**TASK**

If you break a stick uniformly in two places, you will be left with three segments. Write an algorithm for computing the probability that the three segments form a triangle. This algorithm is supposed to employ Metropolis-Hastings ideas and serve as an independent verification of our theoretical calculations.

**SOLUTION**

First, we note that the probability can be calculated quite easily on a piece of paper...

Let A and B be the two break points falling on a stick of length L. To distinguish between the break points, we will order them chronologically, A being the older one. Two cases are to be considered:

**CASE 1:** $0 \leq A \leq B \leq L$

and

**CASE 2:** $0 \leq A < B \leq L$.

The three pieces will form a triangle if none is longer than the sum of the others. In terms of A and B, the conditions are the following.

**CASE 1:**

\[
A < L - A \quad \text{<=======>} \quad A < L/2;
\]

\[
L - B < B \quad \text{<=======>} \quad B > L/2;
\]

\[
B - A < A + (L - A) \quad \text{<=======>} \quad B < A + L/2.
\]

**CASE 2:** $B < L/2$, $A > L/2$ and $A < B + L/2$.

Now we are capable of plotting the acceptable region on the plane. We see that it consists of two small triangles: one triangle corresponds to case 1 and the other triangle corresponds to case 2. The area of the acceptable region can be calculated as

\[
\frac{1}{2} \times (L/2)^2 + \frac{1}{2} \times (L/2)^2.
\]

The space of all elementary possibilities is the square $[0,L] \times [0,L]$. It has the area of $L^2$. Since $(A,B)$ are uniformly distributed on $[0,L] \times [0,L]$, the probability of the three segments forming a triangle equals

\[
\frac{\text{the area of the acceptable region}}{\text{the area of the space of all elementary possibilities}} = \frac{1/2 \times (L/2)^2 + 1/2 \times (L/2)^2}{L^2} = 1/4.
\]

Next, we are going to build an algorithm to verify our theoretical result...

Now let $LF = \min(A,B)$ be the left break point and $RT = \max(A,B)$ be the right breakpoint. It is a straightforward exercise to determine conditional and marginal distributions of $LF$ and $RT$. First we focus on marginal distributions:

\[
\text{Fm}_{LF}(x) = P(LF \leq x) = 1 - P(LF > x) = 1 - P(\min(A,B) > x) = 1 - P(A>x,B>x) = \]
\[ = 1 - P(A > x)^*P(B > x) = 1 - (1-x)/L)^2. \]

\[ Fm_{RT}(y) = P(RT \leq y) = P(\max(A,B) \leq y) = P(A \leq y, B \leq y) = P(A \leq y) * P(B \leq y) = \]

\[ = (y/L)^2. \]

And now we are ready to calculate conditional distributions. For any \( x \leq y, \)

\[ Fc_{LF}(x | y) = P(LF \leq x \mid RT = y) = P(\min(A,B) \leq x \mid \max(A,B) = y) = \]

\[ = 1/2 \times P(\min(A,B) \leq x \mid \max(A,B) = y, A < B) + 1/2 \times P(\min(A,B) \leq x \mid \max(A,B) = y, A \leq B) = \]

\[ = 1/2 \times P(A \leq x \mid A < y) + 1/2 \times P(B \leq x \mid B \leq y) = x/y. \]

Similarly, for any \( x \leq y, \)

\[ Fc_{RT}(y | x) = P(RT \leq y \mid LF = x) = 1 - P(RT > y \mid LF = x) = 1 - (1-y)/(1-x). \]

Using functions \( Fc_{LF}() \) and \( Fc_{RT}() \), random variables \( LF \) and \( RT \) can be simulated one from the other. Here we employ the rule:

\[ \text{if } F(x) \text{ is a cumulative distribution function (cdf) of a given distribution, then random variable } F_{-1}(U) \text{ has this distribution, where } U \text{ is uniformly distributed on } [0,1]. \] (***)

NOTE: of course, we did not have to derive conditional distributions of TF and RT to simulate the three random segments of the line. We could have easily simulated the marginals of \( A \) and \( B \) and seen if the three segments form a triangle. Focusing on \( LF \) and \( RT \) was necessitated by the requirement to use Metropolis algorithm.

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The algorithm below employs Gibbs sampling, which says: to simulate a joint distribution of \((LF,RT)\), we can simulate \( LF \) given \( RT \) and \( RT \) given \( LF \) long enough.

\%
\% INITIALIZATION
\% Counter = 0
\% Random.Seed(0)

\% for(S = 1:Sample_Number)
\% \% SIMULATING INITIAL VALUES OF Z AND W
\% U = Simulated_Uniform(0,1)
\% LF = Fm^{-1}_{LF}(U) \% Using the marginal cdf of LF and rule (***) to simulate LF.
\% U = Simulated_Uniform(0,1)
\% RT = Fm^{-1}_{RT}(U) \% Using the marginal cdf of RT and rule (***) to simulate RT.

\% for(iter = 1:(Burn.In+1))
\% \% THE MAGIC OF GIBBS SAMPLING.

\% \% Randomly selecting LF or RT.
\% U = Simulated_Uniform(0,1)
\% if( U <= 1/2 )
\% \% LF = Fc^{-1}_{LF}(U \mid RT) \% Simulating LF using its conditional cdf
\% \% and the current value of RT.
\% \% else
\% \%
U = Simulated_Uniform(0,1)
RT = Fc^{-1}_RT(U | LF) \% Simulating RT using its conditional cdf
% and the current value of LF.

% CHECKING IF ONE CAN MAKE A TRIANGLE
% OUT OF THE SIMULATED SEGMENTS
if(LF < L/2 & RT > L/2 & RT < LF + L/2)
    Counter = Counter + 1
end

Prob_Of_Triangle = Counter / Sample_Number.

The proposed computational algorithm uses Gibbs sampling. So how is our work related to the ideas of
Metropolis?... It turns out that the employed version of Gibbs sampling is a particular case of the
Metropolis-Hastings algorithm. Let us denote W = (LF,RT).

- Gibbs sampling simulates a Markov chain of different realizations of W in multiple steps (just like
  in the Metropolis-Hastings algorithm).
- At each step we have a current value of W and propose a new value W' [just like in Metropolis].
- We propose the new value W' with the proposal density Q(w' | w), which is based on the following
two-stage procedure. First, a single dimension i of W is chosen randomly. Second, the proposed
value W' is identical to W, except for its value along the i-dimension W_i (which is either LF or
RT). W_i is sampled from the conditional distribution P(W_i | W_{-i}), where W_{-i} is the other
dimension (if W_i = LF, then W_{-i} = RT, and the other way around). Therefore

Q(W' | W) = P(W'_i | W_{-i}).

- The new value is accepted with probability

( P(W') * Q(W | W') ) / ( P(W) * Q(W' | W) )

(just like in the Metropolis-Hastings algorithm). We note that, due to the specific
construction of Q(w,w'), the acceptance probability equals

( P(W') * Q(W | W') ) / ( P(W) * Q(W' | W) ) =
= ( P(W) * P(W_i | W_{-i}) ) / ( P(W) * P(W_i | W_{-i}) ) =
= ( P(W_{-i}) * P(W_i | W_{-i}) ) / P(W_{-i}) = 1.

So we always accept the new realization W'.

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