ISSN 2977-6163

International Research and Innovation Journal
https://irijournal.org
IRIJ 1(1): 134-149

Received: 25 July, 2024
Revised: 8 October, 2024
Accepted: 8 December, 2024
Published: 16 December, 2024

OPTIMAL CONTROL PROBLEMS CONSTRAINED BY NON-LINEAR ORDINARY DIFFERENTIAL EQUATIONS

Orji Josiah Chukwuebuka[1], Miswanto Miswanto [2] and Afolabi Ayodeji Sunday[1]

[1] Department of Mathematical Sciences, Federal University of Technology, Akure, P.M.B. 704, Akure, Ondo State, Nigeria

[2] Department of Mathematics, Faculty of Sciences and Technology, Airlangga University, Surabaya, Indonesia

Corresponding Email: miswanto@fst.unair.ac.id[2]

ABSTRACT

This research focuses on addressing optimal control problems (OCP) constrained by nonlinear ordinary differential equations. The Variational Iteration Method (VIM) addresses these problems since it allows for determining extreme circumstances obtained from Pontryagin's Maximum Principle (PMP). Consequently, this results in formulating a nonlinear problem involving two boundary points. Unlike standard numerical approaches, VIM does not need to be discretized, linearized, or perturbed. Moreover, the method has gained widespread application for solving nonlinear problems, and various enhancements have been suggested to overcome potential limitations in the solution process. Furthermore, one significant area in VIM that has proven valuable is solving Riccati equations, making it an indispensable tool within control theory. The method can reduce computational dimensions and effectively overcome challenges associated with perturbation techniques. Therefore,

this research presents examples and highlights the strong capabilities of the VIM approach to demonstrate its effectiveness. Thus, by these examples, the study illustrates the potential and efficacy of the VIM technique.

1. INTRODUCTION

A subfield of mathematical optimization known as optimal control theory is concerned with determining a dynamic system's time-varying control so as as to optimize an objective function. Furthermore, an optimal control problem is typically characterized as a type of optimization problem. There is a significant distinction between the two concepts. In optimal control theory, the optimizer is a function rather than just a value. This function responsible for optimizing is known as the optimal control. The goal of mathematical optimization is to locate extreme points of functions involving real variables. On the other hand, the goal of optimal control theory is to find a control rule that satisfies a certain optimality condition for a given dynamical system.. In addition to its ability to formulate real-world problems mathematically, optimal control theory is valuable because it has opened up long-term research prospects in a variety of humanities fields. In recent years, the increasing use of this technology in multiple fields has captured the attention of numerous researchers. A variety of practical issues in our everyday lives can be described as optimal control problems.

In mathematics, a representation that exists in a number of states and undergoes constant change is called a dynamical system. Different structural forms, such as ordinary differential equations, partial differential equations, stochastic differential equations, integro-difference equations, and discrete-difference equations, can be adopted by such systems. A measurable indicator of system performance is known as an objective function. A desirable control that maximizes system performance is referred to as optimal control. In business and economics, a typical objective function provides a relevant measure of variables like profits or revenue. If minimizing costs is the aim, then the inverse of cost should be the target function to maximize.

. Mathematically, let:

$$J = \int_0^T F(x(t), u(t), t)dt + S(x(t), T)dt \quad (1)$$

denote the cost function, where the functions $F : \Re^n \times \Re^m \times \Re^1 \rightarrow \Re^1$ and $S : \Re^n \times \Re^1 \rightarrow \Re^1$ are assumed to be continuously differentiable. $S(x, T)$ can stand for the value derived from having x is the final state of the system at time T, while $F(x, u, t)$ might stand for the immediate rate of return in a typical commercial scenario.

Solving optimal control issues can be difficult, especially if they don't lend themselves to numerical or programming techniques. Only simple and fundamental optimal control problems could be addressed until the 1950s, when digital computers were introduced. The advent of digital computers has made it possible to apply optimal control theory and methods to solve challenging problems. Even with the advancements in software, solving optimum control issues using a typical package like MATLAB is still a difficult process. In addition to having a solid understanding of the overall structure of the solution method and the different solvers needed to implement it, one has to be proficient in programming.

Generally, an optimum control problem that is constrained and dynamic can be defined as

$$\text{Minimize } J(t, x(t), u(t)) = \int_{t_0}^{t_f} f(t, x(t), u(t))dt \tag{2}$$

$$\text{Subject to} \quad \dot{x}(t) = h(t, x(t), u(t)) \tag{3}$$

$$x(t_0) = x_0, \quad t_0 \leq t \leq t_f \tag{4}$$

where the beginning and final times are denoted by t_0 and t_f, respectively, and t is the independent time variable. . The state variable vector is $x(t) \in \Re^n$, while the control variable vector is $u(t) \in \Re^m$ which are going to be optimized. $f : \Re \times \Re^n \times \Re^m \to \Re$ is the functional and $h : \Re \times \Re^n \times \Re^m \to \Re^p$ is a smooth vector field. f and h are functions that can be continuously differentiated, that is, $f \in C^2[t_0, t_f]$ and $h \in C^1[t_0, t_f]$. The known beginning state and end state are denoted by x_0 and $x(t_f)$ respectively. $x(t_f)$ could be free (unrestricted) or fixed ($x(t_f) = x_f$).

One cannot emphasize enough the significance of the mathematical theory of optimal control in applied mathematics. The key findings of this theory serve as essential instruments in the field of applied mathematics. Research in these fields is increasing rapidly. due to their numerous applications in various disciplines. Optimal control is utilized in various sectors including science, engineering, operations research, economics, finance, and management science, and remains a valuable area of study in control theory.

Numerous nonlinear problems have been solved using the variational iteration approach. Furthermore, its adaptability and capacity for precise and simple solution of nonlinear equations are its salient features. The present study deals with a system of nonlinear equations associated with the Two Point Boundary Value problem, which stems from Pontryagin's Maximum Principle using the variational iteration approach. Both linear and nonlinear equations may be directly handled by this approach with effectiveness. It is clear that this approach yields rapidly converging continuous approximations without necessitating any restricted transformations or presumptions that could alter the physical nature of the original problem. The method's efficiency is demonstrated by the satisfactory approximations achieved with a minimal number of iterations.

Control theory is one of the most interdisciplinary areas of research and has received great practical applications in different areas of study. It has been a discipline where many mathematical ideas and methods are used. Many researchers have examined optimal control problems and different authors have suggested various solution methods.

The He's variational iteration approach was extended by [7] to offer close solution for non-linear differential-difference equations. The extended variational iteration approach's effectiveness and significant potential in resolving nonlinear differential equations were demonstrated using simple but illustrative examples. Consequently, the outcomes of the approach suggest that the process is effective and uncomplicated. Moreover, [22] introduced a technique for addressing nonlinear boundary value problems, which uses He's polynomials to combine the shooting technique with the variational iteration approach.

[17] reviewed that the VIM method has been witnessing noteworthy advancements and emerging trends in its application. In several technical disciplines, nonlinear wave equation, nonlinear challenges, nonlinear fluctuations, and nonlinear fractional differential equations were all extensively studied in this work. The researcher also explained the fundamental conceptual basis of the variational iteration technique when applied to nonlinear problems. The applicability and drawbacks of this method were examined in particular with regard to approximating nonlinear equation solutions. A new iteration approach was proposed to address the shortcomings. Additionally, a practical method for estimating the period of a nonlinear oscillator was recommended. Instances were provided to demonstrate how the solution process works.

A framework for precisely solving diffusion equations, both linear and nonlinear, was provided by [1]. For several diffusion systems with power law diffusivities, he developed precise solutions. The variational iteration technique that He developed was utilized to derive exact solutions for these equations. This robust VIM method was effective in directly solving both linear and nonlinear equations effortlessly. [21] investigated the use of the Variational Iteration Method

(VIM) in the numerical solution of the non-linear Burgers equation. The technique changed several factors that influenced the quantity of mistakes in order to examine the convergence of Burgers equation solutions. Variational Iteration technique was used to do experimental calculations on the Burgers equation. The solution found showed that with each iteration, the rate of convergence falls as the parameter values grow. However, the approximation solution converged to the analytical answer more quickly when the number of iterations increased.

2. MATERIALS AND METHODS

An optimal control problem's generic form is provided as :

$$\begin{cases} \underset{u}{Max} \quad J = \int_0^T f(t, x(t), u(t))dt, \\ subject\ to \\ x'(t) = g(t, x(t), u(t)) \\ x(0) = x_0 \end{cases} \tag{5}$$

$u^*(t)$ represents the control that maximizes (or reduces) the objective functional. When $u^*(t)$ is substituted in the state differential equation, $x^*(t)$, the associated optimal state, is obtained. As a result, the optimum set is $(u^*(t), x^*(t))$. .

In the 1950s, Lev Pontryagin and others established the basic ideas of optimal control theory. These requirements are satisfied if $(u^*(t), n; x^*(t))$ indicates an ideal combination. In order to integrate the differential equations into the cost function, Pontryagin invented the idea of adjoint functions. These adjoint functions help constrain the function with many variables for optimization, much like calculus's Lagrange multipliers do. We will use two distinct situations to analyze optimum control issues in this analysis.

Case 1: Quadratic Optimal Control Problem Constrained by Non-Linear Ordinary Differential Equation

The general form of the problem is given as

$$Min\ J(t, x, u) = \int_{t_0}^{t_f} (ax^2(t) + bu^2(t))dt \tag{6}$$

subject to the nonlinear system ,

$$\dot{x}(t) = cf(t, x(t)) + dg(t, x(t))u(t), \qquad t \in [t_0, t_f]$$
$$x(t_0) = x_0, \quad x(t_f) = x_f \tag{7}$$

$$a, b, c, d \in \mathbb{R}; \ a, b > 0$$

where $u(t)$ is the control variable that sets the system's direction and $x(t)$ is the state variable that characterizes the system.

Case 2: Generalized Quadratic Optimal Control Problem Constrained by Non Linear Ordinary Differential Equation

$$Min\ J[t, x, u] = \int_{t_0}^{t_f} (x^T(t)Qx(t) + u^T(t)Ru(t))dt \tag{8}$$

subject to the nonlinear system ,

$$\dot{x}(t) = f(t, x(t)) + g(t, x(t))u(t), \quad t \in [t_0, t_f]$$
$$x(t_0) = x_0, \quad x(t_f) = x_f \tag{9}$$

Where t is independent variable, $x(t) \in \mathrm{R}^n$ is the state variable, $u(t) \in \mathrm{R}^m$ is the control variable, x_0 is the initial state at t_0, x_f is the final state at t_f, Q$\in \mathrm{R}^{nxn}$ is a positive semi-definite matrix, R$\in \mathrm{R}^{mxm}$ is a positive definite matrix. $f(t, x(t)) \epsilon\, \mathrm{R}^n$ and $g(t, x(t)) \in \mathrm{R}^{n\times m}$ are two continuously differentiable functions in all arguments.

Pontryagin's Maximum Principle

If f and g are continuously differentiable with respect to their arguments, then Pontryagin's Maximum Principle may be used to express the first order required conditions in their most basic form.

Theorem: *(Pontryagin's Maximum Principle). For problem (6)-(7), if $u^*(t)$ and $x^*(t)$ are optimum, then there exists a piecewise differentiable adjoint variable $\lambda(t)$ such that*

$$H(t, x^*(t), u(t), \lambda(t)) \leq H(t, x^*(t), u^*(t), \lambda(t)), \tag{10}$$

for all $u \in U$ at each time t, where the Hamiltonian, H, is

$$H = f(t, x(t), u(t) + \lambda(t)g(t, x(t), u(t)) \tag{11}$$

and

$$\lambda'(t) = -\frac{\partial H(t, x^*(t), u(t), \lambda(t))}{\partial x} \tag{12}$$

$$\lambda(T) = 0 \tag{13}$$

The transversality requirement represents the last temporal restriction on the adjoint variable. The process of determining the optimal control that optimizes the functional objective while accounting for the state ODE and initial condition is transformed into maximizing the Hamiltonian at each point with the use of Pontryagin's Maximum Principle. A different viewpoint about the Hamiltonian is

$$H = f(t, x(t), u(t) + \lambda(t)g(t, x(t), u(t))$$

The necessary conditions can be derived by differentiating H with respect to u(t) at $u^*(t)$, which are as follows.

$$\frac{\partial H}{\partial u} = 0 \implies f_u + \lambda g_u = 0 \quad (\textit{optimality equation}) \tag{3.13a}$$

$$\dot{\lambda} = -\frac{\partial H}{\partial x} \implies \dot{\lambda} = -(f_x + g_x) \quad (\textit{adjoint equation}), \textit{ and} \tag{3.13b}$$

$$\lambda(t_f) = 0 \quad (\textit{transversality condition})$$

The sufficiency condition, for a maximization(concave) problem is given by

$$\frac{\partial^2 H}{\partial u^2} \leq 0 \quad at\ u^*(t)\ for\ all\ t \in [0, T] \tag{3.13c}$$

while a minimization(convex) problem is given by

$$\frac{\partial^2 H}{\partial u^2} \geq 0 \quad at\ u^*(t)\ for\ all\ t \in [0, T] \tag{3.13d}$$

In addition, the PMP approach can be applied to a variety of states and controls, which requires the introduction of corresponding adjoint variables. This is demonstrated by the utilization of n-variables,

$$\dot{x}_1(t) = g_1(t, x_1(t), \ldots, x_n(t), u(t))$$

$$\vdots$$

$$\dot{x}_n(t) = g_n(t, x_1(t), \ldots, x_n(t), u(t))$$

and matching starting points, then introduce adjoint functions,$\lambda_1(t), \ldots, \lambda_n(t)$. Thus the objective functional becomes,

$$\max \int_0^T f(t, x_1(t), \ldots, x_n(t), u(t))$$

and, the Hamiltonian is

$$\begin{aligned} H = & f(t, x_1(t), \ldots, x_n(t), u(t)) \\ & + \lambda_1(t)g_1(t, x_1(t), \ldots, x_n(t), u(t)) + \ldots \\ & + \lambda_n(t)(g_n(t, x_n(t), \ldots, x_n(t), u(t))) \end{aligned}$$

As a result, the adjoint equations, transversality conditions, and pertinent optimality equations are obtained. The i-th adjoint ordinary differential equation serves as an illustration of this (ODE).

$$\lambda_i = -\frac{\partial H}{\partial x_i}$$

Essentially, add an adjoint variable, $\lambda(t)$, to the simplest case with two unknowns, $u^*(t)$ and $x^*(t)$. Consequently, we must find the optimality equation by setting three unknowns and solving for them.

$$\frac{\partial H}{\partial u} = 0$$

and determining $u^*(t)$; this will be described in terms of $\lambda(t)$ and $x^*(t)$. Note that limits on the controls are necessary for many real-world application situations, like

$$a \le u(t) \le b$$

and that PMP still holds.

The state equations, adjoint differential equations, and control characteristics make up the optimality system. Although it is frequently impossible to solve optimality system solutions explicitly, numerical approximation techniques can be applied.

To keep things simple, describe the sides of the right hand of 3.13a as follows:

$$\begin{aligned} \phi(t, x, \lambda) &:= f(t, x(t)) + g(t, x(t))u(t), \\ \psi(t, x, \lambda) &:= -\frac{\partial H}{\partial x} \implies \dot{\lambda} = -(f_x + g_x) \end{aligned} \quad (14)$$

Thus the Two-Point Boundary Value Problem (TPBVP) in 3.13a changes to:

$$\begin{aligned} \dot{x} &= \phi(t, x, \lambda), \\ \dot{\lambda} &= \psi(t, x, \lambda) \\ x(t_0) &= x_0, \, x(t_f) = x_f \end{aligned} \quad (15)$$

Unfortunately, there is no exact analytical solution available for the nonlinear TPBVP mentioned above. Finding analytic approximation solutions for it is therefore essential. Various approximation methods, both analytical and numerical, have been presented recently to solve such ordinary differential equation problems. The goal of research has been to develop trustworthy techniques that, without limiting the variables, can handle a broad variety of linear and nonlinear differential and integral equations. Beyond the limits of conventional methods, innovative approaches that can handle both linear and nonlinear equations have made substantial development in recent years. Two of these more modern approaches are the variational iteration approach and the Adomian decomposition method. The Variational Iteration Method (VIM) will be used in this work to provide numerical solutions for the TPBVP.

Variation Iteration Method(VIM)

Since its introduction by Ji-Huan He in 1999, the variational iteration approach has been widely used in several research to solve both linear and nonlinear models. This method is known for its reliability and its ability to offer analytical solutions to a wide range of problems, whether they are homogeneous or non-homogeneous. One key benefit of the VIM method is its capacity to reduce the complexity of calculations without compromising the accuracy of the results. Several researchers have attested to the effectiveness and efficiency of this approach across different scientific disciplines. Additionally, it has been demonstrated to outperform other established techniques like the Adomian method and perturbation method. In cases where an exact solution is available, the VIM method generates successive approximations that quickly converge to the true solution.

Many researchers are presently using the variational iteration technique (VIM), it effectively resolves a range of differential equations, both linear and nonlinear. The method's flexibility and adaptability allow it to be applied to situations where the solution is not known in advance, which is common in fields like applied sciences and engineering. The VIM provides a dependable technique for performing numerical simulations for practical scientific applications and locating analytic approximation solutions. In contrast to the Adomian decomposition method, which typically requires computational algorithms to handle nonlinear terms, the VIM does not rely on restrictive assumptions for these terms, thus simplifying the analytic calculations. The VIM addresses both linear and nonlinear problems without differentiation in its approach. To demonstrate the fundamental idea behind the approach, consider a general nonlinear system:

$$L(u(t)) + N(u(t)) = g(t),$$

N is a nonlinear operator, L is a linear operator and the specified continuous function is g(t). The fundamental nature of the method involves the creation of a correction function:

$$u_{n+1}(t) = u_n(t) + \int_{t_0}^{t} \lambda[Lu_n(s) + N\tilde{u}_n(s) - g(s)]ds$$

where u_n represents the nth approximation solution, $\tilde{u}_n$ indicates a confined variation, and the generic Lagrange multiplier λ may be properly determined by the use of variational theory. , i.e., $\delta\tilde{u} = 0$. When solving a linear problem, the Lagrange multiplier may be precisely calculated, allowing for the exact solution to be reached in just one iteration step. To begin with, it is necessary to determine the optimal identification of the Lagrangian multiplier (λ) through integration by parts. By utilizing the obtained Lagrangian multiplier and selecting an appropriate function u_0, the successive approximations $u_{n+1}(x, t)$, where $n \geq 0$, can be easily obtained to approximate the solution u(x, t). Generally, the selective zeroth approximation u_0 is used with starting values u(x, 0) and $u_t(x, 0)$. After determining λ, it is possible to find many approximations, $u_j(x, t)$, where $j \leq 0$. Consequently, the following offers the solution:

$$u = \lim_{n\to\infty} u_n$$

Solving TPBVP Based On VIM

Within this section, we employ a technique similar to the shooting method combined with the Variational Iteration Method (VIM) to solve the Two-Point Boundary Value Problem (TP-BVP) stated in equation (15). Hence, our initial step involves solving the subsequent initial value problem utilizing the VIM.

$$\dot{x} = \phi(t, x, \lambda),$$
$$\dot{\lambda} = \psi(t, x, \lambda) \qquad (16)$$
$$x(t_0) = x_0, \lambda(t_0) = \alpha$$

The parameter α, which belongs to the set of real numbers, is an unknown quantity. After a satisfactory number of iterations of VIM, the parameter will be identified, as stated hereinafter

The Variational Iteration Method (VIM) is renowned for its ability to generate successively convergent approximations of the exact solution. This holds true when an exact solution exists; otherwise, the approximations can be employed for numerical purposes. When tackling an initial value problem like equation (16), a correction functional can be constructed in the following manner:

$$x_{n+1}(t) = x_n(t) + \int_{t_0}^{t} \Lambda_1(s)[\dot{x}_n(s) - \phi(s, x_n(s), \lambda_n(s))]ds$$
$$\lambda_{n+1}(t) = \lambda_n(t) + \int_{t_0}^{t} \Lambda_2(s)[\dot{\lambda}_n(s) - \psi(s, x_n(s), \lambda_n(s))]ds \qquad (17)$$

with n≥0, and the initial values $x_0(t)=x(t_0)=x_0$ and $\lambda_0(t) = \lambda(t_0) = \alpha$. By calculating the variation with respect to both x_n and λ_n and considering the constrained variations

$$\delta\phi(s, \tilde{x}_n, \tilde{\lambda}_n) = \delta\psi(s, \tilde{x}_n, \tilde{\lambda}_n) = 0 \text{ we get}$$

$$\delta x_{n+1}(t) = \delta x_n(t) + \Lambda_1(s)\delta x_n(s)|_{s=t} - \int_{t_0}^{t} \dot{\Lambda}_1(s)\delta x_n(s)ds = 0,$$
$$\delta\lambda_{n+1}(t) = \delta\lambda_n(t) + \Lambda_2(s)\delta\lambda_n(s)|_{s=t} - \int_{t_0}^{t} \dot{\Lambda}_2(s)\delta\lambda_n(s)ds = 0,$$

which leads to the following stationary conditions:

$$1 + \Lambda_1(s)|_{s=t} = 0, \quad \dot{\Lambda}_1(s) = 0$$
$$1 + \Lambda_2(s)|_{s=t} = 0, \quad \dot{\Lambda}_2(s) = 0$$

It is easily recognizable that $\Lambda_1(s)=\Lambda_2(s)=-1$. Consequently, the VIM equation transforms into:

$$x_{n+1}(t) = x_n(t) - \int_{t_0}^{t} \{\dot{x}_n(s) - \phi(s, x_n(s), \lambda_n(s))\}ds$$
$$\lambda_{n+1}(t) = \lambda_n(t) - \int_{t_0}^{t} \{\dot{\lambda}_n(s) - \psi(s, x_n(s), \lambda_n(s))\}ds \quad (18)$$

with n ≥ 0, $x_0(t) = x_0$, and $\lambda_0(t) = \alpha$.

where,

$$x^*(t, \alpha) = \lim_{n\to\infty} x_n(t)$$

$$\lambda^*(t, \alpha) = \lim_{n\to\infty} \lambda_n(t)$$

Subsequently, implementing this set of optimal control $u^*(t)$ and state ($x^*(t, \alpha^*)$ to the objective functional (1), this gives the optimal objective, J^* which is given as;

$$J^*[x, u] = \int_{t_0}^{t_f} (x^*(t, \alpha^*)^T Qx^*(t, \alpha^*) + u^*(t)^T Ru^*(t))dt.$$

In evaluating the accuracy, the following criterion should be taken into account. The optimal control is considered accurate if, for any given positive values of ϵ_1 and ϵ_2, the following two conditions are simultaneously satisfied,

$$\frac{J^*_N - J^*_{N-1}}{J^*_N} < \epsilon_1, \qquad ||x(t_f) - x_f|| < \epsilon_2$$

Theorem: *Assume that $\{x_n(t)\}$ and $\{\lambda_n(t)\}$ are the sequences of solutions generated by the VIM formula (18) converge individually to $x^*(t, \alpha)$ and $\lambda^*(t, \alpha)$ as $n \to \infty$ then $x^*(t, \alpha)$,$\lambda^*(t, \alpha)$ are the exact solutions of (15) when α^* is the real root of $x^*(t_f) - x_f = 0$.*

Proof: By taking limits on both sides of the equation (18) as $n \to \infty$ since $\lim_{n\to\infty} x_n(t) = \lim_{n\to\infty} x_{n+1}(t) = x^*(t, \alpha)$ and $\lim_{n\to\infty} \lambda_n(t) = \lim_{n\to\infty} \lambda_{n+1}(t) = \lambda^*(t, \alpha)$ we have

$$\int_{t_0}^{t} \{\dot{x}_n(s, \alpha) - \phi(s, x_n(s, \alpha), \lambda_n(s, \alpha))\}ds$$
$$\int_{t_0}^{t} \{\dot{\lambda}_n(s, \alpha) - \psi(s, x_n(s, \alpha), \lambda_n(s, \alpha))\}ds \quad (19)$$

Taking the derivative with respect to t on both sides results in

$$\dot{x}_n(s, \alpha) = \phi(s, x_n(s, \alpha), \lambda_n(s, \alpha))$$
$$\dot{\lambda}_n(s, \alpha) = \psi(s, x_n(s, \alpha), \lambda_n(s, \alpha) \quad (20)$$

Furthermore if $t = t_0$ then from (18), $x_{n+1}(t_0) = x_n(t_0)$, $\lambda_{n+1}(t_0) = \lambda_n(t_0)$ for every $n \geq 0$. Thus $x_n(t_0) = x_0(t_0) = x_0$ *and* $\lambda_n(t_0) = \lambda_0(t_0) = \alpha$

or equivalently, $x^*(t,\alpha) = x_0$ and $\lambda^*(t,\alpha) = \alpha$. Hence $x^*(t,\alpha)$ and $\lambda^*(t,\alpha)$ are the exact solutions of (15). Moreover, these solutions correspond to the exact solutions of equation (16), but only when the final state condition $x(t_f) = x_f$ is fulfilled. Therefore, it is simple to select the unidentified variable $\alpha \in \mathbb{R}$, such that $x^*(t_f,\alpha) = x_f$. Representing this real root of $x^*(t_f,\alpha) - x_f = 0$ *by* α^* finalizes the proof..

Theorem: *Given the assumptions of Theorem 1, the sequences* $\{u_n(t)\}$ *and* $\{J_n\}$ *are defined as follows,*

$$u_n(t) = -R^{-1}g^T(t, x_n(t))\lambda_n(t) \tag{21}$$

$$J_n = \int_{t_0}^{t_f} (x_n^T(t)Qx_n(t) + u_n^T(t)Ru_n(t))dt \tag{22}$$

converge towards the optimal control law and achieve the optimal objective value, respectively.

Proof: Given that g is a continuous function, by considering the limit from (21) and putting $\alpha = \alpha^*$, results in

$$u^*(t) := \lim_{n\to\infty} u_n(t) = -R^{-1}g^T(t, \lim_{n\to\infty} x_n(t), \lim_{n\to\infty} \lambda_n(t))$$

$$= -R^{-1}g^T(t, x^*(t,\alpha^*))\lambda^*(t,\alpha^*)$$

the optimal control law, denoted as $u^*(t)$, is derived from the exact solutions $x^*(t,\alpha^*)$ and $\lambda^*(t,\alpha^*)$ of the extreme conditions in equation (29). Similarly, by considering the limit of equation (22), we obtain .

$$J^* := \lim_{n\to\infty} J_n = \lim_{n\to\infty} \int_{t_0}^{t_f} (x_n^T(t)Qx_n(t) + u_n^T(t)Ru_n(t))dt$$

$$= \int_{t_0}^{t_f} \lim_{n\to\infty} x_n^T(t)Q \lim_{n\to\infty} x_n^T + \lim_{n\to\infty} u_n^T(t)R \lim_{n\to\infty} u_n(t)dt$$

$$= \int_{t_0}^{t_f} (x^*(t,\alpha^*)Qx^*(t,\alpha^*) + u^*(t)Ru^*(t))dt$$

Consequently, J^* can be recognized as the optimal objective value, signifying the completion of the proof.

3. RESULTS

Example 1. *Consider the non-linear optimal control problem based on the nonlinear system ,*

$$Minimize\ \ J(x,u) = \int_0^1 u^2(t) + 4dt \tag{23}$$

Subject to

$$\dot{x}(t) = x^2(t) + 2u(t), \quad t \in [0,1]$$
$$x(0) = 0.5, \quad x(1) = 1 \tag{24}$$

Solution 1. *The Hamiltonian, H is given as*

$$H = u^2(t) + 4 + \lambda\ x^2(t) + 2u(t) \tag{25}$$

By employing the necessary and sufficient extreme conditions, we have,

$$\frac{\partial H}{\partial u} = 0 = 2u(t) + 2\lambda \qquad (26)$$

$$\dot{\lambda} = -\frac{\partial H}{\partial x} = -2x(t)\lambda(t) \qquad (27)$$

From Equation (26), the optimal control law is $u(t) = -\lambda(t)$. Therefore the state equation (24) becomes

$$\begin{aligned} \dot{x}(t) &= x^2(t) - 2\lambda(t), \quad t \in [0,1] \\ x(0) &= 0, \quad x(1) = 1 \end{aligned} \qquad (28)$$

Combining Equation (28) and (27), we have

$$\begin{aligned} \dot{x}(t) &= x^2(t) - 2\lambda(t), \quad t \in [0,1] \\ \dot{\lambda} &= -2x(t)\lambda(t) \\ x(0) &= 0, \quad x(1) = 1 \end{aligned} \qquad (29)$$

Solving this TPBVP in equation (29), substituting the final state condition.

$$x(1) = 1 \; by \; \lambda(0) = \alpha$$

Then, the Two Point Boundary Value Problem(TPBVP) in equation (29) becomes

$$\begin{aligned} \dot{x}(t) &= x^2(t) - 2\lambda(t), \quad t \in [0,1] \\ \dot{\lambda} &= -2x(t)\lambda(t) \\ x(0) &= 0, \quad \lambda(0) = \alpha \end{aligned} \qquad (30)$$

where $\alpha \in$ R is an unknown parameter. To find a solution for this problem with given initial conditions, we make use of the variation Iteration Method explained above. The correctional functional (18) becomes

$$\begin{aligned} x_{n+1}(t) &= x_n(t) - \int_{t_0}^{t} \{s^{\cdot}(t) - x^2(s) + 2\lambda(s)\}ds \\ \lambda_{n+1}(t) &= \lambda_n(t) - \int_{t_0}^{t} \{\dot{\lambda} + 2x(t)\lambda(t)\}ds \end{aligned} \qquad (31)$$

where $x_0(t) = 0$, $\lambda_0(t) = \alpha$, $n \geq 0$. Using Maple software we arrive at

$$x^*(t) \approxeq x_5(t) = 0.8944320000\, t + 1.600000000 \times 10^{-11} t^3 - 0.00203178971\, t^7 + 0.07155532945\, t^5 + 0.0001032818612\, t^{15} + 0.0002493127749\, t^{13} + 0.0002692663434\, t^{11} + 0.006966308040\, t^9 + 1.527041524 \times 10^{-12} t^{31} + 7.075455410 \times 10^{-11} t^{29} + 0.000000001245245847\, t^{27} + 0.00000001098273612\, t^{25} + 0.00000007661909085\, t^{23} + 0.0000006809099985\, t^{21} + 0.000003972011163\, t^{19} + 0.00001357334258\, t^{17}$$

$$\begin{aligned} u^*(t) \approxeq u_5(t) = -\lambda_5(t) = +0.447216 - 0.4000043013\, t^2 - \\ + 0.0003267316249\, t^{14} - 0.0003430262869\, t \\ + 2.366914363 \times 10^{-11} t^{30} + 0.00000000092 \\ + 0.00000007951333985\, t^{24} + 0.0000004379 \\ + 0.00001149833913\, t^{18} + 0.0000136626003 \end{aligned} \qquad (32)$$

To calculate the optimal objective value, substitute the control function in (32) into (23) then integrate i.e

$$J^*(x, u) = \int_0^1 u^2(t) + 4dt$$

$$J^* \approxeq 4.34243$$

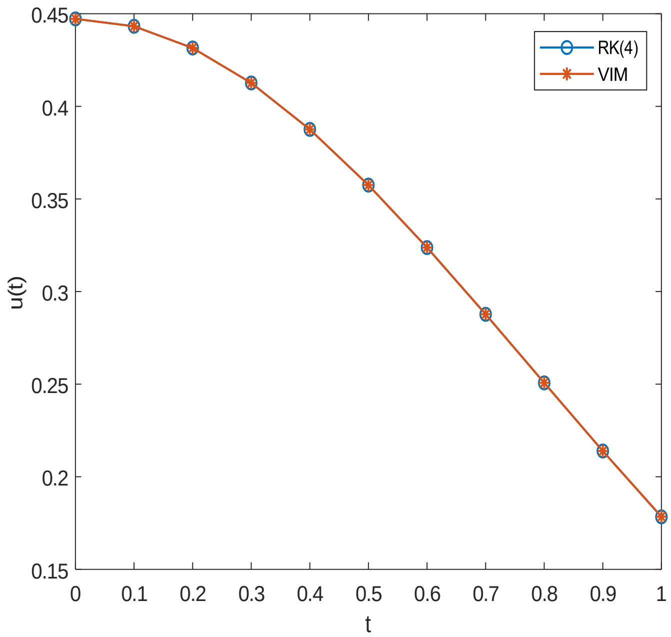

Figure 1: The optimal control, Example 1.

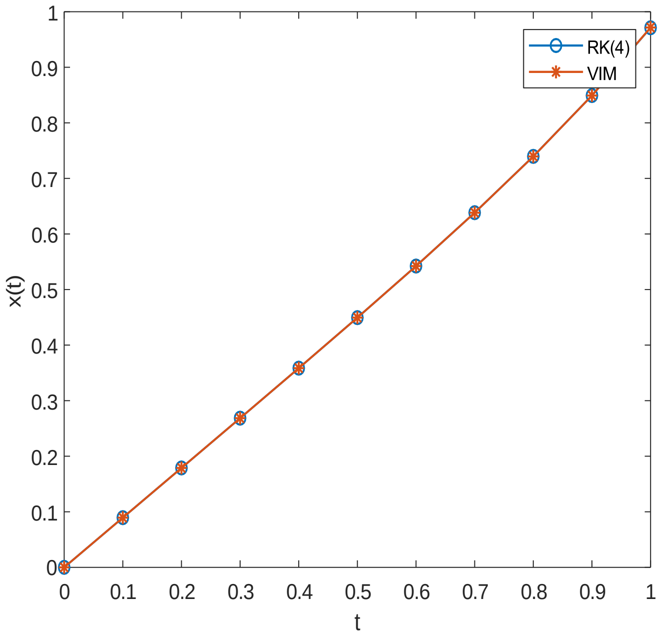

Figure 2: The optimal state, Example 1.

Example 2. *Consider the non-linear optimal control problem based on the nonlinear system*

,

$$Minimize \quad J(x, u) = \frac{1}{2}\int_0^1 x_1^2(t) + x_2^2(t) + u^2(t)dt \tag{33}$$

Subject to

$$\begin{aligned} \dot{x}_1(t) &= x_2 + x_1x_2 \\ \dot{x}_2(t) &= -x_1 + x_2 + u + x_2^2 \\ x_1(0) &= -0.8, \quad x_2(0) = 0 \end{aligned} \tag{34}$$

Solution 2. *The Hamiltonian, H is given as*

$$H = \frac{1}{2}\left(x_1^2+x_2^2+u^2\right)+\lambda_1(x_2+x_1x_2)+\lambda_2(-x_1+x_2+u+x_2^2) \tag{35}$$

By utilizing the extreme necessary and sufficient conditions we have,

$$\frac{\partial H}{\partial u} = 0 = u + \lambda_2 \quad (36)$$

$$\dot{\lambda}_1 = -x_1 - \lambda_1 x_2 + \lambda_2 \quad (37)$$

$$\dot{\lambda}_1 = \; - x_2 + \lambda_1(1 + x_1) + \lambda_2(1 + 2x_2) \quad (38)$$

From Equation (36), the optimal control law is $u(t) = -\lambda_2(t)$. Therefore the state equation (34) becomes,

$$\begin{aligned} \dot{x}_1(t) &= x_2 + x_1 x_2 \\ \dot{x}_2(t) &= -x_1 + x_2 - \lambda_2 + x_2^2 \\ x_1(0) &= -0.8, \quad x_2(0) = 0 \end{aligned} \quad (39)$$

The extreme conditions are,

$$\begin{aligned} \dot{\lambda}_1 &= -x_1 - \lambda_1 x_2 + \lambda_2 \\ \dot{\lambda}_2 &= \; - x_2 + \lambda_1(1 + x_1) + \lambda_2(1 + 2x_2) \\ \dot{x}_1(t) &= x_2 + x_1 x_2 \\ \dot{x}_2(t) &= -x_1 + x_2 - \lambda_2 + x_2^2 \\ x_1(0) &= -0.8, \quad x_2(0) = 0 \\ \lambda_1(1) &= \lambda_2(1) = 0 \end{aligned} \quad (40)$$

In order to address this problem, we utilize the variation Iteration Method(VIM) explained in the previous chapter. The correctional functional (18) becomes:

Letting $a = x_1$, $b = x_2$, $c = \lambda_1$ and $d = \lambda_2$

$$\begin{aligned} a_{n+1} &= a_n - \int_0^t \dot{a}_n - b_n - a_n b_n ds \\ b_{n+1} &= b_n - \int_0^t \dot{b}_n + a_n - b_n + d_n - b_n^2 ds \\ c_{n+1} &= c_n - \int_0^t \dot{c}_n + a_n + c_n b_n - d_n ds \\ d_{n+1} &= d_n - \int_0^t \dot{d}_n + b_n + c_n(1 + a_n) + d_n(1 + 2b_n) ds \end{aligned} \quad (41)$$

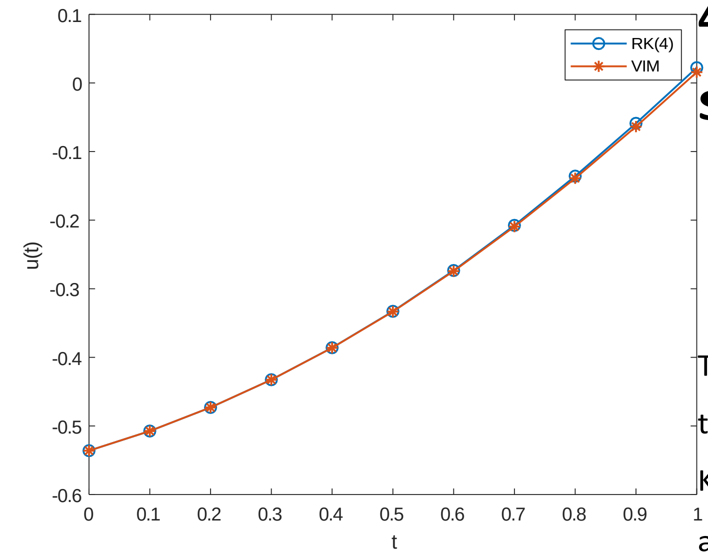

Figure 3: The optimal control, Example 2.

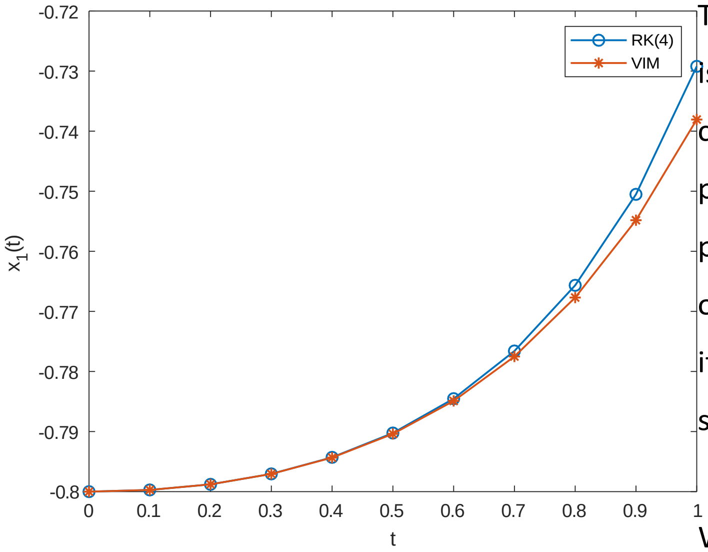

Figure 4: The optimal state $x_1(t)$, Example 2.

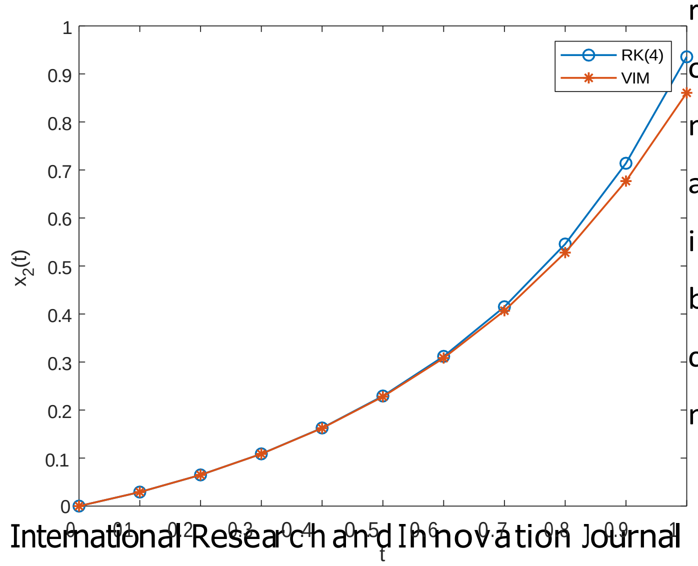

Figure 5: The optimal state $x_2(t)$, Example 2.

4. DISCUSSION OF RESULTS

The results obtained using the Variational Iteration Method (VIM) and the fourth-order Runge-Kutta (RK4) method show a striking similarity, as seen in the overlapping plots. This strong agreement suggests that VIM can achieve results comparable to the well-established RK4 method. The nearly identical plots confirm that VIM is highly accurate in solving nonlinear optimal control problems and can replicate the solutions produced by RK4, a method renowned for its precision in numerical integration. Both methods also show rapid convergence, with VIM's iterative approach closely aligning with the results from RK4.

While the outcomes are similar, VIM offers distinct advantages over RK4. Unlike RK4, which requires discretization of the time domain, VIM operates without this constraint, allowing for a more continuous and accurate solution. VIM's ability to handle nonlinear terms without needing linearization or perturbation is a significant benefit, especially in solving nonlinear optimal control problems where such terms are common. Additionally, VIM simplifies the computational process by eliminating the need for complex polynomial expansions, resulting in a more straightforward and less computationally demanding procedure. The close agreement between VIM and RK4 validates VIM's effectiveness as a legitimate tool for solving nonlinear optimal control problems and underscores its potential as a reliable alternative, particularly in cases where traditional numerical methods may be cumbersome or impractical.

5. CONCLUSION

A class of optimal control problems that are nonlinear is effectively handled. by the application of the variational iteration technique. Using the previously offered explanatory examples as a guide, the VIM technique solved the TPBVP, which was derived from Pontryagin's Maximum Principle. The VIM method's effectiveness and validity were shown by the explanatory examples that were taken into consideration. It is evident that the approach produces quick, sequential, and convergent approximations without limiting conjecture or alteration that may take the role of physical behavior of the problem. Nonlinear terms do not specifically need to be handled in this manner. Numerous nonlinear optimal control problems may be resolved with this approach. Multiple subsequent estimates are generated by iterating the correction functional and applying the variational iteration approach. Furthermore, the VIM makes computations simpler, leading to an easy-to-understand iteration procedure. Future efforts will focus on adapting the VIM methodology to solve the Two-Point Boundary Value Problem in an efficient manner. The VIM approach is very effective and legitimate.

References

[1] Abdul-Majid, W. (2006),"The variational iteration method: A powerful scheme for handling linear and nonlinear diffusion equations", *Computers and Mathematics with Applications*, Volume 54, Issues 7–8

[2] Abdul-Majid, W. (2007),"The variational iteration method: A reliable analytic tool for solving linear and nonlinear wave equations", *Computers and Mathematics with Applications*, Volume 54, Issues 7–8.

[3] Abdul-Majid, W. (2014),"The variational iteration method for solving linear and nonlinear ODEs and scientific models with variable coefficients", *Central European Journal of Engineering*, Volume 4, pages 64-71

[4] Adekunle, A.I. and Olotu, O. (2012) "A primal- dual augmented Lagrangian method for optimal control linear-quadratic problem with time-varying systems."*Journal of Nige-*

rian Association of Mathematical Physics, Volume 20, pages 165-174

[5] Afolabi, A. S. (2022). On the constrained optimal control problems with mixed constraints. *Intentional Journal of Mathematical Analysis and Modelling*, Vol. 5, pp. 25-35.

[6] Afolabi A.S and Olotu O. (2020). Analytical Solutions of Optimal Control Problems with Mixed Constraints. *Iconic Research and Engineering Journals*, Vol. 3, pp. 37-43.

[7] Ahmet Y. (2008), ”Applying He’s Variational Iteration Method for Solving Differential-Difference Equation”,*Hindawi Publishing Corporation Mathematical Problems in Engineering* Volume 2008, Article ID 869614, pages 1-7

[8] Bahaa,G.M.(2017), ”Fractional optimal control problem for differential system with delay argument”, *Bahaa Advances in Difference Equations*,69

[9] Dan, W. (2015), “Analytic Solutions to Optimal Control Problems with Constraints”, *Scientific Research Publishing, Applied Mathematics*, Vol. 6, No. 14, pp. 2326-2339.

[10] Fassino, S.L.(2013), ”Optimal Control of Differential Equations with Pure State Constraints. ” *Master’s Thesis*, University of Tennessee.

[11] Habibolla,L and He,J.H. (2020),”A general numerical algorithm for nonlinear differential equations by the variational iteration method”,*Nonlinear Science Letters A*, vol. 1, pp. 1-30, 2020.

[12] He, J.H. (1999), “Variational iteration method-a kind of non-linear analytical technique: some examples,” *International Journal of Non-linear Mechanics*, vol. 34, pp. 699-708.

[13] Ogunniran, M. O., Tayo, O. A. ., Haruna, Y., & Adebisi, A. F. (2020). Linear Stability Analysis of Runge-Kutta Methods for Singular Lane-Emden Equations. *Journal of the Nigerian Society of Physical Sciences*, 2(3), 134–140. https://doi.org/10.46481/jnsps.2020.87

[14] Olotu, O. (2010), ”Quadratic Convergence Ratio of Constrained Optimal Control Problems,” *Journal of Mathematics and Statistics*, Volume 6, pages 42-46.

[15] Olotu, O. and Afolabi, A.S. (2014), ”A New Scheme Using Penalty Functions with B-Spline Function for Constrained Optimal Control Problems”. *International Journal of Numerical Mathematics (IJNM)*, 7(2), 282-300, Nigeria (40%).

[16] Olotu, O. and Afolabi, A.S. (2017), ”On The Optimal Control Problems Con-

strained by Delay Differential Equation Using Penalty Functions with B-Spline Function". *Journal of the Nigerian Association of Mathematical Physics, Benin, Nigeria*, 39(1), 119–128, Nigeria (40%).

[17] He, J.H. and Wu, X.H. (2007), "Variational iteration method: new development and applications", Computers and Mathematics with Applications, Vol. 54, pp. 881-894.

[18] Heinkenschloss, A. (2005), "A Time-domain Decomposition Iterative Method for the Solution of Distributed Linear Quadratic Optimal Control Problems", *Journal of Computational and Applied Mathematics*, Vol. 173, No.1, pp. 169-198.

[19] Jaddu, H. (2002), "Spectral Method for Constrained Linear Quadratic Optimal Control", *Mathematics and Computers in Simulation*, Vol. 58, No. 2, pp. 159-169.

[20] Jingang, Z., Teng, N., Jianxiong, Y. and Enmin, F.(2017), "Optimal control of nonlinear switched system with mixed constraints and its parallel optimization algorithm", *Nonlinear Analysis: Hybrid Systems*,Volume 25,Pages 21-40

[21] Mamadu,E.J., and Njoseh, I.N.(2016),"Variation Iteration Method for The Approximate Solution of Nonlinear Burgers Equation",*Nigerian Journal of Basic and Applied Science*,Volume 24,pages 70-75

[22] Matinfar, M. and Ghasemi, M.(2013) "Solving BVPs with shooting method and VIMHP", *Journal of the Egyptian Mathematical Society* ,Vol. 21, pp. 354–360

[23] Sargent, R. W. H. (2000), "Optimal Control", *Journal of Computational and Applied Mathematics*, Vol. 124, pp. 361-371.

[24] Yousefi, S. A. , Dehghan, Mehdi and Lotfi, A.(2010) "Finding the optimal control of linear systems via He's variational iteration method",*International Journal of Computer Mathematics*, 87: 5, 1042 — 1050

[25] Yusuf, T. T., Abidemi, A., Afolabi, A. S., & Dansu, E. J. (2022). Optimal Control of the Coronavirus Pandemic with Impacts of Implemented Control Measures. *Journal of the Nigerian Society of Physical Sciences*, 4(1), 88–98. https://doi.org/10.46481/jnsps.2022.414