Excel Coaching Classes

Test Name: Differentiation

Subjects: Mathematics and Statistics

Marks: 25

Standard: XII Science English Maharashtra State Board

Duration: 1hr15min

SECTION A

1. **Select and write the correct answer for the following multiple choice type of questions:** 4

i. **If $y = x \log(x^2 - 3)$, then $\frac{dy}{dx} =$**

- [x] (A) $\frac{2x^2}{x^2-3} + \log(x^2-3)$
- [ ] (B) $\frac{x^2}{x^2-3} + \log(x^2-3)$
- [ ] (C) $\frac{2x}{x^2-3} + \log(x^2-3)$
- [ ] (D) $\frac{x}{x^2-3} + \log(x^2-3)$

Ans: Differentiating w.r.t. *x*, we get

$\frac{dy}{dx} = \log(x^2 - 3) + \frac{x}{x^2-3} \frac{d}{dx}(x^2 - 3)$

$\frac{dy}{dx} = \log(x^2 - 3) + \frac{2x^2}{x^2-3}$

ii. **If $f(x) = x \tan^{-1} x$, then $f'(1) =$**

- [x] (A) $\frac{1}{2} + \frac{\pi}{4}$
- [ ] (B) $-\frac{1}{2} + \frac{\pi}{4}$
- [ ] (C) $-\frac{1}{2} - \frac{\pi}{4}$
- [ ] (D) $\frac{1}{2} - \frac{\pi}{4}$

Ans: Differentiating w.r.t. *x*, we get

$f'(x) = \tan^{-1} x + \frac{x}{1+x^2}$

$\therefore f'(1) = \tan^{-1}(1) + \frac{(1)}{1+(1)^2}$

$= \frac{\pi}{4} + \frac{1}{2}$

2. **Answer the following questions:** 3

i. **Differentiate the following w.r.t. *x*.**
***y* = sin(log*x*)**

Let u = log *x* then *y* = sin u, where *y* is a differentiable function of u and u is a differentiable function of *x* then

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$ ...(I)

Now, *y* = sin u

Differentiate w.r.t. u

$\frac{dy}{du} = \frac{d}{du}(\sin u) = \cos u$ and u = log *x*

Differentiate w.r.t. *x*

$\frac{du}{dx} = \frac{d}{dx}(\log x) = \frac{1}{x}$

Now, equation (I) becomes,

$\frac{dy}{dx} = \cos u \times \frac{1}{x} = \frac{\cos(\log x)}{x}$

Alternate Method:

We have *y* = sin (log *x*)

Differentiate w.r.t. *x*

$\frac{dy}{dx} = \frac{d}{dx}[\sin(\log x)]$

[Treat log *x* as u in mind and use the formula of derivative of sin u]

$\frac{dy}{dx} = \cos(\log x) \times \frac{d}{dx}(\log x)$

$\frac{dy}{dx} = \cos(\log x) \cdot \frac{1}{x}$
$\frac{dy}{dx} = \frac{\cos(\log x)}{x}$

ii. **If $x = t^2$ and $y = t^3$, find $\frac{dy}{dx}$.**

$x = t^2$
Differentiating w. r. t. t, we get
$\frac{dx}{dt} = \frac{d}{dt}\left(t^2\right) = 2t$
$y = t^3$
Differentiating w. r. t. t, we get
$\frac{dy}{dt} = \frac{d}{dt}\left(t^3\right) = 3t^2$
$\therefore \quad \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{2t}{3t^2} = \frac{2}{3t}$

iii. **Suppose that the functions f and g and their derivatives with respect to x have the following values at x = 0 and x = 1.**

x	f(x)	g(x)	f ′(x)	g′(x)
0	1	1	5	$\frac{1}{3}$
1	3	–4	$-\frac{1}{3}$	$-\frac{8}{3}$

The derivative of g[f(x)] w. r. t. x at x = 0 is ......

$\frac{d}{dx} g(f(x))\Big|_{x=0} = g'(f(x)) \cdot f'(x)\Big|_{x=0}$
$= g'(f(0)) \cdot f'(0) = g'(1) f'(0)$
$= -\frac{8}{3} \times 5 = -\frac{40}{3}$

SECTION B

Attempt any FOUR of the following questions: **8**

3. **Find the second order derivative of $2x^5 - 4x^3 - \frac{2}{x^2} - 9$.**

Let $y = 2x^5 - 4x^3 - \frac{2}{x^2} - 9$
Differentiating w. r. t. x, we get
$\frac{dy}{dx} = \frac{d}{dx}\left(2x^5 - 4x^3 - \frac{2}{x^2} - 9\right)$
$= 2(5x^4) - 4(3x^2) - 2(-2x^{-3})$
$\therefore \quad \frac{dy}{dx} = 10x^4 - 12x^2 + \frac{4}{x^3}$
Again, differentiating w. r. t. x, we get
$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(10x^4 - 12x^2 + \frac{4}{x^3}\right)$
$= 10(4x^3) - 12(2x) + 4(-3x^{-4})$
$\therefore \quad \frac{d^2y}{dx^2} = 40x^3 - 24x - \frac{12}{x^4}$

4. **Differentiate the following w.r.t. x**
$\cos^{-1}\left(2x\sqrt{1-x^2}\right)$

Let $y = \cos^{-1}\left(2x\sqrt{1-x^2}\right)$
Put $x = \sin\theta$
$\therefore \quad \theta = \sin^{-1} x$
$\therefore \quad y = \cos^{-1}\left(2\sin\theta\sqrt{1-\sin^2\theta}\right)$
$y = \cos^{-1}\left(2\sin\theta\sqrt{\cos^2\theta}\right)$
$y = \cos^{-1}(2\sin\theta\cos\theta) = \cos^{-1}(\sin 2\theta)$

$y=\cos^{-1}\left[\cos\left(\frac{\pi}{2}-2\theta\right)\right]=\frac{\pi}{2}-2\theta$

$\therefore\ \ y=\frac{\pi}{2}-2\sin^{-1}x$

Differentiate w.r.t. x

$\frac{dy}{dx}\ =\frac{d}{dx}\left(\frac{\pi}{2}-2\sin^{-1}x\right)$

$\frac{dy}{dx}=0-\frac{2\times1}{\sqrt{1-x^2}}$

$\therefore\quad \frac{dy}{dx}=-\frac{2}{\sqrt{1-x^2}}$

5. **Differentiate the following w.r.t. *x*.**

$y=\left(x^3+2x-3\right)^4\left(x+\cos x\right)^3$

$y = (x^3 + 2x - 3)^4 (x + \cos x)^3$

Differentiate w.r.t. x

$\frac{dy}{dx}=\frac{d}{dx}\left[\left(x^3+2x-3\right)^4\left(x+\cos x\right)^3\right]$

$=\left(x^3+2x-3\right)^4\cdot\frac{d}{dx}\left(x+\cos x\right)^3+\left(x+\cos x\right)^3\cdot\frac{d}{dx}\left(x^3+2x-3\right)^4$

$=\left(x^3+2x-3\right)^4\cdot3\left(x+\cos x\right)^2\cdot\frac{d}{dx}(x+\cos x)+\left(x+\cos x\right)^3\cdot4\left(x^3+2x-3\right)^3\cdot\frac{d}{dx}\left(x^3+2x-\right.$

$=\left(x^3+2x-3\right)^4\cdot3\left(x+\cos x\right)^2(1-\sin x)+\left(x+\cos x\right)^3\cdot4\left(x^3+2x-3\right)^3\left(3x^2+2\right)$

$\therefore\ \frac{dy}{dx}=3\left(x^3+2x-3\right)^4\left(x+\cos x\right)^2(1-\sin x)+4\left(3x^2+2\right)\left(x^3+2x-3\right)^3\left(x+\cos x\right)^3$

6. **Differentiate $(\sin x)^x$ w. r. t. *x*.**

Let $y = (\sin x)^x$

Taking log on both sides, we get

$\log y = x \log (\sin x)$

Differentiating w. r. t. x, we get

$\frac{d}{dx}(\log y) = x\ .\ \frac{d}{dx}[\log(\sin x)] + \log(\sin x)\ .\ \frac{d}{dx}(x)$

$\therefore\ \ \frac{1}{y}\cdot\frac{dy}{dx} = x\ .\ \frac{1}{\sin x}.\ \frac{d}{dx}(\sin x) + \log(\sin x)\ .\ 1$

$\therefore\ \ \frac{1}{y}\cdot\frac{dy}{dx} = \frac{x}{\sin x}\ .\ \cos x + \log(\sin x)$

$\therefore\ \ \frac{dy}{dx} = y\,[x\cot x + \log(\sin x)]$

$\therefore\ \ \frac{dy}{dx} = (\sin x)^x\,[x\cot x + \log(\sin x)]$

7. **If $y=\sqrt{\log x+\sqrt{\log x+\sqrt{\log x+\ldots\infty}}}$, then show that $\frac{dy}{dx}=\frac{1}{x(2y-1)}$.**

$y=\sqrt{\log x+\sqrt{\log x+\sqrt{\log x+\ldots.\infty}}}$

$\therefore\ \ y^2 = \log x + \sqrt{\log x+\sqrt{\log x+\ldots.\infty}}$

$\therefore\ \ y^2 = \log x + y$

Differentiating w.r.t. x, we get

$2y\cdot\frac{dy}{dx} = \frac{1}{x} + \frac{dy}{dx}$

$\therefore\ \ (2y-1)\frac{dy}{dx} = \frac{1}{x}$

$\therefore\ \ \frac{dy}{dx} = \frac{1}{x(2y-1)}$

8. **Find the n^{th} derivative of log (a*x* + b)**

Let $y = \log(ax + b)$

Differentiating w. r. t. x, we get

$\frac{dy}{dx}=\frac{d}{dx}[\log(ax+\ b)]$

$=\frac{1}{ax+b}\cdot\frac{d}{dx}(ax+\ b)$

$=\frac{a}{ax+b}$

Differentiating w. r. t. x, we get

$\frac{d}{dx}\left(\frac{dy}{dx}\right)= a\frac{d}{dx}\left(\frac{1}{ax+b}\right)$

$\therefore \quad \frac{d^2y}{dx^2} = a\left[-\frac{1}{(ax+b)^2}\right]\cdot\frac{d}{dx}(ax+\ b)$

$\therefore \quad \frac{d^2y}{dx^2} = -\frac{a^2}{(ax+b)^2}$

Differentiating w. r. t. x, we get

$\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = -a^2\ \frac{d}{dx}\left[\frac{1}{(ax+b)^2}\right]$

$= -a^2\left[\frac{-2}{(ax+b)^3}\right]\cdot\frac{d}{dx}(ax+\ b)$

$= \frac{2a^3}{(ax+b)^3}$

In general, n^{th} order derivative will be

$\frac{d^ny}{dx^n} = \frac{(-1)^{n-1}(n-1)!a^n}{(ax+b)^n}$

SECTION C

Attempt any TWO of the following questions: **6**

9. **Differentiate $\sin^{-1}\left(\frac{4^{x+\frac{1}{2}}}{1+2^{4x}}\right)$ w. r. t. x.**

Let $y = \sin^{-1}\left(\frac{4^{x+\frac{1}{2}}}{1+2^{4x}}\right)$, $x < 0$

$= \sin^{-1}\left(\frac{2\times 2^{2x}}{1+(2^{2x})^2}\right)$

Put $2^{2x} = \tan\theta$, then $\theta = \tan^{-1} 2^{2x}$

$\therefore \quad y = \sin^{-1}\left(\frac{2\,tan}{1+tan^2}\right)$

$= \sin^{-1}(\sin(2\theta))$

$= 2\theta$

$= 2\tan^{-1}(2^{2x})$

Differentiating w. r. t. x, we get

$\frac{dy}{dx} = \frac{d}{dx}[\,2\tan^{-1}(2^{2x})]$

$= \frac{2}{1+(2^{2x})^2}\cdot\frac{d}{dx}\left(2^{2x}\right)$

$= \frac{2\left(2^{2x}\,log\,4\right)}{1+4^{2^x}}$

$= \frac{2(4^x\,log\,4)}{1+4^{2^x}}$ or $\frac{4^{x+\frac{1}{2}}\,log\,4}{1+4^{2^x}}$

10. **Differentiate the following w.r.t. x**

$\left(\frac{(x^2+3)^2\sqrt[3]{(x^3+5)^2}}{\sqrt{(2x^2+1)^3}}\right)$

Let $y = \left(\frac{(x^2+3)^2\sqrt[3]{(x^3+5)^2}}{\sqrt{(2x^2+1)^3}}\right)$

Taking log of both the sides we get,

$\log y = \log\left(\frac{(x^2+3)^2\sqrt[3]{(x^3+5)^2}}{\sqrt{(2x^2+1)^3}}\right) = \log\left[\frac{(x^2+3)^2(x^3+5)^{\frac{2}{3}}}{(2x^2+1)^{\frac{3}{2}}}\right]$

$= \log\left[\left(x^2+3\right)^2\left(x^3+5\right)^{\frac{2}{3}}\right] - \log\left(2x^2+1\right)^{\frac{3}{2}}$

$= \left[\log\left(x^2+3\right)^2 + \log\left(x^3+5\right)^{\frac{2}{3}}\right] - \log\left(2x^2+1\right)^{\frac{3}{2}}$

$\log y = 2\log\left(x^2+3\right) + \frac{2}{3}\log\left(x^3+5\right) - \frac{3}{2}\log\left(2x^2+1\right)$

Differentiate w.r.t. x

$\frac{d}{dx}(\log y) = \frac{d}{dx}\left[2\log\left(x^2+3\right) + \frac{2}{3}\log\left(x^3+5\right) - \frac{3}{2}\log\left(2x^2+1\right)\right]$

$\frac{1}{y}\frac{dy}{dx} = 2\cdot\frac{d}{dx}\left[\log\left(x^2+3\right)\right] + \frac{2}{3}\cdot\frac{d}{dx}\left[\log\left(x^3+5\right)\right] - \frac{3}{2}\cdot\frac{d}{dx}\left[\log\left(2x^2+1\right)\right]$

$= \frac{2}{x^2+3}\cdot\frac{d}{dx}\left(x^2+3\right) + \frac{2}{3(x^3+5)}\cdot\frac{d}{dx}\left(x^3+5\right) - \frac{3}{2(2x^2+1)}\cdot\frac{d}{dx}\left(2x^2+1\right)$

$\frac{dy}{dx} = y\left[\frac{2}{x^2+3}(2x) + \frac{2}{3(x^3+5)}\left(3x^2\right) - \frac{3}{2(2x^2+1)}(4x)\right]$

$\therefore \quad \frac{dy}{dx} = \frac{(x^2+3)^2\sqrt[3]{(x^3+5)^2}}{\sqrt{(2x^2+1)^3}}\left[\frac{4x}{x^2+1}+\frac{2x^2}{(x^3+5)}-\frac{6x}{2x^2+1}\right]$

11. **Find $\frac{dy}{dx}$, if $x+\sqrt{xy}+y=1$**

$x + \sqrt{xy} + y = 1$

Differentiating w.r.t. x, we get

$1 + \frac{1}{2\sqrt{xy}}\frac{d}{dx}(xy) + \frac{dy}{dx} = 0$

$\therefore \quad 1 + \frac{1}{2\sqrt{xy}}\left(x\frac{dy}{dx}+y\frac{d}{dx}\left(x\right)\right) + \frac{dy}{dx} = 0$

$\therefore \quad 1 + \frac{1}{2}\sqrt{\frac{x}{y}}\frac{dy}{dx} + \frac{1}{2}\sqrt{\frac{y}{x}} + \frac{dy}{dx} = 0$

$\therefore \quad \left(\frac{1}{2}\sqrt{\frac{x}{y}}+1\right)\frac{dy}{dx} = -\frac{1}{2}\sqrt{\frac{y}{x}} - 1$

$\therefore \quad \left(\frac{\sqrt{x}+2\sqrt{y}}{2\sqrt{y}}\right)\frac{dy}{dx} = \frac{-\sqrt{y}-2\sqrt{x}}{2\sqrt{x}}$

$\therefore \quad \frac{dy}{dx} = -\frac{\sqrt{y}\left(\sqrt{y}+2\sqrt{x}\right)}{\sqrt{x}\left(\sqrt{x}+2\sqrt{y}\right)}$

SECTION D

Attempt any ONE of the following questions: **4**

12. **If $y = \cos(m\cos^{-1} x)$ then show that $\left(1-x^2\right)\frac{d^2y}{dx^2}-x\frac{dy}{dx}+m^2y=0$.**

Given that $y = \cos(m\cos^{-1} x)$

$\therefore \quad \cos^{-1} y = m\cos^{-1} x$

Differentiate (I) w.r.t. x

$\frac{d}{dx}\left(\cos^{-1}y\right) = m\frac{d}{dx}\left(\cos^{-1}x\right)$

$-\frac{1}{\sqrt{1-y^2}}\cdot\frac{dy}{dx} = -\frac{m}{\sqrt{1-x^2}}$

$\sqrt{1-x^2}\cdot\frac{dy}{dx} = m\sqrt{1-y^2}$

Squaring both sides

$\left(1-x^2\right)\cdot\left(\frac{dy}{dx}\right)^2 = m^2\left(1-y^2\right)$

Differentiate w.r.t. x

$\left(1-x^2\right)\frac{d}{dx}\left(\frac{dy}{dx}\right)^2 + \left(\frac{dy}{dx}\right)^2\frac{d}{dx}\left(1-x^2\right) = m^2\frac{d}{dx}\left(1-y^2\right)$

$\left(1-x^2\right)\cdot 2\left(\frac{dy}{dx}\right)\cdot\frac{d}{dx}\cdot\left(\frac{dy}{dx}\right) + \left(\frac{dy}{dx}\right)^2(-2x) = m^2(-2y)\frac{dy}{dx}$

$2\left(1-x^2\right)\cdot\frac{dy}{dx}\cdot\frac{d^2y}{dx^2} - 2x\left(\frac{dy}{dx}\right)^2 = -2m^2y\frac{dy}{dx}$

Dividing throughout by $2\frac{dy}{dx}$ we get,

$\left(1-x^2\right)\cdot\frac{d^2y}{dx^2} - x\frac{dy}{dx} = -m^2y$

$\therefore \quad \left(1-x^2\right)\cdot\frac{d^2y}{dx^2} - x\frac{dy}{dx} + m^2y = 0$

13. **If $y = f(x)$ is a differentiable function of x such that $\frac{dy}{dx} \neq 0$ and $x = f^{-1}(y)$ exists, then $x = f^{-1}(y)$ is a differentiable function of y and $\frac{d}{dy}[f^{-1}(y)] = \frac{1}{f'(x)}$ i.e. $\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$.**

'y' is a differentiable function of 'x'.

Let there be a small change δx in the value of 'x'.

Correspondingly, there should be a small change δy in the value of 'y'.

As $\delta x \to 0$, $\delta y \to 0$

Consider, $\frac{\delta x}{\delta y} \times \frac{\delta y}{\delta x} = 1$

$\therefore \quad \frac{\delta x}{\delta y} = \frac{1}{\frac{\delta y}{\delta x}}, \ \frac{\delta y}{\delta x} \neq 0$

Taking $\lim\limits_{\delta x\to 0}$ on both sides, we get

$\lim\limits_{\delta x\to 0}\left(\frac{\delta x}{\delta y}\right) = \frac{1}{\lim\limits_{\delta x\to 0}\left(\frac{\delta y}{\delta x}\right)}$

Since 'y' is a differentiable function of 'x',

$\lim\limits_{\delta x \to 0}\left(\frac{\delta y}{\delta x}\right) = \frac{dy}{dx}$

As $\delta x \to 0$, $\delta y \to 0$

$\lim\limits_{\delta y \to 0}\left(\frac{\delta x}{\delta y}\right) = \frac{1}{\lim\limits_{\delta x \to 0}\left(\frac{\delta y}{\delta x}\right)}$ …(i)

∴ limit on R.H.S. of (i) exists and is finite.

Hence, limit on L.H.S. of (i) also should exist and be finite.

∴ $\lim\limits_{\delta y \to 0}\left(\frac{\delta x}{\delta y}\right) = \frac{dx}{dy}$ exists and is finite.

∴ $\frac{dx}{dy} = \frac{1}{\left(\frac{dy}{dx}\right)}$, $\frac{dy}{dx} \neq 0$

Alternate Proof:

We know that $f^{-1}[f(x)] = x$ …[Identity function]

Taking derivative on both the sides, we get

$\frac{d}{dx}[f^{-1}[f(x)]] = \frac{d}{dx}(x)$

∴ $(f^{-1})'[f(x)]\frac{d}{dx}[f(x)] = 1$

∴ $(f^{-1})'[f(x)]\, f'(x) = 1$

∴ $(f^{-1})'[f(x)] = \frac{1}{f'(x)}$ …(i)

So, if $y = f(x)$ is a differentiable function of x and $x = f^{-1}(y)$ exists and is differentiable then

$(f^{-1})'[f(x)] = (f^{-1})'(y) = \frac{dx}{dy}$ and $f'(x) = \frac{dy}{dx}$

∴ Equation (i) becomes

$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$ where $\frac{dy}{dx} \neq 0$