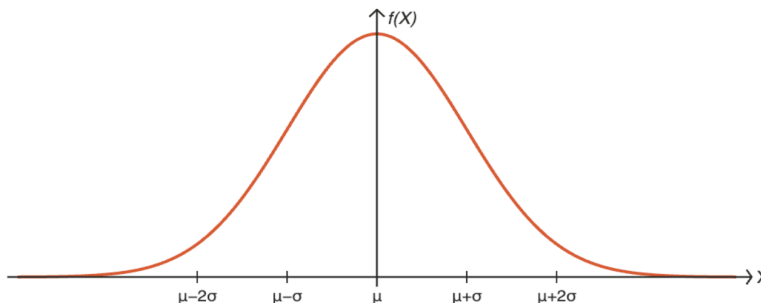


## Derivation of the Normal Distribution Curve



$$f(X) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2}$$

**Let**

$g(r)$  = probability that a point is at distance  $r$  from origin  
 $f(x)$  = probability that a point is at distance  $x$  from origin

**Assume:**

1. Rotational symmetry (probability only depends on  $r$ )
2.  $x$  and  $y$  are independent

So

$$g(r) \Delta x \Delta y = f(x) \Delta x \cdot f(y) \Delta y$$

$$g(r) = f(x) f(y)$$

$$\frac{d}{d\theta} g(r) = \frac{d}{d\theta} (f(x) f(y))$$

$$0 = \frac{df(x)}{d\theta} f(y) + f(x) \frac{df(y)}{d\theta}$$

$$0 = \frac{df(x)}{dx} \frac{dx}{d\theta} f(y) + f(x) \frac{df(y)}{dy} \frac{dy}{d\theta}$$

Using  $x = r \cos \theta$ ,  $y = r \sin \theta$ :

$$0 = -f'(x) r \sin \theta f(y) + f(x) f'(y) r \cos \theta$$

$$0 = -f'(x) y f(y) + f(x) f'(y) x$$

$$f'(x) f(y) y = f(x) f'(y) x$$

$$\frac{f'(x)}{x f(x)} = \frac{f'(y)}{y f(y)}$$

Since  $x$  and  $y$  are independent,

$$\frac{f'(x)}{x f(x)} = c, \quad \text{where } c \text{ is a constant}$$

Rearranging and integrating,

$$\int \frac{f'(x)}{f(x)} dx = \int c x dx$$

$$\ln f(x) = \frac{c x^2}{2} + D, \quad \text{where } D \text{ is a constant}$$

$$\boxed{f(x) = A e^{cx^2/2}} \quad \text{where } A \text{ is a constant}$$

### Finding the constant $A$

Since  $f(x)$  is a probability distribution function, we assume  $\int_{-\infty}^{\infty} f(x) dx = 1$ , or

$$\int_0^{\infty} f(x) dx = \int_0^{\infty} A e^{\frac{c}{2}x^2} dx = \frac{1}{2}, \quad \text{so } c \text{ must be } < 0$$

Thus

$$\int_0^{\infty} e^{\frac{c}{2}x^2} dx = \int_0^{\infty} e^{\frac{c}{2}y^2} dy = \frac{1}{2A}$$

and their product can be written:

$$\int_0^{\infty} e^{\frac{c}{2}x^2} dx \int_0^{\infty} e^{\frac{c}{2}y^2} dy = \frac{1}{4A^2}$$

$$\int_0^{\infty} \int_0^{\infty} e^{\frac{c}{2}(x^2+y^2)} dx dy = \frac{1}{4A^2}$$

Converting to polar coordinates:

$$\int_0^{\pi/2} \int_0^{\infty} e^{\frac{c}{2}r^2} r dr d\theta = \frac{1}{4A^2}$$

Let  $u = r^2$ , so  $du = 2r dr$ :

$$\begin{aligned} \frac{1}{4A^2} &= \frac{1}{2} \int_0^{\pi/2} \int_0^\infty e^{\frac{c}{2}u} du d\theta \\ &= \frac{1}{2} \int_0^{\pi/2} \left. \frac{2}{c} e^{\frac{c}{2}u} \right|_0^\infty d\theta \\ &= \frac{1}{2} \int_0^{\pi/2} \frac{2}{c} [0 - 1] d\theta \\ &= -\frac{1}{c} \cdot \frac{\pi}{2} \end{aligned}$$

$$A^2 = -\frac{c}{2\pi} \implies \boxed{A = \sqrt{\frac{-c}{2\pi}}, \quad c < 0}$$

So

$$\int_0^\infty e^{\frac{c}{2}x^2} dx = \frac{1}{2A} = \frac{1}{2} \sqrt{\frac{2\pi}{-c}}$$

$$\boxed{\int_0^\infty e^{\frac{c}{2}x^2} dx = \sqrt{\frac{-\pi}{2c}}, \quad c < 0}$$

### Identifying $c$ via the variance

Let  $\sigma^2$  be the variance defined as:

$$\sigma^2 = \int_{-\infty}^\infty x^2 f(x) dx = 2 \int_0^\infty x^2 A e^{\frac{c}{2}x^2} dx = 2\sqrt{\frac{-c}{2\pi}} \int_0^\infty x^2 e^{\frac{c}{2}x^2} dx$$

Integrate by parts with  $u = x$ ,  $dv = x e^{\frac{c}{2}x^2} dx$ , so  $du = dx$ ,  $v = \frac{1}{c} e^{\frac{c}{2}x^2}$ :

$$\begin{aligned} \sigma^2 &= 2\sqrt{\frac{-c}{2\pi}} \left[ \left. \frac{x}{c} e^{\frac{c}{2}x^2} \right|_0^\infty - \frac{1}{c} \int_0^\infty e^{\frac{c}{2}x^2} dx \right] \\ &= 2\sqrt{\frac{-c}{2\pi}} \left[ 0 - 0 - \frac{1}{c} \sqrt{\frac{-\pi}{2c}} \right] = -\frac{2}{c} \sqrt{\frac{1}{4}} = -\frac{1}{c} \end{aligned}$$

Therefore:

$$\sigma^2 = -\frac{1}{c} \implies \boxed{c = -\frac{1}{\sigma^2}}$$

$$A = \sqrt{\frac{-c}{2\pi}} = \sqrt{\frac{1}{2\pi\sigma^2}} = \frac{1}{\sigma\sqrt{2\pi}}, \quad \text{where } \sigma \text{ is the standard deviation}$$

Substituting these values for  $A$  and  $c$  into  $f(x)$ :

$$f(x) = A e^{\frac{c}{2}x^2} = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2}$$

Finally, transforming  $x \rightarrow X - \mu$ , where  $X$  is the observed value, and  $\mu$  is the average of these observed values, we arrive at the following equation for the normal distribution curve:

$$f(X) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2}$$