{y} \right)$

(But, According to given form;) → trial part

$$I.F. = e^{\int P dx} = e^{\int 1 dx} = e^{x}.$$

$$\boxed{I.F = e^{x}} \rightarrow 4>$$

$\left(\boxed{I.F. = e^{y}} \rightarrow 3>\right)$ → trial Equati

STEP - 4 :- Now Find It's general Solution

as :-

Trial Solution

The general solution of given differential Equations :-

$$Y \cdot (I.F.) = \int \{P \cdot (I.F.)\} dy + C. \quad \text{------} \quad \textcircled{5} \ (\to 4?)$$

$$Y (e^y) = \int \{1 \cdot e^y\} dy + C.$$

$$Ye^y = \int e^y dy + C.$$

$$\boxed{Ye^y = e^y + C} \ (\Rightarrow \underline{Ans}). \Rightarrow \text{Bonous Equ } \textcircled{1}$$

But the given general form is;

$$\frac{dy}{dx} + Py = Q. \Rightarrow \text{Bonous Equ } \textcircled{2}$$

So;

general solution of given differencial Equation is,

$$Y(IF) = \int \{Q(IF)\} dx + C. \to \textcircled{6}$$

$$Ye^x = \int \{1 (e^x)\} dx + C.$$

$$Ye^x = \int e^x dx + C.$$

$$Ye^x = e^x + C.$$

Or; $\quad Y = e^x + Ce^{-x}.$

or $\Rightarrow \quad \boxed{Y = 1 + Ce^{-x}.} \Rightarrow \underline{Ans}.$

<ii> $\left[\left(\frac{dy}{dx} \Rightarrow \text{format}\right) / \frac{dx}{dy} \Rightarrow \text{format}\right]$

$\Rightarrow \sqrt{1-y^2} \cdot dx = \{\sin^{-1}(y) - x\} dy.$

Solⁿ ⇒ STEP - 1 :- Write the given differential Equation.

$$\sqrt{1-y^2}\ dx = \{\sin^{-1}(y) - x\}\ dy. \rightarrow 1)$$

$$\sqrt{1-y^2}\ \frac{dx}{dy} = \{\sin^{-1}(y) - x\} \rightarrow 2)$$

[∵ Deviding Equation (1) by dy]

$$\boxed{\frac{dx}{dy} = \frac{\sin^{-1}y}{\sqrt{1-y^2}} - \frac{x}{\sqrt{1-y^2}}} \rightarrow (3)$$

[∵ Deviding Equation (2) by $\sqrt{1-y^2}$].

STEP - 2 Compare the given differential Equation with standard form of differential Equation :-

$$\frac{dx}{dy} + Px = Q \rightarrow (4)$$

$$\boxed{\frac{dx}{dy} + \frac{1}{\sqrt{1-y^2}} \cdot x = \frac{1}{\sqrt{1-y^2}} \cdot \sin^{-1}y.} \rightarrow (5)$$

So that, we have ;

$P = \frac{1}{\sqrt{1-y^2}}$; $Q = \frac{\sin^{-1}y}{\sqrt{1-y^2}}$.

STEP-3 :- Determine integrating factor.

$I.F. = e^{\int P\,dy}$

$= e^{\int \frac{1}{\sqrt{1-y^2}}dy} = e^{\sin^{-1}y}$.

$\left[= e^{\tan^{-1}y} . \right]$

STEP-4 :- Determine general solution.

$x(I.F.) = \int \{Q \cdot (I.F.)\}\,dy + C \rightarrow (6)$

trial Equation. $\left[x e^{\tan^{-1}y} = \int \left\{ \frac{\sin^{-1}y}{\sqrt{1-y^2}} e^{\tan^{-1}y} \right\} dy + C \rightarrow (7) \right]$

$x e^{\sin^{-1}y} = \int \left\{ \frac{\sin^{-1}y}{\sqrt{1-y^2}} e^{\sin^{-1}y} \right\} dy + C \rightarrow (7)$

Now; let ; $e^{\sin^{-1}y} = t$, so ; $\frac{1}{\sqrt{1-y^2}}dy = dt$

$x e^{\sin^{-1}y} = \int \{t e^t\,dt\} + C \rightarrow (8)$

$\rightarrow$ [Solve equ. (8) using formulae ; $\int \mathrm{I}\,\mathrm{II}\,dx = \mathrm{I}\int \mathrm{II}\,dx - \mathrm{II}\int \mathrm{I}\,dx$]

we get ; $x e^{\sin^{-1}y} = e^{\sin^{-1}y}(\sin^{-1}y - 1) + C$.

or, $\boxed{x = (\sin^{-1}y - 1) + Ce^{-\sin^{-1}y}.} \Rightarrow$ Ans.

(iii) $\frac{dy}{dx} + y\tan x = \sec x.$

Sol^n $\rightarrow \frac{dy}{dx} + y\tan x = \sec x$ [Step 1: write given diff. equ.].

$\Rightarrow P = \tan x;\ Q = \sec x$ [Compare equ (1) with standard form; $\frac{dy}{dx} + Py = Q$].

$\Rightarrow I.F. = e^{\int P dx} = e^{\int \tan x\, dx}$ [∵ Find integrating Factor].

$= e^{\log \sec x}$

$\boxed{I.F. = \sec x.}$

$\Rightarrow y \cdot I.F. = \int \{Q \cdot (I.F.)\} dx + C$ [∵ Find general Solution]

$\Rightarrow y\sec x = \int (\sec x \times \sec x) dx + C$

$\Rightarrow y\sec x = \int \sec^2 x\, dx + C$

$\Rightarrow \boxed{y\sec x = \tan x + C}$

or; $\frac{y}{\cos x} = \frac{\sin x}{\cos x} + C.$

or, $\boxed{y = \sin x + C\cos x} \Rightarrow$ Ans.

Some important problems :-

Try on own :-

Solve the following differential equation :-

<i> $\frac{dy}{dx} - y = e^{3x}$.

<ii> $(1+x^2)\frac{dy}{dx} + 2xy - 4x^2 = 0$.

<iii> $(1+y^2)\frac{dx}{dy} = (\tan^{-1}y - x)$.

Try on own 2 :-

<i> $(1+x^2)\frac{dy}{dx} + 2xy = \cos x$.

<ii> $\frac{dy}{dx} + \frac{2y}{x} = \sin x$.

<iii> $(1+y^2)\,dx = (\tan^{-1}y - x)\,dy$.

<iv> $(1+y^2) + (x \cdot e^{-\tan^{-1}y})\frac{dy}{dx} = 0$.

<v> $\sin 2x \frac{dy}{dx} = y + \tan x$.

<vi> $\sec x \frac{dy}{dx} = y + \sin x$.

Solutions for try yourself-1 :-

(i) $\frac{dy}{dx} - y = e^{3x}$.

Soln ⇒ $\frac{dy}{dx} - y = e^{3x}$. → (1)

⇒ $\frac{dy}{dx} + Py = Q$ → (2)

⇒ Compairing Equ (1) & (2) we get;

$P = -1; \quad Q = e^{3x}$.

⇒ Now; integrating factor;

$$I.F. = e^{\int P \, dx}$$
$$= e^{\int -1 \, dx}$$
$$= e^{-\int 1 dx}$$
$$= e^{-x}.$$

⇒ Now find it's general Solution :-

$$y\{I.F.\} = \int \{Q \cdot I.F.\} \, dx + c.$$

On putting values & arrangging the Equation:-

$$y(e^{-x}) = \int \{-1(e^{-x})\} dx + C$$

$$y(e^{-x}) = -\int e^{-x} dx + c.$$

$y(e^{-x}) = -e^{-x} + C.$

or; $y = -e^{-x} + Ce^{x}.$

or; $\boxed{y = 1 + Ce^{x}}$ => Ans.

<(ii)> $(1+x^2)\frac{dy}{dx} + 2xy - 4x^2 = 0.$ -> (1)

solⁿ => $(1+x^2)\frac{dy}{dx} + 2xy = 4x^2.$ -> (2)

=> $\frac{dy}{dx} + \frac{(2y)x}{(1+x^2)} = \frac{4x^2}{1+x^2}$ -> (3)

=> Compairing eqⁿ (3) with :-
$\frac{dy}{dx} + Py = Q;$

We get ; $P = \frac{2x}{1+x^2}$; $Q = \frac{4x^2}{1+x^2}.$

=> Now find integrating factor;
I.F. $= e^{\int P dx}$
I.F. $= e^{\int 2x/1+x^2 dx}.$
I.F. $= 1+x^2.$

$1+x^2 = t$; $0+2x = \frac{dt}{dx}$; $dx = \frac{dt}{2x}$

$e^{\int \frac{2x}{t} \frac{dt}{2x}} = e^{\int 1/t\, dt}$

$= e^{\log t} = t.$
$= 1+x^2$

Then finding it's general Solution as;

$$y(I.F.) = \int \{Q \cdot (I.F.)\} dx + C$$

$$y(1+x^2) = \int \left\{ \frac{4x}{\cancel{(1+x^2)}} \cancel{(1+x^2)} \right\} dx + C$$

$$y(1+x^2) = 4\int x\,dx + C$$

$$y(1+x^2) = \cancel{4}2 \frac{x^2}{\cancel{2}1} + C$$

$$y = \frac{2x^2}{(1+x^2)} + C.$$

or,

$$\boxed{y = 2x^2 + C(1+x^2).} \Rightarrow \text{ans}$$

Date ___/___/___

Questions 3-

Q Solve the differencial equation:-

$$\frac{dy}{dx} - y = e^{3x}.$$

Sol^n The given differencial Equation is;

$$\frac{dy}{dx} - y = e^{3x}.$$

Comparing this Equation with standarad form of linear differencial Equation:-

$$\frac{dy}{dx} + P\cdot y = Q, \quad P = -1; \; Q = e^{3x}.$$

$$I.F. = e^{\int P dx} = e^{-\int 1 dx} = e^{-x}.$$

The general solution is;

$$y \cdot I.F. = \int (Q \cdot I.F.)\, dx + C.$$

$$y \cdot e^{-x} = \int (e^{3x} \cdot e^{-x})\, dx + C.$$

$$y \cdot e^{-x} = \int e^{2x} \cdot dx + C.$$

let $2x = t$

$2 \times 1 = \frac{dt}{dx}$

$dx = \frac{dt}{2}.$

$$ye^{-x} = \frac{e^{2x}}{2} + C.$$

$$y = \frac{1}{2} e^{2x} \cdot e^{x} + C e^{x}$$

ans: $\boxed{y = 1/2\, e^{3x} + C e^{x}.}$

Date __ / __ / __

Q. Solve the following differencial Equation.

$$(1+x^2)\frac{dy}{dx} + 2xy - 4x^2 = 0.$$

Sol^n The given differencial Equation can be written in the following form:-

$$(1+x^2)\frac{dy}{dx} + 2xy - 4x^2 = 0.$$

$$(1+x^2)\frac{dy}{dx} + 2xy = 4x^2.$$

$$\frac{dy}{dx} + \frac{2x}{(1+x^2)}\cdot y = \frac{4x^2}{(1+x^2)} \rightarrow (1)$$

Comparing Eqn (1) with given standard differencial Equation; i.e.;

$$\frac{dy}{dx} + P.y. = Q.$$

We have; $P = \frac{2x}{(1+x^2)}$; $Q = \frac{4x^2}{(1+x^2)}$.

Now;

$$I.F. = e^{\int P\,dx};$$
$$= e^{\int \frac{2x}{1+x^2}\,dx}$$

Let $1+x^2 = t$; $0+2x = \frac{dt}{dx}$; $dx = \frac{dt}{2x}$

$$\Rightarrow e^{\int \frac{\cancel{2x}}{t}\cdot\frac{dt}{\cancel{2x}}} \Rightarrow e^{\int \frac{1}{t}\,dt}$$

Saathi

Date 20/3/23

$= e^{\log t}$

$= e^{\log(1+x^2)}$

$I.F = 1+x^2.$

The general solution is;

$y \cdot (I.F.) = \int (Q \cdot I.F.)\, dx + C.$

$y \cdot (1+x^2) = \int \frac{4x^2}{\cancel{(1+x^2)}} \times \cancel{(1+x^2)}\, dx + C.$

$= 4\int x^2\, dx + C.$

$y(1+x^2) = 4 \cdot \frac{x^3}{3} + C.$

Q. $(1+x^2)\frac{dy}{dx} + 2xy = \cos x.$

Sol: $\frac{dy}{dx} + \frac{2x}{(1+x^2)} y = \frac{\cos x}{(1+x^2)}$ → (1)

$\frac{dy}{dx} + Py = Q$ → (2)

$P = \frac{2x}{1+x^2}$; $Q = \frac{\cos x}{1+x^2}.$

Now; $I.F. = e^{\int P dx}$

$I.F. = 1+x^2.$

$y.(I.F) = \sin x + C.$] Ans.

Date ___ / ___ / ______

Q. Solve the following differencial Equations -

$$\frac{dy}{dx} + \frac{2y}{x} = \sin x.$$

Sol^n The given differencial Equation is;

$$\frac{dy}{dx} + \frac{2y}{x} = \sin x. \rightarrow (1)$$

Comparing this differencial Equation with Standarad form of linear differencial equation;

$$\frac{dy}{dx} + Py = Q.$$

So;

$$P = \frac{2y}{x} \; ; \; Q = \sin x.$$

Now;

$$I.F. = e^{\int P\,dx}$$

$$= e^{\int \frac{2y}{x} dx}$$

$$= e^{2\int \frac{1}{x} dx}$$

$$= e^{2\log x}$$

$$= e^{\log x^2}$$

$$= x^2.$$

The general solution is;

$$(y \cdot I.F) = \int (Q \cdot I.F.)\,dx + C$$

$$yx^2 = \int \sin x \cdot x^2\,dx + C$$

$$yx^2 = x^2 \int \sin x\,dx - \int \frac{d}{dx} x^2 \cdot \left\{\int \sin x\,dx\right\} dx +$$

$$yx^2 = x^2 (-\cos x) \cdot \int 2x \cdot (-\cos x)\,dx + C$$

Date ___ / ___ / ___

Saathi

$$yx^2 = -\cos x \cdot x^2 + 2\int x \cos x \, dx + C \rightarrow (11)$$

$I = \int x \cos x \, dx$

$I = x \sin x - \int 1 \cdot \sin x \, dx$

$I = x \sin x - (-\cos x)$

$I = (x \sin x + \cos x)$

Now putting the value of $x \cos x$ in eqn (1) we get.

$yx^2 = -x^2 \cos x + 2x \sin x + 2\cos x + C$

$\hookrightarrow$ ans.

Q. Solve the following differencial Equation;

$$(1+y^2)\, dx - (\tan^{-1} y - x)\, dy.$$

sol: The given differencial Equation is;

$$(1+y^2)\, dx = (\tan^{-1} y - x)\, dy. \rightarrow (1)$$

Now; $(1+y^2)\dfrac{dx}{dy} = (\tan^{-1} y - x).$

then; rearrange; we get;

$$(1+y^2)\frac{dx}{dy} + x = \tan^{-1} y \rightarrow (2)$$

Devide eqn. (2) by $(1+y^2)$; we get;

Date ___/___/_____

$$\frac{dx}{dy} + \frac{1}{(1+y^2)} \cdot x = \frac{\tan^{-1} y}{(1+y^2)} \rightarrow (3)$$

Now; comparing eqn. (3) with standared equation

$$\frac{dy}{dx} + Px = Q \rightarrow (4)$$

we get;

$$P = \frac{1}{(1+y^2)}$$ and;

$$Q = \frac{\tan^{-1} y}{(1+y^2)}.$$

$$I.F = e^{\int P dy} = e^{\int \frac{1}{1+y^2} dy}$$

$$= e^{\tan^{-1} y}.$$

The general solution is;

$$(x \cdot I.F) = \int Q (I.F) \, dy + C$$

$$x \cdot e^{\tan^{-1} y} = \int \left(\frac{\tan^{-1} y}{1+y^2} \right) \cdot e^{\tan^{-1} y} \, dy + c$$

Let $\tan^{-1} y = t$;

$$\frac{1}{1+y^2} = \frac{dt}{dy}$$

$$dy = (1+y^2) \, dt.$$

Saathi

Date ___/___/___

$$xe^{t} = \int \frac{t}{(1+y^2)} \cdot e^{t} \cdot (1+y^2)\, dt + C.$$

$$xe^{t} = \int te^{t} \cdot dt + C.$$

$$= e^{t} \cdot t - \int 1 \cdot e^{t} \cdot dt + C.$$

$$xe^{x} = te^{t} - e^{t} + C.$$

$$xe^{t} = e^{t}(t-1) + C.$$

$$x = \frac{e^{t}}{e^{t}}(t-1) + \frac{C}{e^{t}}.$$

$$x = (t-1) + C \cdot e^{-x}$$

$$\boxed{x = \tan^{-1}y - 1 + C \cdot e^{-\tan^{-1}y}}$$

$\hookrightarrow$ ans.

Q. Solve the following differencial equation;

$$(1+y^2) + (x \cdot e^{-\tan^{-1}y}) \cdot \frac{dy}{dx} = 0.$$

Ans: The given differencial equation;

$$(1+y^2) + x \cdot e^{-\tan^{-1}y} \cdot \frac{dy}{dx} = 0 \quad \rightarrow (1)$$

Deviding the whole equation by $\frac{dy}{dx}$,

we get:

$$\frac{dx}{dy}(1+y^2) + xe^{\tan^{-1}y}$$

$$\frac{dx}{dy}(1+y^2) + x = e^{-\tan^{-1}y}$$

$$\frac{dx}{dy} + \left(\frac{1}{1+y^2}\right)x = \left(\frac{e^{-\tan^{-1}y}}{1+y^2}\right)$$

compare with,

$$\frac{dx}{dy} + Px = Q. \to (3)$$

$$P = \frac{1}{1+y^2}; \quad Q = \frac{e^{-\tan^{-1}y}}{1+y^2}$$

$$I.F. = e^{\int P\,dy}$$

$$= e^{\int \frac{1}{1+y^2}dy}$$

$$= e^{\tan^{-1}y}.$$

Date __/__/__

The general solution is;

$x \cdot (I \cdot F) = \int (Q \cdot I \cdot F)\, dy + C.$

on solving we get,

$x e^{\tan^{-1} y} = \int \frac{e^{-\tan^{-1} y}}{1+y^2} \cdot e^{\tan^{-1}} dy + c$

$= \int \frac{1}{1+y^2} dy + c.$

$x e^{\tan^{-1} y} = \tan^{-1} y + c.$

$x = e^{-\tan^{-1} y} (t \tan^{-1} y)$ ⇒ ans

Q. Solve the following differencial Equation. -

i) $\sin 2x \frac{dy}{dx} = y + \tan x.$

ii) $\sec x \cdot \frac{dy}{dx} = y + \sin x.$