Unit 1 Lesson 0 Lesson Notes: An Overview of Calculus

Purpose

In this lesson, we'll discuss what calculus allows us to do, as well as two problems that differentiate calculus from algebra, geometry, and trigonometry.

Lesson Outcomes

By the end of this lesson, you will

- Articulate the relationship between calculus, algebra, and geometry.
- Engage your problem-solving skills through
    - Investigating a historical problem, estimating the value of π, and
    - Investigating one case of the tangent line problem, to find the velocity of an object at an instant.

Textbook Sections and Other Resources

The outcomes of our course are not defined by a textbook. However, there are many excellent resources that can supplement the material you'll be studying in our course. The official textbook used by our college at this time for this course is the Larson and Edwards calculus text. This lesson is related to material found in

- Section 2.1, Larson and Edwards' *Calculus: Early Transcendental Functions*, 7^{th} edition, and
- Section 2.1, Stewart's Calculus Early Transcendentals, 7^{th} edition.

We will also be reading from *Infinite Powers* by Steven Strogatz. Related material includes the introduction, chapter 1, and chapter 2.

Lesson Notes

Introduction: Without Calculus, and With Calculus

Calculus is a beautiful subject. It's been said that calculus takes the static objects and numbers we studied in arithmetic, algebra, geometry, and trigonometry, and makes them dynamic.

Speed at an Instant

Without calculus, we can compute an average rate of change. If I travel 50 miles over the course of one hour, my average speed for that time interval is 50 miles per hour. We calculated that by computing

$$Average\ Speed = \frac{Distance\ Travelled}{Change\ in\ Time} = \frac{50\text{ miles}}{1\text{ hour}} = 50\text{ mph}.$$

That's easy enough, but what if I wanted to know my speed at an instant? What if I wanted to calculate what my speedometer should read? This may seem like a simple idea, but it amounts to taking a tiny change in distance and dividing it by a tiny time interval, the length of which would be approaching zero. For example, if I want to know my speed at time $t = 2$ hours, I might compute my average speed from $t = 2$ hours to $t = 3$ hours, and then $t = 2$ hours to $t = 2.5$ hours, and then $t = 2$ hours to $t = 2.1$ hours. As the time interval gets smaller and smaller, the distance traveled over that time interval gets smaller and smaller too – so I end up with a quantity that is nearly zero divided by a time interval that is nearly zero. We know from precalculus that we can't do that! It's not legal to divide by zero. And please don't tell me that 0/0 is 1, because we can't cancel the zeros, and get one either. In calculus, we call such a quantity an indeterminate form. Is the quantity that is approaching 0/0 equal to 0, or is it infinite? Is it a finite number? Our physical intuition and life experience tells us that our instantaneous speed at $t = 2$ hours is not zero, if we were moving at all at that time, and it's not infinity. We weren't going infinitely fast at that instant. It's much more likely that we were travelling at a reasonable speed, bound by the speed limits in our hometown. If we drove 200 miles over the course of 4 hours, our average speed is still

$$Average\ Speed = \frac{Distance\ Travelled}{Change\ in\ Time} = \frac{200 \text{ miles}}{4 \text{ hour}} = 50 \text{ mph},$$

but at exactly $t = 2$ hours, if we were still on the road, we were probably traveling at some speed in the vicinity of 50. Maybe we were on a highway at that time, and we were driving 75 miles per hour, on our way to Dallas, Texas, or maybe we passed through a small town, and we had to slow way down to the local speed limit which was somewhere near 30 miles per hour. We can't be sure, but if we were on the road and we were moving, we can be sure that we didn't have an instantaneous speed of zero miles per hour, and we didn't have an instantaneous speed of infinity. (If we had, we certainly would have felt that!) In this case, our speed, which can be represented by the 0/0 indeterminate form was approaching some finite value. We might call that a finite *limiting* value. We can't be sure what that value is without calculus, but we can compute an approximation, by considering what happens to our average speed as the time interval gets shorter and shorter. If our measurements are accurate, the calculations of our average speed over shorter and shorter time intervals should reflect the reality of what we see on the speedometer.

People say that calculus is dynamic whereas precalculus is static, because we can imagine what happens as the time interval repeatedly shrinks. We can almost see it, like we're flipping through the pages of a flip book, watching the patterns change.

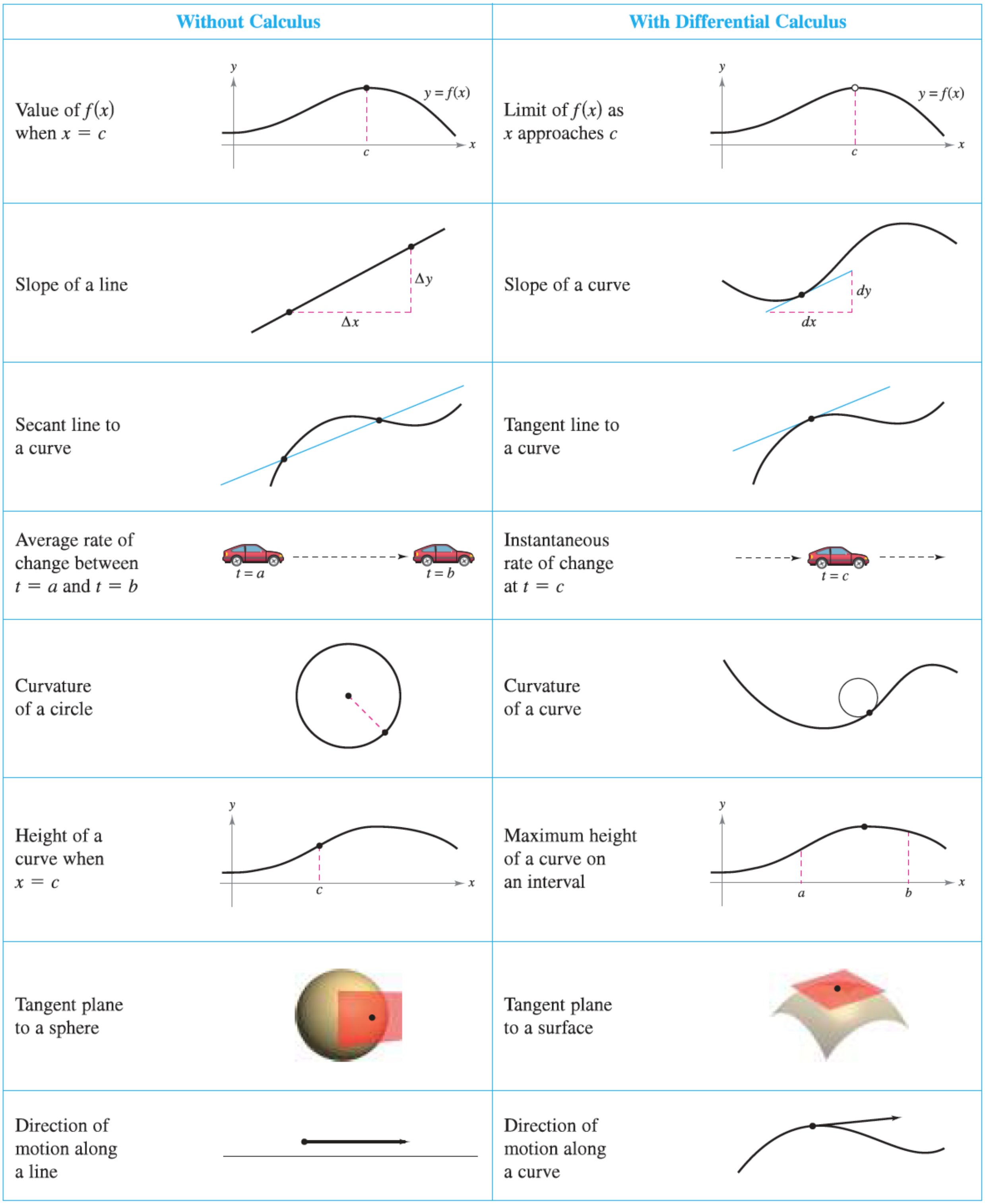
Approximating Pi

We also see this idea when we consider Archimedes trying to find the value of a beautiful ratio, the ratio of the circumference to a circle to its diameter. We have come to know this ratio as π. Approximating this value is difficult, because it's difficult to measure distance or perimeter of some 2D object when the edges are curved. Here is what Archimedes did. In calculus today, we would call it numerical approximation of a limit, but in Archimedes's day, they called it the **method of exhaustion**. Archimedes inscribed polygons in circles, and computed their area, and used that area as an approximation for the area of a circle. We'll do something similar – we'll use perimeter to estimate circumference. Of course, this consistently yields an underestimate for the circumference, because the shortest distance between two points is a straight line, and the circle takes a roundabout way from one point to another. Knowing that his inscribed polygons would yield an underestimate of the area, he also computed an overestimate. Rather than inscribing polygons in a circle, he began inscribing his circle inside of the polygons. We'll start with inscribed and circumscribed hexagons, and find their perimeters, but if we want to be more accurate, we need regular polygons that have *more sides*. If we want to be completely accurate (perfect!), if we want to know the *exact* value of that ratio of the circumference to the diameter, we need polygons with infinitely many sides, with each side having an infinitely tiny length. Steven Strogatz discussed this more thoroughly in chapter 2 in *Infinite Powers*.

This limiting process, in which we take something that we can compute using known formulas and ideas such as the Law of Cosines, the length of a line segment, the perimeter of a regular polygon, and something as simple as average speed, and push it to the limit until it can't be pushed anymore, is the essence of what calculus is about. In the first chapter, we'll study limits. We'll see that limits make the rest of calculus possible. Limits make it possible to find the rate of change of some function instantaneously, and find areas of regions with curved boundaries, among other things.

What can we do with differential calculus?

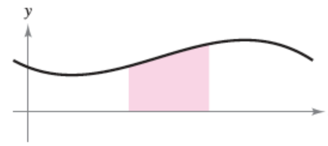
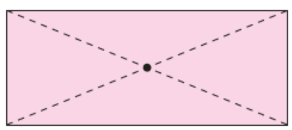
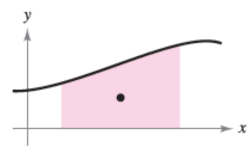
Calculus is generally split into two parts: differential and integral calculus. The following image came from the Larson Calculus text, 9th edition, and shows you what differential calculus can do. We'll use limits to prepare us for differential calculus. Keep these applications in mind as you read Steven Strogatz's interpretation of differential calculus.



Without Calculus	With Differential Calculus
Value of $f(x)$ when $x = c$	Limit of $f(x)$ as x approaches c
Slope of a line	Slope of a curve
Secant line to a curve	Tangent line to a curve
Average rate of change between $t = a$ and $t = b$	Instantaneous rate of change at $t = c$
Curvature of a circle	Curvature of a curve
Height of a curve when $x = c$	Maximum height of a curve on an interval
Tangent plane to a sphere	Tangent plane to a surface
Direction of motion along a line	Direction of motion along a curve

What can we do with integral calculus?

The following image came from the Larson Calculus text, 9th edition, as well, and shows you what integral calculus can do. We'll introduce integral calculus after differential calculus, and we will show the surprising connection between the two branches of calculus. Keep these ideas in mind as you read Steven's interpretation of integral calculus. Can you see how integral calculus might be appropriate for these problems?

Without Calculus		With Integral Calculus	
Area of a rectangle		Area under a curve	
Work done by a constant force		Work done by a variable force	
Center of a rectangle		Centroid of a region	
Length of a line segment		Length of an arc	
Surface area of a cylinder		Surface area of a solid of revolution	
Mass of a solid of constant density		Mass of a solid of variable density	
Volume of a rectangular solid		Volume of a region under a surface	
Sum of a finite number of terms	$a_1 + a_2 + \cdots + a_n = S$	Sum of an infinite number of terms	$a_1 + a_2 + a_3 + \cdots = S$

Active Learning Activity and Reading Assignment

We're going to take these two ideas, and explore them a bit further, using knowledge of average rate of change, the perimeter of a regular polygon, the Law of Cosines, and our scientific calculators. In the process, we're going to be practicing what we call *finding a limit numerically*. I'll also have you read, and reflect on your readings, as described on the handout.

In Unit 1, we will focus extensively on limits. If you want to understand intuitively how limits come about and why they're important, hopefully, this exercise will help you to understand, while encouraging you to exercise your problem-solving skills and review skills, facts, and concepts that are necessary for success in our course.