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SCIENTIFIC NOTES

NUMERICAL INVESTIGATION OF WARING'S PROBLEM FOR CUBES

JAN BOHMAN and CARL-ERIK FRÖBERG

Abstract.

Numerical results on Waring's problem for cubes are given. In particular strong evidence is presented indicating the truth of the conjecture $G(3)=4$, i.e. that every sufficiently large number can be written as the sum of at most 4 positive cubes.

Introduction.

Waring's problem is concerned with the representation of integers as sums of powers of integers. More specifically, for $k=2,3,4,\ldots$ there exists an integer $g(k)$ such that every positive number n can be written as the sum of at most $g(k)$ kth powers. Similarly there exists a number $G(k)$ such that all sufficiently large numbers n, e.g. $n \geqq N(k)$, can be written as the sum of not more than $G(k)$ kth powers. Good expository treatments of this subject are provided by Ellison [4] (with a comprehensive reference list), Graham [5], and Hardy–Wright [6], Ch. XXI.

It has long been known that $g(2)=G(2)=4$, and further it follows from a theorem by Davenport [2] that $G(4)=16$. There is also an algorithm for computation of $g(k)$, $k \geqq 6$ [6]. Further, Dickson [3] proved that $g(3)=9$, Chen [1] that $g(5)=37$; the value of $g(4)$ is not known, but there are overwhelming arguments in favor of $g(4)=19$. It is believed that the general formula is $g(k) = \lfloor (3/2)^k \rfloor + 2^k - 2$. Hence the solution with respect to $g(k)$ is practically complete.

In contrast, no values of $G(k)$ are known, except when $k=2$ and $k=4$ as stated above. In this paper we will report on extensive numerical calculations on Waring's problem for cubes. The best theoretical result so far seems to be a theorem by Davenport [2] that the number of integers $\leqq N$ requiring 5 cubes is $O(N^{1-1/30+\varepsilon})$ with $\varepsilon > 0$. Hence almost all integers can be represented as sums of 4 cubes. Amazingly accurate and extensive numerical computations were performed by Alfred E. Western [7] already in 1926, long before the advent of automatic computers. At least two of his conjectures are supported by our investigation.

In this paper we shall present convincing numerical evidence that $G(3)=4$, and we also suggest definite results (some of which are previously known) for the number of members in the classes C_6, C_7, C_8 and C_9. Here C_k is the set of numbers

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which can be represented as sums of k but not less than k cubes. Hence, $216 = 6^3 \in C_1$ although $3^3+4^3+5^3=216$. Our conjecture $G(3)=4$ rests on determination of the density of numbers requiring 5 cubes up to $4\cdot 10^{11}$ and use of an extrapolation technique suggested by Western [7]. For obvious reasons it has been impossible to investigate all numbers $< 4\cdot 10^{11}$ which would indeed demand computers at least 1000 times as powerful as the present ones. Instead we have taken samples, first of length 20,000, then (around $10^{10}-2\cdot 10^{11}$) of length 100,000, and finally for $4\cdot 10^{11}$ of length 10^6.

Numerical algorithm for investigation of $G(3)$.

A number N which can be represented as the sum of not more than 4 cubes can be split into two parts a and b, both belonging to a set of numbers consisting of cubes and sums of two cubes. With this observation we prepared a vector by adding all pairs of cubes (including zero). This was done by selecting all possible pairs with sum in a certain interval, and when a number appeared, a corresponding digit in a bit pattern was put equal to 1. In this way we could cope with the problems that the sequence formed did not necessarily come in ascending order, and further that some numbers could appear several times (note e.g. the well-known example $10^3+9^3=12^3+1^3$). Once the binary representation was clear it was an easy matter to convert it to the vector mentioned above, and we could avoid a difficult and time-consuming sorting. Successive parts of the vector were moved to disk storage; the final vector contained more than 26 million elements.

Using this vector we tried to form all numbers in a certain interval by adding two elements (a and b) starting with one small and one large element. The number of failures as well as the corresponding numbers, i.e. those which required 5 cubes, were recorded. The computations were performed on a UNIVAC 1100/80. The program, written in FORTRAN and assembly code, took about 8 hours for forming the vector while a run of a sample of 10^6 numbers (actually from $4\cdot 10^{11}$ to $4.00001\cdot 10^{11}$) took about 7 hours. The total computing time was about 25 hours. Unfortunately, the capacity of the computer puts a limit at about $4\cdot 10^{11}$ and we hope that better opportunities will be offered by future more powerful computers. Our impression is that a sample of 10-100 million numbers close to 10^{12} or possibly $2\cdot 10^{12}$ would be desirable.

Results.

As mentioned above we have computed the density $\varrho(N)$ of numbers which require 5 cubes, by taking samples of varying lengths for different values of $N \leqq 4\cdot 10^{11}$. The numerical values are presented in Table 1, and further we have plotted $N\varrho(N)\cdot 10^{-6}$ together with $\ln\ln\varrho(N)^{-1}$ against $\ln N$ in Fig. 1. The shape of curve I indicates that $\varrho(N)$ drops off considerably faster than N^{-1}, and this would imply that *the number of elements in the class C_5 is finite*. A numerical

Table 1.

Estimated densities of numbers which cannot be expressed as sums of 4 cubes. Very few events were obtained in the last cases which accounts for the somewhat irregular behaviour of the density values. The two largest numbers $\in C_5$ *observed in our last sample are* 400000468109 *and* 400000802954.

N	$\ln N$	$\varrho(N)$	$N\varrho(N)\cdot 10^{-6}$	$\ln\ln \varrho(N)^{-1}$
10^4	9.21	0.4994	0.005	−0.365
$2\cdot 10^4$	9.90	0.4538	0.009	−0.236
$5\cdot 10^4$	10.82	0.4040	0.020	−0.098
10^5	11.51	0.3593	0.036	0.023
$2\cdot 10^5$	12.21	0.3206	0.064	0.129
$3.5\cdot 10^5$	12.77	0.2868	0.100	0.222
$5\cdot 10^5$	13.12	0.2646	0.132	0.285
10^6	13.82	0.2302	0.230	0.384
$2\cdot 10^6$	14.51	0.1980	0.396	0.482
$3\cdot 10^6$	14.91	0.1845	0.554	0.525
$4\cdot 10^6$	15.20	0.1678	0.671	0.579
$2.7\cdot 10^7$	17.11	0.0955	2.58	0.854
$6.4\cdot 10^7$	17.97	0.0690	4.42	0.983
$2.16\cdot 10^8$	19.19	0.0407	8.79	1.164
$3.007\cdot 10^8$	19.52	0.0352	10.58	1.208
$7.464\cdot 10^8$	20.43	0.0201	15.00	1.363
10^9	20.72	0.0169	16.90	1.406
$2\cdot 10^9$	21.42	0.00948	18.96	1.539
$4\cdot 10^9$	22.11	0.00541	21.64	1.652
$8\cdot 10^9$	22.80	0.00261	20.88	1.783
$9.77\cdot 10^9$	23.00	0.00202	19.74	1.825
$1.6\cdot 10^{10}$	23.50	0.00098	15.68	1.936
$3.2\cdot 10^{10}$	24.19	0.00048	15.36	2.034
$6.4\cdot 10^{10}$	24.88	0.00011	7.04	2.210
$1.28\cdot 10^{11}$	25.58	0.00002	2.56	2.381
$2\cdot 10^{11}$	26.02	0.000005	1.0	2.502
$4\cdot 10^{11}$	26.71	0.000003	1.2	2.543

integration gave the result 112 millions. Western [7] has suggested a relationship of the form

$$\ln\ln \varrho(N)^{-1} \cong A + B \ln N \,,$$

and in Fig. 1 we can see that the linearity is very good for not too large values ($N < 10^9$). However, for still larger values the curve seems to rise faster. It can be argued that when $\varrho(N)$ gets less than N^{-1} we have probably exhausted the class C_5. By straight-forward although somewhat uncertain graphical extrapolation we find $N\varrho \cong 10$ for $N = 5\cdot 10^{12}$ and $N\varrho \cong 0.05$ for $N = 10^{13}$. Hence we believe that 10^{13} is a reasonably safe estimate of the threshold value, and if so, the largest number not expressible as the sum of four cubes would be less than 10^{13}. This is in good agreement with a conjecture by Western [7].

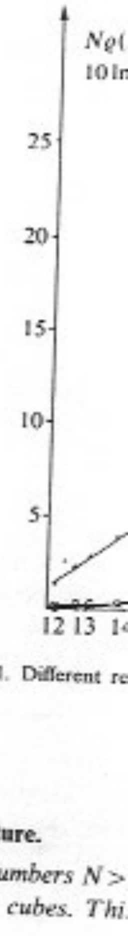

Figure 1. Different re

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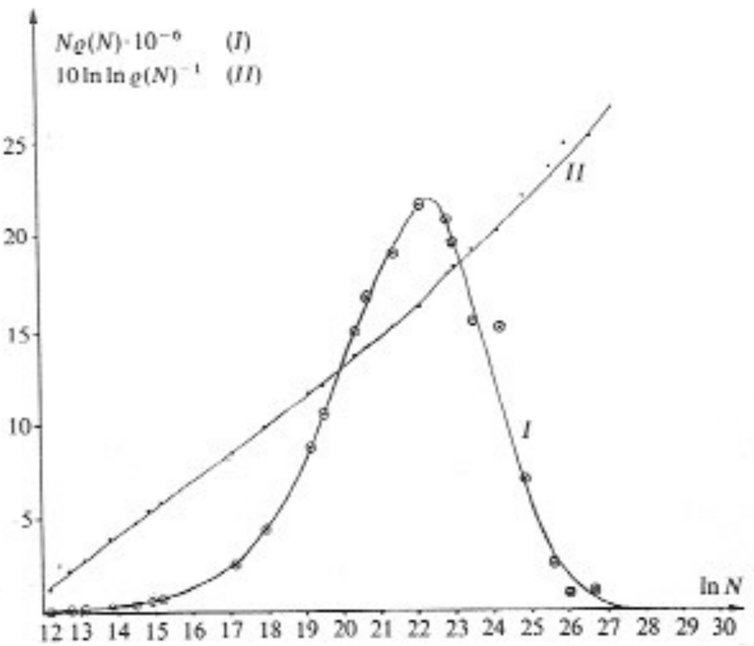

Figure 1. Different representations of the density of numbers in the class C_5 plotted against $\ln N$. Further see Table 1.

Conjecture.

All numbers $N > M$ where $M \simeq 10^{13}$ can be expressed as sums of not more than 4 positive cubes. This is equivalent to $G(3) = 4$.

Investigations of related problems.

We have also performed extensive computer runs in order to find all elements in the classes C_6, C_7, C_8 and C_9, at least partially known before. The most interesting is the class C_6; we believe that it contains 3922 elements ranging from 6, 13, 20, 34, 39, ... to 607477, 626351, 644117, 718196, 864455, 1026545, 1083983, 1290740. Runs were performed up to 2685000. Western [7] reports the number 607477 as the largest number $\equiv 4 \pmod{9}$ belonging to C_6 that he has found. He conjectures that if there are any more, they will be very few. In fact, the following 7 numbers, which we believe are the last ones in C_6, are all $\equiv 5 \pmod{9}$.

Similarly our results indicate that C_7 contains 121 elements ranging from 7, 14, 21, 42, 47, 49, 61, ... to 5279, 5305, 5306, 5818, 8042. The number 8042 is also quoted in [4] as the largest number in C_7. Further, C_8 contains 15 elements (15, 22, 50, 114, 167, 175, 186, 212, 231, 238, 303, 364, 420, 428, 454) and C_9 2 elements (23, 239).

Table 2.

Characteristics of the classes C_k, $k = 5(1)9$, as defined in the introduction.

Class	Number of elements	Largest element
C_5	$1.12 \cdot 10^8$ (?)	$\sim 10^{13}$ (?)
C_6	3922	1290740
C_7	121	8042
C_8	15	454
C_9	2	239

As pointed out by Western [7] the larger numbers in C_5 are $\equiv \pm 4 \pmod 9$ and $\equiv \pm 3 \pmod 7$. No other congruential relations have been observed.

Additional results.

Our computations indicate that there are 4763 numbers which cannot be written as sums of at most 6 *different* cubes, the largest of them being 60,575. Further, the largest number not expressible as the sum of *different* positive cubes is 12758 (also noted by Graham [5]).

Acknowledgements.

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DEPARTMENT OF COMPUTER SCIENCE
LUND UNIVERSITY
LUND, SWEDEN

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Abstract.

We obtain a recurrence rel

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1. Introduction.

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may also be required in
of Section 2.

Now we can easily
solution of (1) in the
evaluation of integrals

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