Name: Key　　　　　　Date: November 7, 2018　　　　　　Class Time:______

Analytic Geometry & Calculus I | Tulsa Community College

Quiz #8: L'Hopital's Rule, Concavity & Curve Sketching, and Applied Optimization

Remember to get full credit, you need to show all work, clearly and neatly. Remember, this isn't just about you getting the answer, but you *showing someone else* how you got the answer.

CALCULATOR POLICY: You *may* use a calculator on this assessment. I want you to easily test points in your sign analyses! However, with the exception testing points for your sign analyses, you must do **all other work algebraically to be eligible for scores of 3 or higher**.

1. Evaluate the following limits. Use of L'Hopital's Rule may or may not be necessary.

(a) $\lim_{x\to 0} \frac{8x^2}{\cos x - 1}$　　"$\frac{0}{0}$" indeterminate form　　Recognize the limit as a $\frac{0}{0}$ ind. form.

$\stackrel{H}{=} \lim_{x\to 0} \frac{\frac{d}{dx}[8x^2]}{\frac{d}{dx}[\cos x - 1]}$　　Apply L'Hopital's Rule.

$= \lim_{x\to 0} \frac{16x}{-\sin x}$　"$\frac{0}{0}$"　　Recognize the limit as a $\frac{0}{0}$ ind. form.

$\stackrel{H}{=} \lim_{x\to 0} \frac{\frac{d}{dx}[16x]}{\frac{d}{dx}[-\sin x]}$　　Apply L'Hopital's Rule.

$= \lim_{x\to 0} \frac{16}{-\cos x} = \frac{16}{-\cos 0} = \boxed{-16}$　　Evaluate the limit.

(b) $\lim_{x\to 0^+} x^2 \ln x$　　"$0 \cdot -\infty$" indet. form　　"$0 \cdot -\infty$" indet. form.

$= \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x^2}}$　"$\frac{-\infty}{\infty}$"　　Write as $\frac{-\infty}{\infty}$ indet. form.

$\stackrel{H}{=} \lim_{x\to 0^+} \frac{\frac{d}{dx}[\ln x]}{\frac{d}{dx}[x^{-2}]} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{-2x^{-3}}$　　Apply L'Hopital's Rule.

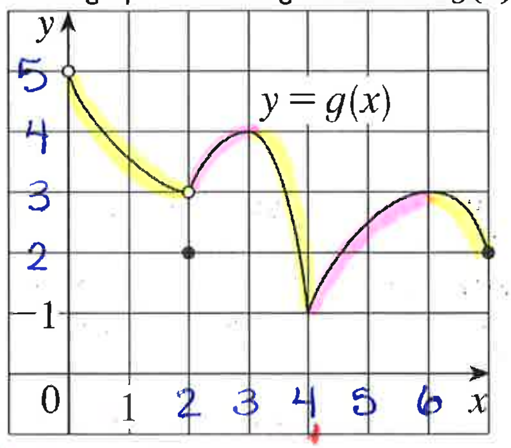
$= \lim_{x\to 0^+} \frac{1}{x} \cdot \frac{-x^3}{2}$　　Simplify algebraically

$= \lim_{x\to 0^+} -\frac{1}{2}x^2 = -\frac{1}{2}(0)^2 = \boxed{0}$　　Evaluate the limit.

Skill #: A20
Score:

2. **Part I:** Analyze the function below and answer the questions in interval notation, when appropriate. You may have to estimate points.

This is a graph of the original function $g(x)$



(a) On which open interval(s) is the function increasing?

$(2,3), (4,6)$

y-values go up from left to right

(b) On which open interval(s) is the function decreasing?

$(0,2), (3,4), (6,7)$

y-values go down from left to righ

(c) On which open interval(s) is the graph of the function concave upward?

$(0,2)$

Recall: concave upward ⇒ ↺

(d) On which open interval(s) is the graph of the function concave downward?

$(2,4), (4,7)$

Recall: Concave downward ⇒

(e) Estimate the x-coordinate of all inflection points, if any exist.

$x = 2$

because the function switches from concave upward to concave downward at this x-value.

(f) What is/are the relative maxima of the function on [0, 7]? State both the value and the x-coordinate where it occurs. If the function has an absolute maximum on [0, 7], identify which of the relative maxima is the absolute maximum.

$y=4$, at $x=3$

$y=3$, at $x=6$

The function does not have an absolute max. on [0, 7], because of the open circle at $(x,y) = (0,5)$.

(g) What is/are the relative minima of the function on [0, 7]? State both the value and the x-coordinate where it occurs. If the function as an absolute minimum on [0, 7], identify which of the relative minima is the absolute minimum.

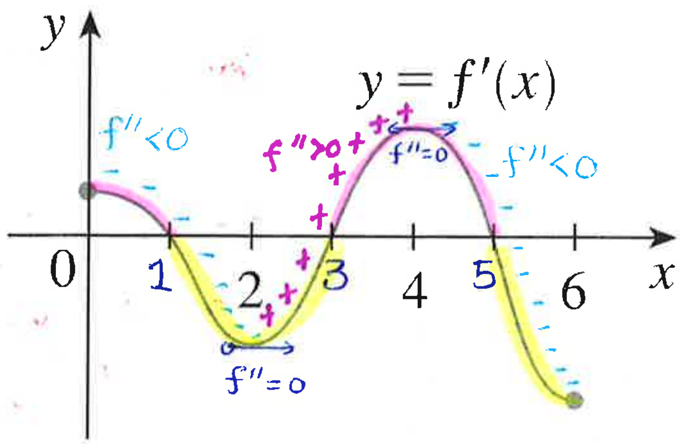
$y = 2$ at $x = 2$

$y = 1$ at $x = 4$ ← Abs. minimum.

$y = 2$ at $x = 7$

Part II. Analyze the function below and answer the questions in interval notation. PAY ATTENTION TO FACT THAT THE GRAPH IS **A GRAPH OF THE DERIVATIVE**.

This is a graph of the DERIVATIVE of $f(x)$



(a) On which open interval(s) is the function $f(x)$ increasing?

f is increasing when $f'(x)$ is positive, so f is increasing on $(0,1)$ and $(3,5)$

(b) On which open interval(s) is is the function $f(x)$ decreasing?

f is decreasing when f' is negative, so f is decreasing on $(1,3)$ and $(5,6)$.

(c) On which open interval(s) is the graph of the function $f(x)$ concave upward?

f is concave upward when $f''>0 \Leftrightarrow f'$ is increasing; so f is concave upward on $(2,4)$.

(d) On which open interval(s) is the graph of the function $f(x)$ concave downward?

f is concave downward when $f''<0 \Leftrightarrow f'$ is decreasing; so f is concave downward on $(0,2)$ & $(4,6)$.

(e) Estimate the x-coordinate of the inflection points of $f(x)$, if any exist.
Hint: You might want to make a sign chart for f'' in order to answer this question.

f'': − + − on 0, 2, 4, 6

f'' changes sign at $x=2$, & $x=4$, so those are the x-coordinates of the points of inflection.

(f) Estimate the x-coordinate(s) corresponding to relative maximum(a) of the function $f(x)$ on (0, 6).
Hint: You might want to make a sign chart for f' in order to answer this question and use the same sig chart to answer part (g).

f': + − + − on 0, 1, 3, 5, 6

f has a relative max at $x=1$ & $x=5$ by the 1st derivative test.

(g) Estimate the x-coordinate(s) corresponding to relative minimum(a) of the function $f(x)$ on (0, 6).

f has a rel. min. at $x=3$ by the 1st derivative test (See sign chart in (f))

3. Find the intervals where the following function is increasing, decreasing, concave up, and concave down. Find all relative maxima and minima and points of inflection (if any), and put your answers on the table on the next page. Be sure to perform sign analyses of f' and f''. Then sketch the graph of the function on the next page.

$$f(x) = -3x^4 + 16x^3 - 18x^2$$

Compute f'.

$$f'(x) = -12x^3 + 48x^2 - 36x$$
$$= -12x(x^2 - 4x + 3)$$
$$= -12x(x-3)(x-1)$$

Factor f'.

$f'(x) = 0$ when $\frac{-12x}{-12} = \frac{0}{-12}$, $x - 3 = 0$, & $x - 1 = 0$

$x = 0$; $\quad x - 3 + 3 = 0 + 3$, $x = 3$; $\quad x - 1 + 1 = 0 + 1$, $x = 1$

Set factors $= 0$ & solve for x to find critical values.

	$(-\infty, 0)$	$(0, 1)$	$(1, 3)$	$(3, \infty)$
f'	+ ↗	− ↘	+ ↗	− ↘
$-12x$	+	−	−	−
$x-3$	−	−	−	+
$x-1$	−	−	+	+

$\Rightarrow$ f is increasing on $(-\infty, 0)$ & $(1, 3)$ & f is decreasing on $(0, 1)$ & $(3, \infty)$.

Make a sign chart. Use sign chart to determine intervals where f is increasing or decreasing.

$$f'' = \frac{d}{dx}[-12x^3 + 48x^2 - 36x]$$
$$= -36x^2 + 96x - 36$$
$$= -12(3x^2 - 8x + 3)$$

Compute f''.

$f'' = 0$ when $3x^2 - 8x + 3 = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-(-8) \pm \sqrt{(-8)^2 - 4(3)(3)}}{2(3)}$$
$$= \frac{8 \pm \sqrt{64 - 36}}{6}$$
$$= \frac{8}{6} \pm \frac{2\sqrt{7}}{6} = \frac{4}{3} \pm \frac{\sqrt{7}}{3} \Rightarrow x = \frac{4 - \sqrt{7}}{3} \text{ \& } x = \frac{4 + \sqrt{7}}{3}.$$

$\Leftrightarrow x \approx 0.45$ & $x \approx 2.22$

Set $f'' = 0$ & solve for x.

(When in doubt, use the quadratic formula.)

Sign chart for f''.

f'': $-$ | $+$ | $-$

-12 $-$ $\frac{4-\sqrt{7}}{3}$ $-$ $\frac{4+\sqrt{7}}{3}$ $-$

$3x^2-8x+3$ $+$ $-$ $+$

Concave up: $\left(\frac{4-\sqrt{7}}{3}, \frac{4+\sqrt{7}}{3}\right)$

Concave down: $\left(-\infty, \frac{4-\sqrt{7}}{3}\right) + \left(\frac{4+\sqrt{7}}{3}, \infty\right)$

$f(0) = -3(0)^4 + 16(0)^3 - 18(0)^2 = 0$

$f(1) = -3(1)^4 + 16(1)^3 - 18(1)^2 = -5$

$f(3) = -3(3)^4 + 16(3)^3 - 18(3)^2 = 27$

$f\left(\frac{4-\sqrt{7}}{3}\right) = -3\left(\frac{4-\sqrt{7}}{3}\right)^4 + 16\left(\frac{4-\sqrt{7}}{3}\right)^3 - 18\left(\frac{4-\sqrt{7}}{3}\right)^2 \approx -2.32$

$f\left(\frac{4+\sqrt{7}}{3}\right) = -3\left(\frac{4+\sqrt{7}}{3}\right)^4 + 16\left(\frac{4+\sqrt{7}}{3}\right)^3 - 18\left(\frac{4+\sqrt{7}}{3}\right)^2 \approx 13.36$

Find y-values at each rel. min, rel. max, & point of inflection.

PUT ANSWERS HERE & use your answers to sketch the graph below.

increasing: $(-\infty, 0)$ & $(1, 3)$

decreasing: $(0, 1)$ & $(3, \infty)$

concave up: $\left(\frac{4-\sqrt{7}}{3}, \frac{4+\sqrt{7}}{3}\right) \approx (0.45, 2.22)$

concave down: $\left(-\infty, \frac{4-\sqrt{7}}{3}\right)$ & $\left(\frac{4+\sqrt{7}}{3}, \infty\right)$

relative min.: -5 at $x = 1$

relative max.: 0 at $x = 0$; 27 at $x = 3$

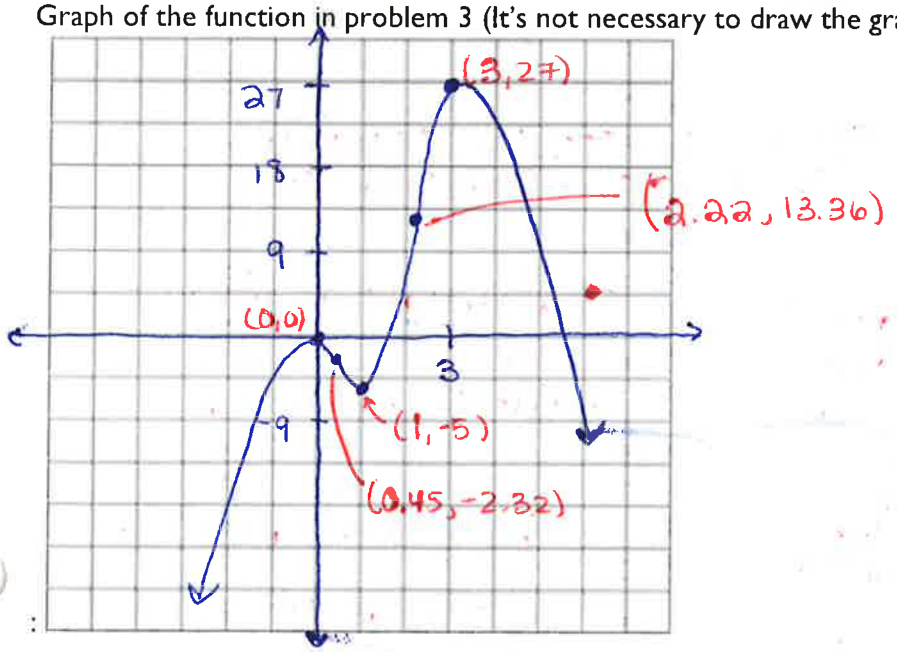
absolute min.: None

absolute max.: 27 at $x = 3$

point(s) of inflection: $\left(\frac{4-\sqrt{7}}{3}, \approx -2.32\right) \approx (0.45, -2.32)$

& $\left(\frac{4+\sqrt{7}}{3}, \approx 13.36\right) \approx (2.22, 13.36)$

Graph of the function in problem 3 (It's not necessary to draw the graph to scale.)



Skill #: A24
Score:

4. You have been contracted to design and build a squared-based, open-top, rectangular steel tank. The tank is to be made by welding steel plates together along their edges. As a production engineer, your job is to find the dimensions for the base and height of the tank that make the tank weigh as little as possible (which means you want to use [redacted] as little [redacted] steel as possible [redacted] to make the tank). What dimensions do you tell the shop to use to construct the tank, if the tank is required to hold 1000 ft³ ?

Hint: You may want to use the quadratic formula. The solutions of $ax^2 + bx + c = 0$ are $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.
(There is more space on the next page.)

Note: This problem came directly from the HW.

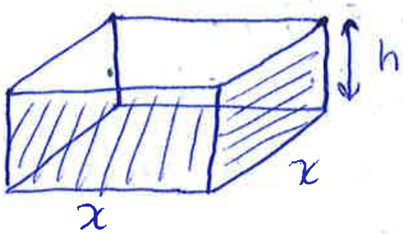
Given: volume held = 1000 ft3

minimize surface area

Square-based, open-top

State given info.

Draw a picture.



$V = x^2 h = 1000$ Constraint equation

$S = x^2 + 4x\,h$ Objective function

$\frac{\cancel{x^2} h}{\cancel{x^2}} = \frac{1000}{x^2}$ Solve constraint equation for h as a function of x.

$S(x) = x^2 + 4\cancel{x}\left(\frac{1000}{x^{\cancel{2}}}\right)$ Substitute h into S to write S as a function of x.

$S(x) = x^2 + 4000x^{-1}$

$S'(x) = 2x - 4000x^{-2}$

$= 2x - \frac{4000}{x^2} \overset{\text{set}}{=} 0$

Find crit. points.
① Compute S'.
② Set S' = 0 & solve for x.

$$\left(2x - \frac{4000}{x^2}\right)x^2 = 0 \cdot x^2$$

Clear fractions.

$$2x^3 - \cancel{4000} = 0$$
$$+\cancel{4000} \quad +4000$$

Isolate x^3 term.

$$\frac{\cancel{2}x^3}{\cancel{2}} = \frac{4000}{2}$$

$$\sqrt[3]{x^3} = \sqrt[3]{2000}$$

Take the cube root.

$$x = 10\sqrt[3]{2} \approx 12.6 \text{ ft.}$$

$$S''(x) = \frac{d}{dx}\left[2x - 4000x^{-2}\right]$$

Determine whether the crit value represents a min or a max.

$$= 2 + 8000x^{-3}$$

$$= 2 + \frac{8000}{x^3}$$

You can use the 1st deriv. test or the 2nd deriv. test. I used the 2nd deriv. test.

$$S''(\sqrt[3]{2000}) = 2 + \frac{8000}{(\sqrt[3]{2000})^3}$$

$$= 2 + \frac{8000}{2000} = 6 > 0$$

Compute h.

$\Rightarrow$

State dimensions.

$\Rightarrow$ $x = \sqrt[3]{2000} = 10\sqrt[3]{2} \approx 12.6$ ft corresponds to a minimum.

$$\Rightarrow h = \frac{1000}{(\sqrt[3]{2000})^2} = \frac{10}{\sqrt[3]{4}} \approx 6.3 \text{ ft}$$

$\Rightarrow$ $\boxed{\text{Dimensions: } 12.6' \times 12.6' \times 6.3'}$

INTEGRITY STATEMENT:

On my personal integrity, I have not given, nor received, nor witnessed any unauthorized assistance on this exam.

____

(signature)

If you can't sign this in good conscience, please don't. Come speak to me.

Skill #: G1
Score:

Skill #: [redacted]
Score: A23