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PHILOSOPHICAL TRANSACTIONS,

GIVING SOME

ACCOUNT

OF THE

Present Undertakings, Studies, *and* Labours,

OF THE

INGENIOUS,

IN MANY

Considerable Parts of the WORLD.

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M.DCC.LXIV.

As to the quantity of water taken off, I find it to amount, upon the neareſt calculation, to twenty-four pints at each operation; for though the firſt time produced only twelve pints, and in ſeveral of the latter operations the quantity fell ſhort of twenty-four pints, yet I may venture to ſtate it at leaſt at twenty-four pints or three gallons on an average, as in many of the operations I took off from twenty-eight to thirty pints. The number of times I tapped her was in all 155, which brings out in the whole 3720 pints, being 465 gallons, not far ſhort of ſeven hogſheads and an half. As to the authenticity of the whole, your connections with the family, and frequent opportunities of ſeeing this young lady during her illneſs, will put it beyond a doubt. I have therefore no more to add, than my wiſh that the caſe may prove acceptable to the Society.

I am, &c.

VII. *Problems concerning Interpolations.* *By* Edward Waring, *M. D. F. R. S. and of the Inſtitute of* Bononia, Lucaſian *Profeſſor of Mathematics in the Univerſity of* Cambridge.

Read Jan. 9, 1779.

MR. BRIGGS was the firſt perſon, I believe, that invented a method of differences for interpolating logarithms at ſmall intervals from each other: his principles were followed by REGINALD and MOVTON in France. Sir ISAAC NEWTON, from the ſame principles, diſcovered a general and elegant ſolution of the abovementioned problem: perhaps a ſtill more elegant one on ſome accounts has been ſince diſcovered by Meſſ. NICHOLE and STIRLING. In the following theorems the ſame problem is reſolved and rendered ſomewhat more general, without having any recourſe to finding the ſucceſſive differences.

THEOREM I.

Aſſume an equation $a+bx+cx^2+dx^3 \ldots . x^{n-1}=y$, in which the co-efficients a, b, c, d, e, &c. are invariable;

 let

let $\alpha, \beta, \gamma, \delta, \varepsilon$, &c. denote n values of the unknown quantity x, whoſe correſpondent values of y let be repreſented by $s^\alpha, s^\beta, s^\gamma, s^\delta, s^\varepsilon$, &c. Then will the equation $a + bx + cx^2 + dx^3 + ex^4 \ldots x^{n-1} = y =$

$$\frac{\overline{x-\beta}\times\overline{x-\gamma}\times\overline{x-\delta}\times\overline{x-\varepsilon}\times\&c.}{\overline{\alpha-\beta}\times\overline{\alpha-\gamma}\times\overline{\alpha-\delta}\times\overline{\alpha-\varepsilon}\times\&c.}\times s^\alpha + \frac{\overline{x-\alpha}\times\overline{x-\gamma}\times\overline{x-\delta}\times\overline{x-\varepsilon}\times\&c.}{\overline{\beta-\alpha}\times\overline{\beta-\gamma}\times\overline{\beta-\delta}\times\overline{\beta-\varepsilon}\times\&c.}\times s^\beta$$

$$+\frac{\overline{x-\alpha}\times\overline{x-\beta}\times\overline{x-\delta}\times\overline{x-\varepsilon}\times\&c.}{\overline{\gamma-\alpha}\times\overline{\gamma-\beta}\times\overline{\gamma-\delta}\times\overline{\gamma-\varepsilon}\times\&c.}\times s^\gamma + \frac{\overline{x-\alpha}\times\overline{x-\beta}\times\overline{x-\gamma}\times\overline{x-\varepsilon}\times\&c.}{\overline{\delta-\alpha}\times\overline{\delta-\beta}\times\overline{\delta-\gamma}\times\overline{\delta-\varepsilon}\times\&c.}\times s^\delta$$

$$+\frac{\overline{x-\alpha}\times\overline{x-\beta}\times\overline{x-\gamma}\times\overline{x-\delta}\times\&c.}{\overline{\varepsilon-\alpha}\times\overline{\varepsilon-\beta}\times\overline{\varepsilon-\gamma}\times\overline{\varepsilon-\delta}\times\&c.}\times s^\varepsilon + \&c.$$

DEMONSTRATION.

Write α for x in the equation $y =$

$$\frac{\overline{x-\beta}\times\overline{x-\gamma}\times\overline{x-\delta}\times\overline{x-\varepsilon}\times\&c.}{\overline{\alpha-\beta}\times\overline{\alpha-\gamma}\times\overline{\alpha-\delta}\times\overline{\alpha-\varepsilon}\times\&c.}\times s^\alpha + \frac{\overline{x-\alpha}\times\overline{x-\gamma}\times\overline{x-\delta}\times\overline{x-\varepsilon}\times\&c.}{\overline{\beta-\alpha}\times\overline{\beta-\gamma}\times\overline{\beta-\delta}\times\overline{\beta-\varepsilon}\times\&c.}\times s^\beta +$$

&c.; and all the terms but the firſt in the reſulting equation will vaniſh, for each of them contains in its numerator a factor $x-\alpha=\alpha-\alpha=o$; and the equation will become $y=\frac{\overline{\alpha-\beta}\times\overline{\alpha-\gamma}\times\overline{\alpha-\delta}\times\overline{\alpha-\varepsilon}\times\&c.}{\overline{\alpha-\beta}\times\overline{\alpha-\gamma}\times\overline{\alpha-\delta}\times\overline{\alpha-\varepsilon}\times\&c.}\times s^\alpha = s^\alpha$. In the ſame manner, by writing $\beta, \gamma, \delta, \varepsilon$, &c. ſucceſſively for x in the given equation it may be proved, that when x is equal to $\beta, \gamma, \delta, \varepsilon$, &c. then will y become reſpectively $s^\beta, s^\gamma, s^\delta, s^\varepsilon$, which was to be demonſtrated.

2. Aſſume $y=ax^r+bx^{r+s}+cx^{r+2s}+dx^{r+3s} \ldots x^{r+\overline{n-1}s}$; and when x becomes $\alpha, \beta, \gamma, \delta, \varepsilon$, &c. let y become reſpectively

ſpectively $s^\alpha, s^\beta, s^\gamma, s^\delta, s^\varepsilon$, &c.; then will $y =$

$$\frac{x^r\times\overline{x^s-\beta^s}\times\overline{x^s-\gamma^s}\times\overline{x^s-\delta^s}\times\overline{x^s-\varepsilon^s}\times\&c.}{\alpha^r\times\overline{\alpha^s-\beta^s}\times\overline{\alpha^s-\gamma^s}\times\overline{\alpha^s-\delta^s}\times\overline{\alpha^s-\varepsilon^s}\times\&c.}\times s^\alpha$$

$$+\frac{x^r\times\overline{x^s-\alpha^s}\times\overline{x^s-\gamma^s}\times\overline{x^s-\delta^s}\times\overline{x^s-\varepsilon^s}\times\&c.}{\beta^r\times\overline{\beta^s-\alpha^s}\times\overline{\beta^s-\gamma^s}\times\overline{\beta^s-\delta^s}\times\overline{\beta^s-\varepsilon^s}\times\&c.}\times s^\beta$$

$$+\frac{x^r\times\overline{x^s-\alpha^s}\times\overline{x^s-\beta^s}\times\overline{x^s-\delta^s}\times\overline{x^s-\varepsilon^s}\times\&c.}{\gamma^r\times\overline{\gamma^s-\alpha^s}\times\overline{\gamma^s-\beta^s}\times\overline{\gamma^s-\delta^s}\times\overline{\gamma^s-\varepsilon^s}\times\&c.}\times s^\gamma + \&c.$$

This may be demonſtrated in the ſame manner as the preceding theorem, by writing $\alpha, \beta, \gamma, \delta, \varepsilon$, &c. ſucceſſively for x.

PROBLEM.

Let there be n values $\alpha, \beta, \gamma, \delta, \varepsilon$, &c. of the quantity x, to which the n values $s^\alpha, s^\beta, s^\gamma, s^\delta, s^\varepsilon$, &c. of the quantity y correſpond; ſuppoſe theſe quantities to be found by any function X of the quantity x; let π, ρ, σ, τ, &c. be values of the quantities x, to which $s^\pi, s^\rho, s^\sigma, s^\tau$, &c. values of the quantity y correſpond: for x ſubſtitute its abovementioned values π, ρ, σ, τ, &c. in the function X, and let the quantities reſulting be $s^\pi, s^\rho, s^\sigma, s^\tau$, &c. not equal to the preceding $s^\pi, s^\rho, s^\sigma, s^\tau$, &c. reſpectively; to find a quantity which added to the function X ſhall not only give the true values of the quantity y correſponding to the values $\alpha, \beta, \gamma, \delta, \varepsilon$, &c. of the quantity x, but alſo cor-

correſponding to the values π, ρ, σ, τ, &c. of the above-mentioned quantity x.

Aſſume $s^{\pi}-S^{\pi}=T^{\pi}$, $s^{\rho}-S^{\rho}=T^{\rho}$, $s^{\sigma}-S^{\sigma}=T^{\sigma}$, $s^{\tau}-S^{\tau}=T^{\tau}$, &c.; then the errors of the function X will be reſpectively T^{π}, T^{ρ}, T^{σ}, T^{τ}, &c.; and the correcting quantity ſought may be

$$\frac{\overline{x-\alpha}\times\overline{x-\beta}\times\overline{x-\gamma}\times\overline{x-\delta}\times\overline{x-\varepsilon}\times\text{\&c.}}{\overline{\pi-\alpha}\times\overline{\pi-\beta}\times\overline{\pi-\gamma}\times\overline{\pi-\delta}\times\overline{\pi-\varepsilon}\times\text{\&c.}}\times\frac{\overline{x-\rho}\times\overline{x-\sigma}\times\overline{x-\tau}\times\text{\&c.}}{\overline{\pi-\rho}\times\overline{\pi-\sigma}\times\overline{\pi-\tau}\times\text{\&c.}}\times T^{\pi}$$

$$+\frac{\overline{x-\alpha}\times\overline{x-\beta}\times\overline{x-\gamma}\times\overline{x-\delta}\times\overline{x-\varepsilon}\times\text{\&c.}}{\overline{\rho-\alpha}\times\overline{\rho-\beta}\times\overline{\rho-\gamma}\times\overline{\rho-\delta}\times\overline{\rho-\varepsilon}\times\text{\&c.}}\times\frac{\overline{x-\pi}\times\overline{x-\sigma}\times\overline{x-\tau}\times\text{\&c.}}{\overline{\rho-\pi}\times\overline{\rho-\sigma}\times\overline{\rho-\tau}\times\text{\&c.}}\times T^{\rho}$$

$$+\frac{\overline{x-\alpha}\times\overline{x-\beta}\times\overline{x-\gamma}\times\overline{x-\delta}\times\overline{x-\varepsilon}\times\text{\&c.}}{\overline{\sigma-\alpha}\times\overline{\sigma-\beta}\times\overline{\sigma-\gamma}\times\overline{\sigma-\delta}\times\overline{\sigma-\varepsilon}\times\text{\&c.}}\times\frac{\overline{x-\pi}\times\overline{x-\rho}\times\overline{x-\tau}\times\text{\&c.}}{\overline{\sigma-\pi}\times\overline{\sigma-\rho}\times\overline{\sigma-\tau}\times\text{\&c.}}\times T^{\sigma}$$

$$+\frac{\overline{x-\alpha}\times\overline{x-\beta}\times\overline{x-\gamma}\times\overline{x-\delta}\times\overline{x-\varepsilon}\times\text{\&c.}}{\overline{\tau-\alpha}\times\overline{\tau-\beta}\times\overline{\tau-\gamma}\times\overline{\tau-\delta}\times\overline{\tau-\varepsilon}\times\text{\&c.}}\times\frac{\overline{x-\pi}\times\overline{x-\rho}\times\overline{x-\sigma}\times\text{\&c.}}{\overline{\tau-\pi}\times\overline{\quad-\quad}\times\overline{\tau-\sigma}\times\text{\&c.}}\times T^{\tau}$$

+ &c.

Aliter.

Let $\overline{x-\alpha}\times\overline{x-\beta}\times\overline{x-\gamma}\times\overline{x-\delta}\times\overline{x-\varepsilon}\times\text{\&c.}\times\overline{x-\pi}\times\overline{x-\rho}\times\overline{x-\sigma}\times\overline{x-\tau}\times\text{\&c.}=N$; $\overline{\pi-\alpha}\times\overline{\pi-\beta}\times\overline{\pi-\gamma}\times\overline{\pi-\delta}\times\overline{\pi-\varepsilon}\times\text{\&c.}\times\overline{\pi-\rho}\times\overline{\pi-\sigma}\times\overline{\pi-\tau}\times\text{\&c.}=\Pi$; $\overline{\rho-\alpha}\times\overline{\rho-\beta}\times\overline{\rho-\gamma}\times\overline{\rho-\delta}\times\overline{\rho-\varepsilon}\times\text{\&c.}\times\overline{\rho-\pi}\times\overline{\rho-\sigma}\times\overline{\rho-\tau}\times\text{\&c.}=P$; $\overline{\sigma-\alpha}\times\overline{\sigma-\beta}\times\overline{\sigma-\gamma}\times\overline{\sigma-\delta}\times\overline{\sigma-\varepsilon}\times\text{\&c.}\times\overline{\sigma-\pi}\times\overline{\sigma-\rho}\times\overline{\sigma-\tau}\times\text{\&c.}=\Sigma$; $\overline{\tau-\alpha}\times\overline{\tau-\beta}\times\overline{\tau-\gamma}\times\overline{\tau-\delta}\times\overline{\tau-\varepsilon}\times\text{\&c.}\times\overline{\tau-\pi}\times\overline{\tau-\rho}\times\overline{\tau-\sigma}\times\text{\&c.}=T$, &c.; then may the correcting quantity ſought be $N\left(\frac{T^{\pi}}{\Pi\times\overline{x-\pi}}+\frac{T^{\rho}}{P\times\overline{x-\rho}}+\frac{T^{\sigma}}{\Sigma\times\overline{x-\sigma}}+\frac{T^{\tau}}{T\times\overline{x-\tau}}+\text{\&c.}\right)$.

This

This problem may be demonſtrated in the ſame manner as the preceding theorems, by writing for x in the correcting quantity ſucceſſively its values π, ρ, σ, τ, &c.

2. For the correcting quantity ſought may be aſſumed the quantity

$$\frac{\overline{x'-\alpha'}\times\overline{x'-\beta'}\times\overline{x'-\gamma'}\times\overline{x'-\delta'}\times\text{\&c.}\times x'\times\overline{x'-\rho'}\times\overline{x'-\sigma'}\times\overline{x'-\tau'}\times\text{\&c.}}{\overline{\pi'-\alpha'}\times\overline{\pi'-\beta'}\times\overline{\pi'-\gamma'}\times\overline{\pi'-\delta'}\times\text{\&c.}\times\pi'\times\overline{\pi'-\rho'}\times\overline{\pi'-\sigma'}\times\overline{\pi'-\tau'}\times\text{\&c.}}\times T^{\pi}+\frac{\overline{x'-\alpha'}\times\overline{x'-\beta'}\times\overline{x'-\gamma'}\times\overline{x'-\delta'}\times\text{\&c.}\times x'\times\overline{x'-\pi'}\times\overline{x'-\sigma'}\times\overline{x'-\tau'}\times\text{\&c.}}{\overline{\rho'-\alpha'}\times\overline{\rho'-\beta'}\times\overline{\rho'-\gamma'}\times\overline{\rho'-\delta'}\times\text{\&c.}\times\rho'\times\overline{\rho'-\pi'}\times\overline{\rho'-\sigma'}\times\overline{\rho'-\tau'}\times\text{\&c.}}\times T^{\rho}+\text{\&c.}$$

3. In general, let Z be any quantity which is $=0$, when x becomes either $\alpha, \beta, \gamma, \delta, \varepsilon$, &c.: let Z become ſucceſſively A, B, C, D, &c. when x becomes π, ρ, σ, τ, &c. reſpectively. When x either $=\rho, \sigma, \tau$, &c. let $\Pi=0$; but if $x=\pi$, let $\Pi=p$: in the ſame manner when x either $=\pi, \sigma, \tau$, &c. let $P=0$; but when $x=\rho$ let $P=r$: and ſimilarly, let $\Sigma=0$ when x is either π, ρ, τ, &c.; but when $x=\sigma$ let $\Sigma=s$: and likewiſe, when x is either π, ρ, σ, &c. let $T=0$; but when $x=\tau$ let $T=t$: &c. then for the correcting quantity ſought may be aſſumed $\frac{Z}{A}\times\frac{\Pi}{p}\times T^{\pi}+\frac{Z}{B}\times\frac{P}{r}\times T^{\rho}+\frac{Z}{C}\times\frac{\Sigma}{s}\times T^{\sigma}+\frac{Z}{D}\times\frac{T}{t}\times T^{\tau}+\text{\&c.}$

THE-

THEOREM.

Assume (n) quantities $\alpha, \beta, \gamma, \delta, \varepsilon$, &c. then will the sum of all the (n) quantities of the following kind

$$\frac{\alpha^m}{\overline{\alpha-\beta}\times\overline{\alpha-\gamma}\times\overline{\alpha-\delta}\times\overline{\alpha-\varepsilon}\times\&c.}+\frac{\beta^m}{\overline{\beta-\alpha}\times\overline{\beta-\gamma}\times\overline{\beta-\delta}\times\overline{\beta-\varepsilon}\times\&c.}$$

$$+\frac{\gamma^m}{\overline{\gamma-\alpha}\times\overline{\gamma-\beta}\times\overline{\gamma-\delta}\times\overline{\gamma-\varepsilon}\times\&c.}+\frac{\delta^m}{\overline{\delta-\alpha}\times\overline{\delta-\beta}\times\overline{\delta-\gamma}\times\overline{\delta-\varepsilon}\times\&c.}$$

$+\frac{\varepsilon^m}{\overline{\varepsilon-\alpha}\times\overline{\varepsilon-\beta}\times\overline{\varepsilon-\gamma}\times\overline{\varepsilon-\delta}\times\&c.}+\&c.=0$, if m be any whole number less than $n-1$; but if $m=n-1$, then will the above mentioned sum $=1$. In general, the sum of the n terms $\frac{\alpha^m(\beta\gamma\delta\&c.+\beta\gamma\varepsilon\&c.+\beta\delta\varepsilon+\gamma\delta\varepsilon\&c.+\&c.)}{\overline{\alpha-\beta}\times\overline{\alpha-\gamma}\times\overline{\alpha-\delta}\times\overline{\alpha-\varepsilon}\times\&c.}+$

$$\frac{\beta^m(\alpha\gamma\delta\&c.+\alpha\gamma\varepsilon\&c.+\alpha\delta\varepsilon\&c.+\gamma\delta\varepsilon\&c.+\&c.)}{\overline{\beta-\alpha}\times\overline{\beta-\gamma}\times\overline{\beta-\delta}\times\overline{\beta-\varepsilon}\times\&c.}+$$

$$\frac{\gamma^m(\alpha\beta\delta\&c.+\alpha\beta\varepsilon\&c.+\alpha\delta\varepsilon\&c.+\beta\delta\varepsilon\&c.+\&c.)}{\overline{\gamma-\alpha}\times\overline{\gamma-\beta}\times\overline{\gamma-\delta}\times\overline{\gamma-\varepsilon}\&c.}+$$

$$\frac{\delta^m(\alpha\beta\gamma\&c.+\alpha\beta\varepsilon\&c.+\alpha\gamma\varepsilon\&c.+\&c.)}{\overline{\delta-\alpha}\times\overline{\delta-\beta}\times\overline{\delta-\gamma}\times\overline{\delta-\varepsilon}\times\&c.}+$$

$$\frac{\varepsilon^m(\alpha\beta\gamma\&c.+\alpha\beta\delta\&c.+\alpha\gamma\delta\&c.+\beta\gamma\delta\&c.+\&c.)}{\overline{\varepsilon-\alpha}\times\overline{\varepsilon-\beta}\times\overline{\varepsilon-\gamma}\times\overline{\varepsilon-\delta}\times\&c.}+\&c.=0,$$

if m be less than n, and $m+r$ not equal to $n-1$, where r is equal to the number of letters contained in each of the contents above mentioned $\beta\gamma\delta$, &c. $\beta\gamma\varepsilon$, &c. $\beta\delta\varepsilon$, &c. $\gamma\delta\varepsilon$, &c. &c. &c. respectively: but if $m+r=n-1$, then will the above mentioned sum $=\pm 1$; it will be $+1$ if r be an even number, otherwise -1.

DEMONSTRATION.

Suppose $a+b\alpha+c\alpha^2+d\alpha^3+e\alpha^4+\&c.=s^\alpha$,
$a+b\beta+c\beta^2+d\beta^3+e\beta^4+\&c.=s^\beta$,
$a+b\gamma+c\gamma^2+d\gamma^3+e\gamma^4+\&c.=s^\gamma$,
$a+b\delta+c\delta^2+d\delta^3+e\delta^4+\&c.=s^\delta$,
$a+b\varepsilon+c\varepsilon^2+d\varepsilon^3+e\varepsilon^4+\&c.=s^\varepsilon$, multiply these equations into A, B, C, D, E, &c. unknown co-efficients to be investigated, and there result

$$A\times s^\alpha=Aa+Ab\alpha+Ac\alpha^2+Ad\alpha^3+Ae\alpha^4+\&c.$$
$$B\times s^\beta=Ba+Bb\beta+Bc\beta^2+Bd\beta^3+Be\beta^4+\&c.$$
$$C\times s^\gamma=Ca+Cb\gamma+Cc\gamma^2+Cd\gamma^3+Cd\gamma^4+\&c.$$
$$D\times s^\delta=Da+Db\delta+Dc\delta^2+Dd\delta^3+Dd\delta^4+\&c.$$
$$E\times s^\varepsilon=Ea+Eb\varepsilon+Ec\varepsilon^2+Ed\varepsilon^3+Ee\varepsilon^4+\&c.\ \&c.\ \&c.$$

Now suppose $As^\alpha+Bs^\beta+Cs^\gamma+Ds^\delta+Es^\varepsilon+\&c.=a+bx+cx^2+dx^3+ex^4+\&c.$ and the correspondent parts respectively equal to each other; that is, $a(A+B+C+D+E+\&c.)=a$; $b(A\alpha+B\beta+C\gamma+D\delta+E\varepsilon+\&c.)=bx$; $A\alpha^2+B\beta^2+C\gamma^2+D\delta^2+E\varepsilon^2+\&c.=x^2$; $A\alpha^3+B\beta^3+C\gamma^3+D\delta^3+E\varepsilon^3+\&c.=x^3$; $A\alpha^4+B\beta^4+C\gamma^4+D\delta^4+E\varepsilon^4+\&c.=x^4$, &c.: But it follows from Theorem 1. that (if $As^\alpha+Bs^\beta+Cs^\gamma+Ds^\delta+Es^\varepsilon+\&c.=a+bx+cx^2+dx^3+ex^4+\&c.$) $A=\frac{\overline{x-\beta}\times\overline{x-\gamma}\times\overline{x-\delta}\times\overline{x-\varepsilon}\times\&c.}{\overline{\alpha-\beta}\times\overline{\alpha-\gamma}\times\overline{\alpha-\delta}\times\overline{\alpha-\varepsilon}\times\&c.}$,

$$B=\frac{\overline{x-\alpha}\times\overline{x-\gamma}\times\overline{x-\delta}\times\overline{x-\epsilon}\times\&c.}{\overline{\beta-\alpha}\times\overline{\beta-\gamma}\times\overline{\beta-\delta}\times\overline{\beta-\epsilon}\times\&c.},\quad C=\frac{\overline{x-\alpha}\times\overline{x-\beta}\times\overline{x-\delta}\times\overline{x-\epsilon}\times\&c.}{\overline{\gamma-\alpha}\times\overline{\gamma-\beta}\times\overline{\gamma-\delta}\times\overline{\gamma-\epsilon}\times\&c.},$$

$$D=\frac{\overline{x-\alpha}\times\overline{x-\beta}\times\overline{x-\gamma}\times\overline{x-\epsilon}\times\&c.}{\overline{\delta-\alpha}\times\overline{\delta-\beta}\times\overline{\delta-\gamma}\times\overline{\delta-\epsilon}\times\&c.},\quad E=\frac{\overline{x-\alpha}\times\overline{x-\beta}\times\overline{x-\gamma}\times\overline{x-\delta}\times\&c.}{\overline{\epsilon-\alpha}\times\overline{\epsilon-\beta}\times\overline{\epsilon-\gamma}\times\overline{\epsilon-\delta}\times\&c.},$$

&c.: ſubſtitute theſe values for A, B, C, D, E, &c. reſpectively in the preceding equations ($A+B+C+D+E+\&c.=1$, $A\alpha+B\beta+C\gamma+D\delta+E\epsilon+\&c.=x$, $A\alpha^2+B\beta^2+C\gamma^2+D\delta^2+E\epsilon^2+\&c.=x^2$, $A\alpha^3+B\beta^3+C\gamma^3+D\delta^3+E\epsilon^3+\&c.=x^3$, &c.) and there reſult the equations (1) $\frac{\overline{x-\beta}\times\overline{x-\gamma}\times\overline{x-\delta}\times\overline{x-\epsilon}\times\&c.}{\overline{\alpha-\beta}\times\overline{\alpha-\gamma}\times\overline{\alpha-\delta}\times\overline{\alpha-\epsilon}\times\&c.}$

$$+\frac{\overline{x-\alpha}\times\overline{x-\gamma}\times\overline{x-\delta}\times\overline{x-\epsilon}\times\&c.}{\overline{\beta-\alpha}\times\overline{\beta-\gamma}\times\overline{\beta-\delta}\times\overline{\beta-\epsilon}\times\&c.}+\frac{\overline{x-\alpha}\times\overline{x-\beta}\times\overline{x-\delta}\times\overline{x-\epsilon}\times\&c.}{\overline{\gamma-\alpha}\times\overline{\gamma-\beta}\times\overline{\gamma-\delta}\times\overline{\gamma-\epsilon}\times\&c.}+\&c.=1;$$

$$(2)\ \alpha\times\frac{\overline{x-\beta}\times\overline{x-\gamma}\times\overline{x-\delta}\times\overline{x-\epsilon}\times\&c.}{\overline{\alpha-\beta}\times\overline{\alpha-\gamma}\times\overline{\alpha-\delta}\times\overline{\alpha-\epsilon}\times\&c.}+\beta\times\frac{\overline{x-\alpha}\times\overline{x-\gamma}\times\overline{x-\delta}\times\overline{x-\epsilon}\times\&c.}{\overline{\beta-\alpha}\times\overline{\beta-\gamma}\times\overline{\beta-\delta}\times\overline{\beta-\epsilon}\times\&c.}$$
$$+\gamma\times\frac{\overline{x-\alpha}\times\overline{x-\beta}\times\overline{x-\delta}\times\overline{x-\epsilon}\times\&c.}{\overline{\gamma-\alpha}\times\overline{\gamma-\beta}\times\overline{\gamma-\delta}\times\overline{\gamma-\epsilon}\times\&c.}+\&c.=x;$$

$$(3)\ \alpha^2\times\frac{\overline{x-\beta}\times\overline{x-\gamma}\times\overline{x-\delta}\times\overline{x-\epsilon}\times\&c.}{\overline{\alpha-\beta}\times\overline{\alpha-\gamma}\times\overline{\alpha-\delta}\times\overline{\alpha-\epsilon}\times\&c.}+\beta^2\times\frac{\overline{x-\alpha}\times\overline{x-\gamma}\times\overline{x-\delta}\times\overline{x-\epsilon}\times\&c.}{\overline{\beta-\alpha}\times\overline{\beta-\gamma}\times\overline{\beta-\delta}\times\overline{\beta-\epsilon}\times\&c.}$$
$$+\gamma^2\times\frac{\overline{x-\alpha}\times\overline{x-\beta}\times\overline{x-\delta}\times\overline{x-\epsilon}\times\&c.}{\overline{\gamma-\alpha}\times\overline{\gamma-\beta}\times\overline{\gamma-\delta}\times\overline{\gamma-\epsilon}\times\&c.}=x^2;$$ and in general,

$$\alpha^m\times\frac{\overline{x-\beta}\times\overline{x-\gamma}\times\overline{x-\delta}\times\overline{x-\epsilon}\times\&c.}{\overline{\alpha-\beta}\times\overline{\alpha-\gamma}\times\overline{\alpha-\delta}\times\overline{\alpha-\epsilon}\times\&c.}+\beta^m\times\frac{\overline{x-\alpha}\times\overline{x-\gamma}\times\overline{x-\delta}\times\overline{x-\epsilon}\times\&c.}{\overline{\beta-\alpha}\times\overline{\beta-\gamma}\times\overline{\beta-\delta}\times\overline{\beta-\epsilon}\times\&c.}$$
$$+\gamma^m\times\frac{\overline{x-\alpha}\times\overline{x-\beta}\times\overline{x-\delta}\times\overline{x-\epsilon}\times\&c.}{\overline{\gamma-\alpha}\times\overline{\gamma-\beta}\times\overline{\gamma-\delta}\times\overline{\gamma-\epsilon}\times\&c.}+\delta^m\times\frac{\overline{x-\alpha}\times\overline{x-\beta}\times\overline{x-\gamma}\times\overline{x-\epsilon}\times\&c.}{\overline{\delta-\alpha}\times\overline{\delta-\beta}\times\overline{\delta-\gamma}\times\overline{\delta-\epsilon}\times\&c.}$$

$+\&c.=x^m$, whatever may be the values of the quantities x; $\alpha, \beta, \gamma, \delta, \epsilon$, &c.: reduce all theſe fractions into terms, proceeding according to the dimenſions of the quantity x, and it is evident, that the ſum of all the fractions multiplied

tiplied into any dimenſion of x not equal to m will be $=0$; but the ſum of all the fractions multiplied into x^m will be $=1$: from this propoſition the theorem is eaſily deduced.

I have invented and demonſtrated from different principles to the preceding the firſt part of this theorem, a particular caſe of which was publiſhed by me many years ago.

From this theorem may eaſily be deduced ſeveral others of a ſimilar nature.

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PART I.

LONDON,

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MDCCLXXIX.

IX. *On the General Refolution of Algebraical Equations.*

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IN the year 1757 I fent fome papers to the Royal Society, which papers were printed in the year 1759, and copies of them delivered to feveral perfons; thefe papers fomewhat corrected, with the addition of a fecond part on the properties of curve lines, were publifhed in the year 1762. In the years 1767, 1768 and 1769 I printed, and publifhed in the beginning of the year 1770, the fame papers with additions and emendations under the title of Meditationes Algebraicæ. In thefe papers were contained, with many other inventions, the moft general refolution of algebraical equations known, as it contains the refolution of every algebraical equation, of which the general refolution has been given, viz. the refolution of quadratic, cubic and biquadratic, the refolution of Mr. DE MOIVRE's and Mr. BEROUT's (fince publifhed) equations; it difcovers the refolution

refolution of an equation of n dimenfions, of which the n roots are given, and alfo deduces innumerable equations of n dimenfions, which contain $n-1$ independent coefficients. From whence it feems probable, that this new method of mine may contain the moft general refolution of algebraical equations that ever has, or, perhaps, ever will be invented.

The general refolution is $x = a\sqrt[n]{p} + b\sqrt[n]{p^2} + c\sqrt[n]{p^3} + d\sqrt[n]{p^4} \ldots + r\sqrt[n]{p^{n-3}} + s\sqrt[n]{p^{n-2}} + t\sqrt[n]{p^{n-1}} + \frac{A}{n}$, if the equation be $x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + Dx^{n-4} - \&c. = 0$.

I fhall add the refolution of fome particular equations from this method, and then fubjoin the equation to which $x = a\sqrt[n]{p} + b\sqrt[n]{p^2} + c\sqrt[n]{p^3} + \&c.$ is the general refolution.

1. Let the refolution be $x = a\sqrt[3]{p} + b\sqrt[3]{p^2}$, and the correfpondent equation free from radicals will be found $x^3 - 3abpx - a^3p - b^3p^2 = 0$. Let $x^3 - Px - Q = 0$ be a cubic equation whofe refolution is required, which fuppofe the fame as the equation found above, and confequently their correfpondent terms equal, i. e. $P = 3abp$ and $Q = a^3p + b^3p^2$, whence $p = \frac{P}{3ab}$, which value being fubftituted for p in the fecond equation, there refults $Q = \frac{Pa^2}{3b} + \frac{bP^2}{9a^2}$. In this equation for a or b may be affumed unity, or any other

other quantity whatever, and there will refult an equation of the formula of a quadratic from which the other b or a may be found, whence from the equation ($p = \frac{P}{3ab}$) p may be deduced, and confequently the refolution of the cubic required.

In the fame manner for p may be affumed any quantity whatever, and in the equation $Q = a^3p + b^3p^2$ for b fubftitute its value $\frac{P}{3ap}$, or for a its value $\frac{P}{3bp}$, and there refult the equations $Q = a^3p + \frac{P^3}{27a^3p^2}$, and $Q = \frac{P^3}{27b^3p^3} + b^3p^2$, which have the formula of a quadratic, from which may be deduced the refolution of the cubic required.

2. Let the refolution affumed be $x = a\sqrt[4]{p} + b\sqrt[4]{p^2} + c\sqrt[4]{p^3}$; exterminate the irrational quantities, and there refults the equation $x^4 - (2b^2 + 4ac)px^2 - 4(a^2bp + bc^2p^2)x - a^4p + b^4p^2 - c^4p^3 + 2a^2c^2p^2 - 4ab^2cp^2 = 0$; fuppofe $p = 1$, and the given equation $x^4 + qx^2 - rx + s = 0$, let the correfpondent terms of the given and refulting equations be refpectively made equal to each other, and there refult the three equations $2b^2 + 4ac = -q$, $4b(a^2 + c^2) = r$, and $a^4 - b^4 + c^4 - 2a^2c^2 + 4ab^2c = -s$; reduce thefe equations into one, fo that the unknown quantities a and c may be exterminated, and there refults the equation $4b^6 + 2qb^4 + (\frac{q^2}{4} - s)b^2 - \frac{r^2}{16} = 0$ of the formula of a cubic, from which the

the unknown quantity b may be found, which being fubftituted for its value (b) in the preceding equations, from the equations thence enfuing may be found the unknown quantities a and c, and confequently the refolution of the given biquadratic $x^4 + qx^2 - rx + s = 0$.

From the fame principles can be deduced different refolutions of the above-mentioned biquadratic $x^4 + qx^2 - rx + s = 0$.

3. I. Let $x = a\sqrt[2n]{p} + b\sqrt[2n]{p^2}$, then will the equation free from radicals be $x^{2n} - 2b^np x^n - \frac{n}{1.2} \cdot 2nb^{n-1}a^2px^{n-1} - \frac{n \times \overline{n^2-4}}{1.2.3.4} \times 2nb^{n-2}a^4px^{n-2} - \frac{n \times \overline{n^2-1} \times \overline{n^2-4}}{1.2.3.4.5.6} \times 2nb^{n-3}a^6px^{n-3} - \frac{n.\overline{n^2-1}.\overline{n^2-4}.\overline{n^2-9}}{1.2.3.4.5.6.7.8} \times 2nb^{n-4}a^8px^{n-4} \ldots\ldots - \frac{n.\overline{n^2-1} \times \overline{n^2-4} \times \overline{n^2-9}.\overline{n^2-16}\ldots\overline{n^2-\overline{n-2}^2}}{1.2.3.4.5.6.7\ldots 2n-2} \times 2na^{2n-1}bpx = a^np - b^np^2$.

This equation may be deduced from the following principles. Let α, β, γ, δ, ε, &c. be the $2n$ roots of the equation $z^{2n} - 1 = 0$, then (by Prop. XXIII. of my Meditat. Algebraicæ) the equation free from radicals will be the product of the following quantities $(x - a\alpha\sqrt[2n]{p} - b\alpha^2\sqrt[2n]{p^2})(x - a\beta\sqrt[2n]{p} - b\beta^2\sqrt[2n]{p^2})(x - a\gamma\sqrt[2n]{p} - b\gamma^2\sqrt[2n]{p^2})(x - a\delta\sqrt[2n]{p} - b\delta^2\sqrt[2n]{p^2})(x - a\varepsilon\sqrt[2n]{p} - b\varepsilon^2\sqrt[2n]{p^2})$ &c. $= 0$: multiply thefe quantities into each other, and from the refulting product, by Prob. III. of the Meditat. Algebr. eafily can be be deduced the equation free from radicals which was to be found.

3. II. Let $x = a\sqrt[2n+1]{p} + b\sqrt[2n+1]{p^2}$, then will the correſpondent equation free from radicals be $x^{2n+1} - \overline{2n+1}\, b^n a p x^n - \frac{n.\overline{n+1}}{1.2.3} \times \overline{2n+1}\, b^{n-1} a^3 p x^{n-1} - \frac{n \times \overline{n^2-1} \times \overline{n+2}}{1.2.3.4.5} \times \overline{2n+1}\, b^{n-2} a^5 p x^{n-2} - \frac{n \times \overline{n^2-1} \times \overline{n^2-4} \times \overline{n+3}}{1.2.3.4.5.6.7} \times \overline{2n+1}\, b^{n-3} a^7 p x^{n-3} - \frac{n.\overline{n^2-1}.\overline{n^2-4}.\overline{n^2-9}.\overline{n+4}}{1.2.3.4.5.6.7.8.9} \times \overline{2n+1}\, b^{n-4} a^9 p x^{n-4} - \dots - \frac{n.\overline{n^2-1}.\overline{n^2-4}.\overline{n^2-9}.\overline{n^2-16} \dots \overline{n^2-\overline{n-1}^2} \times \overline{2n-1}}{1.2.3.4.5.6.7.8.9 \dots 2n-1} \times \overline{2n+1}\, b a^{2n-1} p x = a^{2n+1} p + b^{2n+1} p^2$.

This may be derived from the ſame principles as the preceding.

3. III. In general let the equation be $x = a\sqrt[m]{p} + b\sqrt[m]{p^2}$, then will the equation free from radicals become $x^m - m a^{m-1} b p x - m.\frac{m-3}{2} a^{m-4} b^2 p x^2 - m.\frac{m-4}{2}.\frac{m-5}{3} a^{m-6} b^3 p x^3 - m.\frac{m-5}{2}.\frac{m-6}{3}.\frac{m-7}{4} a^{m-8} b^4 p x^4 - m.\frac{m-6}{2}.\frac{m-7}{3}.\frac{m-8}{4}.\frac{m-9}{5} a^{m-10} b^5 p x^5 - \&c. = a^m p \pm b^m p^2$; if m denotes an even number, it will be $-b^m p^2$, but if an odd number, it will be $+b^m p^2$.

4. I. Let n denote an odd number, and $x = a\sqrt[n]{p} + b\sqrt[n]{p^3}$, then will $x^n - p\,(n a^{n-3} b x^2 + n.\frac{n-5}{2} a^{n-6} b^2 x^4 + n.\frac{n-7}{2}.\frac{n-8}{3} a^{n-9} b^3 x^6 + n.\frac{n-9}{2}.\frac{n-10}{3}.\frac{n-11}{4} a^{n-12} b^4 x^8 + n.\frac{n-11}{2}.\frac{n-12}{3}.\frac{n-13}{4}.\frac{n-14}{5} a^{n-15} b^5 x^{10} + n.\frac{n-13}{2}.\frac{n-14}{3}.\frac{n-15}{4}.\frac{n-16}{5}.\frac{n-17}{6} a^{n-18} b^6 x^{12} + n.$

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$\frac{n-15}{2}.\frac{n-16}{3}.\frac{n-17}{4}.\frac{n-18}{5}.\frac{n-19}{6}.\frac{n-20}{7} a^{n-21} b^7 x^{14} + \&c.) \pm p^2\,(m a^{\frac{n-3}{2}} b^{\frac{n+1}{2}} x - \frac{1}{2^2} \times n.\frac{n-5}{2}.\frac{n-7}{3} a^{\frac{n-9}{2}} b^{\frac{n+3}{2}} x^3 + \frac{1}{2^4} \times n.\frac{n-7}{2}.\frac{n-9}{3}.\frac{n-11}{4}.\frac{n-13}{5} a^{\frac{n-15}{2}} b^{\frac{n+5}{2}} x^5 - \frac{1}{2^6} \times n.\frac{n-9}{2}.\frac{n-11}{3}.\frac{n-13}{4}.\frac{n-15}{5}.\frac{n-17}{6}.\frac{n-19}{7} a^{\frac{n-21}{2}} b^{\frac{n+7}{2}} x^7 + \frac{1}{2^8} \times n.\frac{n-11}{2}.\frac{n-13}{3}.\frac{n-15}{4}.\frac{n-17}{5}.\frac{n-19}{6}.\frac{n-21}{7}.\frac{n-23}{8}.\frac{n-25}{9} a^{\frac{n-27}{2}} b^{\frac{n+9}{2}} x^9 - \&c.) = a^n p + b^n p^3$.

The quantity $\pm p^2$ denotes $+p^2$ if $\frac{n-3}{4}$ is a whole number, otherwiſe $-p^2$.

4. II. Let n denote an even number, and x as before $= a\sqrt[n]{p} + b\sqrt[n]{p^3}$, then will $x^n - p\,(n a^{n-3} b x^2 + n.\frac{n-5}{2} a^{n-6} b^2 x^4 + n.\frac{n-7}{2}.\frac{n-8}{3} a^{n-9} b^3 x^6 + n.\frac{n-9}{2}.\frac{n-10}{3}.\frac{n-11}{4} a^{n-12} b^4 x^8 + n.\frac{n-11}{2}.\frac{n-12}{3}.\frac{n-13}{4}.\frac{n-14}{5} a^{n-15} b^5 x^{10} + \&c.) \pm p^2\,(\frac{1}{2} n.\frac{n-4}{2} a^{\frac{n-6}{2}} b^{\frac{n+2}{2}} x^2 - \frac{1}{2^3} \times n.\frac{n-6}{2}.\frac{n-8}{3}.\frac{n-10}{4} a^{\frac{n-12}{2}} b^{\frac{n+4}{2}} x^4 + \frac{1}{2^5} \times n.\frac{n-8}{2}.\frac{n-10}{3}.\frac{n-12}{4}.\frac{n-14}{5}.\frac{n-16}{6} a^{\frac{n-18}{2}} b^{\frac{n+6}{2}} x^6 - \frac{1}{2^7} \times n.\frac{n-10}{2}.\frac{n-12}{3}.\frac{n-14}{4}.\frac{n-16}{5}.\frac{n-18}{6}.\frac{n-20}{7}.\frac{n-22}{8} a^{\frac{n-24}{2}} b^{\frac{n+8}{2}} x^8 + \&c.) = a^n p + b^n p^3 \pm 2 a^{\frac{n}{2}} b^{\frac{n}{2}} p^2$.

The quantities $\pm p^2$ and $\pm 2 a^{\frac{n}{2}} b^{\frac{n}{2}} p^2$ denote $+p^2$, and $+2 a^{\frac{n}{2}} b^{\frac{n}{2}} p^2$, if $\frac{n-2}{4}$ is a whole number, otherwiſe they denote $-p^2$ and $-2 a^{\frac{n}{2}} b^{\frac{n}{2}} p^2$ reſpectively.

5. I. Let $x = a\sqrt[n]{p} + b\sqrt[n]{p^{n-1}}$, and n an odd number,

then

then will $x^n - nabpx^{n-2} + n.\frac{n-3}{2}a^2b^2p^2x^{n-4} - n.\frac{n-4}{2}.\frac{n-5}{3}.a^3b^3p^3x^{n-6}+n.\frac{n-5}{2}.\frac{n-6}{3}.\frac{n-7}{4}a^4b^4p^4x^{n-8}$ – &c. $=a^np+b^np^{n-1}$.

5. II. Let $x=a\sqrt[n]{p}+b\sqrt[n]{p^{n-1}}$, and n an even number, then will $x^n - nabpx^{n-2} + n.\frac{n-3}{2}a^2b^2p^2x^{n-4} - n.\frac{n-4}{2}.\frac{n-5}{3}a^3b^3p^3x^{n-6} + n.\frac{n-5}{2}.\frac{n-6}{3}.\frac{n-7}{4}a^4b^4p^4x^{n-8}$ – &c. $= a^np \mp 2a^{\frac{n}{2}}b^{\frac{n}{2}}p^{\frac{n}{2}}+b^np^{n-1}$; it will be $+ 2a^{\frac{n}{2}}b^{\frac{n}{2}}p^{\frac{n}{2}}$ if $n = 4r + 2$; but $- 2a^{\frac{n}{2}}b^{\frac{n}{2}}p^{\frac{n}{2}}$ if $n = 4r$.

6. I. Let $x=a\sqrt[n]{p}+b\sqrt[n]{p^{n-2}}$, and n an odd number, which has not the number 3 for a divisor, then will $x^n-na^2bpx^{n-3}+n.\frac{n-5}{2}a^4b^2p^2x^{n-6} - n.\frac{n-7}{2}.\frac{n-8}{3}a^6b^3p^3x^{n-9} + n.\frac{n-9}{2}.\frac{n-10}{3}.\frac{n-11}{4}a^8b^4p^4x^{n-12} - n.\frac{n-11}{2}.\frac{n-12}{3}.\frac{n-13}{4}.\frac{n-14}{5}a^{10}b^5p^5x^{n-15} + n.\frac{n-13}{2}.\frac{n\times 14}{3}.\frac{n-15}{4}.\frac{n-16}{5}.\frac{n-17}{6}a^{12}b^6p^6x^{n-6}$ – &c. (to m terms, where m is the number either equal to, or the least greater than $\frac{n}{3}$) $- ab^{\frac{n+1}{2}}p^{\frac{n-1}{2}}(nx^{\frac{n-3}{2}} + \frac{1}{4}n.\frac{n-5}{2}.\frac{n-7}{3}a^2bpx^{\frac{n-9}{2}} + \frac{1}{2^4}n.\frac{n-7}{2}.\frac{n-9}{3}.\frac{n-11}{4}.\frac{n-13}{5}a^4b^2p^2x^{\frac{n-15}{2}} + \frac{1}{2^6}\times n.\frac{n-9}{2}.\frac{n-11}{3}.\frac{n-13}{4}.\frac{n-15}{5}.\frac{n-17}{6}.\frac{n-19}{7}a^6b^3p^3x^{\frac{n-21}{2}} + \frac{1}{2^8}\times n.\frac{n-11}{2}.\frac{n-13}{3}.\frac{n-15}{4}.\frac{n-17}{5}.\frac{n-19}{6}.\frac{n-21}{7}.\frac{n-23}{8}.\frac{n-25}{9}a^8b^4p^4x^{\frac{n-27}{2}}$ + &c.) $= \text{A} = a^np+b^np^{n-2}$.

Let n be an odd number divisible by 3, then will the above-

mentioned quantity $= \text{A}=a^np+b^np^{n-2}+3a^{\frac{n}{3}}b^{\frac{2n}{3}}p^{\frac{2n}{3}-1}+3a^{\frac{2n}{3}}b^{\frac{n}{3}}p^{\frac{n}{3}}$.

6. II. Let n be an even number, not divisible by 3, then will $x^n-na^2bpx^{n-3}+n.\frac{n-5}{2}a^4b^2p^2x^{n-6}-n.\frac{n-7}{2}.\frac{n-8}{3}a^6b^3p^3x^{n-9}+n.\frac{n-9}{2}.\frac{n-10}{3}.\frac{n-11}{4}a^8b^4p^4x^{n-12}-n.\frac{n-11}{2}.\frac{n-12}{3}.\frac{n-13}{4}.\frac{n-14}{5}a^{10}b^5p^5x^{n-15}+n.\frac{n-13}{2}.\frac{n-14}{3}.\frac{n-15}{4}.\frac{n-16}{5}.\frac{n-17}{6}a^{12}b^6p^6x^{n-18}$ – &c. to m terms as before $- b^{\frac{n}{2}}p^{\frac{n}{2}-1}(2x^{\frac{n}{2}} + \frac{1}{2}n.\frac{n-4}{2}a^2bpx^{\frac{n}{2}-3}+\frac{1}{2^3}\times n.\frac{n-6}{2}.\frac{n-8}{3}.\frac{n-10}{4}a^4b^2p^2x^{\frac{n}{2}-6}+\frac{1}{2^5}\times n.\frac{n-8}{2}.\frac{n-10}{3}.\frac{n-12}{4}.\frac{n-14}{5}.\frac{n-16}{6}a^6b^3p^3x^{\frac{n}{2}-9}+\frac{1}{2^7}\times n.\frac{n-10}{2}.\frac{n-12}{3}.\frac{n-14}{4}.\frac{n-16}{5}.\frac{n-18}{6}.\frac{n-20}{7}.\frac{n-22}{8}a^8b^4p^4x^{\frac{n}{2}-12}$ + &c.) $=\text{A}=a^np-b^np^{n-2}$.

Let n be an even number divisible by 3, then will the above-mentioned quantity $\text{A}=a^np-b^np^{n-2}-3a^{\frac{2n}{3}}b^{\frac{n}{3}}p^{\frac{n}{3}}+3a^{\frac{n}{3}}b^{\frac{2n}{3}}p^{\frac{2n}{3}-2}$.

In all the preceding cases n, m and r denote whole affirmative numbers.

These equations may be deduced in the same manner as is before given in Case 3. I; or can be demonstrated by writing in the equation free from radicals for the different powers of x their values deduced from the given equation $x=a\sqrt[n]{p}+b\sqrt[n]{p^m}$.

To

To render the ſolution general, it may not be improper to ſubjoin the ſubſequent.

LEMMA.

1. Let $\alpha, \beta, \gamma, \delta, \epsilon, \zeta$, &c. be the reſpective roots of the equation $z^n - 1 = 0$; then will $\alpha^m + \beta^m + \gamma^m + \delta^m + \epsilon^m + \&c. = 0$, unleſs $n = m$, or n is a diviſor of m, in which caſe $\alpha^m + \beta^m + \gamma^m + \delta^m + \epsilon^m + \&c. = n$.

2. The ſum of all quantities of the following kind $\alpha^m\beta^r + \alpha^r\beta^m + \alpha^m\gamma^r + \alpha^r\gamma^m + \beta^m\gamma^r + \beta^r\gamma^m + \alpha^m\delta^r + \&c.$ will be $= 0$; unleſs n be either equal to, or a diviſor of $m + r$, in which caſe the ſum above-mentioned will be $= -n$; except n be either equal to m or r, or a diviſor of them, in which caſe the ſum will be $n^2 - n$; but if $m = r$, then in the former caſe will the above-mentioned ſum $= -\frac{n}{2}$, and in the latter $= \frac{n^2 - n}{2}$.

3. The ſum of all quantities of this kind $\alpha^m\beta^r\gamma^s\delta^t$ &c. + $\alpha^r\beta^m\gamma^s\delta^t$ &c. + $\alpha^m\beta^r\gamma^t\delta^s$ &c. + $\alpha^r\beta^s\gamma^m\delta^t$ &c. + &c. will be $= 0$, unleſs n be either equal to $r + m + s + t$ &c. or a diviſor of it.

Let π be the number of indices m, r, s, t, &c. and n be either equal to $m + r + s + t$ + &c. or a diviſor of it, but n be neither equal to, nor a diviſor of the ſum of any two, three, four, ... $\pi - 3$, $\pi - 2$ or $\pi - 1$ of the above-mentioned quantities; then will the ſum above-mentioned $= \mp 1.2.3.4 \ldots \overline{\pi - 2}.\overline{\pi - 1} \times n$; where it will be +, if π be an odd number; otherwiſe −.

In this caſe, if a indices be m, b indices be r, c indices be s, d indices be t, &c. then will the above-mentioned ſum $= \mp \frac{1.2.3.4 \ldots \overline{\pi - 2} \times \overline{\pi - 1}}{1.2.3 \ldots a \times 1.2.3 \ldots b \times 1.2.3 \ldots c \times 1.2.3 \ldots d \times \&c.} \times n$.

Let n be either equal to, or a diviſor of the ſum of any number ϱ (leſs than π) of the above-mentioned quantities m, r, s, t, &c. and conſequently either equal to, or a diviſor of the ſum of the $(\pi - \varrho)$ remaining quantities: find the ſum of all poſſible quantities of this kind $1.2.3 \ldots \overline{\varrho - 2} \times \overline{\varrho - 1} \times 1.2.3 \ldots \overline{\pi - \varrho - 2} \times \overline{\pi - \varrho - 1} \times n^2$, which ſum call A.

Let n be either equal to, or a diviſor of the ſum of any number (σ) of the above-mentioned quantities m, r, s, t, &c; and alſo equal to, or a diviſor of the ſum of any number (ϱ') of the remaining quantities, and conſequently it will be either equal to, or a diviſor of, the ſum of the $(\pi - \varrho' - \sigma)$ remaining quantities; then find the ſum of all poſſible quantities of this ſort $1.2.3 \ldots \overline{\sigma - 2} \times \overline{\sigma - 1} \times 1.2.3 \ldots \overline{\varrho' - 2} \times \overline{\varrho' - 1} \times 1.2.3 \ldots \overline{\pi - \varrho' - \sigma - 2} \times \overline{\pi - \varrho' - \sigma - 1} \times n^3$, which ſum call B.

In the ſame manner let n be either equal to, or a diviſor of the ſum of any number (τ) of the above-mentioned

tioned quantities m, r, s, t, &c; and ſimilarly let n be either equal to, or a diviſor of the ſum of any number (σ') of the remaining quantities; and alſo let n be either equal to, or a diviſor of the ſum of any number (ϱ'') of the remaining quantities; then will n be either equal to, or a diviſor of the ſum of the ($\pi-\tau-\sigma'-\varrho''$) remaining quantities: find the ſum of all quantities of this ſort $1.2.3..\overline{\tau-2}\times\overline{\tau-1}\times 1.2.3..\overline{\sigma'-2}\times\overline{\sigma'-1}\times 1.2.3..\overline{\varrho''-2}\times\overline{\varrho''-1}\times 1.2.3..\overline{\pi-\tau-\sigma'-\varrho''-2}\times\overline{\pi-\tau-\sigma'-\varrho''-1}\times n^4$, which ſum call C; and ſo on; then will the above-mentioned ſum $\alpha^m\beta^r\gamma^s\delta^t$ &c. + $\alpha^r\beta^m\gamma^s\delta^t$ &c. + $\alpha^m\beta^r\gamma^s\delta^t$ &c. + $\alpha^r\beta^s\gamma^m\delta^t$ &c. + &c. $= \mp(1.2.3..\overline{\pi-2}\times\overline{\pi-1}\times n - A + B - C + D - \&c.)$ where it will be + if π be an odd number, otherwiſe −.

In this caſe, if a indices be m, b indices be r, c indices be s, d indices be t, &c. then will the above-mentioned ſum $= \mp\frac{1.2.3..\overline{\pi-2}\times\overline{\pi-1}.n - A + B - C + D - \&c.}{1.2.3..a\times 1.2.3..b\times 1.2.3..c\times 1.2.3..d\times \&c.}$.

7. Let α, β, γ, δ, ϵ, &c. be the roots of the equation $z^n-1=0$, and the reſolution be $x=a\sqrt[n]{p}+b\sqrt[n]{p^2}+c\sqrt[n]{p^3}+d\sqrt[n]{p^4}.....+h\sqrt[n]{p^\lambda}...+k\sqrt[n]{p^\mu}....+l\sqrt[n]{p^\nu}...+q\sqrt[n]{p^\xi}....+r\sqrt[n]{p^\sigma}....+s\sqrt[n]{p^{n-4}}+t\sqrt[n]{p^{n-3}}+v\sqrt[n]{p^{n-2}}+u\sqrt[n]{p^{n-1}}$; then will the different values of x be reſpectively $a\sqrt[n]{p}\times\alpha+b\sqrt[n]{p^2}\times\alpha^2+c\sqrt[n]{p^3}\times\alpha^3+d\sqrt[n]{p^4}\times\alpha^4....+h\sqrt[n]{p^\lambda}\times\alpha^\lambda...+k\sqrt[n]{p^\mu}\times\alpha^\mu......+s\sqrt[n]{p^{n-4}}\times\alpha^{n-4}+t\sqrt[n]{p^{n-3}}\times\alpha^{n-3}+v\sqrt[n]{p^{n-2}}\times\alpha^{n-2}+u\sqrt[n]{p^{n-1}}\times\alpha^{n-1}$;

a√

$a\sqrt[n]{p}\times\beta+b\sqrt[n]{p^2}\times\beta^2+c\sqrt[n]{p^3}\times\beta^3+d\sqrt[n]{p^4}\times\beta^4...+h\sqrt[n]{p^\lambda}\times\beta^\lambda...+k\sqrt[n]{p^\mu}\times\beta^\mu......+s\sqrt[n]{p^{n-4}}\times\beta^{n-4}+t\sqrt[n]{p^{n-3}}\times\beta^{n-3}+v\sqrt[n]{p^{n-2}}\times\beta^{n-2}+u\sqrt[n]{p^{n-1}}\times\beta^{n-1}$;

$a\sqrt[n]{p}\times\gamma+b\sqrt[n]{p^2}\times\gamma^2+c\sqrt[n]{p^3}\times\gamma^3+d\sqrt[n]{p^4}\times\gamma^4...+h\sqrt[n]{p^\lambda}\times\gamma^\lambda...+k\sqrt[n]{p^\mu}\times\gamma^\mu......+s\sqrt[n]{p^{n-4}}\times\gamma^{n-4}+t\sqrt[n]{p^{n-3}}\times\gamma^{n-3}+v\sqrt[n]{p^{n-2}}\times\gamma^{n-2}+u\sqrt[n]{p^{n-1}}\times\gamma^{n-1}$;

$a\sqrt[n]{p}\times\delta+b\sqrt[n]{p^2}\times\delta^2+c\sqrt[n]{p^3}\times\delta^3+d\sqrt[n]{p^4}\times\delta^4...+h\sqrt[n]{p^\lambda}\times\delta^\lambda...+k\sqrt[n]{p^\mu}\times\delta^\mu......+s\sqrt[n]{p^{n-4}}\times\delta^{n-4}+t\sqrt[n]{p^{n-3}}\times\delta^{n-3}+v\sqrt[n]{p^{n-2}}\times\delta^{n-2}+u\sqrt[n]{p^{n-1}}\times\delta^{n-1}$;

&c. &c. &c. &c.

and conſequently the ſum of the values or roots, which is the coefficient of the ſecond term of the equation ſought, will be $a\sqrt[n]{p}\times\overline{\alpha+\beta+\gamma+\delta+\&c.}\,(o)+b\sqrt[n]{p^2}\times\overline{\alpha^2+\beta^2+\gamma^2+\delta^2+\&c.}\,(o)+c\sqrt[n]{p^3}\times\overline{\alpha^3+\beta^3+\gamma^3+\delta^3+\&c.}\,(o)+....+v\sqrt[n]{p^{n-2}}\times\overline{\alpha^{n-2}+\beta^{n-2}+\gamma^{n-2}+\delta^{n-2}+\&c.}\,(o)+u\sqrt[n]{p^{n-1}}\times\overline{\alpha^{n-1}+\beta^{n-1}+\gamma^{n-1}+\delta^{n-1}+\&c.}\,(o)=0$.

The ſum of the products of every two of the values or roots, which is the coefficient of the third term of the equation ſought, will be $a^2\sqrt[n]{p^2}\times\overline{\alpha\beta+\alpha\gamma+\beta\gamma+\alpha\delta+\beta\delta+\gamma\delta+\&c.}\,(o)+ab\sqrt[n]{p^3}\times\overline{\alpha\beta^2+\beta\alpha^2+\alpha\gamma^2+\gamma\alpha^2+\beta\gamma^2+\gamma\beta^2+\alpha\delta^2+\delta\alpha^2+\&c.}\,(o)$, and in general all the terms will be 0, unleſs $a\times u\times p\times\overline{\alpha\beta^{n-1}+\beta\alpha^{n-1}+\alpha\gamma^{n-1}+\gamma\alpha^{n-1}+\beta\gamma^{n-1}+}$

γβ

$\overline{\gamma\beta^{n-1}+\alpha\delta^{n-1}+\delta\alpha^{n-1}+\beta\delta^{n-1}+\delta\beta^{n-1}+\gamma\delta^{n-1}+\delta\gamma^{n-1}+\&c.}(-n)$ $+b\times v\times p\times\overline{\alpha^2\beta^{n-2}+\beta^2\alpha^{n-2}+\alpha^2\gamma^{n-2}+\gamma^2\alpha^{n-2}+\beta^2\gamma^{n-2}+\gamma^2\beta^{n-2}+\alpha^2\delta^{n-2}+\delta^2\alpha^{n-2}+\beta^2\delta^{n-2}+\delta^2\beta^{n-2}+\&c.}\ (-n)+ctp\times$ $\overline{\alpha^3\beta^{n-3}+\beta^3\alpha^{n-3}+\alpha^3\gamma^{n-3}+\gamma^3\alpha^{n-3}+\beta^3\gamma^{n-3}+\gamma^3\beta^{n-3}+\alpha^3\delta^{n-3}+\delta^3\alpha^{n-3}+\beta^3\delta^{n-3}+\delta^3\beta^{n-3}+\&c.}\ (-n)+dsp\times\overline{\alpha^4\beta^{n-4}+\beta^4\alpha^{n-4}+\alpha^4\gamma^{n-4}+\gamma^4\alpha^{n-4}+\beta^4\gamma^{n-4}+\gamma^4\beta^{n-4}+\&c.}\ (-n)+\&c.=-np(au+bv+ct+ds+\&c.)$

If $n=2\lambda$, then will the coefficient of b^2p be $\frac{n}{2}$, i.e. the above-mentioned coefficient will be $-np\,(au+bv+ct+ds+\ldots+\frac{1}{2}b^2)$.

The sum of the contents of every three of the above-mentioned values or roots, which is the coefficient of the fourth term of the equation required, will be $a^3\sqrt[n]{p^3}\times\overline{\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta+\&c.}\ (0)+a^2b\sqrt[n]{p^4}\times\overline{\alpha\beta\gamma^2+\alpha\gamma\beta^2+\beta\gamma\alpha^2+\alpha\beta\delta^2+\&c.}\ (0)+\&c.+a^2v\sqrt[n]{p^n}\times\overline{\alpha\beta\gamma^{n-2}+\alpha\gamma\beta^{n-2}+\beta\gamma\alpha^{n-2}+\alpha\beta\delta^{n-2}+\alpha\delta\beta^{n-2}+\&c.}\ (\frac{1.2.n}{1.2})+abt\sqrt[n]{p^n}\times\overline{\alpha\beta^2\gamma^{n-3}+\alpha\gamma^2\beta^{n-3}+\beta\alpha^2\gamma^{n-3}+\beta\gamma^2\alpha^{n-3}+\gamma\alpha^2\beta^{n-3}+\gamma\beta^2\alpha^{n-3}+\alpha\beta^2\delta^{n-3}\ \&c.}\ (1.2.n)+\&c$; and in general all the terms (unless the quantity $\sqrt[n]{p^{\theta}}$ contained in the term have this formula $\sqrt[n]{p^n}=p$, or $\sqrt[n]{p^{2n}}=p^2$) will be $=0$; let the general term be denoted by $blk\sqrt[n]{p^{\lambda+\mu+\nu}}\times\overline{\alpha^{\lambda}\beta^{\mu}\gamma^{\nu}+\alpha^{\lambda}\beta^{\nu}\gamma^{\mu}+\alpha^{\mu}\beta^{\lambda}\gamma^{\nu}+\alpha^{\mu}\beta^{\nu}\gamma^{\lambda}+\alpha^{\nu}\beta^{\mu}\gamma^{\lambda}+\alpha^{\nu}\beta^{\lambda}\gamma^{\mu}+\alpha^{\lambda}\beta^{\mu}\delta^{\nu}+\&c.}$ first let $\lambda+\mu+\nu$ neither be equal to n or $2n$, then will the term

term above-mentioned $=0$; if it be equal to n or $2n$, then will the term be $1.2\times n\times bklp$ or $1.2nbklp^2$.

If two of the three indexes λ, μ, ν be equal to each other, then divide the above-mentioned term by 1.2; if the three indexes be equal, i.e. $\lambda=\mu=\nu$, divide it by $1.2.3$: find all quantities of this kind where $\lambda+\mu+\nu$ either is equal to n or $2n$, and add all the term from thence derived, and call the sum of them A.

The sum of the contents of every four of the values or roots above-mentioned, which is the coefficient of the fourth term of the equation required, will be $a^4\sqrt[n]{p^4}\times\overline{\alpha\beta\gamma\delta+\alpha\beta\gamma\delta+\&c.}\ (0)+a^3b\sqrt[n]{p^5}\times\overline{\alpha\beta\gamma\delta^2+\alpha\beta\gamma^2\delta+\alpha\beta^2\gamma\delta+\alpha^2\beta\gamma\delta+\&c.}\ (0)+\&c$: let $bklq\sqrt[n]{p^{\lambda+\mu+\nu+\xi}}\times\overline{\alpha^{\lambda}\beta^{\mu}\gamma^{\nu}\delta^{\xi}+\alpha^{\lambda}\beta^{\mu}\gamma^{\xi}\delta^{\nu}+\alpha^{\lambda}\beta^{\nu}\gamma^{\mu}\delta^{\xi}+\alpha^{\lambda}\beta^{\nu}\gamma^{\xi}\delta^{\mu}+\alpha^{\lambda}\beta^{\xi}\gamma^{\nu}\delta^{\mu}+\alpha^{\lambda}\beta^{\xi}\gamma^{\mu}\delta^{\nu}+\alpha^{\mu}\beta^{\lambda}\gamma^{\nu}\delta^{\xi}+\&c.}$ denote a general term; this term will be $=0$, unless $\lambda+\mu+\nu+\xi$ either $=n$ or $2n$ or $3n$; in which case the term will be either $-1.2.3nbklqp$ or $-1.2.3nbklqp^2$ or $-1.2.3nbklqp^3$; unless $\lambda+\mu=\nu+\xi=n$, when the above-mentioned term will be $-(1.2.3n-n^2)\,bklqp^2$; in this case if $\lambda=\nu$, and consequently $\mu=\xi$, then it will be $-(1.2.3n-1.2n^2)\,bklqp^2$; but if $\lambda=\mu=\nu=\xi=\frac{n}{2}$, then will the term be $-(1.2.3n-3n^2)\,bklqp^2$.

In all these cases, if two of the indexes λ, μ, ν, ξ be equal, then must the term given above be divided by

1.2;

1.2; if three, by 1.2.3; if four, by 1.2.3.4; and laftly if two are equal to each other, and the two remaining indexes equal to each other, but not to the former two, then muft the term aforefaid be divided by 1.2.1.2.

Find the fum of all the poffible terms of this kind, which call B.

In the fame manner from the preceding Lemma may be found the aggregates of the contents of every five, fix, feven, &c. roots or values multiplied into each other, which call refpectively C, D, E, &c; then will the equation required be $x^n * - np(au + bv + ct + ds + \&c.)x^{n-2} - Ax^{n-3} + Bx^{n-4} - Cx^{n-5} + Dx^{n-6} - \&c. = 0$.

From the fame principles may be deduced the moft general reduction yet known of equations to others of inferior dimenfions, *e. g.*

Let (X) $x^n + (A + a\sqrt[m]{p} + b\sqrt[m]{p^2} + c\sqrt[m]{p^3} + \ldots + s\sqrt[m]{p^{m-2}} + t\sqrt[m]{p^{m-1}})x^{n-1} + (B + a'\sqrt[m]{p} + b'\sqrt[m]{p^2} + \ldots + s'\sqrt[m]{p^{m-2}} + t'\sqrt[m]{p^{m-1}})x^{n-2} + (C + a''\sqrt[m]{p} + b''\sqrt[m]{p^2} + \&c.)x^{n-3} + \&c. = 0$; let α, β, γ, δ, &c. be the refpective roots of the equation $x^m - 1 = 0$, then, from the principles before given, may be formed the different values of the equation X, which being multiplied into each other from the propofitions before-mentioned of the Meditationes Algebraicæ, may be deduced an equation of *nm* dimenfions free from radicals,

whofe root is x, and which contains *mn* unknown quantities A, a, b, c, &c. B, a', b', c', &c. C, a'', b'', c'', & p: for one, two or more of thefe unknown quantities may be affumed any quantities whatever, and thence may be deduced equations of *mn* dimenfions, which may be reduced to equations $x^n + (A + a\sqrt[m]{p} + b\sqrt[m]{p^2} + c\sqrt[m]{p^3} + \&c.)x^{n-1} + \&c. = 0$ of n dimenfions.

In the fame manner may be affumed equations, which involve $\sqrt[m]{p}$, $\sqrt[m]{p^2}$, ..., $\sqrt[m]{p^{m-1}}$; $\sqrt[r]{Q}$, $\sqrt[r]{Q^2}$, $\sqrt[r]{Q^3}$, ...$\sqrt[r]{Q^{r-1}}$; $\sqrt[s]{T}$, $\sqrt[s]{T^2}$, $\sqrt[s]{T^3}$, ..., $\sqrt[s]{T^{s-1}}$, &c; and from fo reducing them as to exterminate the irrational quantities, may often be derived equations whofe refolutions or reductions are known.

The method of transforming algebraical equations into others, whofe roots bear any affignable algebraical (but not exponential) relation to the roots of a given algebraical equation firft publifhed by me in the papers fent to the Royal Society, and afterwards in the year 1760; and thirdly in my Mifcellanea Analytica; and laftly in the Meditationes Algebraicæ, and fince publifhed by Mr. LE GRANGE in the Berlin Acts, is perhaps (as Mr. LE GRANGE obferves) more general than Mr. HUDDE's, or any transformation yet invented; it is very ufeful in the refolution of numerous problems; and further has this peculiar advantage over all other transformations yet invented, that it often eaſily diſcovers ſome of the firſt terms of the equation required, from which many elegant Theorems may be derived.

In the works above-mentioned, viz. Miſcell. Analyt. Medit. Algéb. &c. are given ſome problems ſerving to this transformation; the firſt of which is a ſeries, which from the coefficients of a given algebraical equation ($x^n - px^{n-1} + qx^{n-2} -$ &c. $= 0$) finds the ſum of any power of the roots (viz. $\alpha^m + \beta^m + \gamma^m + \delta^m +$ &c. where α, β, γ, δ, &c. denote the roots of the given equation), the law of which ſeries was publiſhed by me many years before that it was given by Mr. EULER. The third Problem often mentioned in this paper is an elegant and uſeful ſeries for finding the ſum of quantities of the following kind, viz. $\alpha^m\beta^r\gamma^s\delta^t$, &c. $+ \alpha^r\beta^m\gamma^s\delta^t$, &c. $+ \alpha^t\beta^m\gamma^r\delta^s$ &c. $+ \alpha^t\beta^m\gamma^r\delta^s$ &c. $+ \alpha^m\beta^t\gamma^s\delta^t$ &c $+$. &c.

Mr. EULER gave the following reſolution, $x = \sqrt[n]{\pi} + \sqrt[n]{\rho} + \sqrt[n]{\sigma} + \sqrt[n]{\tau} +$ &c. where π, ρ, σ, τ, &c. denote the roots of an equation of $n-1$ dimenſions $v^{n-1} - pv^{n-2} + qv^{n-3} -$ &c. $= 0$. It is evident, that in this caſe the equation whoſe root is x will have n^{n-1} dimenſions; for let the roots of the equation $z^n - 1 = 0$ be denoted by α, β, γ, δ, &c. then will the quantity $\sqrt[n]{\pi}$ have the n following values $\alpha\sqrt[n]{\pi}$, $\beta\sqrt[n]{\pi}$, $\gamma\sqrt[n]{\pi}$, &c. and the ſame may be affirmed of the quantities $\sqrt[n]{\rho}$, $\sqrt[n]{\sigma}$, $\sqrt[n]{\tau}$, &c. and conſequently the quantity $\sqrt[n]{\pi} + \sqrt[n]{\rho}$ will have $n \times n$ different values; and in the ſame manner the root $x = \sqrt[n]{\pi} + \sqrt[n]{\rho} + \sqrt[n]{\sigma} + \sqrt[n]{\tau} +$ may be proved to contain $n \times n \times n \times n \times$ &c. $= n^{n-1}$ roots, and conſequently in this reſolution, in equations of ſuperior dimenſions, the number of independent coefficients $(n-1)$ will be very few in proportion to the number of dimenſions n^{n-1}, or (if we reſpect its formula) n^{n-2} of the reſulting equation.

Let $n = 3$, and the equation reſulting will riſe to an equation of nine dimenſions, which has the formula of a cubic; for let $x = \sqrt[3]{\pi} + \sqrt[3]{\rho} = a$ one root, then will $\frac{-1+\sqrt{-3}}{2}a$ & $\frac{-1-\sqrt{-3}}{2}a$ be two other of the nine roots, and conſequently the roots will be $\overline{x^3 - a^3} \times \overline{x^3 - b^3} \times \overline{x^3 - c^3} = 0$, which has the formula of a cubic: and in general the above-mentioned equation of n^{n-1} dimenſions will, for the ſame reaſon, have the formula of an equation of n^{n-2} dimenſions.

Let the reſolution be $x = \sqrt[2]{\pi} + \sqrt[2]{\rho} + \sqrt[2]{\sigma} + \sqrt[2]{\tau} +$ &c. where π, ρ, σ, τ, &c. denote the roots of an equation $x^{n-1} - px^{n-2} + qx^{n-3} -$ &c. $= 0$ of $(n-1)$ dimenſions, then will the reſulting equation free from radicals, whoſe root is x, riſe to 2^{n-1} dimenſions; but as every affirmative

tive has a negative root equal to it, it will have the formula of an equation of 2^{s-1} dimenſions.

Let the reſolution be of this formula $x = \sqrt[r \times m]{\alpha} + \sqrt[r \times m]{\beta} + \sqrt[r \times m]{\gamma} + \sqrt[r \times m]{\delta} +$ &c. if $\alpha, \beta, \gamma, \delta$, &c. be conſidered as the r power of the roots of an equation of s dimenſions, then will the reſulting equation, of which the reſolution is given, riſe only to an equation of the formula of m^{s-1} dimenſions.

In the year 1762 I publiſhed ſome reaſons, for which this method could not extend to the general reſolution of algebraical equations.

XI. *Obſervations on the total (with Duration) and annular Eclipſe of the Sun, taken on the 24th of* June, 1778, *on Board the* Eſpagne, *being the Admiral's Ship of the Fleet of* New Spain, *in the Paſſage from the* Azores *towards* Cape St. Vincent's. *By Don* Antonio Ulloa, F. R. S. *Commander of the ſaid Squadron; communicated by* Samuel Horſley, *LL.D. F. R. S.*

Read December 24, 1778.

THE public prints will have given notice of my arrival at this port, with the fleet under my command, on the 29th of June laſt. A very favourable, though

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Obſervations de l'Eclipſe du Soleil totale avec Retenſion et Annulaire faits le 24ᵉ Juin 1778; *ſur le Vaiſſeau* l'Eſpagne *Commandant l'Eſcadre de la Flotte de la* Nouvelle Eſpagne, *en faiſant le trajet des* Iſles Terceres *vers le* Cap St. Vincent. *Par* Don Antonio de Ulloa *chef d'Eſcadre et Commandant General de la ditte Flotte, Membre de la Société Royale de* Londres.

ON aura appris par les nouvelles publiques, que je ſuis rentré dans ce port le 29ᵉ du mois paſſé avec la flotte de la Nouvelle Eſpagne ſous mon commandement. Le trajet dans mon retour, qui a été long mais très heureux, m'a été favorable