Name: Key **Date:** ____________________

Honors Analytic Geometry & Calculus I | Spring 2020

Unit 1 Lesson 0 Active Learning Activity: An Overview of Calculus

Purpose

In this activity we'll explore two problems that differentiate calculus from algebra, geometry, and trigonometry. One problem is related to differential calculus, and is often referred to as the tangent line problem. It amounts to finding the rate of change of a function at an instant. The second problem is related to integral calculus, as well as evaluating limits numerically. We're going to explore something similar to what Archimedes explored. He was approximating π.

Approximating Pi

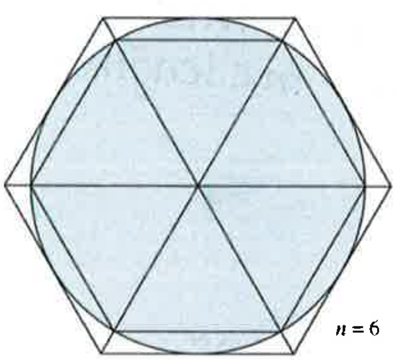
Suppose we wanted to find an approximation of the ratio of the circumference of a circle to its diameter. It turns out that no matter the radius of the circle, the ratio of these is always the same. We call that ratio π. This seems easy now. We have formulas for the circumference ($C = 2\pi r$) and diameter ($d = 2r$). But how might we derive these formulas, and define that ratio when the curved boundary of the circle makes computing length so difficult? If you take Calculus II, you'll use calculus to derive a formula for the length of a curve in that course. With precalculus, the best we can do is find the perimeters of polygons.

Finding perimeters of regular polygons is relatively easy. We can find the perimeter of any polygon by

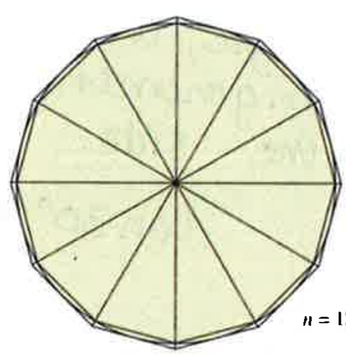
- subdividing it into triangular regions,
- finding the lengths of the sides on those triangles on the outermost edge (those that comprise the perimeter), and
- adding the side lengths together.

If I want to find the length all the way around a circle, I'm going to have to be more creative. I'd really like to place a string on the boundary of the circle, and then stretch it and measure it using a tape measure, but that's not convenient here, or as precise as I would prefer to be. While we can't find the exact length of a curve, we can approximate it, using line segments whose lengths we can calculate.

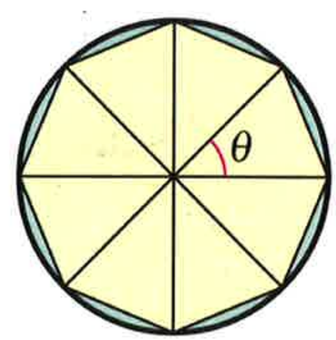
Archimedes used a numerical equivalent of the limiting process to approximate the area of a circle. His method is called the method of exhaustion. He used two polygons, one inscribed in the circle (so its area is smaller than that of the circle) and one circumscribed about the region (so that its area was larger), to zero in on the true area. He knew that the area of the circle would be squeezed between these two areas for given regular polygon with n sides. With two hexagons, he was able to find an upper bound and a lower bound for the area of the circle. He also knew that he could be even more accurate if he used polygons with more sides. If he could somehow create a polygon with infinitely many sides, that had infinitely small side lengths, and find *its* area, he knew he would have computed the exact area of the circle.



Inscribed and circumscribed hexagons
(The number of sides is $n = 6$.)



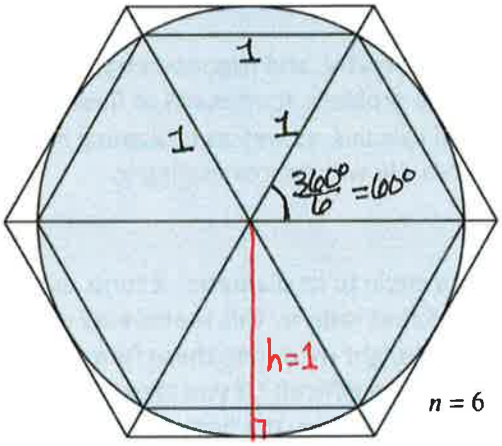
Inscribed and circumscribed dodecagons
(The number of sides is $n = 12$.)



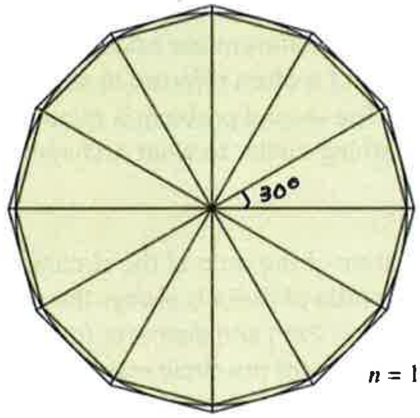
A visual clue of what you might do next

These images were taken from the 9th edition of the Larson Calculus text.

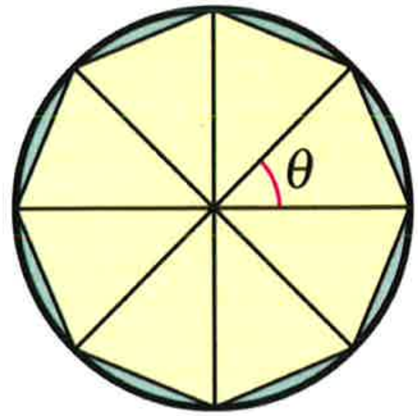
We're going to do something similar to approximate π. Rather than trying to find the area of the circle by finding areas of polygons, we're going to approximate the circumference of the circle with the perimeters of polygons instead. For simplicity, let's assume that the radius of the circle is one.



Inscribed and circumscribed hexagons
(The number of sides is $n = 6$.)



Inscribed and circumscribed dodecagons
(The number of sides is $n = 12$.)



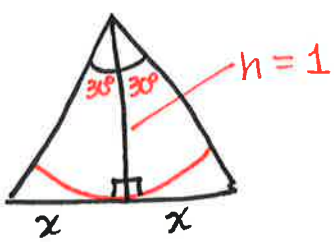
A visual clue of what you might do next

(a) For $n = 6$, find the perimeter of the inscribed and circumscribed hexagons. These are lower and upper bounds for the circumference of the circle, since the closest distance between two points is a straight line.

Since the central angle is $\theta = \frac{360^\circ}{6} = 60^\circ$, the triangles in the inscribed hexagon are equilateral triangles. Since $r = 1$, the side lengths of the triangles are 1-1-1, and the perimeter of the inscribed hexagon is

$$P_6 = 6 \times (\text{side length}) = 6(1) = 6.$$

For the circumscribed hexagon, the triangles are equilateral triangles, but the side length is longer. Since the radius of the circle is 1, $h = 1$. Since we know the height & the central angle, we can use right triangle trigonometry to find the length of one side.



$$\tan 30^\circ = \frac{x}{1}$$

$$\Rightarrow \text{side} = 2x = 2\tan 30^\circ$$

$$\Rightarrow P_6 = 6 \times (\text{side}) = 12 \tan 30^\circ$$

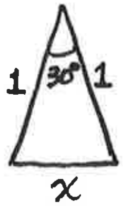
$$= 4\sqrt{3} \approx 6.928.$$

(b) Repeat (a) for $n = 12$. Do you see a pattern?

(Question is at the bottom of page 2.)

You may need to use the Law of Cosines: $c^2 = a^2 + b^2 - 2ab\cos C$.

For $n=12$, the central angle is $\theta = \frac{360°}{12} = 30°$.



For the inscribed polygon, we can use the Law of Cosines to find the length of the 3rd side:

$$x^2 = 1^2 + 1^2 - 2\cdot 1\cdot 1\cos 30°$$

$$x^2 = 2 - \cancel{2}\left(\frac{\sqrt{3}}{\cancel{2}}\right) = 2-\sqrt{3}.$$

$$\Rightarrow \quad x = \sqrt{2-\sqrt{3}}$$

$$\Rightarrow \quad P_{12} = 12 \times (\text{side}) = 12\sqrt{2-\sqrt{3}} \approx 6.212$$

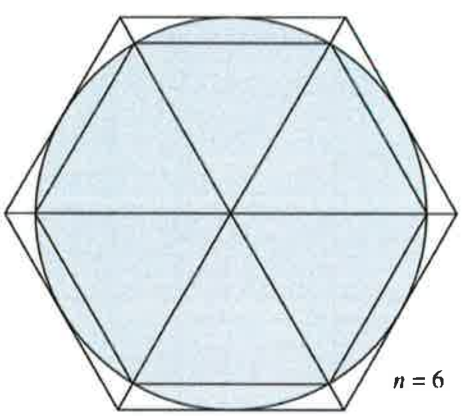
For the circumscribed polygon, we use the same approach we used for $n=6$.

$$\tan 15° = \frac{\text{opp}}{\text{adj}} = \frac{x}{1}$$

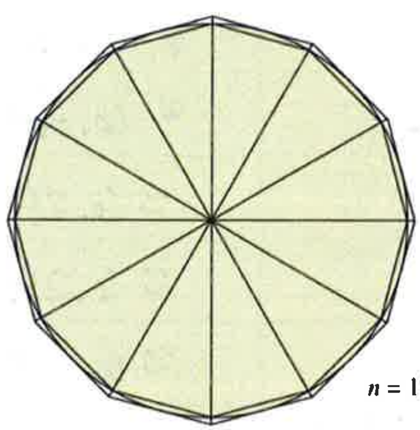
$$\Rightarrow \text{side} = 2x = 2\tan 15°$$

$$\Rightarrow P_{12} = 12 \cdot \text{side} = 24\tan 15°$$

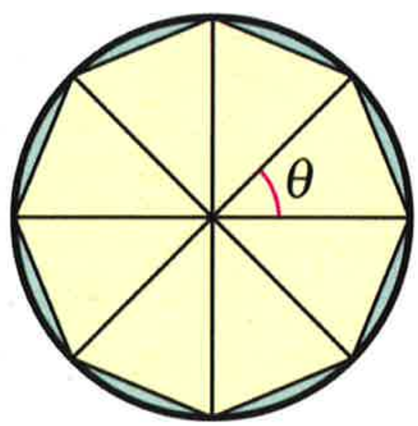
$$= 48 - 24\sqrt{3} \approx 6.431$$



Inscribed and circumscribed hexagons (The number of sides is $n = 6$.)



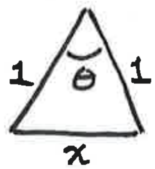
Inscribed and circumscribed dodecagons (The number of sides is $n = 12$.)



A visual clue of what you might do next

(c) Find the perimeter of both the inscribed and circumscribed polygons as a function of the number of sides n. You're looking for an equation. You shouldn't have a value this time. Instead, you should have a formula involving n. You're just doing exactly what you did in parts (a) and (b) with $n = 6$, and $n = 12$, but with a general number of sides n. Let p_n represent the perimeter of the smaller polygon, and let P_n represent the perimeter of the larger polygon.

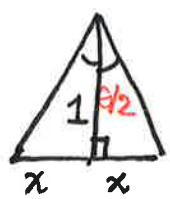
Central $\angle = \theta = \frac{360^\circ}{n}$ or $\frac{2\pi}{n}$ in radians.



By the law of cosines,

$$x^2 = 1^2 + 1^2 - 2 \cdot 1 \cdot 1 \cos\theta$$
$$x^2 = 2 - 2\cos\left(\tfrac{2\pi}{n}\right)$$
$$x = \sqrt{2 - 2\cos\left(\tfrac{2\pi}{n}\right)}$$
$$\Rightarrow \boxed{p_n = n \cdot \text{side} = n\sqrt{2 - 2\cos(2\pi/n)}}$$

For the inscribed polygons



$$\tan\left(\tfrac{\theta}{2}\right) = x$$
$$\Rightarrow \text{side} = 2x = 2\tan\left(\tfrac{\theta}{2}\right) = 2\tan\left(\tfrac{\pi}{n}\right)$$
$$\Rightarrow \boxed{P_n = n \cdot \text{side} = 2n\tan\left(\tfrac{\pi}{n}\right)}$$

For the circumscribed polygo

(d) Use your formula to fill in the table below.

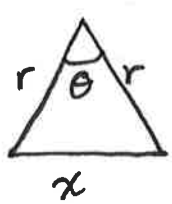
# of sides	p_n	P_n
6	6	≈ 6.928
12	≈ 6.212	≈ 6.431
36	≈ 6.275	≈ 6.299
48	≈ 6.279	≈ 6.292
96	≈ 6.282	≈ 6.285
n	$n\sqrt{2-2\cos(2\pi/n)}$	$2n\tan(\pi/n)$

(e) Use the perimeters you found to state inequalities for the circumference, and the inequalities for π.

# of sides	Bounds for C: $p_n < C < P_n$	Bounds for π: $\frac{p_n}{2r} < \frac{C}{2r} < \frac{P_n}{2r}$
6	$6 < C < 6.928$	$3 < \pi < 3.464$
12	$6.212 < C < 6.431$	$3.106 < \pi < 3.216$
36	$6.275 < C < 6.299$	$3.138 < \pi < 3.150$
48	$6.279 < C < 6.292$	$3.140 < \pi < 3.146$
96	$6.282 < C < 6.285$	$3.141 < \pi < 3.143$
n	$n\sqrt{2-2\cos(\frac{2\pi}{n})} < C < 2n\tan(\frac{\pi}{n})$	$\frac{n\sqrt{2-2\cos(2\pi/n)}}{2} < \pi < n\tan(\frac{\pi}{n})$

Define $\pi = \lim_{n\to\infty} \frac{p_n}{2r} = \lim_{n\to\infty} \frac{P_n}{2r}$.

(f) How would your formulas change if r were not equal to 1?



By the law of cosines,

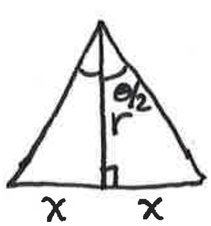
$$x^2 = r^2 + r^2 - 2r\cdot r\cos\theta$$

$$\Rightarrow \text{side} = \sqrt{2r^2 - 2r^2\cos\theta}$$

$$= r\sqrt{2-2\cos\theta}$$

$$\Rightarrow \boxed{p_n = n\cdot\text{side} = nr\sqrt{2-2\cos(2\pi/n)}}$$

Perimeter of the Inscribed Polygon



$$\tan\left(\frac{\theta}{2}\right) = \frac{\text{opp}}{\text{adj}} = \frac{x}{r} \Rightarrow x = r\tan\left(\frac{\theta}{2}\right)$$

$$\Rightarrow \text{side} = 2x = 2r\tan(\theta/2)$$

$$\Rightarrow \boxed{P_n = n\cdot\text{side} = 2nr\tan(\theta/2)}$$

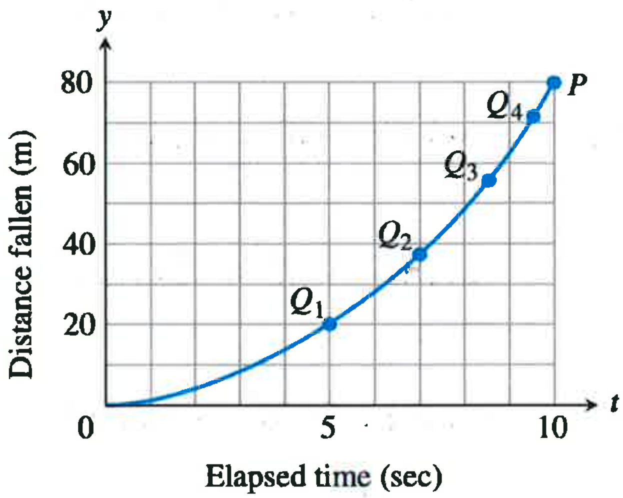
Perimeter of the Circumscribed Polygon

Velocity at an Instant

The accompanying figure shows the plot of distance fallen versus time for an object that fell from the lunar landing module a distance 80 m to the surface of the moon.

a. Estimate the slopes of the secants PQ_1, PQ_2, PQ_3, and PQ_4, arranging them in a table like the one in Figure 2.6.

b. About how fast was the object going when it hit the surface?



This is problem 16 in Thomas' Calculus: Early Transcendentals, section 2.1. Complete the table below, and use it to estimate how fast the object was going when it hit the surface of the moon.

P	Q	$Average\ Speed = \frac{\Delta s}{\Delta t} = \frac{Change\ in\ Distance}{Change\ in\ Time}$
(10, 80)	(5, 20)	$\frac{\Delta s}{\Delta t} = \frac{80-20}{10-5} \approx \frac{60\text{ m}}{5\text{ s}} = 12$ m/s
(10, 80)	(7, 38)	$\frac{\Delta s}{\Delta t} = \frac{80-38}{10-7} \approx 14$ m/s
(10, 80)	(8.6, 56)	$\frac{\Delta s}{\Delta t} = \frac{80-56}{10-8.6} \approx 17.14$ m/s
(10, 80)	(9.5, 72)	$\frac{\Delta s}{\Delta t} = \frac{80-72}{10-9.5} = \frac{8}{0.5} = 16$ m/s

(b) Approximately how fast was the object going when it hit the surface?

≈ 16 m/s

(** Your answers may be different, depending on the estimates you made of the coordinates of the points shown on the graph. As long as your answers are similar, you're on the right track.)

Reading and Reflection

Read the lesson notes, and the introduction and first chapter of *Infinite Powers.* Complete the active learning activity in this handout. Then answer the questions below. Write or type your answers, attach your completed active learning handout, and submit both the active learning activity and your answers to these conceptual questions at the beginning of class on **Thursday, January 23**.

If you can answer these questions, you know you've achieved the goals of this lesson. It's okay if this feels less precise and scientific, and more reflective. There will be plenty of other opportunities for precision in throughout the course. Our goal is for you to articulate your initial impressions of calculus.

1. What is the primary difference between calculus and precalculus?

2. *Infinite Powers* describes the difference between differential and integral calculus. In your own words, what are the differences between the two? The similarities?

3. Steven talks about the infinity principle in his book, and my notes tend to emphasize a limit process. In your own words, describe what Steven seemed to mean by the infinity principle, and then do the same thing for my discussion of limits. After considering Steven's interpretation and my interpretation, what similarities do you see? Differences?

4. You've considered the opinions of others. Take the time to craft a definition of your own. In your own words, what does calculus seem to be about? Imagine that you are visiting with a friend or loved one, discussing your first week of class, and describing this new subject you're studying. How would you describe calculus for them?