Basic Antidifferentiation Rules

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$$\int kf(x)\,dx = k\int f(x)\,dx$$

$$\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$$

$$\int k\,dx = kx + C$$

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C, \qquad \text{where } n \neq -1$$

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int e^x\,dx = e^x + C$$

$$\int a^x\,dx = \frac{1}{\ln a}a^x + C$$

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \arcsin\left(\frac{x}{a}\right) + C$$

$$\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C$$

$$\int \frac{1}{x\sqrt{x^2 - a^2}}\,dx = \frac{1}{a}\operatorname{arcsec}\left(\frac{|x|}{a}\right) + C$$

$$\int \sin x\,dx = -\cos x + C$$

$$\int \cos x\,dx = \sin x + C$$

$$\int \sec^2 x\,dx = \tan x + C$$

$$\int \sec x \tan x\,dx = \sec x + C$$

$$\int \csc^2 x\,dx = -\cot x + C$$

$$\int \csc x \cot x\,dx = -\csc x + C$$

$$\int \tan x\,dx = -\ln|\cos x| + C$$

$$\int \cot x\,dx = \ln|\sin x| + C$$

$$\int \sec x\,dx = \ln|\sec x + \tan x| + C$$

$$\int \csc x\,dx = -\ln|\csc x + \cot x| + C$$

Sometimes we'll need to use algebra or trigonometry to fit the integrand into one of the forms given above. Common techniques include

1. Expanding expressions of the form $(binomial\ or\ trinomial)^2$
2. Separating the numerator and evaluating two integrals
3. Completing the square
4. Using long division when antidifferentiating rational functions. Do this when the rational function is improper – that is, when degree of the polynomial in the numerator is greater than or equal to the degree of the polynomial in the denominator.
5. Add a well-chosen zero to the numerator (this often allows us to avoid using long division)
6. Use trigonometric identities
7. Multiply and divide by the Pythagorean conjugate
8. Multiply and divide by an exponential function
9. Use the symmetry properties of the integrand.

<u>The Goal</u>: To fit the integral into one of the forms above.

Facts from Algebra that You Need to Know

Expression in the Integrand	**Algebraic rewrite** *(so that we can use the power rule for antidifferentiation)*	**Algebraic Rule**
$\frac{1}{x^3}$	$\frac{1}{x^3} = x^{-3}$	$\frac{1}{x^n} = x^{-n}$
$\sqrt{x}$	$\sqrt{x} = \sqrt[2]{x^1} = x^{1/2}$	$\sqrt[n]{x^m} = x^{m/n}$
$\frac{5}{6x^3}$	$\frac{5}{6} \cdot \frac{1}{x^3} = \frac{5}{6}x^{-3}$	$\frac{A}{Bx^n} = \frac{A}{B}x^{-n}$
$-\frac{2\sqrt[3]{x^2}}{5}$	$-\frac{2}{5} \cdot \sqrt[3]{x^2} = -\frac{2}{5}x^{2/3}$	$\frac{Ax^n}{B} = \frac{A}{B}x^n$
$\frac{1}{4x^2}$	$\frac{1}{4} \cdot \frac{1}{x^2} = \frac{1}{4}x^{-2}$	$\frac{A}{Bx^n} = \frac{A}{B}x^{-n}$
$-\frac{x^7}{8}$	$-\frac{1 \cdot x^7}{8} = -\frac{1}{8}x^7$	$\frac{Ax^n}{B} = \frac{A}{B}x^n$
$\frac{6}{x}$	$\frac{6}{1 \cdot x} = \frac{6}{1} \cdot \frac{1}{x} = 6\left(\frac{1}{x}\right)$	Since the power of x is -1, we want to write it as $\frac{1}{x}$ so that we can use the $\int \frac{1}{x}dx$ rule.

Expression	**Algebraic rewrite**	**Algebraic Rule**
$(x^5 + 2x)^2$	$(x^5 + 2x)(x^5 + 2x)$ $= x^{10} + 2x^6 + 2x^6 + 4x^2$ $= x^{10} + 4x^6 + 4x^2$	**FOIL/distribution**
$\frac{3e^{2x}}{4e^x}$	$\frac{3e^{2x}}{4e^x} = \frac{3}{4}e^{2x-x} = \frac{3}{4}e^x$	**Exponent properties**: $\frac{x^n}{x^m} = x^{n-m}$
x^3x^5	$x^3x^5 = x^{3+5} = x^8$	$x^nx^m = x^{n+m}$
$(2^3)^n$	$(2^3)^n = 2^{3n}$	$(x^n)^m = x^{nm}$
$\left(3(x+1)\right)^n$	$(3(x+1))^n = 3^n(x+1)^n$	$(ab)^n = a^nb^n$
$\left(\frac{2x-1}{5}\right)^n$	$\left(\frac{2x-1}{5}\right)^n = \frac{(2x-1)^n}{5^n}$	$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$
$\frac{9e^{4x} - 6e^{2x} + 1}{4e^x}$	$= \frac{9e^{4x}}{4e^x} - \frac{6e^{2x}}{4e^x} + \frac{1}{4e^x}$ $= \frac{9}{4}e^{3x} - \frac{3}{2}e^x + \frac{1}{4}e^{-x}$	**Dividing by a "monomial" + exponent properties**

Note: You can separate terms in the numerator but not the denominator.

$$\frac{1}{x^2+1} \neq \frac{1}{x^2} + \frac{1}{1}.$$

Note: If the x in one of the rules above is replaced by a "kx", then the antidifferentiation rule is almost identical. We simply divide by the constant k. Here are two examples:

$$\int e^{-2x}\,dx = -\frac{1}{2}e^{-2x} + C$$

$$\int \tan(3x)\,dx = -\frac{1}{3}\ln|\cos(3x)| + C$$

Examples:

Let's try to identify the basic integration rule needed for each of these integrals. We will not find the antiderivatives of these functions in class, but we'll identify the technique necessary to find the result. If you're unsure of exactly how/why we would use these techniques, try using the techniques to prove to yourself that the technique works.

Integral	Technique, Basic Rules	Solution
$\int \frac{4}{x^2+9}dx$	Direct application of antiderivative of $\frac{1}{a^2+x^2}$, with a $= 3$	$\frac{4}{3}\arctan\left(\frac{x}{3}\right) + C$
$\int \tan^2(2x)\ dx$	Rewrite the integrand in terms of $\sec^2(2x)$	$\frac{1}{2}\tan(2x) - x + C$
$\int \frac{4}{1-\sin x}\ dx$	Multiply and divide by the Pythagorean conjugate of the denominator, $1+\sin x$, and then use trigonometric identities.	$4(\sec x + \tan x) + C$
$\int_{-\pi/3}^{\pi/3} x^2 \sin x\, dx$	Since the integrand is an odd function and the interval of integration is of the form $[-a, a]$, the integral is 0.	0

The following antiderivatives require the use of u-substitution and/or algebraic manipulation:		
$\int \frac{4x}{x^2+9}\ dx$	u-substitution with $u = x^2 + 9$ and antiderivative of $1/u$	$2\ln(x^2+9) + C$
$\int \frac{x+3}{\sqrt{4-x^2}}\ dx$	Split/separate the fraction into two fractions, and antidifferentiate each part, using u-substitution for the first fraction and the antiderivative of $1/\sqrt{a^2-x^2}$ for the second fraction.	$-(4-x^2)^{\frac{1}{2}} + 3\arcsin\left(\frac{x}{2}\right) + C$
$\int \frac{1}{3+4e^{2x}}\ dx$	Multiply the numerator and denominator by e^{-2x}. Then use u-substitution.	$-\frac{1}{6}\ln(3e^{-2x}+4) + C$

u-Substitution Review

A technique for integrating composite functions

If finding the derivative of function requires the chain rule, then taking the antiderivative of the result will require u-substitution.

Example:

For example, $\frac{d}{dx}[(1+x^2)^4] = 4(1+x^2)^3\, 2x$.

Now if we want to find the antiderivative below, we have to use u-substitution.

$$\int 4(1+x^2)^3\, 2x\, dx$$

If we let $u = 1 + x^2$, then $du = 2x\, dx$.

The integral above can now be rewritten entirely in terms of u, and the resulting integrand is easily integrated using the basic integration rules.

$$\int 4u^3\, du = u^4 + C$$

The original integrand was given in terms of x, so we should give the result in terms of x. The substitution u was only necessary so that we could evaluate the integral.

Thus, the antiderivative can be found by replacing u with $1 + x^2$.

$$\int 4(1+x^2)^3\, 2x\, dx = \int 4u^3\, du = u^4 + C = (1+x^2)^4 + C.$$

In general,

$$\int f'\big(g(x)\big)\, g'(x)\, dx = f\big(g(x)\big) + C$$

To evaluate this integral, we look for an "inside function", and then let that inside function be u.
For the integral above, we choose $u = g(x)$. Then $du = g'(x)\, dx$, so that the antiderivative is evaluated in the following steps.

$$\int f'\big(g(x)\big) g'(x)\, dx = \int f'(u)\, du = f(u) + C = f\big(g(x)\big) + C.$$