NORM ESTIMATES FOR THE FRACTIONAL DERIVATIVE OF MULTIPLE FACTORS

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Abstract. We extend the Kato-Ponce inequality to a product of m functions, proving an estimate currently missing from the literature. This study is motivated by the fact that the 3-factor Kato-Ponce does not follow directly from the 2-factor version in the full range of permissible indices. Our methodology is based upon that in [12] but our extension entails a novel decomposition that elegantly and effectively handles the technical difficulties that arise from the combinatorial complexity of the possibly large number of factors.

1. Introduction

The lack of an explicit Leibniz rule for fractional derivatives leads to the consideration of norm inequalities, most commonly for Lebesgue spaces. Such estimates are known in the literature as Kato-Ponce (KP) inequalities, and they are usually expressed in terms of two factors. In this article we focus on Kato-Ponce estimates for multiple factors. The need to study multiple factors is motivated by the fact that a 3-factor normed Leibniz rule in the full range of indices does not follow from the corresponding 2-factor one by grouping two terms into one.

Let $\widehat{g}$ denote the Fourier transform (precisely defined in Section 2). Let $\widehat{D^s f} := |\cdot|^s \widehat{f}$ and $\widehat{J^s f} := (1+|\cdot|^2)^{\frac{s}{2}} \widehat{f}$ be the fractional Laplacian operators for $f \in \mathcal{S}(\mathbb{R}^n)$, the space of Schwartz functions. If $s > 0$ then D^s and J^s are the homogeneous and inhomogeneous fractional differentiation operators, respectively. Motivated by questions in Euler and Navier-Stokes equations Kato and Ponce [16] obtained L^r norm estimates for the inhomogeneous fractional derivative of a product. Since their work in 1988 there has been a multitude of generalizations and variants of such estimates, which are nowadays known as Kato-Ponce inequalities or fractional Leibniz rules. These estimates have the form

$$\|J^s(fg)\|_{L^r} \leqslant C_{n,s,p_1,p_2,q_1,q_2} \left(\|J^s f\|_{L^{p_1}} \|g\|_{L^{p_2}} + \|f\|_{L^{q_1}} \|J^s g\|_{L^{q_2}} \right), \tag{1.1}$$

where $f, g \in \mathcal{S}(\mathbb{R}^n)$, $p_1, p_2, q_1, q_2 \geqslant 1$, $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{r} = \frac{1}{q_1} + \frac{1}{q_2}$ and s depends on n and r. We say the indices vary when $p_1 \neq q_1$ and $p_2 \neq q_2$.

Earlier variants of the Kato-Ponce inequality were restricted to $1 < r < \infty$, see for instance [16], [4], [14]. Subsequent versions, such as [19], [2], [20], and [12], provide extensions to the range $r \leqslant 1$. It turns out that (1.1) holds exactly when $1/2 < r < \infty$ and $s > \max\{n(\frac{1}{r} - 1), 0\}$ or $s \in 2\mathbb{N}$; The work in [12] provides counterexamples when s is outside that range. These proofs rely on Coifman-Meyer bilinear multipliers for high-low frequency paraproducts (diagonal paraproducts) and also use shifted square function estimates on the high-high frequency paraproducts (off-diagonal paraproducts). Other works on the KP inequality, that use different methodology or provide new estimates, can be found in [11], [3], [1], [8], [18], [22], [21], [15], [7].

The motivation for our study arises from two main obstacles. These are already apparent when one attempts to derive the 3-factor KP from the 2-factor KP inequality: (a) when $r < 1$, applying the 2-factor inequality, we will unavoidably end up with some Hölder indices that are less than one. For instance, in the 3-factor case let $p_1 = p_2 = 3/2, p_3 = 2$ and observe that if q_1, q_2 are such that $\frac{2}{3} + \frac{2}{3} + \frac{1}{2} = \frac{1}{q_1} + \frac{1}{2} = \frac{2}{3} + \frac{1}{q_2}$, then $q_1 < 1$ and $q_2 < 1$. Then (1.1) can not be applied in this case as it requires the indices on the right to be greater than or equal to one; (b) when the indices vary (as defined in Theorem 1.1) then the 2-factor case is not applicable, as it may be the case that a subcollection of two of them is not related to the third index as in Hölder's relationship.

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It should be noted that Theorem 1.1 is not unexpected, as it is stated as Exercise 2.2 on page 76 in [20]. However, its detailed proof is rather technical and was missing from the literature until now.

We now state the precise formulations of our main result. All norms below are over $\mathbb{R}^n$.

Theorem 1.1. *Let $m \in \mathbb{Z}^+$, $\frac{1}{m} < r < \infty$, $1 < p_1, \dots, p_m \leqslant \infty$ satisfy $\frac{1}{r} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$. If $s > \max(\frac{n}{r} - n, 0)$ or $s \in 2\mathbb{N}$, then there exists a constant $C = C(n, q, s, p_1, \dots, p_m) < \infty$ such that for all $f_l \in \mathcal{S}(R^n)$ with $l \in \{1, \dots, m\}$ we have*

$$\|D^s(f_1 \cdots f_m)\|_{L^r} \leqslant C\big(\|D^s f_1\|_{L^{p_1}} \|f_2\|_{L^{p_2}} \cdots \|f_m\|_{L^{p_m}} + \cdots + \|f_1\|_{L^{p_1}} \|f_2\|_{L^{p_2}} \cdots \|D^s f_m\|_{L^{p_m}}\big) \tag{1.2}$$

$$\|J^s(f_1 \cdots f_m)\|_{L^r} \leqslant C\big(\|J^s f_1\|_{L^{p_1}} \|f_2\|_{L^{p_2}} \cdots \|f_m\|_{L^{p_m}} + \cdots + \|f_1\|_{L^{p_1}} \|f_2\|_{L^{p_2}} \cdots \|J^s f_m\|_{L^{p_m}}\big). \tag{1.3}$$

Furthermore, we note that in (1.2), (1.3) any tuple of indices $(p_1, \dots, p_m)$ that appears in a summand on the right of the inequality can be replaced by any other tuple $(q_1, \dots, q_m)$ when $\frac{1}{r} = \frac{1}{q_1} + \cdots + \frac{1}{q_m}$. When this is the case we say the indices vary.

2. Preliminary Material

For a measurable function $f : \mathbb{R}^n \to \mathbb{C}$ and $r > 0$ we use $\|f\|_{L^r}$ to denote the usual Lebesgue L^r norm (or quasi-norm if $r < 1$). For $A, B \in \mathbb{R}$ we use $A \lesssim B$ to mean $A \leqslant CB$ for some constant C. We also define $A \sim B$ when simultaneously $A \lesssim B$ and $B \lesssim A$. We denote elements of $(\mathbb{R}^n)^m$ by $\vec{\xi} = (\xi_1, \dots, \xi_m)$ and $d\vec{\xi} = d\xi_1 \cdots d\xi_m$. The Fourier transform and inverse Fourier transform of a function $f \in L^1(\mathbb{R}^n)$ are respectively defined by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(y) e^{-2\pi i y \cdot \xi} dy \qquad \widecheck{f}(\xi) = \int_{\mathbb{R}^n} f(y) e^{2\pi i y \cdot \xi} dy.$$

The space of Schwartz functions $\mathcal{S}(\mathbb{R}^n)$ is the space of all $C^\infty(\mathbb{R}^n)$ functions that rapidly decay at infinity.

Let $\widehat{\Phi}(\xi)$ be a positive, radially decreasing, and smooth function on $\mathbb{R}^n$ supported in $|\xi| \leqslant 2$ and equal to one on $|\xi| \leqslant 1$. Let $\widehat{\Psi}(\xi) = \widehat{\Phi}(\xi) - \widehat{\Phi}(2\xi)$, which is non-negative and supported in the annulus $\frac{1}{2} \leqslant |\xi| \leqslant 2$. Notice that as the series telescopes we have $\sum_{j \in \mathbb{Z}} \widehat{\Psi}(2^{-j}\xi) = 1$ for $\xi \neq 1$, as well as the useful identity $\sum_{j \leqslant j_0} \widehat{\Psi}(2^{-j}\xi) = \widehat{\Phi}(2^{-j_0}\xi)$ for any $j_0 \in \mathbb{Z}$ and $\xi \neq 0$.

For $\psi, \phi \in \mathcal{S}(\mathbb{R}^n)$ let $\widehat{\psi}$ be supported in an annulus centered about the origin and let $\widehat{\phi}$ be supported in a ball centered at the origin. We denote the Littlewood-Paley frequency projection operators over $\mathbb{R}^n$ by Δ_j^ψ and S_j^ϕ, which are respectively given by convolution with $2^{jn}\psi(2^j \cdot)$ and $2^{jn}\phi(2^j \cdot)$. The shifted Littlewood-Paley operators $\Delta_{j,\mu}^\psi$, $S_{j,\mu}^\phi$ for $\mu \in \mathbb{R}^n$ are respectively given by convolution with $2^{jn}\psi(2^j \cdot + c\mu)$ and $2^{jn}\phi(2^j \cdot + c\mu)$, where the constant c is independent of j, μ. When $\psi = \Psi$ or $\phi = \Phi$ the corresponding operators are denoted by Δ_j and S_j. Lastly, for $s \in \mathbb{R}^+$ and $f \in \mathcal{S}(\mathbb{R}^n)$ we denote the homogeneous and inhomogeneous differentiation operators as $\widehat{D^s f} = |\cdot|^s \widehat{f}$ and $\widehat{J^s f} = (1 + |\cdot|^2)^{\frac{s}{2}} \widehat{f}$ respectively.

The following lemma will be of great use throughout this paper and is the main ingredient in bounding the diagonal paraproducts.

Lemma 2.1. [12] *Let $f \in \mathcal{S}(\mathbb{R}^n)$ and $s > 0$. Then for any $\gamma \in [0, 1]$, there exists a constant $C(n, s, f)$ independent of γ, such that*

$$|(\gamma^2 I - \Delta)^{\frac{s}{2}} f(x)| \leqslant C(n, s, f)(1 + |x|)^{-n-s}.$$

The following multiplier result was proved by Coifman and Meyer [5] when $r \geqslant 1$ and was extended to the case $r < 1$ by Kenig and Stein [17] and by Grafakos and Torres [13]; for a proof see [10].

Theorem 2.2. *(Coifman-Meyer Multiplier Theorem). Suppose that $h(\xi_1, \dots, \xi_m)$ is a C^∞ function on $(\mathbb{R}^n)^m \backslash \{0\}$ which satisfies*

$$|\partial_{\xi_1}^{\beta_1} \cdots \partial_{\xi_m}^{\beta_m} h(\xi_1, \dots, \xi_m)| \leqslant C_{\beta_1, \dots, \beta_m} (|\xi_1| + \cdots + |\xi_m|)^{-(|\beta_1| + \cdots + |\beta_m|)} \tag{2.1}$$

for all multi-indices $\beta_1, \dots, \beta_m$. Let T be given by

$$T(f_1, \dots, f_m)(x) = \int_{\mathbb{R}^{nm}} h(\xi_1, \dots, \xi_m) \widehat{f_1}(\xi_1) \cdots \widehat{f_m}(\xi_m) e^{2\pi i (\xi_1 + \cdots + \xi_m) \cdot x} d\vec{\xi}$$

for $f_1, \dots, f_m \in \mathcal{S}(\mathbb{R}^n)$. *Then* T *is a bounded linear operator from* $L^{p_1} \times \cdots \times L^{p_m}$ *into* L^r *when* $1 < p_i \leqslant \infty$ *and*

$$\frac{1}{p_1} + \cdots + \frac{1}{p_m} = \frac{1}{r}.$$

We need to define some notation that will be used in the following theorem. The space $L^r\big(\mathbb{R}^n, \ell^\infty_L\big) = L^r\ell^\infty_L$ is all L-tuples of complex valued measurable functions defined on $\mathbb{R}^n$, $\{f_j\}_{1\leqslant j\leqslant L}$, such that

$$\big\|\{f_j\}\big\|_{L^r\ell^\infty_L} = \Big\| \sup_{1\leqslant j\leqslant L} |f_j| \Big\|_{L^r} < \infty.$$

The following theorem is the vector valued Calderón-Zygmund Theorem applied to the Banach spaces $L^r(\mathbb{R}^n)$ and $L^r\big(\mathbb{R}^n, \ell^\infty_L\big)$. Its proof can be found in [9, Theorem 5.6.1].

Theorem 2.3. *Let* $1 < r \leqslant \infty, L \in \mathbb{Z}^+$. *Suppose that* $K_1, \dots, K_L$ *are integrable functions defined on* $\mathbb{R}^n$ *that satisfy*

$$|K_j| \leqslant \frac{A_j}{|x|^n} \text{ for almost all } x \in \mathbb{R}^n \backslash \{0\} \text{ for some } A_j > 0, \text{ and,} \tag{2.2}$$

$$\sup_{y\neq 0} \int_{|x|\geqslant 2|y|} \sup_{1\leqslant j\leqslant L} |K_j(x-y) - K_j(x)|dx \leqslant A \text{ for some } A > 0. \tag{2.3}$$

Define

$$\vec{S}(f)(x) = \Big((K_1 * f)(x), \dots, (K_L * f)(x)\Big), \quad x \in \mathbb{R}^n,$$

where $f \in \bigcup_{1<p\leqslant\infty} L^p(\mathbb{R}^n)$. *Assume that* $\vec{S}$ *is a bounded linear operator from* $L^r\big(\mathbb{R}^n\big)$ *to* $L^r\big(\mathbb{R}^n, \ell^\infty_L\big)$ *with norm* $B_\star$. *Then there exists a constant* C_n *such that*

$$\Big\|\vec{S}(f)\Big\|_{L^p\ell^\infty_L} \leqslant C_n \max\big(p, (p-1)^{-1}\big)(A + B_\star)\|f\|_{L^p}$$

for all f *in* $L^p\big(\mathbb{R}^n\big)$, *whenever* $1 < p < \infty$.

Next we have a lemma that will be useful to us.

Lemma 2.4. *Let* $\mu \in \mathbb{R}^n$ *and* $f \in \mathcal{S}(\mathbb{R}^n)$ *then for all* $1 < p < \infty$ *the following estimates hold*

$$\Bigg\| \sqrt{\sum_{j\in\mathbb{Z}} |\Delta^\psi_{j,\mu} f|^2} \Bigg\|_{L^p} \leqslant C_{n,\psi} \max(p, (p-1)^{-1}) \ln(2 + |\mu|)\|f\|_{L^p} \tag{2.4}$$

$$\Big\| \sup_{j\in\mathbb{N}} |S^\phi_{j,\mu} f| \Big\|_{L^p} \leqslant C_{n,\phi} \max(p, (p-1)^{-1}) \ln(2 + |\mu|)\|f\|_{L^p} \tag{2.5}$$

$$\Big\| \sup_{j\in\mathbb{N}} |\Delta^\psi_{j,\mu} f| \Big\|_{L^p} \leqslant C_{n,\psi} \max(p, (p-1)^{-1}) \ln(2 + |\mu|)\|f\|_{L^p}\,. \tag{2.6}$$

Proof. The proof of (2.4) is given in [12] and omitted. For (2.5) we apply Theorem 2.3 where $K_j(x) = 2^{jn}\phi(2^j x + \mu)$, that is the operator

$$\vec{S}(f) = \big(2^n\phi(2\cdot + \mu) * f(x), \dots, 2^{Ln}\phi(2^L \cdot + \mu) * f(x)\big), \quad x \in \mathbb{R}^n,$$

where $f \in \mathcal{S}(\mathbb{R}^n)$ and $L \in \mathbb{N}$. For the size condition (2.2) we have

$$|K_j| = |2^{jn}\phi(2^j x + \mu)| \leqslant \frac{c_n}{(2^{-j} + |x + 2^{-j}\mu|)^n} \leqslant \frac{c_{n,j}}{|x|^n}$$

since ϕ is a Schwartz function. The smoothness estimate, (2.3) follows from the inequality

$$\sup_{y\neq 0} \int_{|x|\geqslant 2|y|} \sum_{1\leqslant j\leqslant L} 2^{jn}|\phi(2^j(x-y) + \mu) - \phi(2^j x + \mu)|dx \leqslant C_n \ln(2 + |\mu|). \tag{2.7}$$

In [12], (2.7) is proven with ψ in place of ϕ. However the same proof applies here as the only property of ψ used in the proof is that it is a Schwartz function. Thus, we obtain the smoothness estimate (2.3) with $A = C_n \ln(2+|\mu|)$. Lastly, observe that $\vec{S}$ is a bounded operator from $L^\infty(\mathbb{R}^n)$ to $L^\infty\big(\mathbb{R}^n, \ell^\infty_L\big)$ with $B_\star = \|\phi\|_1$. Thus we satisfy the hypothesis of (2.3), giving

$$\Big\| \sup_{1\leqslant j\leqslant L} |\mathcal{S}^{\phi}_{j,\mu} f| \Big\|_{L^p} \leqslant C_n \max(p,(p-1)^{-1}) \ln(2+|\mu|) \|f\|_{L^p} .$$

Applying Lebesgue monotone convergence theorem and letting $L \to \infty$ we obtain (2.5). A similar argument can be made for (2.6). □

We will use Khintchine's inequalities in the proof of Theorem 1.1 to obtain a variant of the following vector-valued version of the Littlewood-Paley theorem for $1 < r < \infty$:

$$\Big\| \sqrt{\sum_{k\in\mathbb{Z}} \sum_{j\geqslant 0} |\Delta_k F_j|^2} \Big\|_{L^r} \lesssim \Big\| \sqrt{\sum_{j\geqslant 0} |F_j|^2} \Big\|_{L^r} .$$

Theorem 2.5. *(Khintchine's Inequalities) For $j = 0,1,2,\dots$ let r_j be the jth Rademacher function, that is $r_j(t) = \operatorname{sgn}(\sin(2^j\pi t))$. For any $0 < p < \infty$ and for any real valued square summable sequences $\{a_j\}$ and $\{b_j\}$ we have*

$$B_p \Big(\sum_{j=0}^{\infty} |a_j + ib_j|^2 \Big)^{\frac12} \leqslant \Big\| \sum_{j=0}^{\infty} (a_j + ib_j) r_j \Big\|_{L^p([0,1])} \leqslant A_p \Big(\sum_{j=0}^{\infty} |a_j + ib_j|^2 \Big)^{\frac12} \tag{2.8}$$

where $0 < A_p, B_p < \infty$ depend only on p.

Define $F_n(t_1,\dots,t_n) = \sum_{j_1=0}^{\infty} \cdots \sum_{j_n=0}^{\infty} c_{j_1,\dots,j_n} r_{j_1}(t_1)\dots r_{j_n}(t_n)$ for $t_j \in [0,1]$, where $c_{j_1,\dots,j_n}$ is a sequence of complex numbers and F_n is a function defined on $[0,1]^n$. For any $0 < p < \infty$ and for any complex-valued square summable sequence of n variables $\{c_{j_1,\dots,j_n}\}_{j_1,\dots,j_n}$, we have the following inequalities for F_n:

$$B_p^n \Big(\sum_{j_1=0}^{\infty} \cdots \sum_{j_n=0}^{\infty} |c_{j_1,\dots,j_n}|^2 \Big)^{\frac12} \leqslant \|F_n\|_{L^p([0,1]^n)} \leqslant A_p^n \Big(\sum_{j_1=0}^{\infty} \cdots \sum_{j_n=0}^{\infty} |c_{j_1,\dots,j_n}|^2 \Big)^{\frac12}. \tag{2.9}$$

3. Proof of Theorem 1.1

As highlighted in the introduction, this proof constitutes an extension of the work presented in [12]. While many of the methods employed align with those in the referenced work, the technical intricacies are different. For this reason we provide all the details.

We begin with a decomposition of $\mathbb{Z}^m$. For $\vec{\eta} = (\eta_1,\dots,\eta_m) \in \{0,1\}^m \backslash \{0\}$ let

$$\begin{aligned}\mathscr{B}_{\vec{\eta}} :=\{&(j_1,\dots,j_m)\in\mathbb{Z}^m : \text{If } \eta_t = 1 \text{ for some } 1\leqslant t\leqslant m \text{ then, } \max\{j_1,\dots,j_m\} = j_t \text{ and } j_t > 0.\\ &\text{If } \eta_t = 0 \text{ then } \max\{j_1,\dots,j_m\} > j_t.\}.\end{aligned}$$

Notice that

$$\mathbb{Z}^m = \Big(\bigsqcup_{\vec{\eta}\in\{0,1\}^m\backslash\{0\}} \mathscr{B}_{\vec{\eta}} \Big) \bigsqcup (\mathbb{Z}_{\leqslant 0})^m, \tag{3.1}$$

where $\mathbb{Z}_{\leqslant 0} = \{0,-1,-2,\dots\}$. Observe for $f_k \in \mathcal{S}(\mathbb{R}^n)$ we have

$$\begin{aligned} &J^s(f_1 f_2\cdots f_m)(x)\\ &= \int_{\mathbb{R}^{mn}} (1+|\xi_1+\cdots+\xi_m|^2)^{\frac{s}{2}} \widehat{f}_1(\xi_1)\widehat{f}_2(\xi_2)\cdots\widehat{f_m}(\xi_m) e^{2\pi i(\xi_1+\cdots+\xi_m)\cdot x} d\xi_1 d\xi_2\cdots d\xi_m\\ &= \sum_{\vec{j}\in\mathbb{Z}^m} \int_{\mathbb{R}^{mn}} (1+|\xi_1+\cdots+\xi_m|^2)^{\frac{s}{2}} \widehat{\Psi}(2^{-j_1}\xi_1)\cdots\widehat{\Psi}(2^{-j_m}\xi_m)\widehat{f}_1(\xi_1)\cdots\widehat{f_m}(\xi_m) e^{2\pi i(\xi_1+\cdots+\xi_m)\cdot x} d\vec{\xi}. \qquad (3.2)\end{aligned}$$

For ease of notation let

$$u_{\vec{j}}(\vec{\xi},x) := (1+|\xi_1+\cdots+\xi_m|^2)^{\frac{s}{2}} \widehat{\Psi}(2^{-j_1}\xi_1)\cdots\widehat{\Psi}(2^{-j_m}\xi_m)\widehat{f}_1(\xi_1)\cdots\widehat{f_m}(\xi_m) e^{2\pi i(\xi_1+\cdots+\xi_m)\cdot x}.$$

Breaking the sum over $\mathbb{Z}^m$ into the 2^m disjoint sets we may write (3.2) as

$$
\begin{aligned}
J^s(f_1 f_2 \cdots f_m)(x) &= \sum_{\vec{j}\in\mathbb{Z}^m} \int_{\mathbb{R}^{mn}} u_{\vec{j}}(\vec{\xi},x)d\vec{\xi} \\
&= \sum_{\vec{\eta}\in\{0,1\}^m\setminus\{0\}} \sum_{\vec{j}\in\mathscr{B}_{\vec{\eta}}} \int_{\mathbb{R}^{mn}} u_{\vec{j}}(\vec{\xi},x)d\vec{\xi} + \sum_{\vec{j}\in(\mathbb{Z}_{\leqslant 0})^m} \int_{\mathbb{R}^{mn}} u_{\vec{j}}(\vec{\xi},x)d\vec{\xi}. \qquad (3.3)
\end{aligned}
$$

For the second term in (3.3) we have

$$
\begin{aligned}
&\sum_{\vec{j}\in(\mathbb{Z}_{\leqslant 0})^m} \int_{\mathbb{R}^{mn}} u_{\vec{j}}(\vec{\xi},x)d\vec{\xi} \\
&= \int_{\mathbb{R}^{nm}} (1+|\xi_1+\cdots+\xi_m|^2)^{\frac{s}{2}} \widehat{\Phi}(\xi_1)\widehat{f_1}(\xi_1)\cdots\widehat{\Phi}(\xi_m)\widehat{f_m}(\xi_m)e^{2\pi i(\xi_1+\cdots+\xi_m)\cdot x}d\vec{\xi}. \qquad (3.4)
\end{aligned}
$$

Let $w := (S_0 f_1)\cdots(S_0 f_m)$. Note that $\widehat{w}$ is supported in $|\xi| \leqslant 2m$. Thus we have

$$\widehat{w}(\xi) = \widehat{\Phi}(2^{-m}\xi)\widehat{w}(\xi),$$

since $\widehat{\Phi}(2^{-m}\cdot)$ equals 1 on the support of $\widehat{w}$. It follows that (3.4) can be written as

$$\int_{\mathbb{R}^n} (1+|\xi|^2)^{\frac{s}{2}} \widehat{\Phi}(2^{-m}\xi)\widehat{w}(\xi)e^{2\pi i\xi\cdot x}d\xi. \qquad (3.5)$$

Since $(1+|\xi|^2)^{\frac{s}{2}}\widehat{\Phi}(2^{-m}\xi)$ is a smooth function with compact support we can expand it in Fourier series,

$$(1+|\xi|^2)^{\frac{s}{2}}\widehat{\Phi}(2^{-m}\xi) = \chi_{[-2^{m+1},2^{m+1}]^n}(\xi) \sum_{\mu\in\mathbb{Z}^n} c_\mu e^{2\pi i\xi\cdot\frac{\mu}{2^{m+2}}}$$

where the Fourier coefficients decay rapidly in μ. It follows

$$
\begin{aligned}
\left\| \int_{\mathbb{R}^n} \sigma(\xi)\widehat{w}(\xi)e^{2\pi i\xi\cdot x}d\xi \right\|_{L^r(dx)} &= \left\| \sum_{\mu\in\mathbb{Z}^n} \int_{\mathbb{R}^n} c_\mu e^{2\pi i\xi\cdot\frac{\mu}{2^{m+2}}}\widehat{w}(\xi)e^{2\pi i\xi\cdot x}d\xi \right\|_{L^r(dx)} \\
&\leqslant \sum_{\mu\in\mathbb{Z}^n} |c_\mu| \left\| (S_0 f_1)(x+2^{-(m+2)}\mu)\cdots(S_0 f_m)(x+2^{-(m+2)}\mu) \right\|_{L^r(dx)} \\
&\lesssim \|S_0 f_1\|_{L^{p_1}}\cdots\|S_0 f_m\|_{L^{p_m}} \\
&\lesssim \left\|J^{-s}J^s f_1\right\|_{L^{p_1}}\|f_2\|_{L^{p_2}}\cdots\|f_m\|_{L^{p_m}} \\
&\lesssim \|J^s f_1\|_{L^{p_1}}\|f_2\|_{L^{p_2}}\cdots\|f_m\|_{L^{p_m}}.
\end{aligned}
$$

The last line above is justified by the fact that the Bessel potential J^{-s} is an L^{p_1} Fourier multiplier operator. Now to bound the first term in (3.3), that is

$$
\begin{aligned}
\sum_{\vec{\eta}\in\{0,1\}^m\setminus\{0\}} \sum_{\vec{j}\in\mathscr{B}_{\vec{\eta}}} \int_{\mathbb{R}^{mn}} &(1+|\xi_1+\cdots+\xi_m|^2)^{\frac{s}{2}}\widehat{\Psi}(2^{-j_1}\xi_1)\cdots\widehat{\Psi}(2^{-j_m}\xi_m) \\
&\times\widehat{f_1}(\xi_1)\cdots\widehat{f_m}(\xi_m)e^{2\pi i(\xi_1+\cdots+\xi_m)\cdot x}d\vec{\xi}. \qquad (3.6)
\end{aligned}
$$

From (3.6) we see it is sufficient to consider an $\vec{\eta}$ with exactly b ones. Moreover, by symmetry it is sufficient to only consider when the first b entries are ones, specifically let

$$\vec{\eta}_0 = (\underbrace{1,1,\ldots,1}_{b},0,0,\ldots,0).$$

Notice that $\mathscr{B}_{\vec{\eta}_0}$ is the elements of $\mathbb{Z}^m$ where the first b entries are the same, positive and strictly bigger then the remaining entries. It follows to estimate (3.6) it is enough to only consider the term

$$\int_{\mathbb{R}^{mn}} \sum_{\vec{j}\in\mathscr{B}_{\vec{\eta}_0}} u_{\vec{j}}(\vec{\xi},x)d\vec{\xi}$$

$$= \sum_{j\in\mathbb{N}} \int_{\mathbb{R}^{mn}} (1 + |\xi_1 + \cdots + \xi_m|^2)^{\frac{s}{2}} \widehat{\Psi}(2^{-j}\xi_1) \cdots \widehat{\Psi}(2^{-j}\xi_b)\widehat{\Phi}(2^{-j+1}\xi_{b+1}) \cdots \widehat{\Phi}(2^{-j+1}\xi_m) \times \widehat{f}_1(\xi_1) \cdots \widehat{f_m}(\xi_m) e^{2\pi i(\xi_1+\cdots+\xi_m)\cdot x} d\vec{\xi}. \tag{3.7}$$

We will break the proof into two cases; when $b = 1$, which we call off-diagonal terms and when $b > 1$, which we call diagonal terms.

3.1. b = 1: Off-Diagonal Term

Fix $a \in \mathbb{N}$ to be determined later. When $b = 1$ (3.7) is equal to

$$= \sum_{j\in\mathbb{N}} \int_{\mathbb{R}^{mn}} (1 + |\xi_1 + \cdots + \xi_m|^2)^{\frac{s}{2}} \widehat{\Psi}(2^{-j}\xi_1) \Big(\sum_{j-a<j_2<j} \widehat{\Psi}(2^{-j_2}\xi_2) + \widehat{\Phi}(2^{-j+a}\xi_2) \Big) \cdots \times \Big(\sum_{j-a<j_m<j} \widehat{\Psi}(2^{-j_m}\xi_m) + \widehat{\Phi}(2^{-j+a}\xi_m) \Big) \widehat{f}_1(\xi_1) \cdots \widehat{f_m}(\xi_m) e^{2\pi i(\xi_1+\cdots+\xi_m)\cdot x} d\vec{\xi}. \tag{3.8}$$

Multiplying out the product in (3.8) we see it is equal to

$$\sum_{j\in\mathbb{N}} \int_{\mathbb{R}^{mn}} (1 + |\xi_1 + \cdots + \xi_m|^2)^{\frac{s}{2}} \widehat{\Psi}(2^{-j}\xi_1) \widehat{f}_1(\xi_1) \widehat{\Phi}(2^{-j+a}\xi_2) \widehat{f}_2(\xi_2) \cdots \times \widehat{\Phi}(2^{-j+a}\xi_m) \widehat{f_m}(\xi_m) e^{2\pi i(\xi_1+\cdots+\xi_m)\cdot x} d\vec{\xi}. \tag{3.9}$$

plus finitely many other terms of the form

$$\sum_{j\in\mathbb{N}} \int_{\mathbb{R}^{mn}} (1 + |\xi_1 + \cdots + \xi_m|^2)^{\frac{s}{2}} \widehat{\Psi}(2^{-j}\xi_1) \widehat{f}_1(\xi_1) V_j^2(\xi_2) \widehat{f}_2(\xi_2) V_j^3(\xi_3) \widehat{f}_3(\xi_3) \times \cdots \cdots \times V_j^m(\xi_m) \widehat{f_m}(\xi_m) e^{2\pi i(\xi_1+\cdots+\xi_m)\cdot x} d\vec{\xi}. \tag{3.10}$$

where at least one V_j^k is $\widehat{\Psi}(2^{-k}\cdot)$ with $k \sim j$ and rest are $\widehat{\Phi}(2^{-j+a}\cdot)$. As $j \sim k$, where the implicit constant depends on a, the finitely many terms of the form in (3.10) will be handled by the same technique used in the $b > 1$ case. Thus for the $b = 1$ case it is sufficient to only consider (3.9).

Now to determine a. Looking at the Fourier transform of (3.9) the idea is to pick a large enough so that we have $|\xi_1| \gg |\xi_k|$ for $k \in \{2, \ldots, m\}$. For a large enough the Fourier transform of a summand of (3.9) is supported in the algebraic sum of an annulus with $m - 1$ relatively much smaller balls, which is a slightly bigger annulus. Specifically, if $a > \log_2(8m)$ then on the Fourier transform side we have that $|\xi_k| \leqslant 2^{j-a+1} < \frac{1}{2m} 2^{j-1} \leqslant \frac{1}{2m}|\xi_1|$, which then implies by the reverse triangle inequality that

$$\frac{1}{2}|\xi_1| \leqslant |\xi_1 + \cdots + \xi_m| \leqslant 2|\xi_1|.$$

Let

$$\Pi(\xi_1, \ldots, \xi_m) = \sum_{j\in\mathbb{N}} \widehat{\Psi}(2^{-j}\xi_1) \widehat{\Phi}(2^{-j+a}\xi_2) \cdots \widehat{\Phi}(2^{-j+a}\xi_m) \tag{3.11}$$

then (3.9) can be expressed by

$$\int_{\mathbb{R}^{mn}} \langle \xi_1 + \cdots + \xi_m \rangle^s \langle \xi \rangle^{-s} \Pi(\xi_1, \ldots, \xi_m) \widehat{J^s f_1}(\xi_1) \cdots \widehat{f_m}(\xi_m) e^{2\pi i(\xi_1+\cdots+\xi_m)\cdot x} d\vec{\xi}.$$

We proceed by showing that

$$\frac{\langle \xi_1 + \cdots + \xi_m \rangle^s}{\langle \xi_1 \rangle^s} \Pi(\xi_1, \ldots, \xi_m) \tag{3.12}$$

is a Coifman-Meyer multiplier, i.e., it satisfies estimates (2.1). First observe that Π is a Coifman-Meyer multiplier. To see this observe for a multi-index $\vec{\beta} = (\beta_1, \ldots, \beta_m)$ and $\partial^{\vec{\beta}} = \partial_{\xi_1}^{\beta_1} \cdots \partial_{\xi_m}^{\beta_m}$ we have

$$|\partial^{\vec{\beta}} \Pi(\xi_1, \ldots, \xi_m)|$$

$$\begin{aligned}
&\leqslant \sum_{j\in\mathbb{Z}} C_{\alpha_1,\dots,\alpha_l} |\partial^{\beta_1}\widehat{\Psi}|(2^{-j}\xi_1)2^{-|\beta_1|j}|\partial^{\beta_2}\widehat{\Phi}|(2^{-j+a}\xi_2)2^{-|\beta_2|j}\cdots|\partial^{\beta_m}\widehat{\Phi}|(2^{-j+a}\xi_m)2^{-|\beta_m|j}\chi_{\mathrm{supp}(\Pi)}(\vec{\xi}).\\
&\lesssim |2^{-j}\xi_1|^{-(|\beta_1|+\cdots+|\beta_m|)}2^{-|\beta_1|j}2^{-|\beta_2|j}2^{-|\beta_m|j}\chi_{\mathrm{supp}(\Pi)}(\vec{\xi})\\
&\lesssim (|\xi_1|+\cdots+|\xi_m|)^{-(|\beta_1|+\cdots+|\beta_m|)} \qquad (3.13)
\end{aligned}$$

where the last inequality is due to $|\xi_1|$ being bigger than all other $|\xi_k|$. Furthermore, partials can be brought into the sum because for any fixed value of ξ, it is a finite sum.

For $y \in \mathbb{R}^n$ define

$$\gamma(y) = (1+|y|^2)^{\frac{q}{2}}$$

where $q \in \mathbb{R}$. We will show

$$|\partial^\beta \gamma(y)| \leqslant C_{n,\beta,q}(1+|y|)^{q-|\beta|} \qquad (3.14)$$

for multi-index β. Let $\gamma_*(t,y) = (t^2+|y|^2)^{\frac{q}{2}}$ where $t \in \mathbb{R}$, notice $\gamma_*(1,y) = \gamma(y)$. Observe that by homogeneity we have $\gamma_*(\lambda t, \lambda y) = \lambda^q \gamma_*(t,y)$ for any $\lambda > 0$, thus for a multi-index β and any $(t,y) \in \mathbb{R}\times\mathbb{R}^n$ we have

$$\lambda^{|\beta|-q}\partial^\beta\gamma_*(\lambda(t,y)) = \partial^\beta\gamma_*(t,y).$$

Letting $\lambda = |(t,y)|^{-1} \neq 0$ we obtain

$$|\partial^\beta\gamma_*(t,y)| \leqslant |(t,y)|^{q-|\beta|}\sup_{(t,y)'\in\mathbb{S}^n}|\partial^\beta\gamma_*((t,y)')|. \qquad (3.15)$$

Letting $t = 1$ and using that γ_* is smooth on the sphere $\mathbb{S}^n$, so bounded there we deduce (3.14).

Let $\gamma_1(\vec{\xi}) = \langle\xi_1+\cdots+\xi_m\rangle^s$, and $\gamma_2(\xi_1) = \langle\xi_1\rangle^{-s}$, then (3.12) is equal to $\Pi\gamma_1\gamma_2$. Let $\beta_1 = \alpha_1^1+\alpha_2^1+\alpha_3^1$ and $\beta_k = \alpha_1^k+\alpha_2^k$ for $k \in \{2,\dots,m\}$. By Leibniz rule $\partial_{\xi_1}^{\beta_1}\cdots\partial_{\xi_m}^{\beta_m}(\gamma_1\gamma_2\Pi)$ is a linear combination of terms of the form

$$[\partial_{\xi_1}^{\alpha_1^m}\cdots\partial_{\xi_m}^{\alpha_1^1}\Pi][\partial_{\xi_1}^{\alpha_2^m}\cdots\partial_{\xi_m}^{\alpha_2^1}\gamma_1][\partial_{\xi_1}^{\alpha_3^1}\gamma_2]. \qquad (3.16)$$

By (3.14) we have the absolute value of (3.16) is bounded by a constant multiple of

$$\begin{aligned}
&(|\xi_1|+\cdots+|\xi_m|)^{-\sum|\alpha_1^k|}(1+|\xi_1+\cdots+\xi_m|)^{\frac{s}{2}-\sum|\alpha_2^k|}(1+|\xi_1|)^{-\frac{s}{2}-|\alpha_3^1|}\chi_{\mathrm{supp}(\Pi)}(\vec{\xi})\\
&\lesssim (|\xi_1|+\cdots+|\xi_m|)^{-(\sum|\alpha_1^k|+\sum|\alpha_2^k|+|\alpha_3^1|)} \qquad (3.17)\\
&= (|\xi_1|+\cdots+|\xi_m|)^{-(|\beta_1|+\cdots+|\beta_m|)}
\end{aligned}$$

where in (3.17) we used $|\xi_1+\cdots+\xi_m| \sim |\xi_1|$ on the support of Π. It follows that (3.12) is a Coifman-Meyer multiplier, therefore applying Theorem 2.2 gives the desired inequality.

3.2. b > 1: Diagonal Term

Now we focus on the diagonal term; (3.7) when $b > 1$ is given by

$$\begin{aligned}
&\sum_{j\in\mathbb{N}}\int_{\mathbb{R}^{mn}}\langle\xi_1+\cdots+\xi_m\rangle^s\widehat{\Psi}(2^{-j}\xi_1)\widehat{f_1}(\xi_1)\widehat{\Psi}(2^{-j}\xi_2)\widehat{f_2}(\xi_2)\cdots\widehat{\Psi}(2^{-j}\xi_b)\widehat{f_b}(\xi_b) \qquad (3.18)\\
&\qquad\times\widehat{\Phi}(2^{-j+1}\xi_{b+1})\widehat{f_{b+1}}(\xi_{b+1})\cdots\widehat{\Phi}(2^{-j+1}\xi_m)\widehat{f_m}(\xi_m)e^{2\pi i(\xi_1+\cdots+\xi_m)\cdot x}d\vec{\xi}\\
&= J^s\big((\Delta_j f_1)(\Delta_j f_2)\cdots(\Delta_j f_b)(S_{j-1}f_{b+1})\cdots(S_{j-1}f_m)\big).
\end{aligned}$$

Case (I)**:** $p_t < \infty$ for at least two $t \in \{1,\dots,b\}$.

Suppose without loss of generality that $p_1 < \infty$, $p_2 < \infty$. Observe that $\widehat{\Phi}((m2^{j+1})^{-1}(\xi_1+\cdots+\xi_m))$ equals 1 on the support of the integrand in (3.18). Let

$$\sigma_j(\xi) := (2^{-2j}+|\xi|^2)^{\frac{s}{2}}\widehat{\Phi}((2m)^{-1}\xi).$$

Expanding σ_j in Fourier series we have

$$\sigma_j(y) = \chi_{[-4m,4m]^n}(y)\sum_{\mu\in\mathbb{Z}^n}c_{j,\mu}e^{2\pi i y\cdot\frac{\mu}{8m}}.$$

By Lemma 2.1 we have $|\widehat{\sigma_j}(\mu)| = |c_{j,\mu}| \lesssim (1+|\mu|)^{-n-s}$ independent of j since $j > 0$. Notice in the case $s \in 2\mathbb{N}$ we have arbitrarily fast decay, that is, $|c_{j,\mu}| \lesssim (1+|\mu|)^l$ independent of $j > 0$ and for any $l \in \mathbb{N}$. Since this simplifies the proof we will assume $s \notin 2\mathbb{N}$. Let $\widehat{\Psi_*}(\xi) = |\xi|^{-s}\widehat{\Psi}(\xi)$ and Δ_j^* by given by convolution with $2^{jn}|2^j\cdot|^{-s}\Psi_*(2^j\cdot)$. Furthermore, let $\Delta_{\mu,j}f(x) := \Delta_j f(x+m^{-1}2^{-j-3}\mu)$, $\Delta_{\mu,j}^* f(x) := \Delta_j^* f(x+m^{-1}2^{-j-3}\mu)$, and $S_{\mu,j}f(x) := S_j f(x+m^{-1}2^{-j-3}\mu)$. It follows (3.18) is equal to

$$\begin{aligned}
&\sum_{j\in\mathbb{N}}\int_{\mathbb{R}^{mn}} 2^{js}\sigma_j(2^{-j}(\xi_1+\cdots+\xi_m))\widehat{\Psi}(2^{-j}\xi_1)\widehat{f}_1(\xi_1)\widehat{\Psi}(2^{-j}\xi_2)\widehat{f}_2(\xi_2)\cdots\widehat{\Psi}(2^{-j}\xi_b)\widehat{f}_b(\xi_b)\\
&\qquad\times\widehat{\Phi}(2^{-j+1}\xi_{b+1})\widehat{f_{b+1}}(\xi_{b+1})\cdots\widehat{\Phi}(2^{-j+1}\xi_m)\widehat{f_m}(\xi_m)e^{2\pi i(\xi_1+\cdots+\xi_m)\cdot x}d\vec{\xi}\\
&=\sum_{j\in\mathbb{N}}\sum_{\mu\in\mathbb{Z}^n} c_{j,\mu}\int_{\mathbb{R}^{mn}}\widehat{\Psi_*}(2^{-j}\xi_1)|\xi_1|^s\widehat{\Psi}(2^{-j}\xi_1)\widehat{f}_1(\xi_1)\widehat{\Psi}(2^{-j}\xi_2)\widehat{f}_2(\xi_2)\cdots\widehat{\Psi}(2^{-j}\xi_b)\widehat{f}_b(\xi_b)\\
&\qquad\times\widehat{\Phi}(2^{-j+1}\xi_{b+1})\widehat{f_{b+1}}(\xi_{b+1})\cdots\widehat{\Phi}(2^{-j+1}\xi_m)\widehat{f_m}(\xi_m)e^{2\pi i(\xi_1+\cdots+\xi_m)\cdot(x+m^{-1}2^{-j-3}\mu)}d\vec{\xi}\\
&=\sum_{j\in\mathbb{N}}\sum_{\mu\in\mathbb{Z}^n} c_{j,\mu}(\Delta_{\mu,j}^*D^s f_1)(x)(\Delta_{\mu,j}f_2)(x)\cdots(\Delta_{\mu,j}f_b)(x)\\
&\qquad\times(S_{\mu,j-1}f_{b+1})(x)\cdots(S_{\mu,j-1}f_m)(x),
\end{aligned}\tag{3.19}$$

where the we can drop the characteristic function due to the support of the integrand. Then taking the absolute value of (3.19) and applying the Cauchy-Schwarz inequality we deduce

$$\begin{aligned}
&\left|\sum_{j\in\mathbb{N}}\sum_{\mu\in\mathbb{Z}^n} c_{j,\mu}(\Delta_{\mu,j}^*D^s f_1)(\Delta_{\mu,j}f_2)\cdots(\Delta_{\mu,j}f_b)(S_{\mu,j-1}f_{b+1})\cdots(S_{\mu,j-1}f_m)\right|\\
&\leqslant\sum_{\mu\in\mathbb{Z}^n}(1+|\mu|)^{-n-s}\Big(\sum_{j\in\mathbb{N}}\Big|\Delta_{\mu,j}^*D^s f_1\Big|^2\Big)^{\frac12}\Big(\sum_{j\in\mathbb{N}}\Big|(\Delta_{\mu,j}f_2)\cdots(\Delta_{\mu,j}f_b)(S_{\mu,j-1}f_{b+1})\cdots(S_{\mu,j-1}f_m)\Big|^2\Big)^{\frac12}\\
&\leqslant\sum_{\mu\in\mathbb{Z}^n}(1+|\mu|)^{-n-s}\Big(\sum_{j\in\mathbb{N}}\Big|\Delta_{\mu,j}^*D^s f_1\Big|^2\Big)^{\frac12}\Big(\sum_{j\in\mathbb{N}}\big|\Delta_{\mu,j}f_2\big|^2\Big)^{\frac12}\\
&\qquad\times\sup_{j\in\mathbb{N}}\big|\Delta_{\mu,j}f_3\big|\cdots\sup_{j\in\mathbb{N}}\big|\Delta_{\mu,j}f_b\big|\sup_{j\in\mathbb{N}}\big|S_{\mu,j-1}f_{b+1}\big|\cdots\sup_{j\in\mathbb{N}}\big|S_{\mu,j-1}f_m\big|.
\end{aligned}\tag{3.20}$$

Let $\tilde{r} = \min\{r,1\}$. In view of the subadditivity property of the expression $\|\cdot\|_{L^r}^{\tilde{r}}$ and Hölder's inequality, applying $\|\cdot\|_{L^r}^{\tilde{r}}$ to (3.20) we obtain the bound

$$\leqslant C_{n,m,s,p_1,\dots,p_l}\sum_{\mu\in\mathbb{Z}^n}(1+|\mu|)^{\tilde{r}(-n-s)}\ln(2+|\mu|)^{m\tilde{r}}\|D^s f_1\|_{L^{p_1}}^{\tilde{r}}\|f_2\|_{L^{p_2}}^{\tilde{r}}\cdots\|f_m\|_{L^{p_m}}^{\tilde{r}},$$

where we used Lemma 2.4. Since D^sJ^{-s} is a L^{p_1} multiplier we obtain $\|D^s f_1\|_{L^{p_1}} \lesssim \|J^s f_1\|_{L^{p_1}}$, so all that remains to show is

$$\sum_{\mu\in\mathbb{Z}^n}(1+|\mu|)^{\tilde{r}(-n-s)}\ln(2+|\mu|)^{m\tilde{r}} < \infty. \tag{3.21}$$

Since $(n+s)\tilde{r} > n$ by hypothesis, α can be picked small enough so that $n < (n+s)\tilde{r} - m\tilde{r}\alpha = n+\epsilon_1$ with $\epsilon_1 > 0$, thus

$$\sum_{\mu\in\mathbb{Z}^n}(1+|\mu|)^{-\tilde{r}(n+s)}(2+|\mu|)^{m\tilde{r}\alpha} \lesssim \sum_{\mu\in\mathbb{Z}^n}\frac{1}{(1+|\mu|)^{n+\epsilon_1}} < \infty$$

completing Case (I).

Case (II)**:** $p_t < \infty$ for only one $t \in \{1,\dots,b\}$.

This case requires a more delicate approach since we can not directly apply the Cauchy-Schwarz inequality. Suppose without loss of generality that $1 < p_1 < \infty$ and $p_t = \infty$ for all $t \in \{2,\dots,m\}$. This implies $1 < r = p_1 < \infty$, thus we can use square function estimates. Thus the L^r norm of (3.18) is bounded by a constant multiple of

$$\lesssim \left\| \sqrt{\sum_{k\in\mathbb{Z}} \left| \Delta_k \Big(\sum_{j\in\mathbb{N}} J^s\big((\Delta_j f_1)(\Delta_j f_2)\cdots(\Delta_j f_b)(S_{j-1}f_{b+1})\cdots(S_{j-1}f_m)\big) \right|^2} \right\|_{L^r}. \tag{3.22}$$

We now consider two subcases where $k > 0$ and $k \leqslant 0$. Let $\sigma_{k,s}(\xi) := \widehat{\Psi}(2^{-k}\xi)(2^{-2k} + |2^{-k}\xi|^2)^{\frac{s}{2}}$ and $\Omega_{k,s}$ be the associated multiplier operator. Let $\widehat{\Psi_*}(\xi) = |\xi|^{-s}\widehat{\Psi}(\xi)$ and Δ_j^* be given by convolution with $2^{jn}|2^j\cdot|^{-s}\Psi_*(2^j\cdot)$. First suppose that $k > 0$. Let $c_0 = \log_2(4m)$, now observe

$$\begin{aligned}
&\Big|\sum_{j\in\mathbb{N}} \Delta_k J^s\big((\Delta_j f_1)(\Delta_j f_2)\cdots(\Delta_j f_b)(S_{j-1}f_{b+1})\cdots(S_{j-1}f_m)\big)\Big| \\
&= \Big| \sum_{j\geqslant \max(1,k-c_0)} 2^{ks}\int_{\mathbb{R}^{mn}} \widehat{\Psi}(2^{-k}(\xi_1+\cdots+\xi_m))(2^{-2k}+|2^{-k}(\xi_1+\cdots+\xi_m)|^2)^{\frac{s}{2}} 2^{-js} \\
&\quad \times |2^{-j}\xi_1|^{-s}\widehat{\Psi}(2^{-j}\xi_1)|\xi_1|^s\widehat{f}_1(\xi_1)\widehat{\Psi}(2^{-j}\xi_2)\widehat{f}_2(\xi_2)\cdots\widehat{\Psi}(2^{-j}\xi_b)\widehat{f}_b(\xi_b) \\
&\qquad \times \widehat{\Phi}(2^{-j+1}\xi_{b+1})\widehat{f_{b+1}}(\xi_{b+1})\cdots\widehat{\Phi}(2^{-j+1}\xi_m)\widehat{f_m}(\xi_m)e^{2\pi i(\xi_1+\cdots+\xi_m)\cdot x}d\vec{\xi}\Big| \\
&\leqslant 2^{ks}\sum_{j\geqslant \max(1,k-c_0)} 2^{-js}\Big|\int_{\mathbb{R}^{mn}} \sigma_{k,s}(2^{-k}(\xi_1+\cdots+\xi_m))\widehat{\Psi}_*(2^{-j}\xi_1)\widehat{D^s f_1}(\xi_1) \\
&\qquad \times \widehat{\Psi}(2^{-j}\xi_2)\widehat{f}_2(\xi_2)\cdots\widehat{\Psi}(2^{-j}\xi_b)\widehat{f}_b(\xi_b) \\
&\qquad\quad \times \widehat{\Phi}(2^{-j+1}\xi_{b+1})\widehat{f_{b+1}}(\xi_{b+1})\cdots\widehat{\Phi}(2^{-j+1}\xi_m)\widehat{f_m}(\xi_m)e^{2\pi i(\xi_1+\cdots+\xi_m)\cdot x}d\xi_1\cdots d\xi_m\Big| \\
&= 2^{ks}\sum_{j\geqslant \max(1,k-c_0)} 2^{-js}\Big|\Omega_{k,s}\big((\Delta_j^* D^s f_1)(\Delta_j f_2)\cdots(\Delta_j f_b)(S_{j-1}f_{b+1})\cdots(S_{j-1}f_m)\big)\Big|
\end{aligned}$$

$$\lesssim \sqrt{\sum_{j\geqslant \max(1,k-c_0)} \Big|\Omega_{k,s}\big((\Delta_j^* D^s f_1)(\Delta_j f_2)\cdots(\Delta_j f_b)(S_{j-1}f_{b+1})\cdots(S_{j-1}f_m)\big)\Big|^2} \tag{3.23}$$

where in the last line we applied the Cauchy-Schwarz inequality.

We now derive a similar estimate for $k \leqslant 0$. Denote $\mathbb{N}_0 = \mathbb{N}\cup\{0\}$. Let $\widehat{\Phi_s} = (1+|\cdot|^2)^{\frac{s}{2}}\widehat{\Phi}$ and S_0^* be the operator given by convolution with Φ_s. Notice since $k < 0$ then $\widehat{\Phi}\widehat{\Psi}(2^{-k}\cdot) = \widehat{\Psi}(2^{-k}\cdot)$ and we have

$$\begin{aligned}
&\Big|\sum_{j\in\mathbb{N}_0} \Delta_k J^s\big((\Delta_j f_1)(\Delta_j f_2)\cdots(\Delta_j f_b)(S_{j-1}f_{b+1})\cdots(S_{j-1}f_m)\big)\Big| \\
&= \Big|\sum_{j\in\mathbb{N}_0}\int_{\mathbb{R}^{mn}} \widehat{\Psi}(2^{-k}(\xi_1+\cdots+\xi_m))\widehat{\Phi_s}(\xi_1+\cdots+\xi_m)2^{-js}|2^{-j}\xi_1|^{-s}\widehat{\Psi}(2^{-j}\xi_1)|\xi_1|^s\widehat{f}_1(\xi_1) \\
&\quad \times \widehat{\Psi}(2^{-j}\xi_2)\widehat{f}_2(\xi_2)\cdots\widehat{\Psi}(2^{-j}\xi_b)\widehat{f}_b(\xi_b) \\
&\qquad \times \widehat{\Phi}(2^{-j+1}\xi_{b+1})\widehat{f_{b+1}}(\xi_{b+1})\widehat{\Phi}(2^{-j+1}\xi_m)\widehat{f_m}(\xi_m)e^{2\pi i(\xi_1+\cdots+\xi_m)\cdot x}d\vec{\xi}\Big| \\
&\leqslant \sum_{j\in\mathbb{N}_0} 2^{-js}\Big|\Delta_k S_0^*\big((\Delta_j^* D^s f_1)(\Delta_j f_2)\cdots(\Delta_j f_b)(S_{j-1}f_{b+1})\cdots(S_{j-1}f_m)\big)\Big|
\end{aligned}$$

$$\lesssim \sqrt{\sum_{j\in\mathbb{N}_0} \Big|\Delta_k S_0^*\big((\Delta_j^* D^s f_1)(\Delta_j f_2)\cdots(\Delta_j f_b)(S_{j-1}f_{b+1})\cdots(S_{j-1}f_m)\big)\Big|^2}, \tag{3.24}$$

where in the last line we applied the Cauchy-Schwarz inequality, since $s > 0$. We define a sequence of measurable functions defined on $\mathbb{R}^n$, $\{f_j\}_{j\in\mathbb{N}_0}$, to be in $L^r\ell^2$ if

$$\left\|\Big(\sum_{j=0}^{\infty}|f_j|^2\Big)^{\frac{1}{2}}\right\|_{L^r} < \infty.$$

It is clear that $\{S_0^*\Delta_k\}_{k\in\mathbb{N}_0} : L^r \to L^r\ell^2$ in (3.24) is bounded for $1 < r < \infty$ by standard Littlewood-Paley theory. Also, $\{\Omega_{k,s}\}_{k\in\mathbb{N}} : L^r \to L^r\ell^2$ in (3.23) is bounded for $1 < r < \infty$ and $k > 0$. To see this let Λ be a $\mathcal{C}^\infty(\mathbb{R}^n)$ function that is 1 on $2^{-1} \leqslant |\xi| \leqslant 2$ and supported in $4^{-1} \leqslant |\xi| \leqslant 4$. Expanding in Fourier series we have

$$(2^{-2k} + |2^{-k}\xi|^2)^{\frac{s}{2}}\Lambda(\xi) = \chi_{[-4,4]^n}(\xi)\sum_{\mu\in\mathbb{Z}^n} c_{\mu,k}e^{2\pi i\xi\cdot\frac{\mu}{8}}$$

where the Fourier coefficients decay rapidly in μ and are uniformly bounded in k due to the support Λ and the fact $k > 0$. Then for $f \in \mathcal{S}(\mathbb{R}^n)$ we obtain

$$\begin{aligned}\Omega_{k,s}(f)(x) &= \int_{\mathbb{R}^n} \sigma_{k,s}(\xi)\widehat{f}(\xi)e^{2\pi i\xi\cdot x}d\xi\\ &= \int_{\mathbb{R}^n} [\Lambda(2^{-k}\xi)(2^{-2k} + |2^{-k}\xi|^2)^{\frac{s}{2}}]\widehat{\Psi}(2^{-k}\xi)\widehat{f}(\xi)e^{2\pi i\xi\cdot x}d\xi\\ &= \sum_{\mu\in\mathbb{Z}^n} c_{\mu,k}\Delta_k f(x + 2^{-k-3}\mu). \qquad (3.25)\end{aligned}$$

Letting $\Delta_{\mu,k}f(x) := \Delta_k f(x + 2^{-k-3}\mu)$ we have

$$\begin{aligned}\Bigg\|\sqrt{\sum_{k\in\mathbb{N}}\Big|\Omega_{k,s}(f)\Big|^2}\Bigg\|_{L^r} &\leqslant \Bigg\|\sqrt{\sum_{k\in\mathbb{Z}}\Big(\sum_{\mu\in\mathbb{Z}^n}|c_{\mu,k}||\Delta_{\mu,k}f|\Big)^2}\Bigg\|_{L^r}\\ &\lesssim \Bigg\|\sum_{\mu\in\mathbb{Z}^n}(1+|\mu|)^{-n-1}\sqrt{\sum_{k\in\mathbb{N}}|\Delta_{\mu,k}f|^2}\Bigg\|_{L^r} \qquad (3.26)\\ &\leqslant \sum_{\mu\in\mathbb{Z}^n}(1+|\mu|)^{-n-1}\Bigg\|\sqrt{\sum_{k\in\mathbb{N}}|\Delta_{\mu,k}f|^2}\Bigg\|_{L^r}\\ &\lesssim \sum_{\mu\in\mathbb{Z}^n}(1+|\mu|)^{-n-1}\ln(2+|\mu|)\|f\|_{L^r} \qquad (3.27)\\ &\lesssim \|f\|_{L^r},\end{aligned}$$

where in (3.26) we used Minkowski's integral inequality with respect to counting measures, and in (3.27) we used Lemma 2.4. Breaking the sum up at $k = 0$ and substituting in (3.23) and (3.24) we have (3.22) is bounded by

$$\begin{aligned}&\lesssim \Bigg\|\sqrt{\sum_{k\geqslant 0}\sum_{j\geqslant 0}\Big|\Omega_{k,s}\big((\Delta_j^* D^s f_1)(\Delta_j f_2)\cdots(\Delta_j f_b)(S_{j-1}f_{b+1})\cdots(S_{j-1}f_m)\big)\Big|^2}\Bigg\|_{L^r}\\ &\quad + \Bigg\|\sqrt{\sum_{k\geqslant 0}\sum_{j\geqslant 0}\Big|\Delta_{-k}S_0^*\big((\Delta_j^* D^s f_1)(\Delta_j f_2)\cdots(\Delta_j f_b)(S_{j-1}f_{b+1})\cdots(S_{j-1}f_m)\big)\Big|^2}\Bigg\|_{L^r}. \qquad (3.28)\end{aligned}$$

Let $F_j := (\Delta_j^* D^s f_1)(\Delta_j f_2)\cdots(\Delta_j f_b)(S_{j-1}f_{b+1})\cdots(S_{j-1}f_m)$ and let G_k be either $\Omega_{k,s}$ or $\Delta_{-k}S_0^*$ then either summand in (3.28) can be expressed as

$$\Bigg\|\sqrt{\sum_{k\geqslant 0}\sum_{j\geqslant 0}|G_k F_j|^2}\Bigg\|_{L^r}.$$

Now to bound this term we have

$$\begin{aligned}&\int_{\mathbb{R}^n}\Big(\sum_{k\geqslant 0}\sum_{j\geqslant 0}\Big|G_k F_j(x)\Big|^2\Big)^{\frac{r}{2}}dx\\ &\leqslant A_r\int_{\mathbb{R}^n}\int_0^1\int_0^1\Big|\sum_{k\geqslant 0}\sum_{j\geqslant 0}G_k F_j(x)r_j(t_1)r_k(t_2)\Big|^r dt_2 dt_1 dx \qquad (3.29)\end{aligned}$$

$$\leqslant A'_r \int_{\mathbb{R}^n} \int_0^1 \left(\sqrt{\sum_{k\geqslant 0} |G_k \sum_{j\geqslant 0} F_j(x) r_j(t_1)|^2} \right)^r dt_1 dx \tag{3.30}$$

$$= A'_r \int_0^1 \int_{\mathbb{R}^n} \left(\sqrt{\sum_{k\geqslant 0} \left|G_k \sum_{j\geqslant 0} F_j(x) r_j(t_1)\right|^2} \right)^r dx dt_1 \tag{3.31}$$

$$\leqslant A_{r,n,s} \int_0^1 \int_{\mathbb{R}^n} \Big| \sum_{j\geqslant 0} F_j(x) r_j(t_1) \Big|^r dx dt_1 \tag{3.32}$$

$$= A_{r,n,s} \int_{\mathbb{R}^n} \int_0^1 \Big| \sum_{j\geqslant 0} F_j(x) r_j(t_1) \Big|^r dt_1 dx \tag{3.33}$$

$$= A'_{r,n,s} \int_{\mathbb{R}^n} \left(\sqrt{\sum_{j\geqslant 0} |F_j(x)|^2} \right)^r dx, \tag{3.34}$$

where in (3.29) we applied (2.9), in (3.30) we applied (2.8) with respect to t_2, (3.31) and (3.33) is just Fubini's theorem, (3.32) we used that $\{G_k\}_{k\in\mathbb{N}_0} : L^r \to L^r\ell^2$ is bounded, and lastly in (3.34) we applied (2.8). Continuing from (3.34) we obtain

$$\begin{aligned}
\left\| \sqrt{\sum_{j\geqslant 0} |F_j|^2} \right\|_{L^r} &= \left\| \sqrt{\sum_{j\geqslant 0} \left| (\Delta_j^* D^s f_1)(\Delta_j f_2)\cdots(\Delta_j f_b)(S_{j-1} f_{b+1})\cdots(S_{j-1} f_m) \right|^2} \right\|_{L^r} \\
&\leqslant \left\| \sqrt{\sum_{j\geqslant 0} |\Delta_j^* D^s f_1|^2} \right\|_{L^r} \prod_{\rho=2}^{b} \left\| \sup_{j\geqslant 0} |\Delta_j(f_\rho)| \right\|_{L^\infty} \prod_{\rho=b+1]}^{m} \left\| \sup_{j\geqslant 0} |S_{j-1}(f_\rho)| \right\|_{L^\infty} \\
&\lesssim \|D^s f_1\|_{L^r} \|f_2\|_{L^\infty} \cdots \|f_m\|_{L^\infty} \\
&\lesssim \|J^s f_1\|_{L^r} \|f_2\|_{L^\infty} \cdots \|f_m\|_{L^\infty},
\end{aligned}$$

as desired. □

4. Homogeneous KP from Inhomogeneous KP

In [12] a dilation argument was used to show the sharpness of the range of s for the inhomogeneous Kato-Ponce inequality. In this section we use a similar dilation argument to derive (1.2) directly from (1.3). This is quite advantageous since a direct proof of the homogeneous case requires a different paraproduct decomposition, and hence a different, albeit similar, proof.

We will use the following proposition to obtain the homogeneous Kato-Ponce inequality from the inhomogeneous one. Though this method is mentioned in the literature [6],[14] it needs some variant of Lemma 2.1 to obtain a uniform upper bound in the application of Lebesgue dominated convergence theorem, as done in the following proposition.

Proposition 4.1. *Let* $0 < r \leqslant \infty$, $f \in \mathcal{S}(\mathbb{R}^n)$, $\widehat{J_R^s f} = (R^{-2} + |\cdot|^2)^{\frac{s}{2}} \widehat{f}$, *and* $s > \max(0, n(1/r - 1))$, *then*

$$\lim_{R\to\infty} \|J_R^s f\|_{L^r} = \|D^s f\|_{L^r}.$$

Proof. First suppose $p < \infty$. By Lebesgue dominated convergence theorem J_R^s converges pointwise to $D^s f$. By Lemma 2.1 we have the estimate $|J_R^s f(x)|^r \lesssim (1 + |x|)^{-(n+s)r}$, thus by Lebesgue dominated convergence theorem again we have $\lim_{R\to\infty} \left\| J_R^s f \right\|_{L^r} = \|D^s f\|_{L^r}$. Now suppose $r = \infty$ and observe that

$$|(J_R^s f - D^s f)(\xi)| = \Big| \int_{\mathbb{R}^n} ((R^{-2} + |y|^2)^{\frac{s}{2}} - |y|^s) \widehat{f}(y) e^{2\pi i y\cdot\xi} dy \Big| \tag{4.1}$$

$$\leqslant \int_{\mathbb{R}^n} ((1 + |y|^2)^{\frac{s}{2}} - |y|^s) |\widehat{f}(y)| dy. \tag{4.2}$$

As (4.2) is a uniform upper bound by Lebesgue dominated convergence theorem we can bring the limit over R inside the integral of (4.1) to obtain the desired result. □

We now derive (1.2) from (1.3): For $f \in \mathcal{S}(\mathbb{R}^n)$ let $f_R := f(R\cdot)$. Let $\frac{1}{m} \leqslant r \leqslant \infty$, $1 \leqslant p_1, \ldots, p_m \leqslant \infty$ for $t \in \{1, \ldots, m\}$ satisfy $\frac{1}{r} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$. Observe,

$$\begin{aligned} J^s(f_R)(\xi) &= \int_{\mathbb{R}^n} (1+|y|^2)^{\frac{s}{2}} R^{-n} f(R^{-1}y) e^{2\pi i y\cdot\xi} dy \\ &= R^s \int_{\mathbb{R}^n} (R^{-2}+|y|^2)^{\frac{s}{2}} f(y) e^{2\pi i y\cdot R\xi} dy \\ &= R^s J^s_R(f)(R\xi). \end{aligned}$$

It follows applying the inhomogeneous KP inequality to the dilated functions $(f_1 \cdots f_m)_R = (f_1)_R \cdots (f_m)_R$ gives

$$\begin{aligned} &\|J^s_R(\,f_1\cdots f_m)(R\cdot)\|_{L^r} \\ &\quad\leqslant C\Big(\|J^s_R f_1(R\cdot)\|_{L^{p_1}}\|f_2\|_{L^{p_2}}\cdots\|f_m\|_{L^{p_m}} + \cdots + \|f_1\|_{L^{p_1}}\|f_2\|_{L^{p_2}}\cdots\|J^s_R f_m(R\cdot)\|_{L^{p_m}}\Big) \end{aligned}$$

where the R^s term cancels from both sides. By a change of variables and using that $\frac{1}{r} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$ we obtain

$$\begin{aligned} &\|J^s_R(\,f_1\cdots f_m)\|_{L^r} \\ &\quad\leqslant C\Big(\|J^s_R f_1\|_{L^{p_1}}\|f_2\|_{L^{p_2}}\cdots\|f_m\|_{L^{p_m}} + \cdots + \|f_1\|_{L^{p_1}}\|f_2\|_{L^{p_2}}\cdots\|J^s_R f_m\|_{L^{p_m}}\Big) \end{aligned}$$

after canceling the $R^{-\frac{n}{r}}$ from both sides. We then deduce (1.2) by letting $R \to \infty$ and using Proposition 4.1. □

References

[1] Bae, Hantaek; Biswas, Animikh Gevrey. *Regularity for a class of dissipative equations with analytic nonlinearity.* Methods Appl. Anal. 22 (2015), no. 4, 377–408.

[2] Bernicot, Frédéric; Maldonado, Diego; Moen, Kabe; Naibo, Virginia. *Bilinear Sobolev-Poincaré inequalities and Leibniz-type rules.* J. Geom. Anal. 24 (2014), no. 2, 1144–1180.

[3] Bourgain Jean; Dong Li. *On an endpoint Kato-Ponce inequality.* Differential Integral Eqns 27 (2014), no.11-12, 1037–1072.

[4] Christ, F. Michael; Weinstein, M. I. *Dispersion of small amplitude solutions of the generalized Korteweg-de Vries equation.* J. Funct. Anal. 100 (1991), no. 1, 87–109.

[5] Coifman, R. Ronald; Meyer, Yves. Commutateurs d' intégrales singulières et opérateurs multilinéaires Ann. Inst. Fourier (Grenoble) 28 (1978), no. 3, xi, 177–202.

[6] Cordero, Elena; Zucco, Davide. *Strichartz estimates for the vibrating plate equation.* J. Evol. Equ. 11 (2011), no. 4, 827–845.

[7] Douglas, Sean; Grafakos, Loukas. *Weighted Kato-Ponce inequalities for multiple factors*, submitted.

[8] Fujiwara, Kazumasa; Georgiev, Vladimir; Ozawa, Tohru. *Higher order fractional Leibniz rule.* J. Fourier Anal. Appl. 24 (2018), 650–665.

[9] Grafakos, Loukas. *Classical Fourier analysis.* Third edition. Graduate Texts in Mathematics, 249. Springer, New York, 2014.

[10] Grafakos, Loukas. *Modern Fourier analysis.* Third edition. Graduate Texts in Mathematics, 250. Springer, New York, 2014.

[11] Grafakos, Loukas; Naibo, Virginia; Maldonado, Diego. *A remark on an endpoint Kato-Ponce inequality.* Differential and Integral Equations 27 (2014), no. 5-6, 415–424.

[12] Grafakos, Loukas; Oh, Seungly. *The Kato-Ponce inequality.* Comm. Partial Differential Equations 39 (2014), no. 6, 1128–1157.

[13] Grafakos, Loukas; Torres, Rodolfo H. *Multilinear Calderón-Zygmund theory.* Adv. Math. 165 (2002), no. 1, 124–164.

[14] Gulisashvili, Archil; Kon, Mark A. *Exact smoothing properties of Schrödinger semigroups.* Amer. J. Math. 118 (1996), no. 6, 1215–1248.

[15] Hale, Elizabeth; Naibo, Virginia. *Fractional Leibniz Rules in the Setting of Quasi-Banach Function Spaces.* J. Fourier Anal. Appl., to appear.

[16] Kato, Tosio; Ponce, Gustavo. *Commutator estimates and the Euler and Navier-Stokes equations.* Comm. Pure Appl. Math. 41 (1988), no. 7, 891–907.

[17] Kenig, Carlos E.; Stein, Elias M. *Multilinear estimates and fractional integration.* Math. Res. Lett. 6 (1999), no. 1, 1–15.

[18] Li, Dong. *On Kato-Ponce and fractional Leibniz.* Rev. Mat. Iberoam. 35 (2019), no. 1, 23–100.

[19] Muscalu, Camil; Pipher, Jill; Tao, Terence; Thiele, Christoph. *Bi-parameter paraproducts.* Acta Math. (2004), 193, 269–296.

[20] Muscalu Camil; Schlag Wilhelm. *Classical and Multilinear Harmonic Analysis Volume 2*, Cambridge Studies in Advanced Mathematics, 138, Cambridge, UK, 2013.

[21] Naibo, Virginia; Thomson, Alexander. *Coifman-Meyer multipliers: Leibniz-type rules and applications to scattering of solutions to PDEs*, Trans. Amer. Math. Soc. 372 (2019), no. 8, 5453–5481.

[22] Oh, Seungly; Wu, Xinfeng. *On L^1 endpoint Kato-Ponce inequality.* Mathematical Research Letters. 27. 1129–1163.

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