Chaos Theory- Origins to Applications

By Karen McCain

Is chaos randomness or disorder? It depends on who is answering the question. When mathematicians speak of chaos it is rather likely not what most people have in mind upon hearing the word. In general, most people when they hear the term "chaos" think it is being used to describe things when there is complete confusion, no order or organization, and things are in disarray. (Chaos, 2018) Chaos theory or non-linear mathematics, when discussed by mathematicians on the other hand, describe things that look like they are in disarray but that actually obey certain laws and have a very elegant simplicity and order to them. Part of what gives objects described by nonlinear mathematics the appearance of being in disarray is the uncertainty principle.

Even though the mathematics Chaos theory are incredibly complex and difficult for most people to understand the principle of things in this world being uncertain and unpredictable is something that people in all fields of study and all profession face and understand quite well.

Uncertainty shows up in sports and is what makes the contest between opponents exciting. Military personnel deal with uncertainty in their campaigns. When lives are on the line and freedom is at stake, they endeavor to minimize the uncertainty they face on the battlefield. Medical professionals are working to understand the patterns created as objects obey the laws of chaos in order to understand and to know how to best help their patients. These modern professionals are reaping the benefits of work done and discoveries made by great thinkers long ago.

Origins of Erratic Behavior – Chaos

Isaac Newton's laws of physics first emerged in the 1680s. They were linear and showed, theoretically, how the universe and other such observable systems run like clockwork. In other words, with such deterministic laws, it seems that if we know enough about the current state of a system then we should be able to accurately predict its future outcome. His equations of motion worked when applied to his model of the universe which looked at the gravitational pull between two bodies. During the Nineteenth Century mathematicians weren't sure about Newton's laws really working on a large scale. One of those mathematicians was Henri Poincare. He applied Newton's equations to three bodies instead of just two and found something unexpected. Adding a third body to the equation changes things. (Oestreicher, 2007) A seemingly deterministic system starts to show chaotic behavior. Slight differences in initial measurements of a system produce unpredictably huge differences in the system's future outcome. He discovered that it was impossible to measure initial conditions with complete determination, therefore, it was impossible to predict the outcomes of complex systems with absolute certainty. He explained it this way:

> A very small cause, which eludes us, determines a considerable effect that we cannot fail to see, and so we say that this effect is due to chance. If we knew exactly the laws of nature and the state of the universe at the initial moment, we could

accurately predict the state of the same universe at a subsequent moment. But even if the natural laws no longer held any secrets for us, we could still only know the state approximately. If this enables us to predict the succeeding state to the same approximation, that is all we require, and we say that the phenomenon has been predicted, that it is governed by laws. But this is not always so, and small differences in the initial conditions may generate very large differences in the final phenomena. A small error in the former will lead to an enormous error in the latter. Prediction then becomes impossible, and we have a random phenomenon. (Oestreicher 2007)

Visionaries - Henri Poincare and others

Henri Poincare was truly a visionary. He put his genius towards studying philosophy and physics as well as mathematics. In mathematics, alone, he studied a number of areas including quantum theory, thermodynamics, electricity, and cosmology. He made contributions in several mathematical fields such as fluid mechanics, the philosophy of science and the special theory of relativity as a co-discoverer with Hendrik Lorentz and Albert Einstein. (O'Connor/Robertson 2003) Poincare's discoveries and contributions to science and mathematics served as steppingstones for those who followed after him.

In 1959 an American meteorologist by the name of Edward Lorenz (1917-2008) fostered a rebirth of Poincare's uncertainty theories. Lorenz was part of the meteorological community that was excited about using the new technology of computers to predict weather patterns more accurately. After running a sequence of numbers that simulate weather conditions through a computer, he decided to run the numbers again. He let the computer work in the simulated weather patterns and left for about an hour. When he returned, he noticed that the resulting numbers were different than the first set. Lorenz realized that the second set of numbers were not exactly the same as the first. The second set were rounded off numbers of the first set. This ever so slight change yielded enormous differences in the outcomes. From this incident, Lorenz discovered the phenomenon of *sensitive dependence on initial conditions* (SDIC). (Sorensen & Zobitz, 2012)

A few years after Lorenz's accidental discovery James A. York and his student, Tien-Lien Li, wrote a paper while at the University of Maryland in 1975. Their paper *Period Three Implies Chaos* appeared in *American Mathematic Monthly*. They used the term *Chaos* to describe the uncertainty of outcomes observed in systems in physics, biology, and mathematics. (Berger/Starbird 2013, pg. 552) As mentioned in the introduction chaos in mathematics is not randomness nor disorder. Mathematical chaos is in fact "an apparent lack of order in a system that nevertheless obeys particular laws or rules; this understanding of chaos is synonymous with dynamical instability…" that was first discovered by Poincare. (Rouse 2016)

Taming the Chaos or At Least Understanding It

In 1972 Lorenz spoke at a meeting of the *American Association for the Advancement of Science* where he explained the *butterfly effect*. The butterfly effect is an illustration of how minute changes in small air mass systems caused by a butterfly flapping its wings interact with and affect larger weather systems. It is said that these imperceptible changes in the beginning cascade into enormous consequences such as a tornado weeks later on the other side of the world. Scientists know that small changes yield "dramatic differences" but they don't yet know what changes trigger which outcomes. (Berger/Starbird 2013, pp. 552-554)

When Lorenz used the butterfly effect to explain Chaos Theory he made it more understandable for the masses. Most people are not going to grasp *the uncertainty principle* as it applies to particles in quantum mechanics but visualizing changes that happen in a system they are familiar with such as weather patterns helps it all make sense!

Mathematics

Chaos is evident in our world in a variety of forms. Scientists in a wide range of disciplines see chaos in the systems they study. Biologists and astronomers, as well as physicists, have all made interesting discoveries.

The Lorenz Model

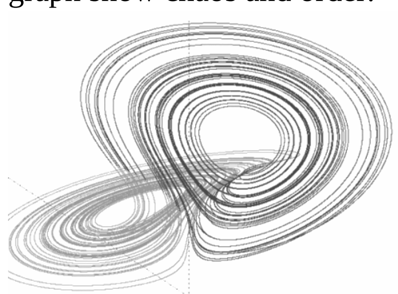
The Lorenz Model is a simplification of a model of convection between two surfaces involving 6 equations. This system of differential equations was much simpler than Barry Saltzman's previous system with 14 or 12 variables. Edward Lorenz simplified Saltzman's system to "a three parameter, three variable system of differential equations that exhibited non-periodic solutions". (Sorensen & Zobitz, 2012, pg.25) This simpler system couldn't be used to predict weather, but it might show how SDIC (*sensitive dependence* on *initial conditions*) develops and leads to chaos. (Sorensen & Zobitz, 2012)

$$dx/dt = P(y - x)$$

$$dy/dt = Rx - y - xz$$

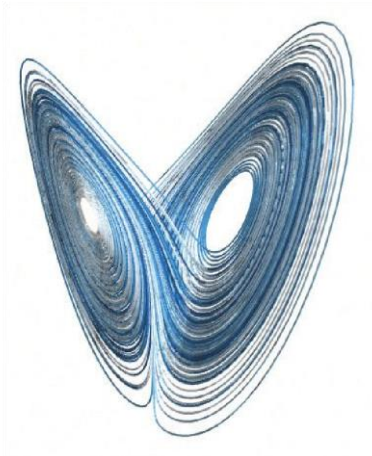
$$dz/dt = xy - Bz$$

When the equations are solved the following graph show chaos and order.



This set of equations is slightly different than the first, therefore, they produce a different graph.

$$\frac{dX}{dt} = -\sigma X + \sigma Y$$
$$\frac{dY}{dt} = -XZ + rX - Y$$
$$\frac{dZ}{dt} = XY - bZ.$$



The results of the equations need to be plotted in three dimensions in order to achieve a three-dimensional looking image. (Sprott, 2010)

Elegant Chaos

Most of the time when words such as "stunning", "beautiful", or "elegant" are used one does not think of mathematics. Usually, these words are used to describe things more easily seen or aesthetically pleasing, for instance, a stunning sunset, a beautiful painting, or a woman in an elegant ball gown. When differential equations of the Lorenz Model are solved and graphed the resulting images are stunningly beautiful and their graceful lines create extremely elegant shapes. Some are reminiscent of butterfly wings and are a reminder of "The Butterfly Effect".

Even though this is the most common use of the word "elegant" mathematicians think of the term in a bit of a different way. Julian C. Sprott in his book *Elegant Chaos: Algebraically Simple Chaotic Flows* notes that the term 'elegance' used in his book is referring to the "form of the equation[s]" and not their "engaging plots". (pg. 38) Mr. Sprott said, "A major theme of this book is to find values of the parameters that make the equations simple but that allow the system to behave chaotically." (pg. 3) He went on to explain the following:

> A system of equations is deemed most elegant if it contains no unnecessary terms or parameters and if the parameters that remain have a minimum of digits. This notion can be quantified by writing an equation such as Eq. (1.9) in its most general form such as $\ddot{x} - (a_1 - a_2x^2 - a_3\dot{x}^2)\dot{x} + a_4 \sin(a_5x + a_6) + a_7x^3 = a_8 \sin a_9t$ (1.10) and adjusting the parameters a_1 through a_9 to achieve this end. Ideally, we want as many of the parameters of be zero as possible while preserving the chaos, and the greatest number of those that remain should be ±1. Note that it is generally possible to linearly rescale the variables (x and t in this case) so that a corresponding number of the parameters are ±1.
> One way to quantify the elegance of Eq. (1.10) is thus to count the number of nonzero parameters and then add to that count the total number of digits including the decimal point but excluding leading and trailing

> zeros for any parameters that are not ±1 so that integer parameters are preferred. The resulting number, which perhaps should be called inelegance, is the quantity to be minimized. By this criterion, Eq. (1.9) has an inelegance of 39, whereas Eq. (1.6), when viewed as a special case of Eq. (1.10) with $a_1 = a_2 = a_3 = a_6 = a_7 = 0$ and $a_4 = a_5 = a_8 = a_9 = 1$, has an inelegance of 4. (pp. 38-39)

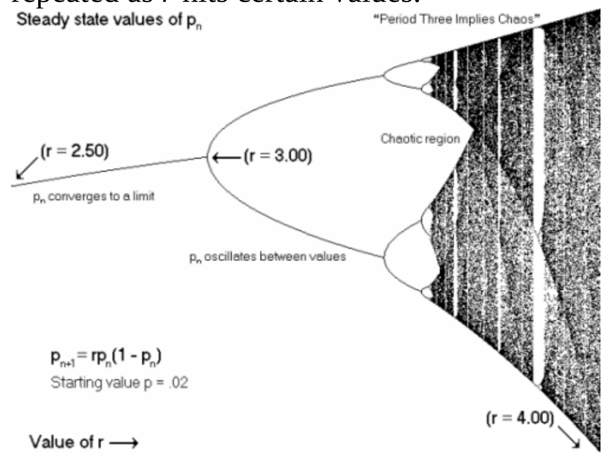
In mathematics simplicity equals elegance and can lead to chaos. The visual manifestation of the solutions of differential equations are also beautiful and contain their own elegance.

The Logistic Equation Chaos

Chaos shows up not only in physical systems but is also observable in biological systems, too. The logistical difference equation is a pretty simple quadratic equation. It has been used by biologists to predict animal populations.

$$x_{n+1} = rx_n(1 - x_n)$$

r = growth rate
x = population size
When *r* has a fixed value that is relatively low an x is also low ($x_1, x_2, . . . x_n$) The equation will produce a single number. When *r* reaches a value of 3.0 bifurcation occurs and we see an oscillation between two values. If *r* eventually reaches 3.57 then x_n passes bifurcations of period eight, sixteen, and on to chaos. These periods of bifurcation and chaos are repeated as *r* hits certain values.



The Three Body Problem

As mentioned in Strand One, Newton's laws of physics explained that it is possible to predict the outcome of a system if we have enough information about its current state. When two celestial bodies interact with one another they are each only influenced in their orbits by the gravitational pull of the other. Therefore, their orbits remain constant.

When a third body is in close enough proximity to influence the other ones its gravitational pull causes changes in the direction and speed of their orbits. This profound effect on their orbits makes it impossible to predict the outcome of the system. Thus, there is uncertainty and chaos. In a system with three or more bodies each body feels and is affected by one another's gravitational pull according to their mass, speed, and proximity to one another. Smaller bodies that orbit a larger one "tend to settle into settle into simple fraction (1/2, 2/3, 1/3, etc.) multiples of each other". (The Physicist, 2011) A two-body system follows Newton's deterministic laws. Our solar system has multiple planets, moons, comets, asteroids, etc. that all chaotically affect one another in their journeys.

Modern Application

When one thinks about Chaos Theory one most likely thinks of physicists, quantum mechanics, the universe and other lofty or complicated subjects. Chaos Theory can be seen in a variety of fields of study and even where one might never think to look, for examplein the sport of Football. Chaos Theory looks at systems and subsystems that interact with one another in unpredictable ways to yield unexpected outcomes. Anyone who has watched a football game knows how unpredictable the final score is.

Team 1 and Team 2 are systems all their own. Each has a group of offensive players and a group of defensive players. A team gets 4 chances or "downs" to move the ball at least 10 yards towards their end zone where they can score. If they succeed in moving the ball at least 10 yards they are allowed another 4 downs. If they fail the other team gets the ball and they then work at moving the ball into scoring position. If each team met at the line of scrimmage and the game went as follows; Team 2's defense waited for Team 1's Quarterback to have the ball hiked to him so he can pass it to a teammate, that teammate is tackled by members of Team 2's defense after moving the ball a few yards when team 1's offense tries unsuccessfully to block them, after 4 downs the ball wouldn't be moved very far, and it would be Team 2's turn to employ its offense against Team1's defense in its attempts to move the ball and score. This kind of game would be extremely boring! What makes Football exciting is the anticipation of the unexpected. Football fans have come to expect the unexpected. In mathematical terms, this is called chaos or uncertainty.

From the very first moments of the game, there was uncertainty. It starts with the kickoff. To a viewer who is unfamiliar with the game of football the kickoff can look like a very random event. With both teams on the field and seemingly spread around without any rhyme or reason. One team kicks the ball, and all the players start running around like ants when their anthill gets stepped on.

These aimless looking actions do have a purpose. The kicking team kicks the ball and runs forward. The receiving team tries to catch the ball and run it as far as they can towards their end zone. There are also several strict rules in the NFL's (National Football League) (NFL, 2017) handbook that give parameters for a kickoff. These rules dictate how far away the receiving team has to be from the ball when it is kicked, how far the ball has to travel to be called fair, etc. As the game continues there is a whole host of rules that must be followed. These rules or parameters form a structure inside which the game is played. This set of rules also sets forth very specific penalties to be applied when a rule is broken.

It seems that within a game with so many parameters there would never be any room for the unexpected. When Team 1's offense has the ball, they devise a play to get past Team 2's defense on the field and move the ball as far as possible. At the same time, Team 2 is trying to anticipate Team 1's actions and perform a counter play that will thwart Team 1's plan. If Team 2 succeeds and Team 1's plan is foiled, they must now improvise the rest of the play and it is called a "broken play". Chaos had set in.

If a pass is thrown too short or too long, it might be intercepted by the other teams and suddenly the direction of play changes as the ball is being raced to the other end of the field. The timing of the player throwing the ball might exactly lineup with the receiver but if the opposing defense gets close enough to the receiver, they could tackle him immediately or stop him before he gets very far down the field. On the other hand, the receiver could even aid the other team's defense, outrun them and score a touchdown. It all depends on an innumerable number of variables; their agility, the running speed of each player, their ability to catch and remain in control of the ball, etc. Even when these traits are finally homed in a player there are still other variables. The sun or stadium lights might shine in a player's eyes making it difficult to see. Moisture on a player's gloves can make it hard to complete a pass. Tired or sore muscles can affect how fast a player is able to run or how well he can throw the ball or even catch it.

When a play does go as planned it might be because the team in possession of the ball is able to trick the opponent into running to the wrong area of the field while they run the other way and are able to score. Another variable is the decisions made by the players about how to handle the ball in each play. If time is short and there is a lot of yardage to cover before a goal is possible a player may throw a Hail Mary (Sporting Charts, 2015). That means that the ball is thrown forward from the line of scrimmage in desperation of one final goal and to run out the clock. A ball being thrown from a great distance has a greater chance of being intercepted by the other team or being fumbled. All these small variables that affect the play of the game eventually have an impact on the final score. That is why a final score can be guessed at but never predicted accurately just by looking at the players or each team as a whole.

At the beginning of each football season on both the college and professional levels, teams are looked at, ranked, analyzed, and almost dissected. People want to know the skill level of the players, their recent injuries, past achievements, all their stats, etc. This deep study of the athletes and their teams is done with the hope of knowing, even in some small way, what kind of season to expect. Yet, somewhere in the back of their minds, the spectators know that in football one should expect the unexpected. Ben Cohen and Jonathan Clegg wrote an article on The Wall Street Journal's website (2015) about the chaos of college football in the 2014 season. They mentioned the just going from 2 teams in the playoffs to 4 teams means that there are more games that add more variables to the whole equation.

Rankings change on a weekly basis because of last-minute victories and unexpected injuries that cause the loss of an instrumental player and the loss of a game. Fans can't just root for their team alone. They understand that the outcomes of matchups between the other teams affect the ranking of their favorite team. Predicting who will be the winning team at the end of the playoffs when it's still September is impossible. It can be speculated about and guessed at but never accurately predicted. In the world of professional football, the saying goes: "Any given

Sunday any team can win". The same is true on Saturdays in the world of college football, too.

Military Science is another field where the concept of Chaos is being applied. It is not just a passing fad. Back in 1996, the U.S. Marine Corps included in their Doctrinal Publication 6 entitled *Command and Control* instructions on understanding Chaos Theory and how it relates to warfare. (U.S. Marine Corps, 1996)

Unpredictability on the battlefield has been growing since the beginning of WWII. In the most recent generation chaos on the battlefield has increased as the use of more sophisticated processing technologies and information gathering has increased.

These new and highly advanced technologies were intended to "reduce confusion on the battlefield". (Pfaff, 2000) In some respects that was the case, however, commanders in several different campaigns saw unexpected outcomes with their use. For example, a gang leader in Somalia who did not possess the same technological advantages as the world's last superpower used the element of surprise to catch them off guard and won. The United States and her allies made up the coalition forces in the Gulf War. They faced an enemy force that equaled them in size and strength. In the end, there weren't even 200 friendly casualties when they expected tens of thousands as the price of victory. In that instance, the unexpected result was a positive one.

In the past, an army sought to gain higher ground. This would give them the advantage over their enemy. By being up high enough to see their approach from a great distance it gave them time to prepare to defend against them, reducing or eliminating the element of surprise. The higher ground also made it harder for the enemy to reach them for an attack. Modern technology uses satellite imaging along with devices that utilize light and thermal amplification to visualize the battlefield and keep track of the enemy. (Pfaff, 2000)

Small changes that occurred on the battlefield were seized by the Germans in WWII when they acted to destabilize the system. They used them to their advantage. The Germans used their aircraft and tanks in fast-moving campaigns and in unexpected ways against the allied forces in their slower moving but well-fortified formations. The Germans also employed the use of misinformation given to the allies by spies. This misinformation caused confusion and led to an opening in the French lines which the Germans used to bring about the fall of France in mere hours. (Pfaff, 2000) The complexity of the system and the use of unpredictable or "irregular actions" (Sun-Tzu, Huang, 1993) led to a German victory.

While new and extremely sophisticated technology has created chaos on the battlefield, some rather simple tactics have been used throughout the ages to gain an advantage over one's enemy in wartime. Sun-Tzu teaches in his ancient text, *The Art of War*, (Haung 1993) how to see that an enemy is weakened through the use of "unconventional means". By utilizing deception, discouragement, evasion, surprise, attacking their weak points, etc. an army can cause uncertainty and unpredictability, thereby gaining an advantage over their enemy. Sun-Tzu didn't call his tactics chaos but causing chaos is certainly what he proposed in several areas of his writings. (Sun-Tzu, 1993)

Psychotherapy

Psychotherapists are finding new approaches to their work of understanding how personality develops by utilizing chaos theory or nonlinear mathematics. They feel that it is not easy to find a symbol to represent the complexity of the material they deal with. Some in the field feel that perhaps the Mandelbrot set with its overwhelming complexity used in conjunction with colored graphics created by a computer could be helpful. A therapist may be able to use these tools to understand how much of the session's material focuses on the heart of the person as opposed to external events.

Another pattern made of fractals and designed by Mandelbrot is called the monkey's tree. Like the Mandelbrot set it is a self-repeating pattern. This aspect of its nature may not be apparent upon first observing it but with closer inspection, one can see that the shape of the monkeys is repeated in positive and negative space, in different angles and sizes.

In her paper entitled *Chaos Theory: A New Paradigm for Psychotherapy?* (1991) Isla Lonie points out that studying the monkey's tree and eventually discovering the self-repeating pattern is very reminiscent of what it can be like working with a patient. At first, there is so much information to take in and make sense of. It can be very complex, even overwhelming. With perseverance and continual study, one starts to see a self-repeating pattern and things start to make more sense. Is the initial time of uncertainty during treatment necessary? Some therapists believe it is. Perhaps it is what builds patience in the therapist. (Lonie, 1991)

Other images created in nonlinear mathematics are the limit cycle attractor, the strange attractor, and the equilibrium attractor. When an observation of a cross-section of one of these attractors (a Poincaré map) is made a pattern is not visible early on. However, just as with the monkey's tree a pattern is eventually noticed. When a therapist discovers behavior that repeats like a limit cycle attractor it is then possible to identify and then start to work at changing it. These are just a few of the ways non-linear mathematics are being used to enhance the field of psychotherapy. (Lonie, 1991)

Whether psychotherapists are battling the unknown to understand and heal the mind of a patient or a military commander is fighting survival on the battlefield, the number of variables they face are innumerable. These variables create uncertainty. People are trying to understand our world and the systems within it. Professionals in science, medicine, sports, and even on the battlefield are working to understand chaos theory and how they can use it to their advantage. Using nonlinear mathematics or Chaos theory gives them a platform from which to observe these and other systems in new and lifesaving ways.

We should understand that mathematics touches more than just computational lessons in a classroom. If we truly grasp this and open our minds to new applications of nonlinear mathematics the future of many scientific, artistic, and social fields will yield beautiful comprehension of concepts never before conceived of.

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