Name: Key Date: October 17, 2018 Class Time:__________

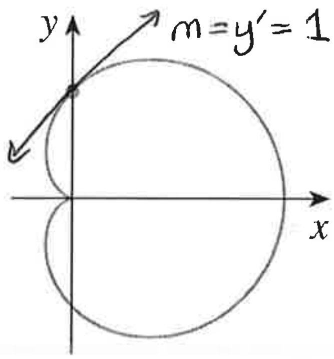
Analytic Geometry & Calculus I | Tulsa Community College

Quiz #6: Implicit and Logarithmic Differentiation, Related Rates, and Inverse Trigonometric Functions

Remember to get full credit, you need to show all work, clearly and neatly. Remember, this isn't just about you getting the answer, but you *showing someone else* how you got the answer.

You may use a scientific, nongraphing calculator on this assessment.

1. The graph of the relation given below, and the point $(x, y) = \left(0, \frac{1}{2}\right)$ are shown.



$$x^2 + y^2 = (2x^2 + 2y^2 - x)^2$$

Find the slope of the curve $\frac{dy}{dx}$ at the point $\left(0, \frac{1}{2}\right)$ using **implicit differentiation**.

$$\frac{d}{dx}[x^2 + y^2] = \frac{d}{dx}[(\underbrace{\boxed{2x^2 + 2y^2 - x}}_{\text{inside function}})^2]$$

Diff. both sides with respect to x.

$$2x + 2y\frac{dy}{dx} = 2(2x^2 + 2y^2 - x)^1 \frac{d}{dx}[2x^2 + 2y^2 - x]$$

use chain rule on RHS.

$$2x + 2y\frac{dy}{dx} = (4x^2 + 4y^2 - 2x)(4x + 4y\frac{dy}{dx} - 1)$$

$$2x + 2y\frac{dy}{dx} = 16x^3 + 16x^2y\frac{dy}{dx} - 4x^2 + 16xy^2 + 16y^3\frac{dy}{dx} - 4y^2 - 8x^2 - 8xy\frac{dy}{dx} + 2x$$

Distribute

$$(2y - 16x^2y - 16y^3 + 8xy)\frac{dy}{dx} = 16x^3 - 4x^2 + 16xy^2 - 4y^2 - 8x^2 + \cancel{2x} - \cancel{2x}$$

Move all dy/dx terms to one side + other terms to the other side.

$$\frac{dy}{dx} = \frac{16x^3 - 4x^2 + 16xy^2 - 4y^2 - 8x^2}{2y - 16x^2y - 16y^3 + 8xy}$$

Divide, to isolate dy/dx.

1

$$\frac{dy}{dx} = \frac{\overset{2}{\cancel{4}}(4x^3 - x^2 + 4xy^2 - y^2 - 2x^2)}{\cancel{2}(y - 8x^2y - 8y^3 + 4xy)}$$

$$\left.\frac{dy}{dx}\right|_{(x,y)=(0,\frac{1}{2})} = \frac{2[4(0)^3 - (0)^2 - 4(0)(1/2)^2 - (1/2)^2 - 2(0)^2]}{(\frac{1}{2}) - 8(0)^2(\frac{1}{2}) - 8(\frac{1}{2})^3 + 4(0)(\frac{1}{2})}$$

$$= \frac{2(-\frac{1}{4})}{\frac{1}{2} - 1} = \frac{-1/2}{-1/2} = \boxed{1}.$$

2. Use **logarithmic differentiation** to find $\frac{dy}{dx}$. Write the derivative entirely in terms of x.

$$y = (\sin x)^{\ln x}$$

$$\ln y = \ln\left[(\sin x)^{\ln x}\right]$$

Take the natural log of both sides.

$$\ln y = \ln x \ln(\sin x)$$

Simplify RHS using $\ln A^n = n \ln A$.

$$\frac{d}{dx}[\ln y] = \frac{d}{dx}[\underbrace{\ln x}_{1st}\ \underbrace{\ln(\sin x)}_{2nd}]$$

Differentiate both sides with respect to x.

$$\frac{1}{y}\frac{dy}{dx} = \frac{d}{dx}[\ln x]\cdot\ln(\sin x) + \frac{d}{dx}[\ln(\underbrace{\boxed{\sin x}}_{\text{inside}})]\cdot\ln x$$

Use product rule on RHS.

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x}\cdot\ln(\sin x) + \frac{1}{\sin x}\frac{d}{dx}[\sin x]\ln x$$

Use chain rule for $\frac{d}{dx}[\ln(\sin x)]$

$$\frac{1}{y}\frac{dy}{dx} = \frac{\ln(\sin x)}{x} + \frac{1}{\sin x}\frac{\cos x}{1}\cdot\ln x$$

Simplify

$$y\cdot\frac{1}{\cancel{y}}\frac{dy}{dx} = \left(\frac{\ln(\sin x)}{x} + \cot x \ln x\right)y$$

Multiply both sides by y.

$$\boxed{\frac{dy}{dx} = \left(\frac{\ln(\sin x)}{x} + \cot x \ln x\right)(\sin x)^{\ln x}}$$

Substitute $y = (\sin x)^{\ln x}$.

Skill #: D15

Score:

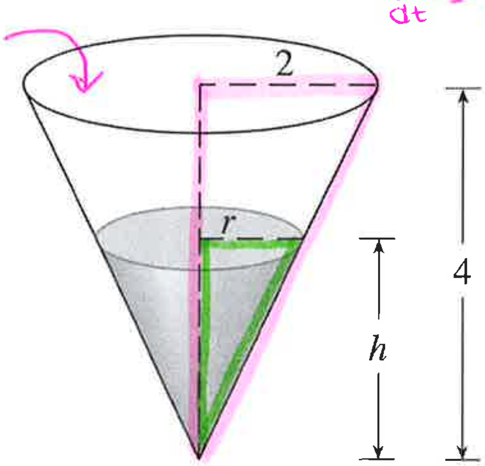
3. **Problem Statement**: A water tank has the shape of an inverted circular cone with a base of radius 2 meters and height of 4 meters. If the water is being pumped into the tank at a rate of 2 m^3/min, find the rate at which the water level is rising when the water is 3 meters deep.

$h = 3$

Note: The volume of a right circular cone is given by $V = \frac{1}{3}\pi r^2 h$.

$\frac{dh}{dt} = ?$

$\frac{2\ m^3}{min.} = \frac{dV}{dt}$



V : volume (in m^3)

h : height = water level (in m)

r : radius (in m)

t : time (in minutes)

(a) In terms of the variables $V, r, h,$ and t, what information is given in this problem? Use mathematical symbols, equations, and variables.

Since m^3 are units of volume, $\boxed{2\ m^3/min = dV/dt}$.
The radius of the cone is 2 m, and the height of the cone is 4 m. Since $dV/dt = 2\ m^3/min$, time is measured in minutes.

(b) In terms of the variables $V, r, h,$ and t, what is the question asking for? Use mathematical symbols and variables.

Since h represents the water level, "the rate at which the water level is rising" is dh/dt.

$\boxed{\frac{dh}{dt} = ? \text{ when } h = 3 \text{ meters.}}$

(c) Use similar triangles to find the relationship between r and h.

The pink & green triangles are similar triangles.

$\frac{r}{h} = \frac{2}{4} \Rightarrow \boxed{r = \frac{h}{2}}$ OR $\boxed{r = \frac{1}{2}h}$

(d) State the equation relating the variables in the problem.

$\boxed{V = \frac{1}{3}\pi r^2 h}$

OR

$V = \frac{1}{3}\pi\left(\frac{1}{2}h\right)^2 h$ since $r = \frac{1}{2}h$

$V = \frac{1}{3}\pi \cdot \frac{1}{4}h^2 h$

$\boxed{V = \frac{1}{12}\pi h^3}$

Since the question asks about dh/dt, we want volume in terms of h.

(e) Differentiate your equation implicitly with respect to t, and use your differentiated equation to answer the question. State your final answer to the question in the form of a complete sentence.

$$\frac{d}{dt}[V] = \frac{d}{dt}\left[\frac{1}{12}\pi h^3\right]$$

Diff. both sides with respect to time t.

$$\frac{dV}{dt} = \frac{1}{12}\pi \cdot 3h^2 \frac{dh}{dt}$$

Simplify

$$\frac{dV}{dt} = \frac{3}{12}\pi h^2 \frac{dh}{dt}$$

$$4\frac{dV}{dt} = \left(\frac{1}{\cancel{4}}\pi h^2 \frac{dh}{dt}\right)\cancel{4}$$

$$\frac{4\, dV/dt}{\pi h^2} = \frac{\cancel{\pi h^2}\, dh/dt}{\cancel{\pi h^2}}$$

Solve for $\frac{dh}{dt}$.

$$\frac{dh}{dt} = \frac{4\frac{dV}{dt}}{\pi h^2}$$

$$\frac{dh}{dt} = \frac{4(2)}{\pi(3)^2} = \boxed{\frac{8}{9\pi}\ m/min}$$

when $h = 3$ m & $\frac{dV}{dt} = 2\ m^3/min$.

Substitute $\frac{dV}{dt} = 2$ & $h = 3$ and simplify.

Skill #: D17
Score:

4. Find the derivative of the function.

(a) $g(x) = \underbrace{(5\sin x + 2x^2)}_{\text{1st}}\ \underbrace{\arccos x}_{\text{2nd}}$

$\boxed{\text{Product Rule}}$

$$g'(x) = \underbrace{\frac{d}{dx}[5\sin x + 2x^2]}\arccos x + \frac{d}{dx}[\arccos x]\cdot(5\sin x + 2x^2)$$

$$= \boxed{(5\cos x + 4x)\arccos x + \frac{-1}{\sqrt{1-x^2}}(5\sin x + 2x^2)}$$

OR

$$= \boxed{(5\cos x + 4x)\arccos x - \frac{5\sin x + 2x^2}{\sqrt{1-x^2}}}$$

(b) $h(x) = \frac{e^{3x}+8x^5}{\arctan\left(\frac{x}{4}\right)}$

$\boxed{\text{Quotient Rule / Chain Rule}}$

$$h'(x) = \frac{\arctan\left(\frac{x}{4}\right)\frac{d}{dx}[e^{3x}+8x^5] - (e^{3x}+8x^5)\frac{d}{dx}\left[\arctan\left(\frac{x}{4}\right)\right]}{\arctan^2\left(\frac{x}{4}\right)}$$

$$= \frac{\arctan\left(\frac{x}{4}\right)\left(\overbrace{e^{3x}\frac{d}{dx}[3x]}^{\text{Chain Rule}} + 40x^4\right) - (e^{3x}+8x^5)\cdot\overbrace{\frac{1}{1+(x/4)^2}\frac{d}{dx}\left[\frac{x}{4}\right]}^{\text{Chain Rule}}}{\arctan^2(x/4)}$$

$$= \boxed{\frac{\arctan\left(\frac{x}{4}\right)(3e^{3x}+40x^4) - (e^{3x}+8x^5)\cdot\frac{1}{(1+x^2/16)}\cdot\frac{1}{4}}{\arctan^2(x/4)}}\ \cdot\frac{4}{4}$$

$$\text{OR} = \boxed{\frac{\arctan\left(\frac{x}{4}\right)(3e^{3x}+40x^4) - (e^{3x}+8x^5)\cdot\frac{4}{16+x^2}}{\arctan^2(x/4)}}$$

(c) $f(x) = \arcsin^3(5x)$ $= (\underbrace{\arcsin(5x)}_{\text{inside function}})^3$

Chain Rule

$$f'(x) = 3(\arcsin(5x))^2 \frac{d}{dx}[\arcsin(\underbrace{\boxed{5x}}_{\text{inside}})]$$

Apply the chain rule.

$$= 3\arcsin^2(5x) \cdot \frac{1}{\sqrt{1-(5x)^2}} \underbrace{\frac{d}{dx}[5x]}_{5}$$

Apply the chain rule.

$$= \boxed{3\arcsin^2(5x) \cdot \frac{1}{\sqrt{1-(5x)^2}} \cdot 5}$$

OR $= \boxed{\dfrac{15\arcsin^2(5x)}{\sqrt{1-25x^2}}}$

Simplified.

Skill #: D18
Score:

INTEGRITY STATEMENT:
On my personal integrity, I have not given, nor received, nor witnessed any unauthorized assistance on this exam.

____
(signature)

If you can't sign this in good conscience, please don't. Come speak to me.

Skill #: G1
Score:

Skill #: G2
Score: