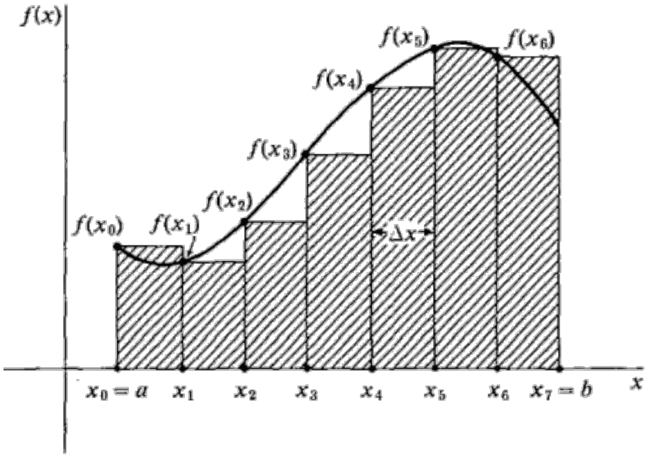
The Riemann Sum



$$\sum_{k=1}^{n} f(x_k^*)\,\Delta x_k^*$$

Suppose we have an arbitrary function $f(x)$ defined on the closed interval $[a, b]$. The function f may have negative as well as positive values.

We **subdivide $[a, b]$ into n subintervals**, not necessarily of equal widths, by choosing $n - 1$ points $x_1, x_2, x_3, \dots, x_{n-1}$ on the interior of the interval, so that

$$a = x_0 < x_1 < x_2 < x_3 < \cdots < x_{n-1} < x_n = b$$

We call the set $P = \{ x_0, x_1, \dots, x_n\}$ a **partition** of $[a, b]$, because it divides $[a, b]$ into n closed subintervals $[x_0, x_1], [x_1, x_2], \dots, [x_{n-1}, x_n]$. The kth subinterval is the interval $[x_{k-1}, x_k]$, where $k = 1, \dots, n$. The **width** of the kth subinterval is denoted $\Delta x_k = x_k - x_{k-1}$.

On each subinterval, select some x-value, and label it x_k^*. Then, on that subinterval, draw a rectangle that stretches from the x-axis to the function value $f(x_k^*)$. Then, form the product $f(x_k^*)\,\Delta x_k^*$ for each subinterval. This product will be positive when $f(x_k^*)$ is positive, negative when $f(x_k^*)$ is negative, and zero when $f(x_k^*) = 0$.

The sum below is called the **Riemann sum** for f on the interval $[a, b]$.

$$f(x_1^*)\,\Delta x_1^* + f(x_2^*)\,\Delta x_2^* + \cdots + f(x_n^*)\,\Delta x_n = \sum_{k=1}^{n} f(x_k^*)\,\Delta x_k^*.$$

It is the sum of the areas of the rectangles above the x-axis, minus the areas of the rectangles of below the x-axis. It is an approximation of the “net area” (area above the x-axis minus the area below the x-axis) between the curve and the x-axis on $[a, b]$.

A Better Approximation

The norm of the partition P, written $\|\Delta\|$ or $\|P\|$, is the largest of all the widths of the subintervals. If we let $\|P\| \to 0$, the rectangles become thinner. If all of the subintervals have the same width, when $\|P\| \to 0$, the number of rectangles $n \to \infty$.