Unit 1 Lesson 3 Detailed Lesson Notes: The Epsilon-Delta Definition of a Limit

Purpose

We've discussed the intuitive definition of a limit, and how to compute limits graphically, numerically, and algebraically. We said that "the limit of $f(x)$ as x approaches a is L" means that as x gets arbitrary close to a, the function values $y = f(x)$ get arbitrarily close to L. If you wondered, "What do you mean by close?" then this lesson is for you. We'll also look at a little of the history in an article called "Who Gave You the Epsilon? Cauchy and the Origins of Rigorous Calculus" by Judith V. Grabiner from the *American Mathematical Monthly*.

These detailed lesson notes exist to supplement the notes in the PowerPoint. This is a typed version of the lecture that goes with the U1 L3 Lesson Notes and Practice Problems PowerPoint.

Lesson Outcomes

By the end of this lesson, you will be able to

- State the epsilon-delta definition of a limit, and explain it in your own words,
- Find a δ for a given ε, given a limit,
- Prove that the limit of a given function as x approaches a given a is L, using the epsilon-delta definition of a limit.

Materials That You'll Need

Before you get started, read the following handouts.

- U1 L3 Detailed Lesson Notes – The Epsilon-Delta Definition of a Limit
- U1 L3 Lesson Notes and Practice Problems PowerPoint – The Formal Definition of a Limit
- "Who Gave You The Epsilon? Cauchy and the Origins of Rigorous Calculus" by Judith V. Grabiner,
- A printable version of this lesson plan.

You'll also need access to HW #1. You may also want to read the section about the formal definition of a limit in Section 2.2 of our text.

Lesson Notes

We've discussed the intuitive definition of a limit, and how to compute limits graphically, numerically, and algebraically. We have said that "the limit of $f(x)$ as x approaches a is L" means that as x gets arbitrary close to a, the function values $y = f(x)$ get arbitrarily close to L. If you've wondered, "What do you mean by close?" then this lesson is for you. Here's the formal definition:

For every $\varepsilon > 0$ (no matter how small), there is a $\delta > 0$ such that
if $0 < |x - a| < \delta$, then $|f(x) - L| < \varepsilon$.

If this initially looks like Greek, it's because it *is* Greek(!) but it's also relatively easy to understand if you think about what each piece of that statement means.

Let's take a look at $|f(x) - L| < \varepsilon$. From precalculus, you know that $|x| < 5$ means that $-5 < x < 5$. Similarly, $|f(x) - L| < \varepsilon$ means that

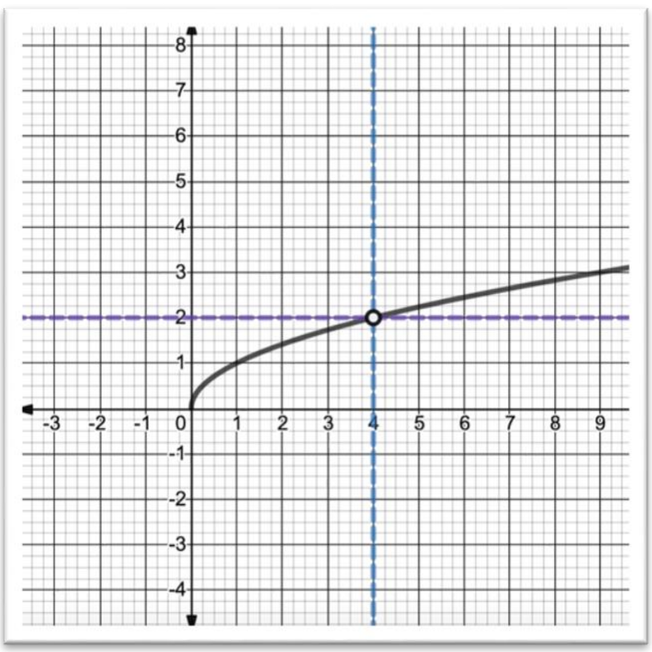
$$-\varepsilon < f(x) - L < \varepsilon,$$

or, if we add L to all three parts, we get

$$L - \varepsilon < f(x) < L + \varepsilon.$$

Other words, the values of the function $y = f(x)$ are between $L - \varepsilon$ and $L + \varepsilon$. Visually, this means we're considering the y-values on the graph of the function between the horizontal lines $y = L - \varepsilon$ and $y = L + \varepsilon$.

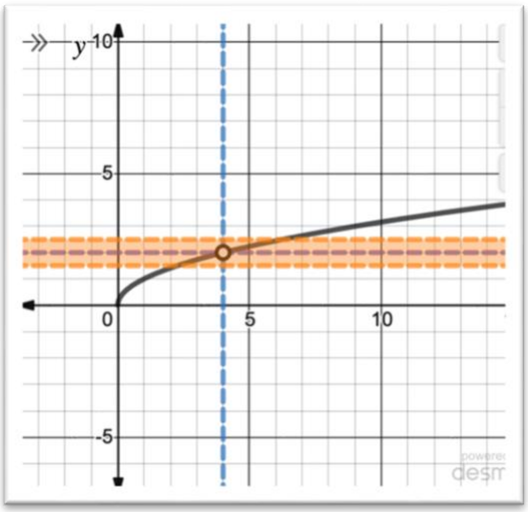
Consider the following limit: $\lim_{x \to 4} f(x) = 2$. Here, $L = 2$ and $a = 4$. There are many functions that satisfy this. Just so that we have a visual, consider the function f shown in the picture below. The function pictured is not defined at $x = 4$, but otherwise, it looks just like $y = \sqrt{x}$. To help draw our attention to both the limiting values a and L, the blue vertical line with equation $x = a = 4$ has been added to the graph, and a purple horizontal line with equation $y = L = 2$ has been sketched as well. All of these images were created using the interactive graph linked here at Desmos.com.



In our example, $|f(x) - L| = |f(x) - 2| < \varepsilon$ is equivalent to the inequality

$$2 - \varepsilon < f(x) < 2 + \varepsilon.$$

In other words, f lies within ε units of 2. Visually, this means that the values of $y = f(x)$ lie in the orange band shown below. The lower horizontal line has the equation $y = 2 - \varepsilon$, for a given ε, and the upper horizontal line is given by $y = 2 + \varepsilon$. From the picture, it looks like $\varepsilon \approx \frac{1}{2}$.



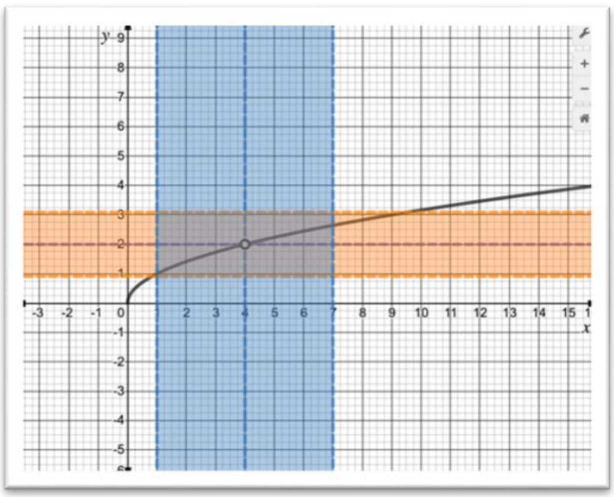
Similarly, $|x - a| < \delta$ means that $-\delta < x - a < \delta$, or equivalently,

$$a - \delta < x < a + \delta.$$

Graphically, this means that our x-values lies between the vertical lines given by $x = a - \delta$ and $x = a + \delta$. In our example, we have $a = 4$, so x lies within δ units of 4.

$$4 - \delta < x < 4 + \delta$$

This means that the x-values lie within the blue band shown in the image below. The vertical line on the left is given by the equation $x = 4 - \delta$, and the vertical line on the right is given by $x = 4 + \delta$, for a specific value of δ. Notice that both vertical lines are an equal distance δ from the vertical line $x = 4$. From the picture, it looks like $\delta \approx 3$, because the band includes x-values that extend from $4 - \delta = 4 - 3 = 1$, to $4 + \delta = 4 + 3 = 7$.



Note that $0 < |x - a|$ just tells us that the absolute value of $(x - a)$ is positive. Initially, that might not seem very insightful. Isn't the absolute value always positive?! If you said yes, you'd be *mostly right*. The absolutely value is always *nonnegative*. In other words, the absolute value is usually positive, except when it's equal to zero. The inequality $0 < |x - a|$ just means that $|x - a| \neq 0$, or in other words, $x \neq a$.

That actually makes a lot of sense when we're discussing limits, because when we're considering the limit as x approaches a, we're only concerned about the y-values on the graph as x gets arbitrarily close to a. We aren't concerned with the function value when $x = a$.

So, putting the pieces together, the inequality $0 < |x - a| < \delta$ means that x lies within δ units of a, and $x \neq a$.

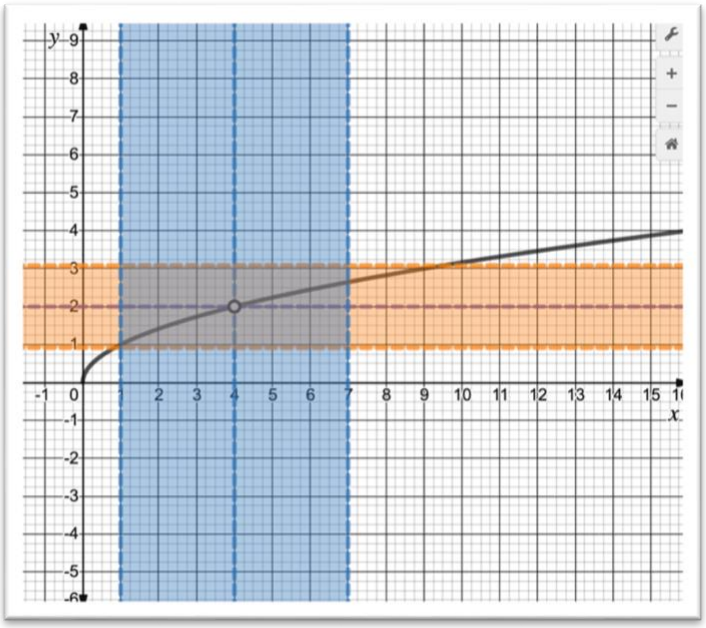
Now that we understand what the two inequalities mean, let's return to the ε-δ definition. It reads

For every $\varepsilon > 0$ (no matter how small), there is a $\delta > 0$ such that
if $0 < |x - a| < \delta$, then $|f(x) - L| < \varepsilon$.

Let's focus on the highlighted part of the sentence. In other words, given an epsilon,

there is a $\delta > 0$ such that if x is within δ units of a and $x \neq a$, then $f(x)$ is within ε units of L.

This means that for any $\varepsilon > 0$, I can choose δ so that the x-values in the blue band, with the exception of $x = a$, all map to the y-values in the orange band.



For $\varepsilon = 1$, an appropriate δ was already chosen above. Do you see how making the interval around $x = 4$ (represented by the blue band) wider would result in y-values outside the orange band? If we made the blue band wider, it would include x-values less than 1. The y-values corresponding to those x-values lie below the orange band. BUT, as long as we stay within the blue band, all of the y-values are between 1 and 3. Thus, it looks like if $\varepsilon = 1$, the corresponding δ is $\delta = 3$.

If $1 < x < 7$ and $x \neq 4$, then $1 < f(x) < 3$.

or in other words,

if $|x - 4| < 3$ and $x \neq 4$, then $|f(x) - 2| < 1$,

or in other words,

if $0 < |x - 4| < 3$, then $|f(x) - 2| < 1$,

or, in still other words,

If $0 < |x - 4| < \delta$, then $|f(x) - 2| < \varepsilon$, where $\varepsilon = 1$ and $\delta = 3$.

At this point, many students see the bands and they understand what I'm saying, but they still don't see how this is equivalent to the statement that

as x gets arbitrarily close to $a = 4$, the function values of $y = f(x)$ get arbitrarily close to $L = 2$.

The key is the very beginning of the definition. Let's look at it again.

For every $\varepsilon > 0$ (*no matter how small*), there is a $\delta > 0$ such that
if x is within δ units of a and $x \neq a$, then $f(x)$ is within ε units of L.

We saw that we could find a δ for $\varepsilon = 1$. The definition above states that if $\lim_{x \to a} f(x) = L$, then we can find the corresponding δ for *any* ε.

Remember, ε determines the width of the horizontal orange band. The end of the definition says, "$f(x)$ is within ε units of L." What if we want to be within $\varepsilon = 0.5$ units of 2? Well, if the limit of the function at the x-value we're interested in is 2, we can find the appropriate δ. What if we want to be within $\varepsilon = 0.0001$ units of 2? In other words, what if we want $1.999 < f(x) < 2.0001$? We can find the appropriate δ for that ε too. No matter how small ε is, or in other words, no matter how narrow we make that orange band, I can always find a blue band of the appropriate width, so that all of the x-values within the blue band, except possibly $x = 4$, map to the y-values in the orange band.

As ε shrinks, both the orange band and the corresponding blue band shrink, so that we're zeroing in on the point $(a, L) = (4, 2)$. This point may or may not actually lie on the graph, but we get closer and closer to that point as ε and the corresponding δ get smaller and smaller. In other words,

for every $\varepsilon > 0$ (no matter how small), there is a $\delta > 0$ such that
if x is within δ units of a and $x \neq a$, then $f(x)$ is within ε units of L,

means

as x gets arbitrarily close to a, the function values $y = f(x)$ get arbitrarily close to L.

The rigorous definition given by Cauchy, and the intuitive definition given at the beginning of the semester agree. Cauchy's definition answers the question, "What do you mean arbitrarily close?" The epsilon measures how close we want $f(x)$ and L to be, and epsilon can be as small as we want it to be! Delta measures how close x and a need to be so that $f(x)$ is within ε units of L.

Notation

If you're saying to yourself, "That's a lot of words to write," mathematicians have an answer for that too. We have notation, a shorthand, that allows us to say the same thing symbolically.

For every $\varepsilon > 0$ (no matter how small), there is a $\delta > 0$ such that
if x is within δ units of a and $x \neq a$, then $f(x)$ is within ε units of L,

can be written this way:

$$\forall \varepsilon > 0 \; \exists \delta > 0 \;\; \text{s.t.} \;\; 0 < |x - a| < \delta \Longrightarrow |f(x) - L| < \varepsilon.$$

Each symbol and its meaning are shown in the table below.

Symbol	Meaning
$\forall \varepsilon > 0$	For every $\varepsilon > 0$
$\exists \delta > 0$	there exists a $\delta > 0$
s.t.	such that
$P \Longrightarrow Q$	**If** P is true, **then** Q is true, or P **implies** Q
$0 < \|x - a\| < \delta$	x is within δ units of a and $x \neq a$
$\|f(x) - L\| < \varepsilon$	$f(x)$ is within ε units of L
$0 < \|x - a\| < \delta \Longrightarrow \|f(x) - L\| < \varepsilon$	**if** x is within δ units of a and $x \neq a$, **then** $f(x)$ is within ε units of L.

Example 1:
From our studies of continuous functions, we know that $\lim_{x \to 2}(3x - 1) = 5$. Fill in the table below by finding the appropriate δ for each ε.

$\boldsymbol{\varepsilon}$	$\boldsymbol{\delta}$
1	
0.5	
0.1	
0.01	
0.001	
General ε	

Solution:
We know that $\lim_{x \to 2}(3x - 1) = 5$ is true, based on our studies of continuous functions. Now we want to prove that its true using the ε-δ definition of the limit, at least for the given epsilons.

Let's start with $\varepsilon = 1$. Since the statement is true, by definition, we can assume the following.

For $\varepsilon = 1$, there is a $\delta =$ ______ (to be determined by us), such that if $0 < |x - 2| < \delta$, we're guaranteed that $|f(x) - L| = |(3x - 1) - 5| < \varepsilon = 1$.

To find this δ, we need to establish a relationship between $|x - 2| < \delta$ and $|3x - 6| < \varepsilon = 1$. What would δ need to be so that $|x - 2| < \delta$ implied that $|3x - 6| < 1$? To figure that out, let's manipulate the last inequality. The inequality $|3x - 6| < 1$ is equivalent to

$$3|x - 2| < 1,$$

which is equivalent to

$$|x-2| < \frac{1}{3}.$$

That wasn't too bad! This is telling us that if $|x-2| < \frac{1}{3}$, I can multiply both sides of the inequality by 3 and get $|3x-6| < 1$, so if we choose $\delta = \frac{1}{3}$, we'll have

$$0 < |x-2| < \delta = \frac{1}{3} \Longrightarrow |3x-6| < \varepsilon = 1.$$

In other words, for $\varepsilon = 1$, choose $\delta = \frac{1}{3}$. Then, if $0 < |x-2| < \delta = \frac{1}{3}$, we're guaranteed that $|3x-6| < \varepsilon = 1$.

Let's do it again, this time for $\varepsilon = 0.5$, or ½.

For $\varepsilon = \frac{1}{2}$, there is a $\delta =$ ______ (to be determined by us), such that if $0 < |x-2| < \delta$, we're guaranteed that $|f(x) - L| = |(3x-1) - 5| < \varepsilon = \frac{1}{2}$.

We need to establish a relationship between $|x-2| < \delta$ and $|3x-6| < \varepsilon = 1/2$. What would δ need to be so that $|x-2| < \delta$ implied that $|3x-6| < \frac{1}{2}$? To figure that out, let's manipulate the last inequality. The inequality $|3x-6| < \frac{1}{2}$ is equivalent to

$$3|x-2| < \frac{1}{2},$$

which is equivalent to

$$|x-2| < \frac{1}{2}\left(\frac{1}{3}\right) = \frac{1}{6}.$$

Thus, if $|x-2| < \frac{1}{6}$, I can multiply both sides of the inequality by 3 and get $|3x-6| < \frac{1}{2}$, so if we choose $\delta = \frac{1}{6}$, we'll have

$$0 < |x-2| < \delta = \frac{1}{6} \Longrightarrow |3x-6| < \varepsilon = \frac{1}{2}.$$

In other words, for $\varepsilon = \frac{1}{2}$, choose $\delta = \frac{1}{6}$. Then, if $0 < |x-2| < \delta = \frac{1}{6}$, we're guaranteed that $|3x-6| < \varepsilon = \frac{1}{2}$.

I think I see the pattern. What if it were any $\varepsilon > 0$?

We need to establish a relationship between $|x-2| < \delta$ and $|3x-6| < \varepsilon$. What would δ need to be so that $|x-2| < \delta$ implied that $|3x-6| < \varepsilon$? To figure that out, let's manipulate the last inequality. The inequality $|3x-6| < \varepsilon$ is equivalent to

$$3|x-2| < \varepsilon$$

which is equivalent to

$$|x-2| < \frac{\varepsilon}{3},$$

so if δ is $\frac{1}{3}$ of ε, then $0 < |x-2| < \delta$ implies that $|3x-6| < \varepsilon$.

Using this formula for δ, we can fill in the other values in our table.

$\boldsymbol{\varepsilon}$	$\boldsymbol{\delta}$
1	$\frac{1}{3}$
$0.5 = \frac{1}{2}$	$\frac{1}{6}$
$0.1 = \frac{1}{10}$	$\frac{1}{30}$
$0.01 = \frac{1}{100}$	$\frac{1}{300}$
$0.001 = \frac{1}{1000}$	$\frac{1}{3000}$
General ε	$\frac{\varepsilon}{3}$

Since we know what δ should be for any given ε, we can prove that $\lim_{x \to 2}(3x-1) = 5$ using the formal definition.

Proof.
Given any $\varepsilon > 0$. Choose $\delta = \frac{\varepsilon}{3}$.
Assume $0 < |x-2| < \delta$.
Then $|(3x-1)-5| = |3x-6| = 3|x-2| < 3\left(\frac{\varepsilon}{3}\right) = \varepsilon$.

Example 2: Use the formal definition of a limit to prove that $\lim_{x\to 1} x^2 = 1$.

To show that $\lim_{x\to 1} x^2 = 1$, given any $\varepsilon > 0$, we need to find a $\delta > 0$ so that when $0 < |x-1| < \delta$, we're guaranteed that $|x^2 - 1| < \varepsilon$. The final structure of our proof is shown below. We just need to find the right δ.

> Let $\varepsilon > 0$. Choose $\delta =$ _______. Assume $0 < |x-1| < \delta =$ ________.
> Then $|x^2 - 1| = |x-1||x+1| < \varepsilon$.

We need to choose δ so that $|x-1||x+1|$ is less than ε.

Notice that we immediately assume that $|x-1| < \delta$. That will ultimately help us. We need to focus on the factor $|x+1|$. I need an upper bound for this as well.

To find an upper bound for $|x+1|$, let's choose $\delta = 1$. Then $|x-1| < \delta$ means that we're within one unit of $a = 1$, so $0 < x < 2$. Add one to all three parts of this inequality, to get $1 < x+1 < 3$. Then, we can say that $|x+1| < 3$. Note that this only holds for $\delta = 1$ though. We need it to hold for any δ.

If we let $\delta = \min\left(1, \frac{\varepsilon}{3}\right)$, then $\delta \le 1$ *and* $\delta \le \frac{\varepsilon}{3}$, so if $|x-1| < \delta$, we have $|x-1| < \frac{\varepsilon}{3}$ and $|x+1| < 3$. That's excellent! I think we have enough to finish our proof.

> Let $\varepsilon > 0$. Choose $\delta = \min\left(1, \frac{\varepsilon}{3}\right)$. Assume $0 < |x-1| < \delta = \min\left(1, \frac{\varepsilon}{3}\right)$.
> Then $|x-1| < \frac{\varepsilon}{3}$ and $|x+1| < 3$, so we have
> $|x^2 - 1| = |x-1||x+1| < \left(\frac{\varepsilon}{3}\right)(3) = \varepsilon$.
> By definition, we have $\lim_{x\to 1} x^2 = 1$.

"How did you know to choose $\delta = \min\left(1, \frac{\varepsilon}{3}\right)$?"

Many students read the example above, and ask, "How did you know to choose $\delta = \min\left(1, \frac{\varepsilon}{3}\right)$?" The short answer is that I've seen proofs like this before. The longer answer is that I knew that the product $|x-1||x+1|$ had to be less than epsilon, and that when $\delta = 1$, I had $|x+1| < 3$.

I knew that I needed $|x-1| < \frac{\varepsilon}{3}$ so that the product $|x-1||x+1|$ would be less than epsilon. I also knew that I needed to choose a δ so that $|x-1| < \frac{\varepsilon}{3}$ and $|x+1| < 3$ *at the same time*.

One way to guarantee that both would be true is to choose $\delta = \min\left(1, \frac{\varepsilon}{3}\right)$. Then $\delta \le 1$ and $\delta \le \frac{\varepsilon}{3}$, and we obtain the desired results for $|x-1|$ and $|x+1|$.