Kato-Ponce Inequality With $A_{\vec{P}}$ Weights

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Abstract

We prove the Kato-Ponce inequality (normed Leibniz rule) for multiple factors in the setting of *multiple weights* ($A_{\vec{P}}$ weights). This improves existing results to the product of m factors and extends the class of known weights for which the inequality holds.

1 Introduction

In the study of Euler and Navier-Stokes equations, Kato and Ponce [7] obtained normed Leibniz rules for fractional derivatives of the form

$$\|J^s(fg)\|_{L^p} \leqslant C_{n,s,p_1,p_2}\left(\|J^s f\|_{L^{p_1}}\|g\|_{L^{p_2}} + \|f\|_{L^{p_1}}\|J^s g\|_{L^{p_2}}\right) \tag{1.1}$$

where $f, g \in \mathcal{S}(\mathbb{R}^n)$, $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$ with some conditions on s, p_1, p_2. Soon after their initial work, Kato-Ponce (denoted KP) inequalities have been used in a variety of applications related to PDE. In view of the great range of applications, it became apparent such inequalities soon merited their own study. Incremental improvements in range of the smoothness index s and the integrablity indices p_1, p_2 were made [1], [6], [3], [2], [11]. For a detailed history of the development of the KP inequality see [10]. The cumulative contribution of these works gave (1.1) with $1 \leqslant p_1, p_2 \leqslant \infty$ and $s > n(1/\min(1,r) - 1)$ or $s \in 2\mathbb{N}$ being the largest range of s. Remarkably, this includes the endpoint cases $L^1 \times L^1 \to L^{1/2}$ and $L^\infty \times L^\infty \to L^\infty$, which is uncommon for bilinear operators.

A weighted KP inequality is an inequality of the form

$$\|J^s(fg)\|_{L^p(w)} \leqslant C_{n,s,p_1,p_2}\left(\|J^s f\|_{L^{p_1}(w_1)}\|g\|_{L^{p_2}(w_2)} + \|f\|_{L^{p_1}(w_1)}\|J^s g\|_{L^{p_2}(w_2)}\right) \tag{1.2}$$

where $f, g \in \mathcal{S}(\mathbb{R}^n)$, $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$, w, w_1, w_2 are weights (non-negative and measurable). In most cases w is related to w_1, w_2 via $w = w_1^{\frac{p}{p_1}} w_2^{\frac{p}{p_2}}$. Naibo and Thomson [13] proved a more general normed Leibniz rule over Triebel-Lizorkin spaces that implies (1.2) when $1 < p_1, p_2 < \infty$, $w_1 \in A_{p_1}$, $w_2 \in A_{p_2}$ and the range of s is sharp based on n, p_1, p_2, w_1, w_2. Oh and Wu [10] proved (1.2) when $1 \leqslant p_1, p_2 \leqslant \infty$, and $w_1 = (1 + |\cdot|)^{\alpha_1}$, $w_2 = (1 + |\cdot|)^{\alpha_2}$, $\alpha_1, \alpha_2 \geqslant 0$ and notably s is independent of the choice of weights.

In this paper we prove the KP inequality with respect to *multiple weights* denoted by $A_{\vec{P}}$. The $A_{\vec{P}}$ class of weights was developed by Lerner, Ombrosi, Pérez, Torres, and Trujillo-González [9], as a natural class of weights for m-linear Calderón-Zygmund operators. While closely related to the tensor product of Muckenhoupt weight classes, the multiple weight class ($A_{\vec{P}}$), is strictly larger (see Defintion 1.2). If $(w_1, \ldots, w_m) \in A_{\vec{P}}$ then w_k may not even be locally integrable. Fortunately, their geometric average is better behaved, indeed $w = \prod_{j=1}^m w_j^{p/p_j}$ is in A_{pm} [9]. Hence we are able to define $\tau_w = \inf\{p : w \in A_p\}$.

In addition, we consider KP inequalities with a product of m functions i.e., we obtain estimates for $J^s(f_1 \cdots f_m)$. This is necessary to point out since the 2-factor KP inequality does not imply the 3-factor KP inequality in the full range of indices. For instance, in the 3-factor case let $p_1 = p_2 = 3/2, p_3 = 2$ and observe that if a_1, a_2 are such that $\frac{2}{3} + \frac{2}{3} + \frac{1}{2} = \frac{1}{a_1} + \frac{1}{2} = \frac{2}{3} + \frac{1}{a_2}$, then $a_1 < 1$ and $a_2 < 1$. It follows the 2-factor inequality can not be applied in this case as it requires the indices to be greater than or equal to one. We now state our main result.

Theorem 1.1. *Let $m \in \mathbb{Z}^+$, $\frac{1}{m} < p < \infty$, $1 < p_1, \ldots, p_m < \infty$ satisfy $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$. Let $\vec{w} \in A_{\vec{P}}$. If $s > n(\frac{1}{\min(p/\tau_w, 1)} - 1)$, then there exists a constant $C = C(w, n, m, s, p_1, \ldots, p_m) < \infty$ such that for all $f_l \in \mathcal{S}(\mathbb{R}^n)$ with $l \in \{1, \ldots, m\}$ we have*

$$\begin{aligned} &\|J^s(f_1 \cdots f_m)\|_{L^p(w)} \leqslant \\ &C\big(\|J^s f_1\|_{L^{p_1}(w_1)}\|f_2\|_{L^{p_2}(w_2)} \cdots \|f_m\|_{L^{p_m}(w_{p_m})} + \cdots + \|f_1\|_{L^{p_1}(w_1)}\|f_2\|_{L^{p_2}(w_2)} \cdots \|J^s f_m\|_{L^{p_m}(w_m)}\big). \end{aligned} \tag{1.3}$$

Furthermore, the same estimate holds with D^s in place of J^s. Moreover, the range of s is sharp.

Regarding the sharpness, Oh and Wu [10] give an example using power weights i.e., weights of the form $|\cdot|^\alpha$, that shows if s is outside of the range given in theorem 1.1 then (1.3) could fail.

1.1 Notation

Cubes in $\mathbb{R}^n$ will be denoted by Q and have sides parallel to the axes. A ball of radius r and center $x \in \mathbb{R}^n$ is denoted by $B(x, r)$. The space $L^p(w)$, where w is a non-negative measurable function and $0 < p < \infty$, is defined as the set of Lebesgue measurable functions on $\mathbb{R}^n$ such that the norm

$$\|f\|_{L^p(w)} := \left(\int_{\mathbb{R}^n} |f(x)|^p w(x)\, dx\right)^{1/p}$$

is finite. For $A, B \in \mathbb{R}$ we use $A \lesssim B$ to mean $A \leqslant CB$ for some positive constant C. The dependence of the constant C on other parameters or constants will be clear from the context and will often be suppressed. We also define $A \sim B$ if and only if $A \lesssim B$ and $B \lesssim A$. In practice the implicit constant will never depend on the functions involved.

For $f \in L^1(\mathbb{R}^n)$ the Fourier transform and inverse Fourier transform are respectively defined by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(y) e^{-2\pi i y \cdot \xi} dy \qquad \check{f}(\xi) = \int_{\mathbb{R}^n} f(y) e^{2\pi i y \cdot \xi} dy.$$

On occasion we will use $\mathcal{F}$ to denote the Fourier transform, precisely $\mathcal{F}(f) = \widehat{f}$ and $\mathcal{F}^{-1}(f) = \check{f}$. The space of Schwartz functions i.e., smooth rapidly decaying functions, is denoted by $\mathcal{S}(\mathbb{R}^n)$.

Let $\widehat{\Phi}(\xi)$ be a radially decreasing bump function on $\mathbb{R}^n$ supported in $|\xi| \leqslant 2$ and equal to one on $|\xi| \leqslant 1$. Let $\widehat{\Psi}(\xi) = \widehat{\Phi}(\xi) - \widehat{\Phi}(2\xi)$, which is non-negative and supported in the annulus $\frac{1}{2} \leqslant |\xi| \leqslant 2$. Notice that $\widehat{\Psi}$ gives rise to a partition of unity i.e., for $\xi \neq 0$ we have

$$\sum_{j \in \mathbb{Z}} \widehat{\Psi}(2^{-j}\xi) = 1, \tag{1.4}$$

as well as the useful identity $\sum_{j \leqslant j_0} \widehat{\Psi}(2^{-j}\xi) = \widehat{\Phi}(2^{-j_0}\xi)$ for any $j_0 \in \mathbb{Z}$. The operators Δ_j and S_j are defined to be convolution with $2^{jn}\Psi(2^j \cdot)$ and $2^{jn}\Phi(2^j \cdot)$ respectively. The operator $\Delta_{j,\mu}$ is defined to be convolution with $2^{jn}\widehat{\Psi}(2^j \cdot + c\mu)$, where c is a constant independent of j and $\mu \in \mathbb{R}^n$. The explicit value of c will depend on the context. Occasionally we will will also use the symbol Δ_{j,μ_1,μ_1}, which denotes convolution with $2^{jn}\widehat{\Psi}(2^j \cdot + c_1\mu_1 + c_2\mu_2)$.

We use M to denote the standard uncentered Hardy-Littlewood maximal operator over cubes. A Muckenhoupt weight or A_p weight is a nonnegative locally integrable function w on $\mathbb{R}^n$ such that for $1 < p < \infty$ there exists a constant $C > 0$ where for all cubes Q in $\mathbb{R}^n$, we have

$$\left(\frac{1}{|Q|}\int_Q w(x)\, dx\right)\left(\frac{1}{|Q|}\int_Q w(x)^{-\frac{1}{p-1}}\, dx\right)^{p-1} \leqslant C.$$

We say $w \in A_1$ if $M(w) \leqslant Cw$ a.e. Let $A_\infty = \bigcup_{p \in (1,\infty)} A_p$ and for $w \in A_\infty$ define $\tau_w = \inf\{p : w \in A_p\}$.

1.2 Inhomogeneous Decomposition

In this subsection we break $J^s(f_1 \cdots f_m)$ into a sum of paraproducts of 2 types; those given by (1.8) and (1.9). Observe for $f_l \in \mathcal{S}(\mathbb{R}^n)$ we have

$$\begin{aligned}
&J^s(f_1 f_2 \cdots f_m)(x)\\
&= \mathcal{F}^{-1}((1+|\cdot|^2)^{\frac{s}{2}}(f_1 f_2 \cdots f_m))(x)\\
&= \int_{\mathbb{R}^n} (1+|\xi_1|^2)^{\frac{s}{2}} (\widehat{f_1} * \cdots * \widehat{f_m})(\xi_1) e^{2\pi i \xi_1 \cdot x} d\xi_1\\
&= \cdots\\
&= \int_{\mathbb{R}^{mn}} (1+|\xi_1|^2)^{\frac{s}{2}} \widehat{f_1}(\xi_1-\xi_2)\widehat{f_2}(\xi_2-\xi_3)\cdots\widehat{f_m}(\xi_m) e^{2\pi i \xi_1 \cdot x} d\xi_1 \cdots d\xi_m\\
&= \int_{\mathbb{R}^{mn}} (1+|\xi_1+\cdots+\xi_m|^2)^{\frac{s}{2}} \widehat{f_1}(\xi_1)\widehat{f_2}(\xi_2)\cdots\widehat{f_m}(\xi_m) e^{2\pi i (\xi_1+\cdots+\xi_m) \cdot x} d\xi_1 \cdots d\xi_m, \qquad (1.5)
\end{aligned}$$

where the last line is from applying a lower triangular orthogonal matrix where all non-zero entries are 1. Using (1.4) and $x \in \mathbb{R}^n$ we can write (1.5) as

$$\begin{aligned}
&\int_{\mathbb{R}^{mn}} \Big(\sum_{\vec{j} \in \mathbb{Z}^m} \widehat{\Psi}(2^{-j_1}\xi_1)\widehat{\Psi}(2^{-j_2}\xi_2)\cdots\widehat{\Psi}(2^{-j_m}\xi_m) \Big)\\
&\qquad \times (1+|\xi_1+\cdots+\xi_m|^2)^{\frac{s}{2}} \widehat{f_1}(\xi_1)\widehat{f_2}(\xi_2)\cdots\widehat{f_m}(\xi_m) e^{2\pi i (\xi_1+\cdots+\xi_m) \cdot x} d\xi_1 \cdots d\xi_m. \qquad (1.6)
\end{aligned}$$

We now decompose $\mathbb{Z}^m$ into 2^m pieces so that we can express (1.6) as a sum of paraproducts. For $\vec{\eta} = (\eta_1, \ldots, \eta_m) \in \{0,1\}^m \backslash \{\vec{0}\}$ let

$$\mathscr{B}_{\vec{\eta}} := \{(j_1, \ldots, j_m) \in \mathbb{Z}^m : \text{if } \eta_t = 1 \text{ for some } 1 \leqslant t \leqslant m \text{ then, } \max\{j_1, \ldots, j_m\} = j_t \text{ and } j_t > 0\}.$$

Notice that

$$\mathbb{Z}^m = \bigsqcup_{\vec{\eta} \in \{0,1\}^m} \mathscr{B}_{\vec{\eta}}, \qquad (1.7)$$

where $\mathscr{B}_{\vec{0}} := (\mathbb{Z}_{\leqslant 0})^m = \{0, -1, -2, \ldots\}^m$. For $\vec{\eta} = (\eta_1, \ldots, \eta_m) \in \{0,1\}^m$ let

$$\Omega_j^{\eta_k}(\xi_k) := \begin{cases} \widehat{\Psi}(2^{-j}\xi_k) & \eta_k = 1 \\ \widehat{\Phi}(2^{-j+1}\xi_k) & \eta_k = 0 \end{cases} \qquad V_j^{\eta_k} := \begin{cases} \Delta_j & \eta_k = 1 \\ S_{j-1} & \eta_k = 0 \end{cases}$$

Observe that

$$\sum_{\vec{j} \in \mathbb{Z}^m} \widehat{\Psi}(2^{-j_1}\xi_1)\widehat{\Psi}(2^{-j_2}\xi_2)\cdots\widehat{\Psi}(2^{-j_m}\xi_m) = \sum_{\vec{\eta} \in \{0,1\}^m} \sum_{j \in \mathbb{N}} (\Omega_j^{\eta_1}(\xi_1)) \cdots (\Omega_j^{\eta_m}(\xi_m)).$$

Substituting this into (1.6) gives

$$\begin{aligned}
&\int_{\mathbb{R}^{mn}} \Big(\sum_{\vec{\eta} \in \{0,1\}^m} \sum_{j \in \mathbb{N}} (\Omega_j^{\eta_1}(\xi_1)) \cdots (\Omega_j^{\eta_m}(\xi_m)) \Big)\\
&\qquad \times (1+|\xi_1+\cdots+\xi_m|^2)^{\frac{s}{2}} \widehat{f_1}(\xi_1)\widehat{f_2}(\xi_2)\cdots\widehat{f_m}(\xi_m) e^{2\pi i (\xi_1+\cdots+\xi_m) \cdot x} d\xi_1 \cdots d\xi_m\\
&= \sum_{\vec{\eta} \in \{0,1\}^m} J^s \Big(\sum_{j \in \mathbb{N}} (V_j^{\eta_1} f_1) \cdots (V_j^{\eta_m} f_m) \Big)\\
&= J^s \Big((S_0 f_1) \cdots (S_0 f_m) \Big) + \sum_{\vec{\eta} \in \{0,1\}^m \backslash \{\vec{0}\}} J^s \Big(\sum_{j \in \mathbb{N}} (V_j^{\eta_1} f_1) \cdots (V_j^{\eta_m} f_m) \Big).
\end{aligned}$$

Note that the coordinates with a 1 in η correspond to a Δ_j operator, while the coordinates with a 0 correspond to a S_{j-1} operator. Let

$$u_j^{\vec{\eta}}(\vec{f}\,) := (V_j^{\eta_1} f_1) \cdots (V_j^{\eta_m} f_m).$$

Define the inhomogeneous paraproduct, $\mathscr{P}^J_{\vec{\eta}}$, for $\vec{\eta} = (\eta_1, \dots, \eta_m) \in \{0,1\}^m$ as

$$\mathscr{P}^J_{\vec{0}}(f_1, \dots, f_m) := (S_0 f_1) \cdots (S_0 f_m)$$
$$\mathscr{P}^J_{\vec{\eta}}(f_1, \dots, f_m) = \sum_{j \in \mathbb{N}} u_j^{\vec{\eta}}(\vec{f}) = \sum_{j \in \mathbb{N}} (V_j^{\eta_1} f_1) \cdots (V_j^{\eta_m} f_m).$$

Notice that $J^s\Big(\mathscr{P}^J_{\vec{\eta}}(\vec{f})\Big)$ is a well defined function; this can be seen by the Lebesgue dominated convergence theorem in conjunction with the fact that f_k are Schwartz functions and the support of $\widehat{\Psi}$. Furthermore, we have $\operatorname{supp}\mathcal{F}(u_j^{\eta}(\vec{f})) \subset B(0, m2^{j+1})$.

Let $l_0 \in \{1, 2, \dots, m\}$. Since there are finitely many $\vec{\eta}$, and $\vec{\eta}$ with exactly l_0 ones can be treated similarly up to permutation, it is enough to show the result for an $\vec{\eta}$ where the first l_0 entries are one, specifically let

$$\vec{\eta}_0 = (\underbrace{1, 1, \dots, 1}_{l_0}, 0, 0, \dots, 0).$$

Thus in the sequel we will focus our attention on the terms

$$J^s\Big(\mathscr{P}^J_{\vec{0}}(f_1, \dots, f_m)\Big) = J^s\Big((S_0 f_1) \cdots (S_0 f_m)\Big) \tag{1.8}$$

$$\begin{aligned} J^s\Big(\mathscr{P}^J_{\vec{\eta}_0}(f_1, \dots, f_m)\Big) &= J^s\Big(\sum_{j \in \mathbb{N}} u_j^{\vec{\eta}_0}(\vec{f})\Big) \\ &= J^s\Big(\sum_{j \in \mathbb{N}} (\Delta_j f_1)(\Delta_j f_2) \cdots (\Delta_j f_{l_0})(S_{j-1} f_{l_0+1}) \cdots (S_{j-1} f_m)\Big). \end{aligned} \tag{1.9}$$

1.3 Homogeneous Decomposition

The decomposition for the homogeneous case $D^s(f_1 \cdots f_m)$ is similar. We decompose $\mathbb{Z}^m$ In the following way. For $\vec{\eta} = (\eta_1, \dots, \eta_m) \in \{0,1\}^m$ let

$$\mathscr{A}_{\vec{\eta}} := \{(j_1, \dots, j_m) \in \mathbb{Z}^m : \text{if } \eta_t = 1 \text{ for some } 1 \leqslant t \leqslant m \text{ then, } \max\{j_1, \dots, j_m\} = j_t\}.$$

Notice that

$$\mathbb{Z}^m = \bigsqcup_{\vec{\eta} \in \{0,1\}^m} \mathscr{A}_{\vec{\eta}}. \tag{1.10}$$

For $\vec{\eta} = (\eta_1, \dots, \eta_m) \in \{0,1\}^m$ let

$$\Omega_j^{\eta_k}(\xi_k) := \begin{cases} \widehat{\Psi}(2^{-j}\xi_k) & \eta_k = 1 \\ \widehat{\Phi}(2^{-j+1}\xi_k) & \eta_k = 0 \end{cases} \qquad V_j^{\eta_k} := \begin{cases} \Delta_j & \eta_k = 1 \\ S_{j-1} & \eta_k = 0 \end{cases}$$

Observe that

$$\sum_{\vec{j} \in \mathbb{Z}^m} \widehat{\Psi}(2^{-j_1}\xi_1)\widehat{\Psi}(2^{-j_2}\xi_2) \cdots \widehat{\Psi}(2^{-j_m}\xi_m) = \sum_{\vec{\eta} \in \{0,1\}^m} \sum_{j \in \mathbb{Z}} \Omega_j^{\eta_1}(\xi_1) \cdots \Omega_j^{\eta_m}(\xi_m).$$

Changing $(1 + |\xi_1 + \cdots + \xi_m|^2)^{\frac{s}{2}}$ to $|\xi_1 + \cdots + \xi_m|^s$ and substituting the above equation into (1.6) we obtain

$$\begin{aligned} &D^s(f_1 f_2 \cdots f_m)(x) \\ &= \int_{\mathbb{R}^{mn}} \Big(\sum_{\vec{\eta} \in \{0,1\}^m} \sum_{j \in \mathbb{Z}} \Omega_j^{\eta_1}(\xi_1) \cdots \Omega_j^{\eta_m}(\xi_m)\Big) \\ &\qquad\qquad \times |\xi_1 + \cdots + \xi_m|^s \widehat{f}_1(\xi_1)\widehat{f}_2(\xi_2) \cdots \widehat{f_m}(\xi_m) e^{2\pi i(\xi_1 + \cdots + \xi_m)\cdot x} d\xi_1 d\xi_2 \cdots d\xi_m \\ &= \sum_{\vec{\eta} \in \{0,1\}^m} D^s\Big(\sum_{j \in \mathbb{Z}} (V_j^{\eta_1} f_1) \cdots (V_j^{\eta_m} f_m)\Big). \end{aligned}$$

Define the homogeneous paraproduct, $\mathscr{P}^D_{\vec{\eta}}$, for $\vec{\eta} = (\eta_1, \ldots, \eta_m) \in \{0,1\}^m$ as

$$\mathscr{P}^D_{\vec{\eta}}(f_1, \ldots, f_m) = \sum_{j\in\mathbb{Z}} \nu_j^{\eta}(\vec{f}) := \sum_{j\in\mathbb{Z}} (V_j^{\eta_1} f_1) \cdots (V_j^{\eta_m} f_m).$$

Let $l_0 \in \{1, 2, \ldots, m\}$. Since there are finitely many $\vec{\eta}$, and $\vec{\eta}$ with exactly l_0 ones can be treated similarly up to permutation, it is enough to show the result for an $\vec{\eta}$ where the first l_0 entries are ones, specifically let

$$\vec{\eta}_0 = (\underbrace{1, 1, \ldots, 1}_{l_0}, 0, 0, \ldots, 0).$$

In the sequel we will focus our attention on the term

$$\begin{aligned} D^s\Big(\mathscr{P}^D_{\vec{\eta}_0}(f_1, \ldots, f_m)\Big) &= D^s\Big(\sum_{j\in\mathbb{Z}} \nu_j^{\vec{\eta}_0}(\vec{f})\Big) \\ &= D^s\Big(\sum_{j\in\mathbb{Z}} (\Delta_j f_1)(\Delta_j f_2) \cdots (\Delta_j f_{l_0})(S_{j-1} f_{l_0+1}) \cdots (S_{j-1} f_m)\Big). \end{aligned} \tag{1.11}$$

1.4 The $A_{\vec{P}}$ Condition

Definition 1.2. *Let $\vec{P} = (p_1, \ldots, p_m)$ with $1 < p_1, \ldots, p_m < \infty$ satisfy $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$. Given $\vec{w} = (w_1, \ldots, w_m)$ set*

$$w = \prod_{j=1}^m w_j^{p/p_j}.$$

We say that $\vec{w}$ satisfies the $A_{\vec{P}}$ condition (or $\vec{w} \in A_{\vec{P}}$) if

$$\sup_Q \Big(\frac{1}{|Q|}\int_Q w\Big)^{1/p} \prod_{j=1}^m \Big(\frac{1}{|Q|}\int_Q w_i^{1-p_i'}\Big)^{1/p_i'} < \infty,$$

where the supremum is taken over all cubes Q with sides parallel to the axes.

It turns out that $w \in A_{mp}$, which will be crucial for our results. The accompanying maximal operator called the *multilinear maximal function* given below is trivially smaller than the m-fold product of Hardy-Littlewood maximal operators.

Definition 1.3. *Given $\vec{f} = (f_1, \ldots, f_m)$ where each entry is measurable, we define the maximal operator $\mathcal{M}$ by*

$$\mathcal{M}(\vec{f})(x) = \sup_{Q\ni x} \prod_{j=1}^m \frac{1}{|Q|}\int_Q |f_j(y_j)|\, dy_j,$$

where the supremum is taken over all cubes Q containing x.

In analogy with the relationship between A_p and the standard Hardy-Littlewood maximal operator we have following relationship between $A_{\vec{P}}$ and the multilinear maximal operator.

Theorem 1.4 ([9])**.** *Let $1 < p_j < \infty$, $j = 1, \ldots, m$, and $\frac{1}{p} = \frac{1}{p_1} + \ldots + \frac{1}{p_m}$. Then the inequality*

$$\|\mathcal{M}(\vec{f})\|_{L^p(w)} \leqslant C \prod_{j=1}^m \|f_j\|_{L^{p_j}(w_j)}$$

holds for every measurable $\vec{f}$ if and only if $\vec{w} \in A_{\vec{P}}$.

We will need the following result.

Proposition 1.5. *Let $\vec{f} = (f_1, \ldots, f_m) \in (\mathscr{S}(\mathbb{R}^n))^m$ and $\varphi^j \in \mathscr{S}(\mathbb{R}^n)$ for $j \in \{1, \ldots, m\}$. For $t \in \mathbb{R}_{>0}$ define the operator Υ_t^j to be convolution with $t^{-n}\varphi^j(t^{-1}\cdot)$, then there is a finite constant independent of t such that*

$$\left|(\Upsilon_t^1 f_1) \cdots (\Upsilon_t^m f_m)\right| \leqslant C_{n,m,\varphi^1,\ldots,\varphi^m} \mathcal{M}(\vec{f}).$$

Proof. In this proof we will use the same decomposition of $\mathbb{Z}^m$ as that for $D^s(f_1 \ldots f_m)$ into paraproducts, namely

$$\mathbb{Z}^m = \bigsqcup_{\vec{\eta} \in \{0,1\}^m} \mathscr{A}_{\vec{\eta}}.$$

First we define some notation. Let $A_{t,k} := \{y : 2^k t \leqslant |y| \leqslant 2^{k+1} t\}$. For $j \in \{1, \ldots, m\}$ define

$$a_k^j(x) = \int_{A_{t,k}} \left|t^{-n}\varphi^j(t^{-1}y_j) f_j(x - y_j)\right| dy_j$$

and

$$b_k^j(x) = \sum_{l \leqslant k} \int_{A_{t,l}} \left|t^{-n}\varphi^j(t^{-1}y_j) f_j(x - y_j)\right| dy_j = \int_{|y_j| \leqslant 2^{k+1}t} \left|t^{-n}\varphi^j(t^{-1}y_j) f_j(x - y_j)\right| dy_j.$$

Observe,

$$\begin{aligned}
|(\Upsilon_t f_1)(x) \cdots (\Upsilon_t f_m)(x)| &\leqslant \left(\int \left|t^{-n}\varphi^1(t^{-1}y_1) f_1(x - y_1)\right| dy_1\right) \cdots \left(\int \left|t^{-n}\varphi^m(t^{-1}y_m) f_m(x - y_m)\right| dy_m\right) \\
&= \sum_{\vec{k} \in \mathbb{Z}^m} a_{k_1}^1(x) \cdots a_{k_m}^m(x) \\
&= \sum_{\vec{\eta} \in \{0,1\}^m} \sum_{\vec{k} \in \mathscr{A}_{\vec{\eta}}} a_{k_1}^1(x) \cdots a_{k_m}^m(x).
\end{aligned}$$

Since there are finitely many $\vec{\eta}$, and $\vec{\eta}$ with exactly l_0 ones are the same up to permutation it is enough to show the result for an $\vec{\eta}$ where the first l_0 entries are one, specifically let

$$\vec{\eta}_0 = (\underbrace{1, 1, \ldots, 1}_{l_0}, 0, 0, \ldots, 0).$$

It follows

$$\begin{aligned}
\sum_{\vec{k} \in \mathscr{A}_{\vec{\eta}_0}} a_{k_1}^1 \cdots a_{k_m}^m &= \sum_{k \in \mathbb{Z}} a_k^1(x) \cdots a_k^{l_0}(x) b_k^{l_0+1}(x) \cdots b_k^m(x) \\
&\leqslant \sum_{k \in \mathbb{Z}} a_k^1(x) b_k^2 b_k^3(x) \cdots b_k^m(x). \qquad (1.12)
\end{aligned}$$

We now estimate a_k^1 and b_k^j for $j \in \{2, \ldots, m\}$. Let $\gamma > n(m-1)$, then

$$\begin{aligned}
a_k^1(x) &= \int_{A_{t,k}} \left|t^{-n}\varphi^1(t^{-1}y_1) f_1(x - y_1)\right| dy_1 \\
&\lesssim t^{-n} \int_{2^k t \leqslant |y| \leqslant 2^{k+1}t} \min\left(\left(\frac{|y_1|}{t}\right)^{\frac{1}{2}}, \left(\frac{|y_1|}{t}\right)^{-\gamma}\right)\left(\frac{|y_1|}{t}\right)^{-n} |f_1(x - y_1)| dy_1 \qquad (1.13) \\
&\lesssim t^{-n} \int_{2^k t \leqslant |y| \leqslant 2^{k+1}t} \min\left(2^{\frac{k}{2}}, 2^{-k\gamma}\right) 2^{-kn} |f_1(x - y_1)| dy_1 \\
&\lesssim \min\left(2^{\frac{k}{2}}, 2^{-k\gamma}\right) \frac{1}{|B(0, 2^{k+1}t)|} \int_{B(0,2^{k+1}t)} |f_1(x - y_1)| dy_1. \qquad (1.14)
\end{aligned}$$

To estimate b_k^j for $j \in \{2, \ldots, m\}$ we have

$$b_k^j(x) = \int_{|y_j| \leqslant 2^{k+1}t} \left|t^{-n}\varphi^j(t^{-1}y_j) f_j(x - y_j)\right| dy_j$$

$$\leqslant \left\|\varphi^j\right\|_{L^\infty} t^{-n}\frac{2^{(k+1)n}}{2^{(k+1)n}}\int_{|y_j|\leqslant 2^{k+1}t}\left|f_j(x-y_j)\right|dy_j$$
$$\lesssim 2^{(k+1)n}\frac{1}{|B(0,2^{k+1}t)|}\int_{B(0,2^{k+1}t)}|f_j(x-y_j)|dy_j. \tag{1.15}$$

Applying estimates (1.14) and (1.15) to (1.12) we obtain

$$\sum_{k\in\mathbb{Z}}\min\left(2^{\frac{k}{2}},2^{-k\gamma}\right)2^{(k+1)n(m-1)}\fint_{B(0,2^{k+1}t)}|f_1(x-y_1)|dy_1\cdots\fint_{B(0,2^{k+1}t)}|f_m(x-y_m)|dy_m\lesssim\mathcal{M}(\vec{f})(x)$$

where the implicit constant is independent of t. □

Remark 1.5.1. *Suppose that in Proposition 1.5 the Υ_t^j were replaced by the shifted operators $\Upsilon_{t,\mu}^j$ defined by convolution with $t^{-n}\varphi^j(t^{-1}\cdot+\mu)$ for $\mu\in\mathbb{R}^n$. Then the final constant grows polynomially in $|\mu|$, i.e. see* (1.16)*. To see this notice that the only part of the proof that this effects is the estimate of a_k^1, specifically at* (1.13)*, where we bound it by a radially decreasing function. We will make use of the simple inequality for $v_1,v_2\in\mathbb{R}^n$*

$$\frac{1}{1+|v_2-v_1|}\leqslant\frac{1+|v_1|}{1+|v_2|}$$

in finding the new estimate. Observe,

$$|\varphi^1(t^{-1}y+\mu)|\lesssim(1+|t^{-1}y+\mu|)^{-n+\frac{1}{2}}\leqslant\frac{(1+|\mu|)^{n-\frac{1}{2}}}{(1+|t^{-1}y|)^{n-\frac{1}{2}}}$$

and

$$|\varphi^1(t^{-1}\cdot+\mu)|\lesssim(1+|t^{-1}y+\mu|)^{-(n+\gamma)}\leqslant\frac{(1+|\mu|)^{n+\gamma}}{(1+|t^{-1}y|)^{n+\gamma}}.$$

It follows that we may estimate $|\varphi(t^{-1}\cdot+\mu)|$ by a constant multiple of

$$(1+|\mu|)^{n+\gamma}\min\left(\left(\frac{|\cdot|}{t}\right)^{\frac{1}{2}},\left(\frac{|\cdot|}{t}\right)^{-\gamma}\right)\left(\frac{|\cdot|}{t}\right)^{-n}.$$

Then continuing the proof we obtain the bound

$$\left|(\Upsilon_{t,\mu}^1f_1)\cdots(\Upsilon_{t,\mu}^mf_m)\right|\leqslant(1+|\mu|)^{n+\gamma}C_{n,m,\varphi^1,\ldots,\varphi^m}\mathcal{M}(\vec{f}). \tag{1.16}$$

In practice there will be Fourier coefficients of arbitrary good decay to cancel out the polynomial growth of $(1+|\mu|)^{n+\gamma}$.

Let $\sigma\in L^\infty(\mathbb{R}^{mn})$. The m-linear Fourier multiplier is defined as

$$T_\sigma(f_1,\ldots,f_m)(x)=\int_{\mathbb{R}^{mn}}e^{2\pi ix\cdot(\xi_1+\cdots+\xi_m)}\sigma(\xi_1,\ldots,\xi_m)\widehat{f_1}(\xi_1)\cdots\widehat{f_m}(\xi_m)\,d\xi_1\ldots d\xi_m.$$

The first use of a bilinear multiplier theorem that employed a Hörmander-type smoothness condition like that in (1.17) was introduced by Tomita [12]. Then Grafakos and Si [4] extend this multiplier theorem to the m-linear case. Soon after the development of $A_{\vec{P}}$-weighted Calderón-Zygmund inequalities in [9], Li and Sun [8] proved the $A_{\vec{P}}$-weighted m-linear multiplier result Theorem 1.6.

Let Λ be a Schwartz function on $\mathbb{R}^{mn}$ satisfying

$$\operatorname{supp}\Lambda\subseteq\left\{(\xi_1,\ldots,\xi_m):\frac{1}{2}\leqslant|\xi_1|+\cdots+|\xi_m|\leqslant 2\right\}\text{ and }\sum_{k\in\mathbb{Z}}\Lambda(2^{-k}\xi_1,\ldots,2^{-k}\xi_m)=1,\forall(\xi_1,\ldots,\xi_m)\neq\vec{0}.$$

Theorem 1.6 ([8])**.** *Let $\vec{P}=(p_1,\ldots,p_m)$ with $1<p_1,\ldots,p_m<\infty$ and $\frac{1}{p_1}+\cdots+\frac{1}{p_m}=\frac{1}{p}$. Suppose that $mn/2<t\leqslant mn$, and $\sigma\in L^\infty(\mathbb{R}^{mn})$ with*

$$\sup_{k\in\mathbb{Z}}\left\|J^t\sigma_k\right\|_{L^2(\mathbb{R}^{mn})}<\infty, \tag{1.17}$$

where

$$\sigma_k(\xi_1, \dots, \xi_m) = \Lambda(\xi_1, \dots, \xi_m)\sigma(2^{-k}\xi_1, \dots, 2^{-k}\xi_m).$$

Let $r_0 := mn/t < p_1, \dots, p_m < \infty$ *and* $\vec{w} \in A_{\vec{P}/r_0}$. *Then*

$$\left\| T_\sigma(\vec{f}) \right\|_{L^p(w)} \lesssim \prod_{i=1}^{N} \|f_i\|_{L^{p_i}(w_i)}. \tag{1.18}$$

1.5 Decay and Summablity

In the proof of Theorem 1.1 the paraproducts of $J^s(f_1 \cdots f_m)$ can be classified into two types; high-high frequency (diagonal) terms and high-low frequency (off-diagonal) terms. The multiplier theorem of the previous section, Theorem 1.6, will deal with the high-low frequency terms. This leaves the high-high frequency terms. In the unweighted bilinear case a technique using Fourier series is used to write $J^s(\sum_{j\in\mathbb{N}} \Delta_j f_1 \Delta_j f_2)$ essentially as a sum of $(\Delta_j J^s f_1)(\Delta_j f_2)$, then the desired estimate follows from the Cauchy-Schwartz inequality, a square function estimate, and Hölder's inequality [6]. In the $A_{\vec{P}}$-weighted case this method will not work for two reasons. First, the Fourier coefficients used in the Fourier series technique may not decay fast enough. Secondly, square function estimates may not work in this weighted setting, indeed recall if $(w_1, w_2) \in A_{\vec{P}}$ then w_1, w_2 may not even be locally integrable functions.

To handle the issue of summablity we will adapt an averaging technique of Oh and Wu [10], but apply Hölder's inequality later in the proof. We will apply a useful theorem of Naibo and Thomson (Theorem 3.2, [13]) that allows us to side step the issue of decay.

Theorem 1.7 ([13]). *Let* $\vec{f} \in (\mathscr{S}(\mathbb{R}^n))^m$, $w \in A_\infty$, $\vec{\eta} \in \{0,1\}^m$, $0 < p < \infty$ *and* $s > n(\min(1, r/\tau_w)^{-1} - 1)$. *Then for the homogeneous paraproduct* $\mathscr{P}^D_{\vec{\eta}}(\vec{f}) = \sum_{j\in\mathbb{Z}} \nu_j^{\vec{\eta}}(\vec{f})$ *we have*

$$\left\| D^s\Big(\sum_{j\in\mathbb{Z}} \nu_j^{\vec{\eta}}(\vec{f})\Big) \right\|_{L^p(w)} \lesssim \left\| \Big(\sum_{j\in\mathbb{Z}} |2^{js}\nu_j^{\vec{\eta}}(\vec{f})|^2\Big)^{\frac{1}{2}} \right\|_{L^p(w)}$$

where the implicit constant depends only on w, n, s, p. *Furthermore, for a inhomogeneous paraproduct with the same parameters we have*

$$\left\| J^s\Big(\sum_{j\in\mathbb{N}} u_j^{\vec{\eta}}(\vec{f})\Big) \right\|_{L^p(w)} \lesssim \left\| \Big(\sum_{j\in\mathbb{N}} |2^{js}u_j^{\vec{\eta}}(\vec{f})|^2\Big)^{\frac{1}{2}} \right\|_{L^p(w)}.$$

Bernstein's inequality says for $1 \leqslant q \leqslant \infty$, $f \in \mathcal{S}(\mathbb{R}^n)$ and $j \in \mathbb{Z}$ that $\|\Delta_j(D^s f)\|_{L^q} \sim 2^{js}\|\Delta_j(f)\|_{L^q}$. We need a way to directly express $\Delta_j(D^s f)$ in terms of $\Delta_j f$ and vice versa without the appearance of a norm. In fact we need such expressions for the shifted Littlewood-Paley operator $\Delta_{j,\mu}$. Recall the operator $\Delta_{j,\mu}$ means convolution with $2^{jn}\widehat{\Psi}(2^j \cdot + c\mu)$, where c is a constant independent of j and $\mu \in \mathbb{R}^n$.

Proposition 1.8. *(Bernstein-type expressions) Let* $s \in \mathbb{R}$, *and let* $\widehat{\psi}$ *be a* $\mathcal{C}^\infty(\mathbb{R}^n)$ *function supported in the annulus* $\frac{1}{2} \leqslant |\xi| \leqslant 2$. *Define* $\Delta_j^\psi f$ *to be convolution with* $2^{jn}\psi(2^j\cdot)$ *for* $f \in \mathcal{S}(\mathbb{R}^n)$ *and let* $j \in \mathbb{Z}$. *Then one has*

$$J^s\Delta_j^\psi f(x) = 2^{js}\sum_{\mu\in\mathbb{Z}^n} c_{j,\mu}\Delta_{j,\mu}^\psi f(x) \text{ and } 2^{js}\Delta_j^\psi f(x) = \sum_{\mu\in\mathbb{Z}^n} c_{j,\mu}\Delta_{j,\mu}^\psi J^s f(x) \tag{1.19}$$

where $|c_{j,\mu}| \lesssim (1+|\mu|)^{-N}$ *for any* $N \in \mathbb{N}$, *when* $j \geqslant 0$, *the implicit constant is independent of* j. *Analogously, for the operator* D^s *and* $j \in \mathbb{Z}$ *we have*

$$D^s\Delta_j^\psi f(x) = 2^{js}\sum_{\mu\in\mathbb{Z}^n} c_\mu\Delta_{j,\mu}^\psi f(x) \text{ and } 2^{js}\Delta_j^\psi f(x) = \sum_{\mu\in\mathbb{Z}^n} c_\mu\Delta_{j,\mu}^\psi D^s f(x), \tag{1.20}$$

where c_μ *is a rapidly decaying constant in* $|\mu|$, *which does not depend on* $j \in \mathbb{Z}$.

Proof. Let $\widehat{\psi_\star}$ be a bump function that is 1 for $\frac{1}{2} \leqslant |\xi| \leqslant 2$ and supported in $\frac{1}{4} \leqslant |\xi| \leqslant 4$. Let $\sigma_j(\xi) \equiv (2^{-2j} + |\xi|^2)^{\frac{s}{2}}\widehat{\psi_\star}(\xi)$ which is a smooth compactly supported function. Furthermore, σ_j and all of its partial derivatives are uniformly bounded in j due to the support of $\widehat{\psi_\star}$ and the fact that $j \geqslant 0$. Expanding in Fourier series we have

$$\sigma_j(\xi) = \chi_{[-4,4]^n}(\xi) \sum_{\mu\in\mathbb{Z}^n} c_{j,\mu} e^{2\pi i\xi\cdot\frac{\mu}{8}} \tag{1.21}$$

where due to σ_j's smoothness the coefficients satisfy

$$|c_{j,\mu}| \lesssim \frac{\left\|(1-\Delta)^N \sigma_j\right\|_{L^1}}{(1+|\mu|^2)^N} \lesssim \frac{1}{(1+|\mu|^2)^N}$$

for any $N \in \mathbb{Z}^+$, where the implicit constant is independent of j and μ. Observe for $j \geqslant 0$,

$$\begin{aligned} J^s\Delta_j^\psi f(x) &= \int (1+|\xi|^2)^{\frac{s}{2}}\widehat{\psi_\star}(2^{-j}\xi)\widehat{\Delta_j^\psi f}(\xi)e^{2\pi i\xi\cdot x}d\xi \\ &= \int 2^{js}(2^{-2j}+|2^{-j}\xi|^2)^{\frac{s}{2}}\widehat{\psi_\star}(2^{-j}\xi)\widehat{\Delta_j^\psi f}(\xi)e^{2\pi i\xi\cdot x}d\xi \\ &= 2^{js}\int \sum_{\mu\in\mathbb{Z}^n} c_{j,\mu}e^{2\pi i\xi\cdot 2^{-j-3}\mu}\widehat{\Delta_j^\psi f}(\xi)e^{2\pi i\xi\cdot x}d\xi \\ &= 2^{js}\sum_{\mu\in\mathbb{Z}^n} c_{j,\mu}\Delta_{j,\mu}^\psi f(x). \end{aligned}$$

To get the other direction observe,

$$\begin{aligned} 2^{js}\Delta_j^\psi f(x) &= \int 2^{js}(1+|\xi|^2)^{-\frac{s}{2}}\widehat{\psi_\star}(2^{-j}\xi)\widehat{\Delta_j^\psi J^s f}(\xi)e^{2\pi i\xi\cdot x}d\xi \\ &= \int (2^{-2j}+|2^{-j}\xi|^2)^{-\frac{s}{2}}\widehat{\psi_\star}(2^{-j}\xi)\widehat{\Delta_j^\psi J^s f}(\xi)e^{2\pi i\xi\cdot x}d\xi \\ &= \int \sum_{\mu\in\mathbb{Z}^n} c_{j,\mu}e^{2\pi i\xi\cdot 2^{-j-3}\mu}\widehat{\Delta_j^\psi J^s f}(\xi)e^{2\pi i\xi\cdot x}d\xi \\ &= \sum_{\mu\in\mathbb{Z}^n} c_{j,\mu}\Delta_{j,\mu}^\psi J^s f(x). \end{aligned}$$

The equality in (1.20) follows by the same technique.

□

Proposition 1.8 combined with the following averaging lemma will allow us to achieve summablity of the diagonal paraproduct.

Lemma 1.9 ([10])**.** *If $a_k \lesssim \min(2^{ka}A, 2^{-kb}B)$ for some $a, b, A, B > 0$ and every $k \in \mathbb{Z}$, then for any $u > 0$, we have $\{a_k\}_{k\in\mathbb{Z}} \in \ell^u(\mathbb{Z})$ and*

$$\|\{a_k\}_{k\in\mathbb{Z}}\|_{\ell^u} \lesssim A^{\frac{b}{a+b}}B^{\frac{a}{a+b}}.$$

In particular, if $\|f_k\|_{L^r(w)} \lesssim |a_k|$ for $0 < r \leqslant \infty$, every $k \in \mathbb{Z}$, and a weight w then

$$\left\|\sum_{k\in\mathbb{Z}} f_k\right\|_{L^r(w)} \lesssim A^{\frac{b}{a+b}}B^{\frac{a}{a+b}}.$$

2 Proof of Theorem 1.1

2.1 Case 1. $l_0 \geqslant 2$

Proof. We need to bound

$$\Big\| J^s\Big(\sum_{j\in\mathbb{N}}(\Delta_j f_1)(\Delta_j f_2)\cdots(\Delta_j f_k)(S_{j-1}f_{k+1})\cdots(S_{j-1}f_m)\Big)\Big\|_{L^p(w)}$$

where at least the first two operators are Δ_j. For $j\in\mathbb{Z}$ define operators

$$T_1^j(f_1,\dots,f_m)=(\Delta_j f_1)(\Delta_j f_2)\cdots(\Delta_j f_{l_0})(S_{j-1}f_{l_0+1})\cdots(S_{j-1}f_m)$$

and

$$T_2^j(f_1,\dots,f_m)=\sum_{\mu_1\in\mathbb{Z}}\sum_{\mu_2\in\mathbb{Z}}c_{j,\mu_1}c_{j,\mu_2}(\Delta_{j,\mu_1}f_1)(\Delta_{j,\mu_2}f_2)(\Delta_j f_3)\cdots(\Delta_j f_{l_0}S_{j-1}f_{l_0+1})\cdots(S_{j-1}f_m)$$

where $c_{j,\mu_l}\lesssim(1+|\cdot|)^{-N}$ for $l=1,2$ any $N\in\mathbb{N}$, and the implicit constant is independent of $j\in\mathbb{Z}$. Since by hypothesis $s>n(\min(1,r/\tau_w)^{-1}-1)$ we apply Theorem 1.7 and the fact that $\sqrt{\sum_{j\in\mathbb{Z}}|t_j|^2}\leqslant\sum_{j\in\mathbb{Z}}|t_j|$ to obtain

$$\begin{aligned}&\Big\| J^s\Big(\sum_{j\in\mathbb{N}}(\Delta_j f_1)(\Delta_j f_2)\cdots(\Delta_j f_{l_0})(S_{j-1}f_{l_0+1})\cdots(S_{j-1}f_m)\Big)\Big\|_{L^p(w)}\\ &\lesssim\Big\|\sum_{j\in\mathbb{N}}2^{js}\big|(\Delta_j f_1)(\Delta_j f_2)\cdots(\Delta_j f_{l_0})(S_{j-1}f_{l_0+1})\cdots(S_{j-1}f_m)\big|\Big\|_{L^p(w)}.\end{aligned}$$

We now obtain two upper bounds on

$$\Big\|2^{js}(\Delta_j f_1)(\Delta_j f_2)\cdots(\Delta_j f_{l_0})(S_{j-1}f_{l_0+1})\cdots(S_{j-1}f_m)\Big\|_{L^p(w)}. \tag{2.1}$$

Trivially, (2.1) is bounded above by

$$2^{js}\Big\|T_1^j(f_1,\dots,f_m)\Big\|_{L^p(w)}. \tag{2.2}$$

Applying Proposition 1.8 twice on the first two Δ_j operators we obtain that (2.1) is bounded above by a constant multiple of

$$2^{-js}\Big\|T_2^j(J^sf_1,J^sf_2,f_3\dots,f_m)\Big\|_{L^p(w)}. \tag{2.3}$$

In view of (2.2), (2.3) Lemma 1.9 with $a=s$, $b=-s$, $A=\sup_j\Big\|T_1^j(f_1,\dots,f_m)\Big\|_{L^p(w)}$,
$B=\sup_j\Big\|T_2^j(J^sf_1,J^sf_2,f_3\dots,f_m)\Big\|_{L^p(w)}$ we obtain

$$\begin{aligned}&\Big\| J^s\Big(\sum_{j\in\mathbb{N}}(\Delta_j f_1)(\Delta_j f_2)\cdots(\Delta_j f_{l_0})(S_{j-1}f_{l_0+1})\cdots(S_{j-1}f_m)\Big)\Big\|_{L^p(w)}\\ &\lesssim\Big(\sup_{j\in\mathbb{N}}\Big\|T_1^j(f_1,\dots,f_m)\Big\|_{L^p(w)}\sup_{j\in\mathbb{N}}\Big\|T_2^j(J^sf_1,J^sf_2,f_3\dots,f_m)\Big\|_{L^p(w)}\Big)^{\frac{1}{2}}.\end{aligned} \tag{2.4}$$

Next using Proposition 1.5 and then Theorem 1.4 we obtain

$$\sup_{j\in\mathbb{N}}\left\|T_1^j(f_1,\dots,f_m)\right\|_{L^p(w)} \lesssim \left\|\mathcal{M}(\vec{f})\right\|_{L^p(w)} \lesssim \|f_1\|_{L^{p_1}(w_1)}\|f_2\|_{L^{p_2}(w_2)}\cdots\|f_m\|_{L^{p_m}(w_m)}. \tag{2.5}$$

Similarly, we have the estimate

$$\sup_{j\in\mathbb{N}}\left\|T_2^j(J^sf_1,J^sf_2,f_3\dots,f_m)\right\|_{L^p(w)} \lesssim \|\mathcal{M}(J^sf_1,J^sf_2,f_3,\dots,f_m)\|_{L^p(w)} \tag{2.6}$$

$$\lesssim \|J^sf_1\|_{L^{p_1}(w_1)}\|J^sf_2\|_{L^{p_2}(w_2)}\|f_3\|_{L^{p_3}(w_3)}\cdots\|f_m\|_{L^{p_m}(w_m)}, \tag{2.7}$$

where (2.6) follows from Proposition 1.5 and Remark 1.5.1. Applying estimates (2.5) and (2.7) to (2.4) and using the AMGM inequality we obtain

$$\begin{aligned}&\Big(\|f_1\|_{L^{p_1}(w_1)}\|f_2\|_{L^{p_2}(w_2)}\cdots\|f_m\|_{L^{p_m}(w_m)}\|J^sf_1\|_{L^{p_1}(w_1)}\|J^sf_2\|_{L^{p_2}(w_2)}\|f_3\|_{L^{p_3}(w_3)}\cdots\|f_m\|_{L^{p_m}(w_m)}\Big)^{\frac{1}{2}}\\ &\lesssim \|J^sf_1\|_{L^{p_1}(w_1)}\|f_2\|_{L^{p_2}(w_2)}\cdots\|f_m\|_{L^{p_m}(w_m)}+\|f_1\|_{L^{p_1}(w_1)}\|J^sf_2\|_{L^{p_2}(w_2)}\|f_3\|_{L^{p_3}(w_3)}\cdots\|f_m\|_{L^{p_m}(w_m)}\end{aligned} \tag{2.8}$$

as desired. This finishes the proof for the diagonal term in the inhomogeneous case. As both Lemma 1.9 and Proposition 1.5 work over $j\in\mathbb{Z}$ we many replace $\mathbb{N}$ with $\mathbb{Z}$ and J^s with D^s to get the homogeneous case by the same proof.

2.2 Case 2. Low Frequency Term

In the inhomogeneous case we also have the term $J^s((S_0f_1)\cdots(S_0f_m))$ which is dealt with in a similar way. Observe,

$$\begin{aligned}J^s((S_0f_1)\cdots(S_0f_m))(x) = \int_{\mathbb{R}^{2n}}&(1+|\xi_1+\cdots+\xi_m|^2)^{\frac{s}{2}}\widehat{\Phi}\Big(\frac{1}{2m}(\xi_1+\cdots+\xi_m)\Big)(1+|\xi_1|^2)^{-\frac{s}{2}}\widehat{\Phi}\Big(\frac{1}{2}\xi_1\Big)\\ &\times\widehat{S_0J^sf}_1(\xi_1)\cdots\widehat{S_0f_m}(\xi_m)e^{2\pi ix\cdot(\xi_1+\cdots+\xi_m)}d\vec{\xi}.\end{aligned} \tag{2.9}$$

Since the following expressions are smooth with compact support we can expand them in Fourier series whose Fouier coefficients, c_{μ_1}, c_{μ_2} have rapid decay in μ_1,μ_2;

$$(1+|\xi|^2)^{\frac{s}{2}}\widehat{\Phi}\Big(\frac{1}{2m}\xi\Big) = \chi_{[-4m,4m]^n}(\xi)\sum_{\mu_1\in\mathbb{Z}^n}c_{\mu_1}e^{2\pi i\xi\cdot\mu_1/8m}$$

$$(1+|\xi|^2)^{-\frac{s}{2}}\widehat{\Phi}\Big(\frac{1}{2}\xi\Big) = \chi_{[-4,4]^n}(\xi)\sum_{\mu_2\in\mathbb{Z}^n}c_{\mu_2}e^{2\pi i\xi\cdot\mu_2/8}.$$

Substituting the Fourier series into (2.9) we obtain

$$J^s(S_0f_1\cdots S_0f_m)(x)$$

$$= \int_{\mathbb{R}^{mn}}\sum_{\mu_1\in\mathbb{Z}^n}c_{\mu_1}e^{2\pi i(\xi_1+\cdots+\xi_m)\cdot\mu_1/8m}\sum_{\mu_2\in\mathbb{Z}^n}c_{\mu_2}e^{2\pi i\xi_1\cdot\mu_2/8}\widehat{S_0J^sf}_1(\xi_1)\cdots\widehat{S_0f_m}(\xi_m)e^{2\pi ix\cdot(\xi_1+\cdots+\xi_m)}d\vec{\xi} \tag{2.10}$$

$$= \sum_{\mu_1\in\mathbb{Z}^n}\sum_{\mu_2\in\mathbb{Z}^n}c_{\mu_1}c_{\mu_2}S_{0,\mu_1,\mu_2}(J^sf_1)(x)S_{0,\mu_1}(f_2)(x)\cdots S_{0,\mu_1}(f_m)(x) \tag{2.11}$$

where the operator S_{0,μ_1,μ_2} corresponds to convolution with $\Phi(\cdot+\mu_1/8m+\mu_2/8)$. By Proposition 1.5, Remark 1.5.1, and Theorem 1.4 applied to (2.11) we deduce

$$\|J^s(S_0f_1\cdots S_0f_m)\|_{L^p(w)} \lesssim \|J^sf_1\|_{L^{p_1}(w_1)}\prod_{j=2}^m\|f_j\|_{L^{p_j}(w_j)}. \tag{2.12}$$

2.3 Case 3. $l_0 = 1$

When $l_0 = 1$ we have a"high-low-$\cdots$-low" frequency multiplier of the form

$$\sum_{j\in\mathbb{N}} \widehat{\Psi}(2^{-j}\xi_1)\widehat{\Phi}(2^{-j+1}\xi_2)\cdots\widehat{\Phi}(2^{-j+1}\xi_m)$$

which corresponds to the term

$$J^s\Big(\sum_{j\in\mathbb{N}}(\Delta_j f_1)(S_{j-1}f_2)\cdots(S_{j-1}f_m)\Big). \tag{2.13}$$

Fix $a \in \mathbb{N}$ to be determined later. We want to replace S_{j-1} by S_{j-a}. We do this by expressing S_{j-1} as $S_{j-a} + \sum_{j-a<k\leqslant j}\Delta_k$ in (2.13) i.e.

$$\begin{aligned}\sum_{j\in\mathbb{N}}(\Delta_j f_1)(S_{j-1}f_2)\cdots(S_{j-1}f_m) &= \sum_{j\in\mathbb{N}}(\Delta_j f_1)\Big(\sum_{j-a<k\leqslant j-1}\Delta_k f_2 + S_{j-a}f_2\Big)(S_{j-1}f_3)\cdots(S_{j-1}f_m)\\ &= \sum_{j\in\mathbb{N}}(\Delta_j f_1)\Big(\sum_{j-a<k\leqslant j-1}\Delta_k f_2\Big)(S_{j-1}f_3)\cdots(S_{j-1}f_m) + \sum_{j\in\mathbb{N}}(\Delta_j f_1)(S_{j-a}f_2)(S_{j-1}f_3)\cdots(S_{j-1}f_m).\end{aligned} \tag{2.14}$$

The first term in (2.14) is a finite sum (over k) of operators from Case 1 since $k \sim j$. Therefore, we focus on the second term in (2.14). Expressing $S_{j-1}f_3$ as $S_{j-a}f_3 + \sum_{j-a<k\leqslant j}\Delta_k f_3$ we obtain

$$\begin{aligned}&\sum_{j\in\mathbb{N}}(\Delta_j f_1)(S_{j-a}f_2)(S_{j-1}f_3)\cdots(S_{j-1}f_m)\\ &= \sum_{j\in\mathbb{N}}(\Delta_j f_1)(S_{j-a}f_2)\Big(\sum_{j-a<k\leqslant j-1}\Delta_k f_3\Big)(S_{j-1}f_4)\cdots(S_{j-1}f_m) \qquad (2.15)\\ &\qquad + \sum_{j\in\mathbb{N}}(\Delta_j f_1)(S_{j-a}f_2)(S_{j-a}f_3)(S_{j-1}f_4)\cdots(S_{j-1}f_m).\end{aligned}$$

Again the term in (2.15) is a finite sum (over k) of operators from Case 1 since $k \sim j$. Continuing in this way we eventually express (2.13) as

$$\sum_{j\in\mathbb{N}}(\Delta_j f_1)(S_{j-a}f_2)(S_{j-a}f_3)\cdots(S_{j-a}f_m)$$

plus finitely many other terms that have already been handled by Case 1. We have reduced matters to showing the desired bound for the off-diagonal term

$$J^s\Big(\sum_{j\in\mathbb{N}}(\Delta_j f_1)(S_{j-a}f_2)(S_{j-a}f_3)\cdots(S_{j-a}f_m)\Big). \tag{2.16}$$

The support of the Fourier transform of $(S_{j-a}f_2)(S_{j-a}f_3)\cdots(S_{j-a}f_m)$ is contained in the ball centered at zero with the radius $(m-1)2^{j-a+1}$.The support of the Fourier transform of $\Delta_j f_1$ in contained in the annulus $2^{j-1} \leqslant |\xi_1| \leqslant 2^{j+1}$. Choose a so that

$$(2m)2^{j-a+1} \leqslant 2^{j-1}$$

for all integers j. Solving the above inequality gives a can be picked to be some integer larger than $\log_2(8m)$. On the Fourier transform side this choice of a gives $|\xi_l| \leqslant \frac{1}{2m}|\xi_1|$ for $l \in \{2,\ldots,m\}$. Hence,

$$2|\xi_1| \geqslant |\xi_1+\cdots+\xi_m| \geqslant |\xi_1| - |\xi_2| - \cdots - |\xi_m| \geqslant |\xi_1| - \frac{(m-1)|\xi_1|}{2m} \geqslant \frac{|\xi_1|}{2},$$

thus $|\xi_1| \sim |\xi_1+\cdots+\xi_m|$.

In order to apply Theorem 1.6 with $t = mn$ to the off-diagonal term in (2.16) we need to show the following Hörmander smoothness condition

$$\sup_{k\in\mathbb{Z}} \sum_{|\alpha|\leqslant nm} \|\partial^\alpha \sigma_k\|_{L^2(\mathbb{R}^{nm})} < \infty \tag{2.17}$$

where

$$\sigma_k = \Lambda(\xi_1, \dots, \xi_m)\langle 2^{-k}\xi_1 + \cdots + 2^{-k}\xi_m\rangle^s \langle 2^{-k}\xi_1\rangle^{-s} \sum_{j>-k} \widehat{\Psi}(2^{-j}\xi_1)\widehat{\Phi}(2^{-j+a}\xi_2)\cdots\widehat{\Phi}(2^{-j+a}\xi_m).$$

If (2.17) holds then theorem 1.6 implies

$$\left\| J^s\Big(\sum_{j\in\mathbb{N}} \Delta_j f_1 S_{j-a} f_2 S_{j-a} f_3 \cdots S_{j-a} f_m\Big)\right\|_{L^p(w)} \lesssim \|J^s f_1\|_{L^{p_1}(w_1)} \prod_{j=2}^m \|f_j\|_{L^{p_j}(w_j)}. \tag{2.18}$$

To start fix a multi-index, $\alpha = (\alpha^1, \dots, \alpha^m) \in (\mathbb{N}_0^n)^m$, such that $0 \leqslant |\alpha| \leqslant nm$ and let $\beta_l = (\beta_l^1, \dots, \beta_l^m) \in (\mathbb{N}_0^n)^m$ for $l \in \{1, 2, \dots, m+3\}$. Since σ_k is a product of $m+3$ functions by Leibniz rule on $\partial^\alpha\sigma_k$ it is enough to consider a summand of the form

$$\begin{aligned}\partial^{\beta_1}[\Lambda(\vec{\xi})]\partial^{\beta_2}[\langle 2^{-k}\xi_1 + \cdots + 2^{-k}\xi_m\rangle^s]\partial^{\beta_3}[\langle 2^{-k}\xi_1\rangle^{-s}]\\ \times \sum_{j>-k} \partial^{\beta_4}[\widehat{\Psi}(2^{-j}\xi_1)]\partial^{\beta_5}[\widehat{\Phi}(2^{-j+a}\xi_2)]\cdots\partial^{\beta_{m+3}}[\widehat{\Phi}(2^{-j+a}\xi_m)]\end{aligned} \tag{2.19}$$

where $\beta_1 + \beta_2 + \cdots + \beta_{m+3} = \alpha$. Also, we can bring the derivative into the sum because locally it is a finite sum. Lets derive an estimate for $\partial^{\beta_2}[\langle 2^{-k}\xi_1 + \cdots + 2^{-k}\xi_m\rangle^s]$ and $\partial^{\beta_3}[\langle 2^{-k}\xi_1\rangle^{-s}]$. Notice that partial derivatives of $\langle 2^{-k}\xi_1 + \cdots + 2^{-k}\xi_m\rangle^s = 2^{-ks}(2^{2k} + |\xi_1 + \cdots + \xi_m|^2)^{\frac{s}{2}}$ are linear combinations of terms of the form

$$2^{-ks}(2^{2k} + |\xi_1 + \cdots + \xi_m|)^{\frac{s}{2}-l}P(\vec{\xi})$$

where $0 \leqslant l \leqslant |\beta_2|$ is a nonnegative integer and P is a polynomial over $\mathbb{R}^{nm}$. Furthermore, on the support of Λ any polynomial over $\mathbb{R}^{nm}$ is bounded by a constant. It follows on the support of Λ that

$$\partial^{\beta_2}[\langle 2^{-k}\xi_1 + \cdots + 2^{-k}\xi_m\rangle^s] \lesssim 2^{-ks}\sum_{l=0}^{|\beta_2|}(2^{2k} + |\xi_1 + \cdots + \xi_m|^2)^{\frac{s}{2}-l}. \tag{2.20}$$

By the same argument we have

$$\partial^{\beta_3}[\langle 2^{-k}\xi_1\rangle^{-s}] \lesssim 2^{ks}\sum_{t=0}^{|\beta_3^1|}(2^{2k} + |\xi_1|^2)^{-\frac{s}{2}-t} \tag{2.21}$$

on the support of Λ.

Applying partial derivatives to each term in (2.19), and taking absolute value, and using estimates (2.20) and (2.21) we obtain that (2.19) is bounded above by a constant multiple of

$$\begin{aligned}\Big|(\partial^{\beta_1}\Lambda)(\vec{\xi})\Big| 2^{-ks}\sum_{l=0}^{|\beta_2|}(2^{2k} + |\xi_1 + \cdots + \xi_m|^2)^{\frac{s}{2}-l} 2^{ks}\sum_{t=0}^{|\beta_3^1|}(2^{2k} + |\xi_1|^2)^{-\frac{s}{2}-t}\chi_{\{|\xi_1|\gg|\xi_l|: l\in\{1,\dots,m\}\}}(\vec{\xi})\\ \times \sum_{j\in\mathbb{Z}} |(\partial^{\beta_4}\widehat{\Psi})(2^{-j}\xi_1)|2^{-j(|\beta_4|+\cdots+|\beta_{m+3}|)}.\end{aligned} \tag{2.22}$$

Using the support of the above expression, namely $|\xi_1| \sim |\xi_1 + \cdots + \xi_m|$ we can further bound (2.22) by

$$\Big|(\partial^{\beta_1}\Lambda)(\vec{\xi})\Big| \sum_{l=0}^{|\beta_2|}\sum_{t=0}^{|\beta_3^1|}(2^{2k} + |\xi_1|^2)^{-(l+t)}\chi_{\{|\xi_1|\gg|\xi_l|: l\in\{1,\dots,m\}\}}(\vec{\xi})\sum_{j\in\mathbb{Z}} |(\partial^{\beta_4}\widehat{\Psi})(2^{-j}\xi_1)|2^{-j(|\beta_4|+\cdots+|\beta_{m+3}|)}. \tag{2.23}$$

Now dropping the 2^{2k} in (2.23) leaves $|\xi_1|^{-2(l+t)}$ which is bounded above by a constant due to the support of $\partial^{\beta_1}\Lambda$ by the observation that $1/2 \leqslant |\xi_1| + \cdots + |\xi_m| \leqslant m|\xi_1|$. Also notice that by the support of $\partial^{\beta_1}\Lambda$ and

$\chi_{\{|\xi_1|\gg|\xi_l|:l\in\{1,\ldots,m\}\}}$ we have $1/2m \leqslant |\xi_1| \leqslant 2$; this forces $-\log_2(m) \leqslant j \leqslant \log_2(m)$. It follows (2.23) can be estimated above by constant multiple of

$$\begin{aligned}&\left|(\partial^{\beta_1}\Lambda)(\vec{\xi})\right|\chi_{\{|\xi_1|\gg|\xi_l|:l\in\{1,\ldots,m\}\}}(\vec{\xi})\sum_{|j|\leqslant c_m}|(\partial^{\beta_4}\widehat{\Psi})(2^{-j}\xi_1)|2^{-j(|\beta_4|+\cdots+|\beta_{m+3}|)}\\&\leqslant C_{m,n}\left|(\partial^{\beta_1}\Lambda)(\vec{\xi})\right|\end{aligned}$$

which has a finite L^2 norm independent of k. This finishes the proof for the off-diagonal term in the inhomogeneous case. The proof for the nondiagonal term in the homogeneous case is the same except estimates (2.20) and (2.21) are replaced by

$$\partial^{\beta_2}[|2^{-k}\xi_1+\cdots+2^{-k}\xi_m|^s] \lesssim 2^{-ks}\sum_{l=1}^{|\beta_2|}|\xi_1+\cdots+\xi_m|^{s-2l}$$

and

$$\partial^{\beta_3}[|2^{-k}\xi_1|^{-s}] \lesssim 2^{ks}\sum_{t=1}^{|\beta_3^1|}|\xi_1|^{-s-2t}$$

on the support of Λ.

□

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