Time: 60min	Differentiation	Marks: 20

SECTION A

Q.1 **Select and write the correct answer for the following multiple choice type of questions:** **(4)**

i **If $y = (5x^3 - 4x^2 - 8x)^9$,then $\frac{dy}{dx} =$**

(A) $9(5x^3 - 4x^2 - 8x)^8 (15x^2 - 8x - 8)$ (B) $9(5x^3 - 4x^2 - 8x)^9 (15x^2 - 8x - 8)$

(C) $9(5x^3 - 4x^2 - 8x)^8 (5x^2 - 8x - 8)$ (D) $9(5x^3 - 4x^2 - 8x)^9 (5x^2 - 8x - 8)$

Ans: ["A"]

Explanation:

$y = (5x^3 - 4x^2 - 8x)^9$

Differentiating both sides w.r.t.x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left[\left(5x^3 - 4x^2 - 8x\right)^9\right]$$

$$= 9(5x^3 - 4x^2 - 8x)^8 \cdot \frac{d}{dx}(5x^3 - 4x^2 - 8x)$$

$$= 9(5x^3 - 4x^2 - 8x)^8 \cdot \left[5\left(3x^2\right) - 4\left(2x\right) - 8\right]$$

$$\therefore \frac{dy}{dx} = 9(5x^3 - 4x^2 - 8x)^8 \cdot (15x^2 - 8x - 8)$$

ii **Differentiate $(\sin x)^x$ w. r. t. x.**

(A) $[x \cot x + \log (\sin x)]$ (B) $(\sin x)^x [x \cot x + \log (\sin x)]$

(C) $(\sin x)^x [x \tan x + \log (\cos x)]$ (D) $[x \tan x + \log (\cos x)]$

Ans: ["B"]

Explanation:

Let $y = (\sin x)^x$

Taking log on both sides, we get

$\log y = x \log (\sin x)$

Differentiating w. r. t. x, we get

$$\frac{d}{dx}(\log y) = x \cdot \frac{d}{dx}[\log(\sin x)] + \log(\sin x) \cdot \frac{d}{dx}(x)$$

$$\therefore \frac{1}{y} \cdot \frac{dy}{dx} = x \cdot \frac{1}{\sin x} \cdot \frac{d}{dx}(\sin x) + \log(\sin x) \cdot 1$$

$$\therefore \frac{1}{y} \cdot \frac{dy}{dx} = \frac{x}{\sin x} \cdot \cos x + \log(\sin x)$$

$$\therefore \frac{dy}{dx} = y\,[x \cot x + \log(\sin x)]$$

$$\therefore \frac{dy}{dx} = (\sin x)^x\,[x \cot x + \log(\sin x)]$$

Q.2 **Answer the following questions:** **(2)**

i **Differentiate the following w.r.t. *x*.**

$y = \sqrt{x^2 + 5}$

Ans: []

Explanation:

Let u = x^2 + 5 then $y = \sqrt{u}$, where y is a differentiable function of u and u is a differentiable function of x then $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$ ...(I)

Now, $y = \sqrt{u}$

Differentiate w.r.t. u

$\frac{dy}{du} = \frac{d}{du}\left(\sqrt{u}\right) = \frac{1}{2\sqrt{u}}$ and u = x^2 + 5

Differentiate w.r.t. x

$\frac{du}{dx} = \frac{d}{dx}\left(x^2 + 5\right) = 2x$

Now, equation (I) becomes,

$\frac{dy}{dx} = \frac{1}{2\sqrt{u}} \times 2x = \frac{x}{\sqrt{x^2+5}}$

Alternate Method:

We have $y = \sqrt{x^2 + 5}$

Differentiate w.r.t. x

$\frac{dy}{dx} = \frac{d}{dx}\left(\sqrt{x^2 + 5}\right)$ [Treat x^2 + 5 as u in mind and use the formula of derivative of $\sqrt{u}$]

$\frac{dy}{dx} = \frac{1}{2\sqrt{x^2+5}} \cdot \frac{d}{dx}\left(x^2 + 5\right)$

$\frac{dy}{dx} = \frac{1}{2\sqrt{x^2+5}}(2x)$

$\frac{dy}{dx} = \frac{x}{\sqrt{x^2+5}}$

ii **Differentiate the following w.r.t. *x*.**

$\sin^{-1}(x^3)$

Ans: []

Explanation:

Let $y = \sin^{-1}(x^3)$

Differentiate w.r.t. x

$\frac{dy}{dx} = \frac{d}{dx}\left(\sin^{-1}\left(x^3\right)\right)$

$= \frac{1}{\sqrt{1-\left(x^3\right)^2}} \cdot \frac{d}{dx}\left(x^3\right)$

$= \frac{1}{\sqrt{1-x^6}}\left(3x^2\right)$

$\therefore \quad \frac{dy}{dx} = \frac{3x^2}{\sqrt{1-x^6}}$

SECTION B

Attempt any TWO of the following questions: **(4)**

Q.3 **Differentiate $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ w. r. t. x.**

Ans: []

Explanation:

Let $y = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$, $x > 0$

Put $x = \tan\theta$, then $\theta = \tan^{-1}(x)$

$\therefore \quad \theta = \tan^{-1}(x)$

$\therefore \quad y = \cos^{-1}\left(\frac{1-tan^2\theta}{1+tan^2\theta}\right)$

$= \cos^{-1}(\cos(2\theta))$

$= 2\theta$

$= 2\tan^{-1} x$

Differentiating w. r. t. x, we get

$\frac{dy}{dx} = 2\,\frac{d}{dx}\left(tan^{-1}x\right) = \frac{2}{1+x^2}$

Q.4 **Find the second order derivative of the following:**

$x^3 + 7x^2 - 2x - 9$

Ans: []

Explanation:

Let $y = x^3 + 7x^2 - 2x - 9$

Differentiate w.r.t. x

$$\frac{dy}{dx}=\frac{d}{dx}\left(x^3+7x^2-2x-9\right)$$

$$\frac{dy}{dx}=3x^2+14x-2$$

Differentiate w.r.t. x

$$\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dx}\left(3x^2+14x-2\right)$$

$$\frac{d^2y}{dx^2}=6x+14$$

Q.5 **Find $\frac{dy}{dx}$, if $e^{x+y}=\cos(x-y)$**

Ans: []

Explanation:

$e^{x+y}=\cos(x-y)$

Differentiating w.r.t. x, we get

$e^{x+y}\cdot\frac{d}{dx}(x+y)=-\sin(x-y)\cdot\frac{d}{dx}(x-y)$

$\therefore\ e^{x+y}\left[1+\frac{dy}{dx}\right]=-\sin(x-y)\left[1-\frac{dy}{dx}\right]$

$\therefore\ e^{x+y}+e^{x+y}\frac{dy}{dx}=-\sin(x-y)+\sin(x-y)\frac{dy}{dx}$

$\therefore\ [e^{x+y}-\sin(x-y)]\frac{dy}{dx}=-[\sin(x-y)+e^{x+y}]$

$\therefore\ \frac{dy}{dx}=-\frac{sin(x-y)+e^{x+y}}{e^{x+y}-sin(x-y)}=\frac{sin(x-y)+e^{x+y}}{sin(x-y)-e^{x+y}}$

SECTION C

Attempt any TWO of the following questions: **(6)**

Q.6 **If $\sec^{-1}\left(\frac{x^3+y^3}{x^3-y^3}\right)=2a$, then show that $\frac{dy}{dx}=\frac{x^2\tan^2a}{y^2}$, where a is a constant.**

Ans: []

Explanation:

Given that: $\sec^{-1}\left(\frac{x^3+y^3}{x^3-y^3}\right)=2a\ldots$ [We will not eliminate a, as answer contains a]

$\therefore\ \cos^{-1}\left(\frac{x^3-y^3}{x^3+y^3}\right)=2a$

$\frac{x^3-y^3}{x^3+y^3}=\cos 2a$

$x^3-y^3=x^3\cos 2a+y^3\cos 2a$

$x^3-x^3\cos 2a=y^3\cos 2a+y^3$

$x^3(1-\cos 2a)=y^3(1+\cos 2a)$

$y^3=\left(\frac{1-\cos 2a}{1+\cos 2a}\right)x^3$

$$y^3 = \left(\frac{2\sin^2 a}{2\cos^2 a}\right)x^3$$

$\therefore\ y^3 = (\tan^2 a)x^3$ .....(I)

Differentiate w.r.t. x

$$\frac{d}{dx}\left(y^3\right) = \left(\tan^2 a\right)\frac{d}{dx}\left(x^3\right)$$

$$3y^2\frac{dy}{dx} = \left(\tan^2 a\right)3x^2$$

$$\therefore\ \frac{dy}{dx} = \frac{x^2\tan^2 a}{y^2}$$

Q.7 **Find** $\frac{d^2y}{dx^2}$**, if** $x = sin\sqrt{t},\ y = e^{\sqrt{t}}$

Ans: []

Explanation:

$$x = \sin\sqrt{t}, y = e^{\sqrt{t}}$$

Differentiating x w.r.t. t, we get

$$\frac{dx}{dt} = \cos\sqrt{t}\frac{d}{\mathrm{dt}}\left(\sqrt{t}\right)$$

$$= \frac{\cos\sqrt{t}}{2\sqrt{t}} \qquad \ldots\text{(i)}$$

Differentiating y w.r.t. t, we get

$$\frac{dy}{dt} = \mathrm{e}^{\sqrt{t}}\frac{d}{\mathrm{dt}}\left(\sqrt{t}\right)$$

$$= \frac{e^{\sqrt{t}}}{2\sqrt{t}} \qquad \ldots\text{(ii)}$$

Dividing the equation (ii) by (i), we get

$$\frac{dy}{dx} = \frac{e^{\sqrt{t}}}{\cos\sqrt{t}}$$

Again differentiating w.r.t. x, we get

$$\frac{d^2y}{dx^2} = \frac{\cos\sqrt{t}\cdot\frac{e^{\sqrt{t}}}{2\sqrt{t}} + \mathrm{e}^{\sqrt{t}}\frac{\sin\sqrt{t}}{2\sqrt{t}}}{\cos^2\sqrt{t}} \times \frac{\mathrm{dt}}{dx}$$

$$= \frac{e^{\sqrt{t}}(\cos\sqrt{t} + \sin\sqrt{t})}{2\sqrt{t}\left(\cos^2\sqrt{t}\right)} \times \frac{2\sqrt{t}}{\cos\sqrt{t}} \quad \ldots\text{[from (i)]}$$

$$\therefore\ \frac{d^2y}{dx^2} = \frac{e^{\sqrt{t}}(\cos\sqrt{t} + \sin\sqrt{t})}{\cos^3\sqrt{t}}$$

Q.8 **If y = A cos (log x) + B sin (log x), show that $x^2 y_2 + xy_1 + y = 0$.**

Ans: []

Explanation:

y = Acos (log x) + Bsin (log x) …(i)

Differentiating w.r.t. x, we get

$\frac{dy}{dx}$ = –A sin (log x). $\frac{d}{dx}(logx)$ + Bcos (log x) $\frac{d}{dx}(logx)$

$\frac{dy}{dx}$ = –Asin (log x)· $\frac{1}{x}$ + Bcos (log x)· $\frac{1}{x}$

∴ $x\frac{dy}{dx}$ = *–Asin (log x) + Bcos (log x)*

Again, differentiating w.r.t. x , we get

$x\ \frac{d^2y}{dx^2} + \frac{dy}{dx}$ = –Acos (log x)· $\frac{1}{x}$ – Bsin (log x)· $\frac{1}{x}$

∴ $x^2\ \frac{d^2y}{dx^2} + x\frac{dy}{dx}$ = *–[Acos (log x) + Bsin (log x)]*

= –y …[From (i)]

∴ $x^2\ \frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$

∴ $x^2y_2 + x\,y_1 + y = 0$

SECTION D

Attempt any ONE of the following questions: **(4)**

Q.9 **If y = cos (mcos^{-1} x) then show that** $\left(1-x^2\right)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + m^2y = 0$**.**

Ans: []

Explanation:

Given that y = cos (m cos^{-1} x)

∴ cos^{-1} y = mcos^{-1} x

Differentiate (I) w.r.t. x

$\frac{d}{dx}\left(\cos^{-1}y\right) = m\frac{d}{dx}\left(\cos^{-1}x\right)$

$-\frac{1}{\sqrt{1-y^2}} \cdot \frac{dy}{dx} = -\frac{m}{\sqrt{1-x^2}}$

$\sqrt{1-x^2} \cdot \frac{dy}{dx} = m\sqrt{1-y^2}$

Squaring both sides

$\left(1-x^2\right) \cdot \left(\frac{dy}{dx}\right)^2 = m^2\left(1-y^2\right)$

Differentiate w.r.t. x

$\left(1-x^2\right)\frac{d}{dx}\left(\frac{dy}{dx}\right)^2 + \left(\frac{dy}{dx}\right)^2\frac{d}{dx}\left(1-x^2\right) = m^2\frac{d}{dx}\left(1-y^2\right)$

$\left(1-x^2\right) \cdot 2\left(\frac{dy}{dx}\right) \cdot \frac{d}{dx} \cdot \left(\frac{dy}{dx}\right) + \left(\frac{dy}{dx}\right)^2(-2x) = m^2(-2y)\frac{dy}{dx}$

$$2\left(1-x^2\right)\cdot\frac{dy}{dx}\cdot\frac{d^2y}{dx^2}-2x\left(\frac{dy}{dx}\right)^2=-2m^2y\frac{dy}{dx}$$

Dividing throughout by $2\frac{dy}{dx}$ we get,

$$\left(1-x^2\right)\cdot\frac{d^2y}{dx^2}-x\frac{dy}{dx}=-m^2y$$

$$\therefore\ \left(1-x^2\right)\cdot\frac{d^2y}{dx^2}-x\frac{dy}{dx}+m^2y=0$$

Q.10 **Find the n^{th} derivative of cos (3 − 2x)**

Ans: []

Explanation:

Let $y = \cos(3 - 2x)$

Differentiating w. r. t. x, we get

$$\frac{dy}{dx}=\frac{d}{dx}[\cos\ (3-2x)]$$

$$= -\sin(3-2x).\ \frac{d}{dx}(3-2x)$$

$$= (-\sin(3-2x))(-2)$$

$$= -2\cos\left(\frac{\pi}{2}+3-2x\right)$$

Differentiating w. r. t. x, we get

$$\frac{d}{dx}\left(\frac{dy}{dx}\right)=-2\frac{d}{dx}\left[\cos\left(\frac{\pi}{2}+3-2x\right)\right]$$

$$=(-2)\left(-\sin\left(\frac{\pi}{2}+3-2x\right)\right).\ \frac{d}{dx}\left(\frac{\pi}{2}+3-2x\right)$$

$$= (-2)\left(-\sin\left(\frac{\pi}{2}+3-2x\right)\right)(-2)$$

$$\therefore\ \frac{d^2y}{dx^2}=(-2)^2\cos\left(\frac{2\pi}{2}+3-2x\right)$$

Differentiating w. r. t. x, we get

$$\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)=(-2)^2\frac{d}{dx}\left[\cos\left(\frac{2\pi}{2}+3-2x\right)\right]$$

$$=(-2)^2\left[-\sin\left(\frac{2\pi}{2}+3-2x\right)\right].\ \frac{d}{dx}\left(\frac{2\pi}{2}+3-2x\right)$$

$$=(-2)^2\left[-\sin\left(\frac{2\pi}{2}+3-2x\right)\right].(-2)$$

$$\therefore\ \frac{d^3y}{dx^3}=(-2)^3\cos\left(\frac{3\pi}{2}+3-2x\right)$$

In general, n^{th} order derivative will be

$$\frac{d^n y}{dx^n} = (-2)^n \cos\left(\frac{n\pi}{2} + 3 - 2x\right)$$