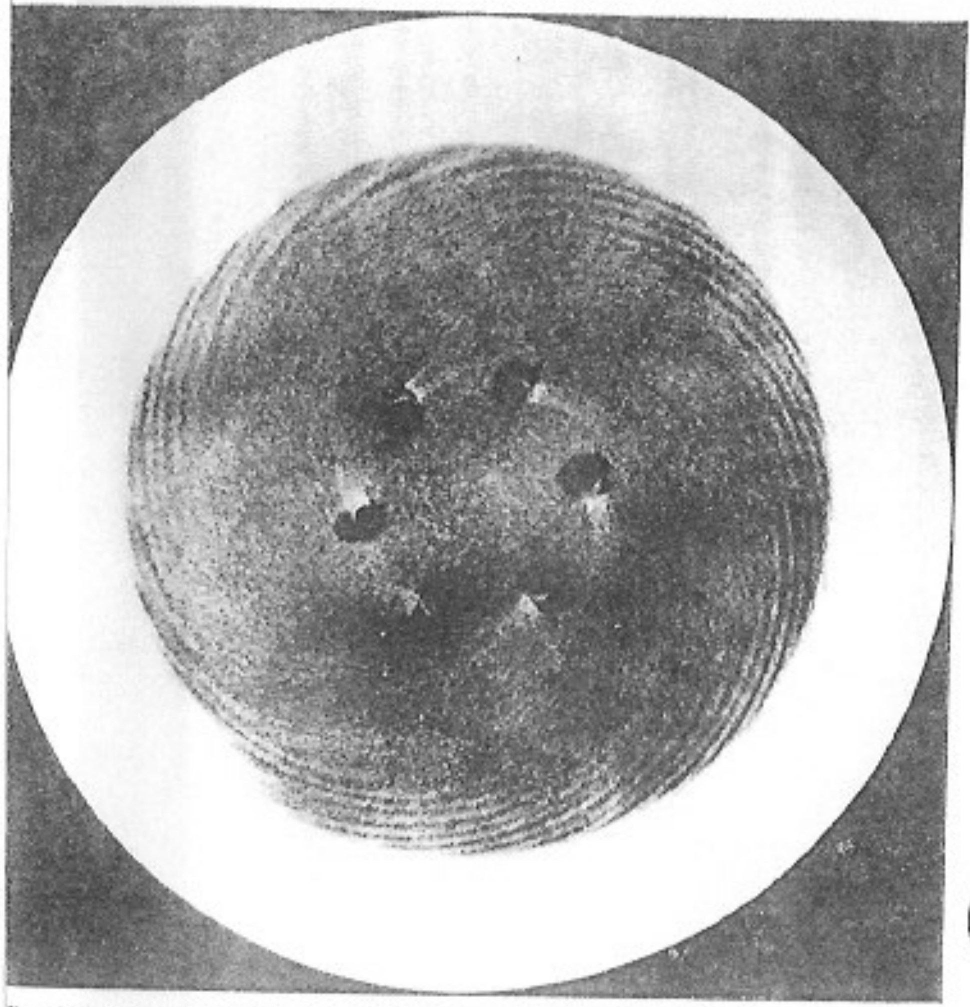

Frontispiece. Boundary-layer flow induced by a rotating disk (china-clay photograph), showing laminar flow near the centre and subsequent instability in the form of spiral vortices which cause transition to turbulent flow near the rim.
(See Gregory, Stuart, and Walker 1955)

FLUID MOTION MEMOIRS

LAMINAR BOUNDARY LAYERS

An Account of the Development, Structure and Stability of Laminar Boundary Layers in Incompressible Fluids, together with a Description of the Associated Experimental Techniques

Editor

L. ROSENHEAD

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III

THE NAVIER-STOKES EQUATIONS OF MOTION

PART I

GENERAL THEORY

1. Analysis of the motion of a fluid element

THE motion of a fluid is completely determined when the velocity vector $\mathbf{v}$ is known as a function of time and position. We now show how the motion of any small element of fluid may be analysed in terms of this function. If the components of $\mathbf{v}$ at the point $P(x,y,z)$ are (u,v,w), and at a neighbouring point $(x+\delta x, y+\delta y, z+\delta z)$ the velocity components are $(u+\delta u, v+\delta v, w+\delta w)$, then, to the first order of small quantities,

$$\begin{aligned} \delta u &= \tfrac{1}{2}(e_{xx}\,\delta x+e_{xy}\,\delta y+e_{xz}\,\delta z)+\tfrac{1}{2}(\eta\,\delta z-\zeta\,\delta y),\\ \delta v &= \tfrac{1}{2}(e_{yx}\,\delta x+e_{yy}\,\delta y+e_{yz}\,\delta z)+\tfrac{1}{2}(\zeta\,\delta x-\xi\,\delta z),\\ \delta w &= \tfrac{1}{2}(e_{zx}\,\delta x+e_{zy}\,\delta y+e_{zz}\,\delta z)+\tfrac{1}{2}(\xi\,\delta y-\eta\,\delta x), \end{aligned} \tag{1}$$

where
$$e_{xx} = 2\frac{\partial u}{\partial x}, \qquad e_{yy} = 2\frac{\partial v}{\partial y}, \qquad e_{zz} = 2\frac{\partial w}{\partial z},$$

$$e_{yz} = e_{zy} = \frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}, \qquad e_{zx} = e_{xz} = \frac{\partial u}{\partial z}+\frac{\partial w}{\partial x},$$

$$e_{xy} = e_{yx} = \frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}, \tag{2}$$

and
$$\xi = \frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}, \qquad \eta = \frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}, \qquad \zeta = \frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}. \tag{3}$$

A small element of fluid initially at P has an over-all translational velocity (u,v,w), but as it moves it is distorted and rotated in accordance with the relative velocities given by (1).

The grouping of the terms in (1) corresponds to their different physical interpretations. The terms $\frac{1}{2}(\eta\,\delta z-\zeta\,\delta y)$, $\frac{1}{2}(\zeta\,\delta x-\xi\,\delta z)$, $\frac{1}{2}(\xi\,\delta y-\eta\,\delta x)$, represent a rotation of the fluid element as if it were a rigid body with angular velocity $(\frac{1}{2}\xi, \frac{1}{2}\eta, \frac{1}{2}\zeta)$. The vector $\boldsymbol{\omega} = (\xi,\eta,\zeta)$ defined by (3) is the curl of $\mathbf{v}$ and is the vorticity of the fluid. The quantities e_{xx} etc., defined in (2) are called the rate-of-strain components, and they

constitute the rate-of-strain tensor. The contribution of each of these components to the motion described by (1) is now considered.

First, suppose that all the components except e_{xx} are zero. Then $\delta u = \frac{1}{2}e_{xx}\,\delta x$, $\delta v = 0$, $\delta w = 0$, which represents an extension of the element at a rate $\frac{1}{2}e_{xx}$ per unit length in the x direction. Similarly, $\frac{1}{2}e_{yy}$ and $\frac{1}{2}e_{zz}$ are the rates of extension of the fluid element in the y- and z-directions. Secondly, if all the components except e_{xy} are zero,

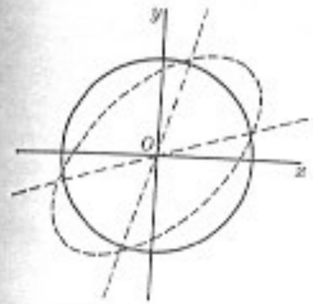


FIG. III. 1. Deformation produced by the rate-of-strain component e_{xy}.

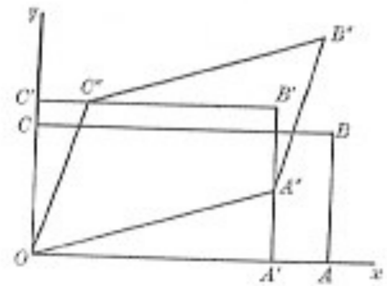


FIG. III. 2. General deformation of a fluid element.

then $\delta u = \frac{1}{2}e_{xy}\,\delta y$, $\delta v = \frac{1}{2}e_{xy}\,\delta x$, $\delta w = 0$. This represents a motion in which the angle between the two lines of particles which lie initially on the x- and y-axes is decreasing at a rate e_{xy}. The position of these two lines of particles after a short time is shown by the broken lines in Fig. III. 1. If, for example, a section of the fluid element is initially circular, it will be deformed into an ellipse under this distortion (see Fig. III. 1). The interpretation of the component e_{xy} may be modified slightly by considering its effect in conjunction with a rigid-body rotation with angular velocity $-\frac{1}{2}e_{xy}$ which reduces the line of particles on the x-axis to rest. Then, we have a simple shear flow parallel to the x-axis with rate of shear equal to e_{xy}.

To exemplify the combined effects of the rate-of-strain components, the distortion, in two-dimensional flow, of a fluid element which is initially rectangular in shape, is shown in Fig. III. 2. The original shape $OABC$ is deformed under the rates of extension e_{xx}, e_{yy}, alone into $OA'B'C'$, and the final shape is $OA''B''C''$.

2. Equation of continuity

Throughout the motion the mass of any element of fluid must be conserved; hence, for incompressible flow, the volume of the fluid element must remain constant. This condition yields the equation of continuity which must be satisfied by the rate-of-strain components at all points of the fluid. (The components of vorticity are not involved since a rigid-body rotation leaves the volume unchanged.) If we consider the fluid initially occupying a small rectangular parallelepiped with faces normal to the coordinate axes, it is clear from the interpretations given above that e_{yz}, e_{zx}, e_{xy} have only a second-order effect on the change of the volume of the element in a small time δt. But the lengths of the sides are increased by factors $1+\frac{1}{2}e_{xx}\delta t$, $1+\frac{1}{2}e_{yy}\delta t$, $1+\frac{1}{2}e_{zz}\delta t$. Hence, the rate of increase of volume per unit volume is $\Delta = \frac{1}{2}(e_{xx}+e_{yy}+e_{zz})$. This quantity Δ is called the dilatation, and for incompressible flow it must vanish. Thus, the equation of continuity is

$$\operatorname{div} \mathbf{v} = \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z} = 0. \tag{4}$$

3. Principal axes of rate of strain

When discussing general properties of the flow, it is often convenient to transform to the special set of rectangular axes for which the components corresponding to e_{yz}, e_{zx}, e_{xy} become zero. The existence of such axes can be shown by appeal to a geometrical argument. If we write

$$\Psi = e_{xx}(\delta x)^2+e_{yy}(\delta y)^2+e_{zz}(\delta z)^2+2e_{yz}\,\delta y\delta z+2e_{zx}\,\delta z\delta x+2e_{xy}\,\delta x\delta y, \tag{5}$$

and regard δx, δy, δz as current coordinates, then Ψ = constant is the equation of a quadric with its centre at P. In any other set of coordinates, (x',y',z') say, if $e_{x'x'}$, $e_{y'z'}$, etc., are defined in terms of the corresponding velocity components (u',v',w') as in (2), Ψ will have exactly the same form as (5) in terms of x', y', z'. This 'invariance' is an essential consequence of the tensor character of the rate-of-strain components. It may be verified directly by going through the coordinate transformation in detail, but it also follows from the following property of the quadric. In the motion described by (1), omitting the rigid-body rotation, the displacements of points on the quadric (5) are normal to the surface of the quadric since δu, δv, δw are proportional to the derivatives of Ψ with respect to δx, δy, δz, respectively. This property is independent of the choice of axes, and it follows that for any coordinate system the coefficients in the equation of the quadric have the meanings given by (1), i.e. they are the rate-of-strain components.

Now, the new axes may be chosen along the principal axes of the quadric, and then Ψ becomes

$$e_{x'x'}(\delta x')^2+e_{y'y'}(\delta y')^2+e_{z'z'}(\delta z')^2.$$

Hence, for these axes, $e_{y'z'}$, $e_{z'x'}$, $e_{x'y'}$ vanish. The rates of change of the angles between lines of particles along the axes are zero, and the distortion of the fluid consists entirely of extensions along the principal axes. For example, in Fig. III. 1 the principal axes are at 45° to the original axes. Quite generally, a fluid element which is initially spherical will be distorted into an ellipsoid with its axes along the principal axes.

The result used above that the terms in (1) which include the rate-of-strain components are proportional to the derivatives of Ψ, is worth noticing for its own sake, since it shows that this part of the motion may be deduced from a potential function. It also leads to the result that, if Q is any point on the rate-of-strain quadric, the rate of change of $(PQ)^2$ is Ψ. For, the rate of increase of PQ is

$$\frac{1}{4}\left\{\frac{\partial\Psi}{\partial(\delta x)}\frac{\delta x}{PQ}+\frac{\partial\Psi}{\partial(\delta y)}\frac{\delta y}{PQ}+\frac{\partial\Psi}{\partial(\delta z)}\frac{\delta z}{PQ}\right\} = \frac{1}{2}\frac{\Psi}{PQ}.$$

Hence, the rate of extension of any line element through P is inversely proportional to the square of the radius vector drawn to the rate-of-strain quadric in the direction of the element.

Finally, it may be shown from (1) (preferably using principal axes for ease of calculation) that the angular momentum of a small sphere of fluid is the same as if it were rotating as a rigid body with angular velocity $\frac{1}{2}\omega = (\frac{1}{2}\xi, \frac{1}{2}\eta, \frac{1}{2}\zeta)$. Thus, the whole of the rotation is given by the second terms on the right-hand side of (1), and the rotation depends only on the vorticity vector. It should be pointed out, however, that this result does not apply to *any* small portion of the fluid with its mass centre at P; it is true only if the principal axes of rate of strain coincide with the principal axes of inertia of the fluid element.

4. Analysis of stress

Let us consider any small element of surface containing a point $P(x,y,z)$, and let us define the direction of the normal to the element, $\mathbf{n}$, as pointing from one side, called 'negative', to the other, called 'positive'. We then consider the stress exerted *on* the fluid on the negative side, *by* that on the positive side; this will be a vector which depends on $\mathbf{n}$. The components of the resultant stress in the direction of the fixed axes Ox, Oy, Oz, will be denoted by p_{nx}, p_{ny}, p_{nz}, respectively. Thus, when the surface is normal to the x-direction, we denote the components of the stress parallel to the x-, y-, z-axes by p_{xx}, p_{xy}, p_{xz}, respectively. Similarly, the stresses across surfaces normal to the y- and z-axes are denoted by (p_{yx}, p_{yy}, p_{yz}) and (p_{zx}, p_{zy}, p_{zz}), respectively.

The stress components p_{xx}, etc., constitute the stress tensor, and the stress across a surface at P with its normal in *any* given direction may be obtained from them. For, consider a small tetrahedron with three of its faces through P and normal to the coordinate axes, and with the fourth face normal to the given direction **n**. The forces exerted across the faces of this tetrahedron are proportional to the areas of the faces; hence, the external body forces and the inertia forces which are proportional to the volume of the tetrahedron are of smaller order. Thus, to this order of approximation, the surface tractions must form a system in equilibrium. Moreover, to the same approximation, the stresses can be taken equal to their values at P. Then, if the stresses across the surface normal to n are (p_{nx}, p_{ny}, p_{nz}), we have by resolving in the x-direction

$$p_{nx}\Delta S_n = p_{xx}\Delta S_x + p_{yx}\Delta S_y + p_{zx}\Delta S_z,$$

using an obvious notation for the areas ΔS of the faces of the tetrahedron. Now

$$\Delta S_x/\Delta S_n = l, \qquad \Delta S_y/\Delta S_n = m, \qquad \Delta S_z/\Delta S_n = n,$$

where (l, m, n) are the direction cosines of the normal **n**. Therefore, we have

$$p_{nx} = lp_{xx} + mp_{yx} + np_{zx}.$$

Two similar equations are found for p_{ny} and p_{nz}.

The same considerations apply for the equilibrium of the stresses exerted across the faces of a cube centred at P with its faces perpendicular to the axes. Then, taking moments of the forces about P, we have

$$p_{yz} = p_{zy}, \qquad p_{zx} = p_{xz}, \qquad p_{xy} = p_{yx}. \tag{6}$$

In the same way as for the rate-of-strain components we can define a stress quadric which is invariant to change of coordinates. Then we can say that there exist principal axes Ox', Oy', Oz', such that $p_{x'x'}$, $p_{y'y'}$, $p_{z'z'}$ are the only non-zero components of stress. The planes perpendicular to the principal axes of stress are called principal planes of stress; the stress across each of them is purely normal, and these three normal stresses are called the principal stresses.

5. Relations between the stress and rate-of-strain components

The simplest assumption for the relations between the two sets of components is that they are linear. That is, each stress component may be expressed as a sum of multiples of the six rate-of-strain components plus a quantity independent of them. If the rate-of-strain components were small this would certainly be a natural approximation, but as explained in Chapter I the relations have in fact been verified for many fluids under a very wide range of conditions when the rate-of-strain components are not small. The assumption that the relations are linear is the basic one in deriving the Navier–Stokes equations for incompressible flow.

In an isotropic fluid the principal axes of stress and rate of strain coincide, and the relations between the stress and rate-of-strain components must be symmetric since there is no preferred direction. Therefore, relative to principal axes, we have

$$\begin{aligned} p_{x'x'} &= -p + \mu e_{x'x'}, \\ p_{y'y'} &= -p + \mu e_{y'y'}, \\ p_{z'z'} &= -p + \mu e_{z'z'}, \end{aligned} \tag{7}$$

where the quantities p and μ are independent of the rate-of-strain components, but may vary from point to point of the fluid. (For compressible flow, in which $\Delta = \frac{1}{2}(e_{x'x'} + e_{y'y'} + e_{z'z'})$ is not zero, a multiple of Δ is added to each of these equations.) The appropriate relations for any other set of Cartesian coordinates (x, y, z) may then be deduced from (7) together with the transformation equations for the stress and rate-of-strain components. Apart from the term $-p$ in (7), the stress components are equal to the rate-of-strain components multiplied by μ; hence, since the transformation equations for p_{xx}, etc., and e_{xx}, etc., are identical, this is true in any set of axes. Therefore, in general,

$$\begin{aligned} p_{xx} &= -p + \mu e_{xx} = -p + 2\mu\frac{\partial u}{\partial x}, \\ p_{yy} &= -p + \mu e_{yy} = -p + 2\mu\frac{\partial v}{\partial y}, \\ p_{zz} &= -p + \mu e_{zz} = -p + 2\mu\frac{\partial w}{\partial z}, \\ p_{yz} &= \mu e_{yz} = \mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right), \\ p_{zx} &= \mu e_{zx} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right), \\ p_{xy} &= \mu e_{xy} = \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right). \end{aligned} \tag{8}$$

Further details of the derivation of these equations are to be found in Lamb (1932, pp. 571–6). A more sophisticated derivation using the full power of the tensor properties is given by Jeffreys (1931, Chapters 7 and 9).

For an incompressible flow it is observed, using the equation of continuity (4), that p is the mean of the normal pressures over three planes mutually at right angles; it is called simply the pressure. It is also seen that in a simple shear flow with $u = u(y)$, $v = w = 0$, we have

$$p_{xy} = \mu \frac{\partial u}{\partial y}. \tag{9}$$

Hence μ is the coefficient of shear viscosity introduced and discussed in Chapter I. For each substance it varies only with temperature, and thus where the temperature is constant, or the temperature variations are small, the coefficient of shear viscosity may be taken to be a constant.

6. The momentum equations

The momentum equations, which must be satisfied by the flow quantities at each point of the fluid, may be deduced by applying Newton's second law of motion to the fluid which occupies a rectangular parallelepiped centred at P with its edges parallel to the coordinate axes. To the second order in the length of its sides, the stresses form a system in equilibrium (as discussed in Section 4). But, we now write down its equation of motion including terms of the third order. The x-component of the net force due to the stresses on the element is

$$\frac{\partial p_{xx}}{\partial x}+\frac{\partial p_{yx}}{\partial y}+\frac{\partial p_{zx}}{\partial z}$$

multiplied by the volume of the element. (This type of result is often required in this chapter; it is deduced as follows: in the distance δx between the two faces normal to the x-axis, p_{xx} increases by $\delta x\,.\,\partial p_{xx}/\partial x$; when multiplied by the face area, this gives a contribution $\partial p_{xx}/\partial x$ multiplied by the volume of the element. Similarly, the other stress components contribute the terms proportional to $\partial p_{yx}/\partial y$ and $\partial p_{zx}/\partial z$.) If the extraneous force $\mathbf{F}$ per unit mass has components (X, Y, Z) and the acceleration $\mathbf{f}$ has components (f_x, f_y, f_z), we have

$$\rho f_x = \rho X+\frac{\partial p_{xx}}{\partial x}+\frac{\partial p_{yx}}{\partial y}+\frac{\partial p_{zx}}{\partial z}, \tag{10}$$

with two similar equations for the y- and z-directions.

Now,

$$f_x = \frac{Du}{Dt} = \frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}$$
$$= \frac{\partial u}{\partial t}+\frac{\partial}{\partial x}(\tfrac{1}{2}\mathbf{v}^2)-(v\zeta-w\eta), \tag{11}$$

with similar expressions for f_y and f_z. The expression

$$\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)$$

represents the contribution to f_x resulting from the change of the position of the fluid element; it, and similar ones, are usually called the 'convection terms'. Substituting (11) in (10) and using the relations (8) together with the continuity equation (4), we find

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x}+X+\nu\nabla^2 u,$$
$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial y}+Y+\nu\nabla^2 v,$$
$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z}+Z+\nu\nabla^2 w, \tag{12}$$

where $\nu = \mu/\rho$, and

$$\nabla^2 u = \operatorname{div}\operatorname{grad} u = \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}, \quad \text{etc.} \tag{13}$$

In vector form the equations may be written

$$\mathbf{f} = \frac{\partial \mathbf{v}}{\partial t}+\operatorname{grad}(\tfrac{1}{2}\mathbf{v}^2)-\mathbf{v}\times\boldsymbol{\omega} = -\frac{1}{\rho}\operatorname{grad} p+\mathbf{F}+\nu\nabla^2\mathbf{v}, \tag{14}$$

where $\nabla^2\mathbf{v}$ is the vector whose components in Cartesian coordinates are $(\nabla^2 u, \nabla^2 v, \nabla^2 w)$. Since $\nabla^2\mathbf{v} = \operatorname{grad}(\operatorname{div}\mathbf{v})-\operatorname{curl}\operatorname{curl}\mathbf{v}$ and $\operatorname{div}\mathbf{v} = 0$ whilst $\operatorname{curl}\mathbf{v} = \boldsymbol{\omega}$, the term $\nabla^2\mathbf{v}$ may be replaced by $-\operatorname{curl}\boldsymbol{\omega}$. If, in addition, the extraneous field of force is potential so that $\mathbf{F} = -\operatorname{grad}\Omega$, we may then write (14) as

$$\frac{\partial \mathbf{v}}{\partial t}-\mathbf{v}\times\boldsymbol{\omega} = -\operatorname{grad}\left(\frac{p}{\rho}+\Omega+\tfrac{1}{2}\mathbf{v}^2\right)-\nu\operatorname{curl}\boldsymbol{\omega}. \tag{15}$$

The equations of motion (12) were obtained by Navier (1823), Poisson (1831), Saint-Venant (1843), and Stokes (1845), and are usually known as the 'Navier-Stokes equations'. A short account of the various methods and hypotheses adopted by these authors is to be found in Stokes (1846).

7. Equations for the vorticity. The rate of change of circulation

The equation for the rate of change of the vorticity vector is obtained by taking the curl of (14). Assuming again that $\mathbf{F} = -\operatorname{grad}\Omega$, we have

$$\frac{\partial \boldsymbol{\omega}}{\partial t}-\operatorname{curl}(\mathbf{v}\times\boldsymbol{\omega}) = \nu\nabla^2\boldsymbol{\omega}. \tag{16}$$

Now, $\operatorname{div}\boldsymbol{\omega} = 0$ since $\boldsymbol{\omega} = \operatorname{curl}\mathbf{v}$, and $\operatorname{div}\mathbf{v} = 0$ from the equation of continuity; hence, the components of $\operatorname{curl}(\mathbf{v}\times\boldsymbol{\omega})$ are $\boldsymbol{\omega}.\operatorname{grad} u - \mathbf{v}.\operatorname{grad}\xi$ and two similar expressions with v and η, w and ζ, respectively, in place of u and ξ. Therefore, the components of (16) are

$$\frac{D\xi}{Dt} = \boldsymbol{\omega}.\operatorname{grad} u + \nu\nabla^2\xi,$$

$$\frac{D\eta}{Dt} = \boldsymbol{\omega}.\operatorname{grad} v + \nu\nabla^2\eta,$$

$$\frac{D\zeta}{Dt} = \boldsymbol{\omega}.\operatorname{grad} w + \nu\nabla^2\zeta, \tag{17}$$

and they may be combined in the abbreviated form

$$\frac{D\boldsymbol{\omega}}{Dt} = \boldsymbol{\omega}.\nabla\mathbf{v} + \nu\nabla^2\boldsymbol{\omega}.$$

These extend Helmholtz's equations to viscous flow.

The first terms on the right of (17) have the same interpretation as they have in ideal flow ($\nu = 0$). They represent a stretching of the vortex lines with a consequent increase in the vorticity. To see this, consider first any line element $\delta\mathbf{r}$ which moves with the fluid. Its rate of change $D(\delta\mathbf{r})/Dt$ is the difference of the velocities at its ends. Hence

$$\frac{D}{Dt}\delta\mathbf{r} = \delta\mathbf{r}.\nabla\mathbf{v},$$

and we observe that $\boldsymbol{\omega}$ and $\delta\mathbf{r}$ satisfy the same relation. Now, suppose that $\delta\mathbf{r}$ is part of a vortex line at $t = 0$, with $\delta\mathbf{r} = \epsilon\boldsymbol{\omega}$, say. The equations for $\delta\mathbf{r}$ and $\boldsymbol{\omega}$ give

$$\frac{D}{Dt}(\delta\mathbf{r} - \epsilon\boldsymbol{\omega}) = (\delta\mathbf{r} - \epsilon\boldsymbol{\omega}).\nabla\mathbf{v},$$

and we want to conclude that $\delta\mathbf{r} - \epsilon\boldsymbol{\omega} = 0$ for all time. This is certainly a solution; it is the only solution provided that the derivatives of $\mathbf{v}$ are bounded (and integrable). This appeals to the well-known uniqueness theorem for ordinary differential equations. (The essential requirement is that the right-hand sides should be Lipschitz continuous in the unknowns; in the present case, this is so if the components of $\nabla\mathbf{v}$ are bounded.) The result that $\delta\mathbf{r} = \epsilon\boldsymbol{\omega}$ for all time shows that vortex lines move with the fluid and the vorticity is proportional to the length $|\delta\mathbf{r}|$ of the line element. If we consider a small vortex tube, its cross-sectional area will be inversely proportional to $|\delta\mathbf{r}|$ as it moves with the fluid. Therefore, the 'strength' of the vortex tube, defined as the vorticity multiplied by the cross-sectional area, remains constant. These results constitute Helmholtz's theorem for ideal flow.

Some remarks should be made about the derivation given here, since it is almost the same as Helmholtz's original one and that has been criticized by Lamb and various subsequent writers as being unrigorous. However, the version rightly contested by Lamb uses only the fact that $D(\delta\mathbf{r} - \epsilon\boldsymbol{\omega})/Dt = 0$ at $t = 0$. Clearly, the result cannot follow from this fact alone, since any function behaving like t^2 has this property. But when the full equation is used as above, there is no gap in the argument. (Notice that t^2 satisfies $df/dt = Af$ with $A \propto f^{-\frac{1}{2}}$ which is *unbounded* near $t = 0$.) After pointing out the flaw in Helmholtz's argument, Lamb discards this method instead of correcting it, and goes on to deduce the results from a different approach. This is a pity because, first of all, the corrected argument gives a very direct derivation of Helmholtz's theorem; secondly, it is desirable to have the physical interpretation of the terms in Helmholtz's equation as a stretching of vortex lines without feeling a little uneasy about its soundness; thirdly, when the viscosity terms are added, their interpretation is immediate from (17) (see below) but not from the alternative approaches. Lamb refers to Stokes (1845), who pointed out a similar flaw in a 'proof' of Lagrange's theorem that $\boldsymbol{\omega} = 0$ for all time if it does so initially. But Stokes, in fact, goes on to propose a genuine proof using exactly the argument given above; he starts from $D\boldsymbol{\omega}/Dt = \boldsymbol{\omega}.\nabla v$ and concludes that $\boldsymbol{\omega} = 0$ for all t if it does so at $t = 0$. This seems to have been overlooked.

The second terms on the right of (17) show the additional variation introduced by the viscosity of the fluid. This variation follows the same law as the variation of temperature in the conduction of heat and represents diffusion of the vorticity.

An alternative approach to the above results uses the expression for the rate of change of circulation round any closed circuit moving with the fluid. By definition the circulation round a closed circuit is

$$\Gamma = \oint \mathbf{v}.d\mathbf{r}, \tag{18}$$

where the integral is taken once completely round the circuit. If the fluid particles making up the circuit are labelled by a parameter σ, the motion of the circuit will be given by a function $\mathbf{r} = \mathbf{r}(\sigma, t)$. Then we may write

$$\Gamma = \oint \mathbf{v}.\frac{\partial\mathbf{r}}{\partial\sigma}\,d\sigma.$$

As the circuit moves with the fluid, the rate of change of circulation is

$$\frac{D\Gamma}{Dt} = \oint \left(\frac{D\mathbf{v}}{Dt}.\frac{\partial \mathbf{r}}{\partial \sigma} + \mathbf{v}.\frac{\partial^2 \mathbf{r}}{\partial \sigma \partial t}\right) d\sigma.$$

Since $D\mathbf{v}/Dt = -\text{grad}(p/\rho+\Omega)-\nu\,\text{curl}\,\boldsymbol{\omega}$ and $\partial\mathbf{r}/\partial t = \mathbf{v}$, this may be written

$$\frac{D\Gamma}{Dt} = -\oint \frac{\partial}{\partial \sigma}\left(\frac{p}{\rho}+\Omega-\tfrac{1}{2}\mathbf{v}^2\right)d\sigma - \nu \oint (\text{curl}\,\boldsymbol{\omega}).d\mathbf{r}.$$

Assuming that Ω is single valued, we have

$$\frac{D\Gamma}{Dt} = -\nu \oint (\text{curl}\,\boldsymbol{\omega}).d\mathbf{r}. \tag{19}$$

Thus, in a viscous fluid, the rate of change of circulation in a circuit moving with the fluid depends only on the kinematic viscosity and the space rates of change of the vorticity components at the contour, so that it is small when the viscosity is small, unless the space rates of change of the vorticity components are large.

In an ideal fluid in which ν is taken to be zero, we have Kelvin's theorem that the circulation remains constant. From this result and Stokes's theorem it is possible to give an alternative proof that vortex lines move with the fluid and the strength of a narrow vortex tube remains constant (Lamb 1932, pp. 203–4). In fact, (17) and (19) are equivalent to each other, and either may be deduced directly from the other.

8. The energy equation

In incompressible motion the equations of continuity and momentum (together with appropriate boundary conditions) are sufficient to determine the pressure and the three components of velocity. Essentially, the energy equation is replaced by the assumption ρ = constant. In any real flow there will be small density and temperature variations and these become determinate when the energy equation and an equation of state between p, ρ, and T are introduced. In the approximation of 'incompressible motion', the equations of continuity and momentum are solved first, neglecting the small variations in density, then the values of $\mathbf{v}$ and p obtained from them may be used in the energy equation to determine the temperature distribution in the flow. In liquids, the density changes may be very much smaller than temperature changes, but in gases (to which the 'incompressible' theory applies when the velocities are small compared to the sound speed) they will be comparable. Thus, for the energy equation, changes in ρ must be

included; however, their explicit appearance may be eliminated in the final form.

First, the rate of dissipation of energy by the viscous forces will be obtained. We consider again a fluid element which at time t occupies a rectangular parallelepiped centred at $P(x, y, z)$ with edges parallel to the coordinate axes. The net rate at which the stresses are doing work on this element is

$$\frac{\partial}{\partial x}(p_{xx}u+p_{xy}v+p_{xz}w)+\frac{\partial}{\partial y}(p_{yx}u+p_{yy}v+p_{yz}w)+\frac{\partial}{\partial z}(p_{zx}u+p_{zy}v+p_{zz}w) \tag{20}$$

per unit volume, and the rate of working of the extraneous forces is

$$\rho(Xu+Yv+Zw) \tag{21}$$

per unit volume. The kinetic energy of this element is increasing at the rate

$$\rho\,\frac{D}{Dt}\{\tfrac{1}{2}(u^2+v^2+w^2)\} \tag{22}$$

per unit volume. Using equation (10), it is found that the total rate of working of the forces, which is equal to (20) plus (21), exceeds the rate of increase of the kinetic energy by

$$\tfrac{1}{2}\{p_{xx}e_{xx}+p_{yy}e_{yy}+p_{zz}e_{zz}+2p_{yz}e_{yz}+2p_{zx}e_{zx}+2p_{xy}e_{xy}\}. \tag{23}$$

(It should be noted that the assumption ρ = constant is not made in the derivation of the momentum equation (10).) Introducing the relations between stress and rate-of-strain components given by (8), (23) can be expressed as

$$-p\Delta+\Phi, \tag{24}$$

where

$$\Phi = \tfrac{1}{2}\mu\{e_{xx}^2+e_{yy}^2+e_{zz}^2+2e_{yz}^2+2e_{zx}^2+2e_{xy}^2\}, \tag{25}$$

and Δ is the dilatation $\tfrac{1}{2}(e_{xx}+e_{yy}+e_{zz})$. Although Δ may be set equal to zero as far as the equations of continuity and momentum are concerned, for gases the term $p\Delta$ is not small compared with Φ. Now, $-p\Delta$ represents the rate at which work is done in compressing the element of fluid, since Δ is the rate of increase of volume per unit volume. Hence Φ given by (25) is the rate of dissipation of energy per unit time per unit volume by the viscous forces.

As noted in Section 3, for compressible flow a multiple of Δ is added to the first three equations of (8), and this leads to an additional term proportional to Δ^2 in the expression for Φ (see Howarth 1953). But for 'incompressible' flow, this term can be neglected.

In addition to the work done by the stresses acting on the fluid element, heat energy is being conducted across its boundaries. If the thermal conductivity is k and we neglect its variation with temperature, the net loss of heat by conduction is

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) = k\nabla^2 T \tag{26}$$

per unit volume. The difference of (24) and (26) is equal to the rate at which the internal energy is increasing. Thus, if E denotes the internal energy per unit mass,

$$\rho\frac{DE}{Dt} = -p\Delta+\Phi+k\nabla^2 T. \tag{27}$$

Introducing the entropy S and the enthalpy $I = E+p/\rho$, this equation may be written in the alternative forms

$$\rho T\frac{DS}{Dt} = \rho\frac{DI}{Dt}-\frac{Dp}{Dt} = \Phi+k\nabla^2 T. \tag{28}$$

Equation (28) gives explicitly the rate of increase of entropy due to viscosity and heat conduction.

In terms of pressure and temperature, the enthalpy is given by

$$dI = C_p\,dT+\left\{1+\frac{T}{\rho}\left(\frac{\partial\rho}{\partial T}\right)_p\right\}\frac{dp}{\rho};$$

for this and other thermodynamical results introduced here, reference may be made to the discussion given by Howarth (1953, chapter 2). An immediate consequence of the result quoted above is that (28) may be written

$$\rho C_p\frac{DT}{Dt}-\frac{T}{\rho}\left(\frac{\partial\rho}{\partial T}\right)_p\frac{Dp}{Dt} = \Phi+k\nabla^2 T. \tag{29}$$

For liquids, the coefficient of expansion, $-\rho^{-1}(\partial\rho/\partial T)_p$, is usually very small and as a consequence the term in Dp/Dt can be omitted. For a perfect gas, in which $p \propto \rho T$, we have $-T\rho^{-1}(\partial\rho/\partial T)_p = 1$. In either case, derivatives of the density do not occur, and in the coefficient of the first term ρ may now be taken as a constant. Thus, when the velocity and pressure have been determined from the equations of continuity and momentum, we have a linear equation for the temperature.

In an important class of problem, the temperature variations are principally due to applied heating (e.g. the wall of a body may be maintained at a given temperature) and the terms involving Dp/Dt and Φ may then be neglected in (29). For liquids the small coefficient of Dp/Dt makes the term negligible in any case, but for gases the coefficient is not small (being unity for a perfect gas) and we must consider the

magnitude of Dp/Dt. From the inertia terms in the momentum equations we see that the pressure changes are $O(\rho U_1^2)$, where U_1 is a typical value of the main-stream velocity. Hence, if T_1 is a typical main-stream temperature, Dp/Dt is small compared with $\rho C_p\,DT/Dt$ if the temperature changes are much greater than U_1^2/C_p. Since $C_p T_1$ is proportional to the square of the sound speed a_1, the condition becomes

$$\frac{T-T_1}{T_1} \gg M_1^2, \tag{30}$$

where M_1 is the Mach number U_1/a_1.

If we estimate the order of magnitude of Φ by the value of $\mu(\partial u/\partial y)^2$ in a boundary layer (where the viscous dissipation is greatest), we see that it is of the order $\mu U_1^2/\delta^2 \approx \rho U_1^3/l$, where δ is the boundary-layer thickness proportional to $(\mu l/\rho U_1)^{\frac{1}{2}}$ and l is a typical length in the stream direction. Thus Φ is also small compared to $\rho C_p\,DT/Dt$ if the relative temperature changes are large compared to M_1^2. Therefore, when (30) is satisfied, we have both for gases and liquids the approximate equation

$$\frac{DT}{Dt} = \frac{k}{\rho C_p}\nabla^2 T = \frac{\nu}{\sigma}\nabla^2 T, \tag{31}$$

where σ is the Prandtl number $\mu C_p/k$. Of course $(T-T_1)/T_1$ must still be small if we are to neglect density changes in the equations of continuity and momentum and the variation of k with T.

9. Dynamical similarity

It has been shown in Chapter I that, as far as the effects of viscosity are concerned, the determining parameter for geometrically similar flow patterns is the Reynolds number. We may now verify this in detail for equations (4) and (12) which determine $\mathbf{v}$ and p. It is assumed that either the external forces are negligible or they are conservative and have been absorbed into the pressure terms in (12) so that p measures the difference of the pressure from its hydrostatic value $-\Omega/\rho$.

We consider the flow of a uniform stream parallel to the x-axis with velocity U past a fixed obstacle of given shape and given orientation whose size is specified by a typical length d. Then we introduce non-dimensional variables by the following scheme:

$$\mathrm{U} = u/U, \quad \mathrm{V} = v/U, \quad \mathrm{W} = w/U,$$
$$\mathrm{X} = x/d, \quad \mathrm{Y} = y/d, \quad \mathrm{Z} = z/d, \quad \mathrm{T} = Ut/d,$$
$$\mathrm{P} = p/\rho U^2.$$

The equation of continuity takes the same form in the new variables, that is,

$$\frac{\partial \mathrm{U}}{\partial \mathrm{X}}+\frac{\partial \mathrm{V}}{\partial \mathrm{Y}}+\frac{\partial \mathrm{W}}{\partial \mathrm{Z}}=0, \tag{32}$$

while the momentum equations (12) become

$$\frac{\partial \mathrm{U}}{\partial \mathrm{T}}+\mathrm{U}\frac{\partial \mathrm{U}}{\partial \mathrm{X}}+\mathrm{V}\frac{\partial \mathrm{U}}{\partial \mathrm{Y}}+\mathrm{W}\frac{\partial \mathrm{U}}{\partial \mathrm{Z}}=-\frac{\partial \mathrm{P}}{\partial \mathrm{X}}+\frac{1}{R}\left(\frac{\partial^2 \mathrm{U}}{\partial \mathrm{X}^2}+\frac{\partial^2 \mathrm{U}}{\partial \mathrm{Y}^2}+\frac{\partial^2 \mathrm{U}}{\partial \mathrm{Z}^2}\right) \tag{33}$$

with two similar equations, where R is the Reynolds number Ud/ν. The boundary conditions $u = U$, $v = 0$, $w = 0$ at infinity, and

$$u = v = w = 0$$

at the boundary of the solid obstacle, become $\mathrm{U} = 1$, $\mathrm{V} = 0$, $\mathrm{W} = 0$ at infinity, and $\mathrm{U} = \mathrm{V} = \mathrm{W} = 0$ at a fixed surface, independent of d, in the (X,Y,Z) space. With these boundary conditions, (32) and the equations of type (33) determine U, V, W, and P. Thus, for fluids of different densities and coefficients of viscosity, for streams of different speeds and obstacles of different sizes, so long as R is the same, u/U, v/U, w/U, and $p/\rho U^2$ will be functions of x/d, y/d, z/d only. Since

$$p_{xx}=-p+2\mu\frac{\partial u}{\partial x}=\rho U^2\left(-\mathrm{P}+\frac{2}{R}\frac{\partial \mathrm{U}}{\partial \mathrm{X}}\right),$$

$$p_{yz}=\mu\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)=\rho\frac{U^2}{R}\left(\frac{\partial \mathrm{W}}{\partial \mathrm{Y}}+\frac{\partial \mathrm{V}}{\partial \mathrm{Z}}\right),$$

the same is true for any stress component divided by ρU^2. Any velocity component divided by U, and any stress component divided by ρU^2 is a function of x/d, y/d, z/d, and R only. Again, the component along the axis of x, for example, of the force on the obstacle (apart from the force of buoyancy) is $\int p_{nx}\,dS$, and since, for a given value of R, p_{nx} varies as ρU^2, this force component will vary as $\rho U^2 S$, where S is some representative area associated with the obstacle. The same is true for any other force component; hence, any force component divided by $\rho U^2 S$ is a function of R alone.

When the temperature is determined by (31) it is clear that, in addition to the Reynolds number, the Prandtl number appears as a similarity parameter for the temperature field.

10. The stream function

In two-dimensional flow the equation of continuity is

$$\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}=0, \tag{34}$$

and it may be solved by introducing the stream function ψ such that

$$u=\frac{\partial \psi}{\partial z}, \qquad w=-\frac{\partial \psi}{\partial x}. \tag{35}$$

The only non-zero component of vorticity is η, and, in terms of ψ, it becomes

$$\eta=\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}=\nabla^2\psi. \tag{36}$$

The equation of motion for ψ may then be established from the appropriate vorticity equation which reduces, since there is no stretching of vortex lines in this case, to

$$\frac{D\eta}{Dt}=\nu\nabla^2\eta. \tag{37}$$

Thus,

$$\frac{\partial}{\partial t}(\nabla^2\psi)-\frac{\partial(\psi,\nabla^2\psi)}{\partial(x,z)}=\nu\nabla^4\psi. \tag{38}$$

When coordinate systems other than the Cartesian one are considered, the continuity equation both for two-dimensional flow and for axi-symmetric flow may still be solved by the introduction of a stream function. The details and the equations corresponding to (36) and (38) are included in the next section.

11. General orthogonal coordinates

The invariant vector form of the equations of motion, that is, equation (15) for the momentum equation and $\operatorname{div}\mathbf{v}=0$ for the continuity equation, apply for any coordinate system. In order to expand them in component form for any particular system we require the formulae for the gradient of a scalar and the divergence and curl of a vector in that system. Here, we write out these formulae for general orthogonal coordinates, x_1, x_2, x_3.

Let the elements of length at (x_1, x_2, x_3) in the directions of increasing x_1, x_2, and x_3 respectively, be $h_1\,dx_1$, $h_2\,dx_2$, and $h_3\,dx_3$. Let (a_1, a_2, a_3) denote the components of a vector $\mathbf{a}$ in the directions of increasing x_1, x_2, and x_3, respectively. Then (see, for example, Weatherburn (1944), pp. 28–30, or Love (1927), pp. 51–55, but note that the quantities h_1, h_2, h_3 used by Love are the reciprocals of those employed here),

$$\operatorname{div}\mathbf{a}=\frac{1}{h_1h_2h_3}\left\{\frac{\partial}{\partial x_1}(h_2h_3a_1)+\frac{\partial}{\partial x_2}(h_3h_1a_2)+\frac{\partial}{\partial x_3}(h_1h_2a_3)\right\}, \tag{39}$$

and the components of $\mathbf{b}=\operatorname{curl}\mathbf{a}$ are given by

$$b_1=\frac{1}{h_2h_3}\left\{\frac{\partial}{\partial x_2}(h_3a_3)-\frac{\partial}{\partial x_3}(h_2a_2)\right\}, \text{ etc.} \tag{40}$$

The components of the gradient of a scalar ϕ are

$$\frac{1}{h_1}\frac{\partial\phi}{\partial x_1}, \qquad \frac{1}{h_2}\frac{\partial\phi}{\partial x_2}, \qquad \frac{1}{h_3}\frac{\partial\phi}{\partial x_3}. \tag{41}$$

With these results the equation of continuity and the three components of (15) may be obtained for any system of orthogonal coordinates. However, since $\operatorname{div}\mathbf{v} = 0$ for incompressible flow, there is an indeterminateness in the equations because expressions involving $\operatorname{div}\mathbf{v}$ may be added. In fact it is usual to retain the form

$$\nu(\operatorname{grad}\operatorname{div}\mathbf{v} - \operatorname{curl}\boldsymbol{\omega})$$

in working out the last term of (15), since this has components $\nabla^2 u$, $\nabla^2 v$, $\nabla^2 w$, in Cartesian coordinates.

It should be noted that in general orthogonal coordinates the components of $\nabla^2\mathbf{a}$ have terms in addition to $\nabla^2 a_1$, $\nabla^2 a_2$, $\nabla^2 a_3$ (where $\nabla^2 a_1 = \operatorname{div}\operatorname{grad} a_1$, etc.); the correct expressions may be obtained using

$$\nabla^2\mathbf{a} = \operatorname{grad}\operatorname{div}\mathbf{a} - \operatorname{curl}\operatorname{curl}\mathbf{a}. \tag{42}$$

This must be used, for example, in deriving the appropriate form of the vorticity equations from (16). Similarly, the vector, usually denoted by $(\mathbf{b}.\nabla)\mathbf{a}$, defined by the components

$$(\mathbf{b}.\operatorname{grad} a_x,\ \mathbf{b}.\operatorname{grad} a_y,\ \mathbf{b}.\operatorname{grad} a_z)$$

in Cartesian coordinates, cannot be expressed immediately in general orthogonal coordinates. We do, in fact, avoid using the formulae for this vector, but since they are often useful and do not appear in the usual reference books, we note the results here. The x_1 component is

$$\mathbf{b}.\operatorname{grad} a_1 + \frac{b_1}{h_1}\left(\frac{a_1}{h_1}\frac{\partial h_1}{\partial x_1} + \frac{a_2}{h_2}\frac{\partial h_1}{\partial x_2} + \frac{a_3}{h_3}\frac{\partial h_1}{\partial x_3}\right) - \\ - \left(\frac{a_1 b_1}{h_1^2}\frac{\partial h_1}{\partial x_1} + \frac{a_2 b_2}{h_2 h_1}\frac{\partial h_2}{\partial x_1} + \frac{a_3 b_3}{h_3 h_1}\frac{\partial h_3}{\partial x_1}\right), \tag{43}$$

and the other components are given by similar expressions.

We now consider the expressions for the rate-of-strain components in general orthogonal coordinates. For Cartesian coordinates with axes in the directions of x_1 increasing, x_2 increasing, and x_3 increasing at P, the rate-of-strain components would be defined exactly as in (2). Thus, even for non-Cartesian orthogonal coordinates, $\frac{1}{2}e_{11}$ is still the rate of extension of a line element in the direction of x_1 increasing, and e_{23} is still the rate of change of the angle between two lines, moving with the fluid, drawn in the directions of x_2 increasing and x_3 increasing. But the directions of x_1 increasing, x_2 increasing, and x_3 increasing are

different at (x_1, x_2, x_3) and at any neighbouring point, so that additional terms are introduced when the rate-of-strain components are expressed in terms of the derivatives of the velocity components. These expressions are (Love (1927), pp. 53, 57)

$$\tfrac{1}{2}e_{11} = \frac{1}{h_1}\frac{\partial v_1}{\partial x_1} + \frac{v_2}{h_1 h_2}\frac{\partial h_1}{\partial x_2} + \frac{v_3}{h_3 h_1}\frac{\partial h_1}{\partial x_3},$$

$$e_{23} = \frac{h_3}{h_2}\frac{\partial}{\partial x_2}\left(\frac{v_3}{h_3}\right) + \frac{h_2}{h_3}\frac{\partial}{\partial x_3}\left(\frac{v_2}{h_2}\right), \tag{44}$$

and the other components are given by similar expressions. The stress components are still given by

$$p_{11} = -p + \mu e_{11}, \qquad p_{23} = p_{32} = \mu e_{23}, \quad \text{etc.}, \tag{45}$$

where p_{23}, for example, is the component in the direction of x_3 increasing of the stress exerted at (x_1, x_2, x_3) across the surface $x_2 = \text{constant}$.

For two-dimensional motion, if x_1 and x_3 are general orthogonal coordinates in the plane of motion, the equation of continuity $\operatorname{div}\mathbf{v} = 0$ becomes

$$\frac{\partial}{\partial x_1}(h_3 v_1) + \frac{\partial}{\partial x_3}(h_1 v_3) = 0.$$

Therefore, we can introduce a stream function ψ such that

$$v_1 = \frac{1}{h_3}\frac{\partial\psi}{\partial x_3}, \qquad v_3 = -\frac{1}{h_1}\frac{\partial\psi}{\partial x_1}.$$

The non-zero component of vorticity, ω_2, is then given by

$$\omega_2 = \frac{1}{h_1 h_3}\left[\frac{\partial}{\partial x_1}\left(\frac{h_3}{h_1}\frac{\partial\psi}{\partial x_1}\right) + \frac{\partial}{\partial x_3}\left(\frac{h_1}{h_3}\frac{\partial\psi}{\partial x_3}\right)\right] = \nabla^2\psi.$$

Substituting this value in the third component of the vorticity equation (16), we have the equation for ψ:

$$\frac{\partial}{\partial t}(\nabla^2\psi) - \frac{1}{h_1 h_3}\frac{\partial(\psi, \nabla^2\psi)}{\partial(x_1, x_3)} = \nu\nabla^4\psi.$$

For axisymmetric flow, if x_1 and x_3 are general orthogonal coordinates in a meridian plane and x_2 is the azimuthal angle, and if all quantities are supposed independent of x_2, the equation of continuity is

$$\frac{\partial}{\partial x_1}(h_2 h_3 v_1) + \frac{\partial}{\partial x_3}(h_1 h_2 v_3) = 0,$$

where h_2 measures distance from the axis of symmetry. Again a stream function may be introduced and in this case we take

$$h_2 v_1 = \frac{1}{h_3}\frac{\partial\psi}{\partial x_3}, \qquad h_2 v_3 = -\frac{1}{h_1}\frac{\partial\psi}{\partial x_1}.$$

This applies whether the velocity v_2 round the axis is zero or not, so long as it is independent of x_2. If it is not zero, we put $h_2 v_2 = \Omega$. Then

$$\omega_1 = -\frac{1}{h_2 h_3}\frac{\partial\Omega}{\partial x_3}, \qquad \omega_3 = \frac{1}{h_2 h_1}\frac{\partial\Omega}{\partial x_1},$$

$$\omega_2 = \frac{1}{h_1 h_3}\left[\frac{\partial}{\partial x_1}\left(\frac{h_3}{h_2 h_1}\frac{\partial\psi}{\partial x_1}\right)+\frac{\partial}{\partial x_3}\left(\frac{h_1}{h_2 h_3}\frac{\partial\psi}{\partial x_3}\right)\right] = \frac{1}{h_2}D^2\psi,$$

where $$D^2 = \frac{h_2}{h_1 h_3}\left[\frac{\partial}{\partial x_1}\left(\frac{h_3}{h_2 h_1}\frac{\partial}{\partial x_1}\right)+\frac{\partial}{\partial x_3}\left(\frac{h_1}{h_2 h_3}\frac{\partial}{\partial x_3}\right)\right].$$

The x_2 component of the equation for the vorticity gives

$$\frac{\partial}{\partial t}(D^2\psi)+\frac{2\Omega}{h_1 h_3 h_2^2}\frac{\partial(\Omega, h_2)}{\partial(x_1, x_3)}-\frac{1}{h_1 h_2 h_3}\frac{\partial(\psi, D^2\psi)}{\partial(x_1, x_3)}+\frac{2D^2\psi}{h_1 h_3 h_2^2}\frac{\partial(\psi, h_2)}{\partial(x_1, x_3)} = \nu D^4\psi.$$

If $v_2 = 0$ then $\Omega = 0$ and this is an equation for ψ. Otherwise, we require another equation, which is provided by the second component of the momentum equation. This gives

$$\frac{\partial\Omega}{\partial t}-\frac{1}{h_1 h_2 h_3}\frac{\partial(\psi,\Omega)}{\partial(x_1, x_3)} = \nu D^2\Omega.$$

12. Cylindrical polar coordinates

With cylindrical polar coordinates r, θ, z such that

$$x = r\cos\theta, \qquad y = r\sin\theta,$$

if r, θ, z are taken as x_1, x_2, x_3, respectively, then

$$h_1 = 1, \qquad h_2 = r, \qquad h_3 = 1.$$

Hence, using subscripts r, θ, z for the components of vectors and tensors, we have

$$\operatorname{div}\mathbf{v} = \frac{1}{r}\frac{\partial}{\partial r}(rv_r)+\frac{1}{r}\frac{\partial v_\theta}{\partial\theta}+\frac{\partial v_z}{\partial z}, \tag{46}$$

$$\omega_r = \frac{1}{r}\frac{\partial v_z}{\partial\theta}-\frac{\partial v_\theta}{\partial z}, \qquad \omega_\theta = \frac{\partial v_r}{\partial z}-\frac{\partial v_z}{\partial r}, \qquad \omega_z = \frac{1}{r}\frac{\partial}{\partial r}(rv_\theta)-\frac{1}{r}\frac{\partial v_r}{\partial\theta}, \tag{47}$$

$$\tfrac{1}{2}e_{rr} = \frac{\partial v_r}{\partial r}, \qquad \tfrac{1}{2}e_{\theta\theta} = \frac{1}{r}\frac{\partial v_\theta}{\partial\theta}+\frac{v_r}{r}, \qquad \tfrac{1}{2}e_{zz} = \frac{\partial v_z}{\partial z},$$

$$e_{\theta z} = \frac{1}{r}\frac{\partial v_z}{\partial\theta}+\frac{\partial v_\theta}{\partial z}, \qquad e_{zr} = \frac{\partial v_r}{\partial z}+\frac{\partial v_z}{\partial r}, \qquad e_{r\theta} = r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right)+\frac{1}{r}\frac{\partial v_r}{\partial\theta}, \tag{48}$$

and the momentum equations are

$$\frac{\partial v_r}{\partial t}+v_r\frac{\partial v_r}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_r}{\partial\theta}+v_z\frac{\partial v_r}{\partial z}-\frac{v_\theta^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r}+\nu\left(\nabla^2 v_r-\frac{v_r}{r^2}-\frac{2}{r^2}\frac{\partial v_\theta}{\partial\theta}\right),$$

$$\frac{\partial v_\theta}{\partial t}+v_r\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial\theta}+v_z\frac{\partial v_\theta}{\partial z}+\frac{v_r v_\theta}{r} = -\frac{1}{\rho}\frac{\partial p}{r\partial\theta}+\nu\left(\nabla^2 v_\theta+\frac{2}{r^2}\frac{\partial v_r}{\partial\theta}-\frac{v_\theta}{r^2}\right),$$

$$\frac{\partial v_z}{\partial t}+v_r\frac{\partial v_z}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_z}{\partial\theta}+v_z\frac{\partial v_z}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z}+\nu\nabla^2 v_z, \tag{49}$$

where $$\nabla^2 = \frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+\frac{\partial^2}{\partial z^2}.$$

13. Spherical polar coordinates

With spherical polar coordinates (r, θ, λ) such that

$$x = r\sin\theta\cos\lambda, \qquad y = r\sin\theta\sin\lambda, \qquad z = r\cos\theta,$$

if r, θ, λ are taken as (x_1, x_2, x_3) respectively, then

$$h_1 = 1, \qquad h_2 = r, \qquad h_3 = r\sin\theta.$$

Hence, using subscripts r, θ, λ for the components of vectors and tensors, we have

$$\operatorname{div}\mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(v_\theta\sin\theta)+\frac{1}{r\sin\theta}\frac{\partial v_\lambda}{\partial\lambda}, \tag{50}$$

$$\omega_r = \frac{1}{r\sin\theta}\left\{\frac{\partial}{\partial\theta}(v_\lambda\sin\theta)-\frac{\partial v_\theta}{\partial\lambda}\right\},$$

$$\omega_\theta = \frac{1}{r\sin\theta}\frac{\partial v_r}{\partial\lambda}-\frac{1}{r}\frac{\partial}{\partial r}(rv_\lambda),$$

$$\omega_\lambda = \frac{1}{r}\frac{\partial}{\partial r}(rv_\theta)-\frac{1}{r}\frac{\partial v_r}{\partial\theta}, \tag{51}$$

$$\tfrac{1}{2}e_{rr} = \frac{\partial v_r}{\partial r}, \qquad \tfrac{1}{2}e_{\theta\theta} = \frac{1}{r}\frac{\partial v_\theta}{\partial\theta}+\frac{v_r}{r},$$

$$\tfrac{1}{2}e_{\lambda\lambda} = \frac{1}{r\sin\theta}\frac{\partial v_\lambda}{\partial\lambda}+\frac{v_r}{r}+\frac{v_\theta\cot\theta}{r},$$

$$e_{\theta\lambda} = \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\frac{v_\lambda}{\sin\theta}\right)+\frac{1}{r\sin\theta}\frac{\partial v_\theta}{\partial\lambda},$$

$$e_{\lambda r} = \frac{1}{r\sin\theta}\frac{\partial v_r}{\partial\theta}+r\frac{\partial}{\partial r}\left(\frac{v_\lambda}{r}\right),$$

$$e_{r\theta} = r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right)+\frac{1}{r}\frac{\partial v_r}{\partial\theta}, \tag{52}$$

and the momentum equations are

$$\frac{\partial v_r}{\partial t}+v_r\frac{\partial v_r}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}+\frac{v_\lambda}{r\sin\theta}\frac{\partial v_r}{\partial \lambda}-\frac{v_\theta^2+v_\lambda^2}{r}$$

$$=-\frac{1}{\rho}\frac{\partial p}{\partial r}+\nu\left(\nabla^2 v_r-\frac{2v_r}{r^2}-\frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}-\frac{2v_\theta\cot\theta}{r^2}-\frac{2}{r^2\sin\theta}\frac{\partial v_\lambda}{\partial \lambda}\right),$$

$$\frac{\partial v_\theta}{\partial t}+v_r\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}+\frac{v_\lambda}{r\sin\theta}\frac{\partial v_\theta}{\partial \lambda}+\frac{v_r v_\theta}{r}-\frac{v_\lambda^2\cot\theta}{r}$$

$$=-\frac{1}{\rho}\frac{\partial p}{r\partial \theta}+\nu\left(\nabla^2 v_\theta+\frac{2}{r^2}\frac{\partial v_r}{\partial \theta}-\frac{v_\theta}{r^2\sin^2\theta}-\frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial v_\lambda}{\partial \lambda}\right),$$

$$\frac{\partial v_\lambda}{\partial t}+v_r\frac{\partial v_\lambda}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\lambda}{\partial \theta}+\frac{v_\lambda}{r\sin\theta}\frac{\partial v_\lambda}{\partial \lambda}+\frac{v_\lambda v_r}{r}+\frac{v_\theta v_\lambda\cot\theta}{r}$$

$$=-\frac{1}{\rho}\frac{1}{r\sin\theta}\frac{\partial p}{\partial \lambda}+\nu\left(\nabla^2 v_\lambda-\frac{v_\lambda}{r^2\sin^2\theta}+\frac{2}{r^2\sin\theta}\frac{\partial v_r}{\partial \lambda}+\frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial v_\theta}{\partial \lambda}\right), \tag{53}$$

where $$\nabla^2\equiv\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial}{\partial \theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial \lambda^2}.$$

PART II

SOME EXACT SOLUTIONS

14. Solutions for which the convection terms vanish

The fundamental difficulty in solving the Navier–Stokes equations (either exactly or approximately) is the non-linearity introduced by the convection terms in the momentum equations (12). There exist, however, non-trivial flows in which the convection terms vanish, and these provide the simplest class of solutions of the equations of motion.

For the equations in Cartesian form, such solutions are obtained by taking all except one of the velocity components equal to zero. If we take $v = w = 0$, an immediate consequence of the continuity equation is that u is independent of x, and it then follows that all the convection terms in (12) vanish. Assuming, as in the remainder of this chapter, that any external field of force may be accounted for by measuring p

from its hydrostatic value, as in Section 9, the equations become

$$\frac{\partial u}{\partial t}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right), \tag{54}$$

$$\frac{\partial p}{\partial y}=0,\qquad \frac{\partial p}{\partial z}=0. \tag{55}$$

Since u is independent of x, we see that $\partial p/\partial x$ must be a function of t alone. This pressure gradient may be prescribed as an arbitrary function of t, then $u(y,z,t)$ is determined by solving the *linear* equation (54). It should be noted that (54) is identical with the equation for heat conduction in two dimensions if the term $-\rho^{-1}\partial p/\partial x$ is interpreted as a uniform distribution of heat sources. Thus, known solutions in the theory of heat conduction may be taken over directly and interpreted as fluid flows.

It is clear that the flows to which this theory applies are parallel to cylindrical surfaces whose generators are in the x-direction. There are two main problems: (i) steady flows through pipes of uniform cross-section with constant pressure gradient, and (ii) unsteady flows produced by the motion of a solid boundary in the x-direction.

In problem (i), $u(y,z)$ satisfies Poisson's equation,

$$\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=\frac{1}{\mu}\frac{dp}{dx}, \tag{56}$$

and the boundary condition $u = 0$ at the walls of the pipe. It is clear that in all cases u may be expressed as $-f(y,z)\mu^{-1}dp/dx$, where $f(y,z)$ depends only on the cross-sectional shape; similarly, the volume flux takes the form $-C\mu^{-1}dp/dx$. The problem can be solved analytically for several special shapes of cross-section, and for a detailed account of these, reference may be made to Love (1927, chapter xiv), where the analogous problem of the torsion of bars of various cross-sections is considered. Here, we note the results for the more important cases.

(*a*) Two-dimensional channel $-c \leqslant z \leqslant c$:

$$f=\tfrac{1}{2}(c^2-z^2),\qquad C=\tfrac{2}{3}c^3.$$

(In this case, of course, C corresponds to the volume flux per unit width.)

(*b*) Circular section of radius c:

$$f=\tfrac{1}{4}(c^2-r^2),\qquad C=\tfrac{1}{8}\pi c^4,$$

where $r^2 = y^2+z^2$.

(c) Annular section $b \leqslant r \leqslant c$:

$$f = \frac{1}{4}\left(b^2 - r^2 + \frac{c^2-b^2}{\log c/b}\log\frac{r}{b}\right), \qquad C = \tfrac{1}{8}\pi\left\{c^4 - b^4 - \frac{(c^2-b^2)^2}{\log c/b}\right\}.$$

(d) Elliptic section $y^2/b^2 + z^2/c^2 \leqslant 1$:

$$f = \frac{b^2c^2}{2(b^2+c^2)}\left(1 - \frac{y^2}{b^2} - \frac{z^2}{c^2}\right), \qquad C = \tfrac{1}{4}\pi\frac{b^3c^3}{b^2+c^2}.$$

(e) Rectangular section $-b \leqslant y \leqslant b$, $-c \leqslant z \leqslant c$:

$$f = \tfrac{1}{2}b^2 - \tfrac{1}{2}y^2 - 2b^2\left(\frac{2}{\pi}\right)^3 \sum_0^\infty \frac{(-1)^n}{(2n+1)^3}\frac{\cosh(2n+1)(\pi z/2b)}{\cosh(2n+1)(\pi c/2b)}\cos(2n+1)\frac{\pi y}{2b},$$

$$C = \tfrac{4}{3}cb^3 - 8b^4\left(\frac{2}{\pi}\right)^5 \sum_0^\infty \frac{1}{(2n+1)^5}\tanh(2n+1)\frac{\pi c}{2b}.$$

When $b = c$, $C = 0{\cdot}5623b^4$; when $c > 3b$,

$$C \sim cb^3\left(\tfrac{4}{3} - 0{\cdot}840\frac{b}{c}\right).$$

When the cross-section is not one of the special shapes for which an analytic solution can be found, the required results may be obtained by making certain measurements on soap films. For, if a soap film is stretched across a hole of the given shape and has a small excess pressure p on one side of it, then the displacement $X(y,z)$ satisfies

$$2T\left(\frac{\partial^2 X}{\partial y^2} + \frac{\partial^2 X}{\partial z^2}\right) + p = 0$$

(where T is the surface tension), together with the boundary condition $X = 0$ on the edges of the hole. Therefore,

$$X = \frac{p}{2T}f(y,z)$$

and is proportional to u. Thus, the velocity distribution can be deduced from measurements of the displacement X. The volume flux is proportional to $\iint X\,dydz$ and this is found by measuring the total volume under the soap film. These measurements are much more easily made than direct measurements of the velocity in the fluid-flow problem or of the displacement in the torsion problem. The experimental technique is described by G. I. Taylor (1937a).

Perhaps the simplest case of problem (ii) is that of an infinite plate which, starting at $t = 0$, is moved in its own plane with constant velocity U through fluid initially at rest. If the plate lies in the plane

$z = 0$, $u(z,t)$ satisfies

$$\frac{\partial u}{\partial t} = \nu\frac{\partial^2 u}{\partial z^2},$$

and the appropriate boundary and initial conditions are $u(0,t) = U$ ($t > 0$), and $u(z,0) = 0$. The analogous problem in the theory of heat conduction is well known and the solution is (Carslaw and Jaeger 1947, p. 43)

$$u = U\left(1 - \operatorname{erf}\frac{z}{2\sqrt{(\nu t)}}\right), \tag{57}$$

where

$$\operatorname{erf}\zeta = \frac{2}{\sqrt{\pi}}\int_0^\zeta e^{-\lambda^2}\,d\lambda. \tag{58}$$

In diffusion problems there is, of course, no true propagation speed, but if the boundary-layer thickness is defined as the distance in which u drops to a given small fraction of U, we may determine its rate of growth. From (57) we see that this boundary-layer thickness is proportional to $\sqrt{(\nu t)}$, and it grows at a rate proportional to $\sqrt{(\nu/t)}$.

Rayleigh (1911) suggested that this comparatively simple solution could be used to give an approximate solution to the problem of steady flow along a semi-infinite flat plate, the flow direction being perpendicular to the edge of the plate. The basic idea is that the disturbance due to the plate spreads out into the stream at the rate given by the unsteady problem, but at the same time it is swept downstream with the fluid. As a rough approximation, it is assumed that the disturbance is convected downstream with the main stream velocity U. Thus, at a distance x along the plate from the leading edge, the boundary-layer thickness and velocity distribution can be found by identifying t in (57) with x/U. Modifying (57) so that u measures downstream velocity relative to the plate, this gives

$$u = U\operatorname{erf}\tfrac{1}{2}z\sqrt{\left(\frac{U}{x\nu}\right)}. \tag{59}$$

Therefore, the boundary-layer thickness is proportional to $\sqrt{(\nu x/U)}$, and the skin friction is

$$\tau = \mu\left(\frac{\partial u}{\partial z}\right)_{z=0} = \frac{\rho U^2}{\sqrt{\pi}}\sqrt{\left(\frac{\nu}{xU}\right)}. \tag{60}$$

Actually, the momentum equation for the x-direction in the steady flow past a flat plate is

$$u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} = \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2}\right),$$

and the physical argument given above is equivalent mathematically to the following approximations in this equation. Firstly, the convection terms on the left are replaced by the approximation $U\,\partial u/\partial x$, and, secondly, the term $\nu\,\partial^2 u/\partial x^2$ is neglected in the viscous terms on the right. In this way, we obtain the diffusion equation for u with t replaced by x/U. The boundary-layer approximation retains the convection terms in full and makes only the second simplification.

The Rayleigh approximation obviously overestimates the convection effects; hence, its prediction of the boundary-layer thickness will be too small and the value of τ too great. As later investigations show (see Section V. 12), the accurate value for τ, obtained from boundary-layer theory, is $0{\cdot}332\rho U^2\sqrt{(\nu/xU)}$, which corresponds to identifying t with $x/(0{\cdot}346U)$. But, in spite of its approximate nature, this so-called 'Rayleigh analogy' offers a method of estimating the skin friction in steady problems for which the accurate formulation proves intractable. As examples of unsteady problems which have been used for this purpose, we may mention the flows resulting from the impulsive motions of a semi-infinite plate parallel to its edge (Howarth 1950), a wedge parallel to its edge (Hasimoto 1951, Sowerby and Cooke 1953), and cylinders of finite cross-section parallel to their generators (Batchelor 1954*a*). These will be discussed in Chapter VII, and again in Chapter VIII where the steady-flow 'analogies' are required.

For the motion of the infinite plate, the solution can be obtained for any prescribed variation of the velocity of the plate with time. One case of special interest arises when the plate oscillates periodically, that is, $u = A\cos\omega t$ at $z = 0$. The solution satisfying this boundary condition is

$$u = Ae^{-kz}\cos(\omega t - kz), \qquad k = \sqrt{(\omega/2\nu)}. \tag{61}$$

It represents waves spreading out from the plate with velocity

$$\omega/k = \sqrt{(2\nu\omega)}$$

and amplitude decaying exponentially with z. When ν is small the damping is heavy and the disturbance is then confined mainly to a thin boundary layer near the plate with thickness of order $\sqrt{(\nu/\omega)}$.

There are, of course, many other solutions of the heat equation which may be applied to fluid flows, but we do not attempt a complete survey here. (Additional examples are given in Schlichting (1955), chapter v.) However, it should be pointed out that the solutions are not confined to the Cartesian form of the equations. For example, in cylindrical polar coordinates (Section 12), we may take $v_r = v_z = 0$, $v_\theta = v_\theta(r,t)$,

$p =$ constant. Then v_θ satisfies

$$\frac{\partial v_\theta}{\partial t} = \nu\left(\frac{\partial^2 v_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r^2}\right),$$

and the vorticity,
$$\omega_z = \frac{1}{r}\frac{\partial}{\partial r}(rv_\theta),$$
satisfies the diffusion equation

$$\frac{\partial \omega_z}{\partial t} = \nu\left(\frac{\partial^2 \omega_z}{\partial r^2} + \frac{1}{r}\frac{\partial \omega_z}{\partial r}\right). \tag{62}$$

A well-known solution of this equation is

$$\omega_z = \frac{\Gamma}{4\pi\nu t}e^{-r^2/4\nu t};$$

in the application to fluid flow it describes the dissolution of a vortex filament which is concentrated at the origin at $t = 0$, and Γ is the initial value of the circulation about the origin.

Another application of (62) is to the motion of fluid contained in or surrounding an infinite cylinder which starts to rotate. Equation (62) may be used to determine how the vorticity, which is initially concentrated at the surface of the cylinder, spreads out into the fluid. Outside the cylinder, $\omega_z \to 0$ as $t \to \infty$ and then $v_\theta \propto 1/r$; inside the cylinder, ω_z tends to a constant value equal to twice the angular velocity of the cylinder, and the fluid rotates like a solid body.

In concluding this section we may refer briefly to a solution which comes under the heading of this section, although it is quite different from the above solutions. It was derived by Taylor (1923*a*) in the following way. In two-dimensional flow the equation for the vorticity η is (Section 10)

$$\frac{\partial \eta}{\partial t} + \frac{\partial \psi}{\partial z}\frac{\partial \eta}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial \eta}{\partial z} = \nu\nabla^2\eta, \tag{63}$$

and η is given in terms of the stream function ψ by $\eta = \nabla^2\psi$. Now the convection terms in (63) vanish when η is a function of ψ. As a special case we take $\eta = -k\psi$; then (63) is satisfied if

$$\frac{\partial \psi}{\partial t} = -k\nu\psi, \qquad \nabla^2\psi = -k\psi.$$

Therefore $\psi = \psi_1(x,z)e^{-k\nu t}$, where $\nabla^2\psi_1 = -k\psi_1$. This equation for ψ_1 arises in the mechanical problem of the vibrating membrane, and we have the useful analogy that the streamlines $\psi_1 =$ constant are given by the contours of the membrane. A particular example is

$$\psi = A\cos\frac{\pi x}{d}\cos\frac{\pi z}{d}\exp\left(-\frac{2\pi^2\nu t}{d}\right), \tag{64}$$

and this flow may be interpreted as a double array of vortices (as shown in Fig. III. 3) which decay exponentially with time.

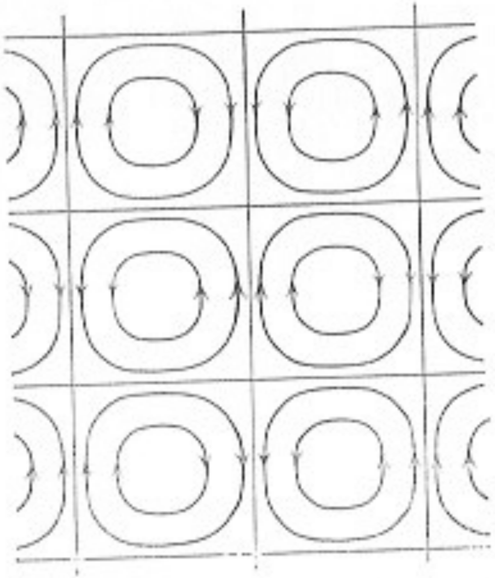


FIG. III. 3. Streamlines of the flow given by equation (64).

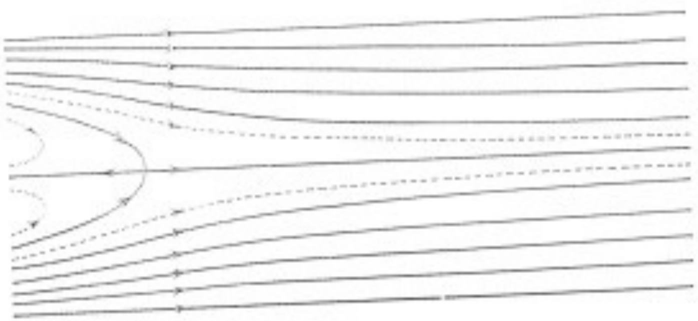


FIG. III. 4. Streamlines of the flow given by equation (65).

Kovasznay (1948) has obtained a steady-flow solution in a similar manner by assuming that η is proportional to $\psi - Uz$, U being the mainstream velocity in the x-direction. The stream function in that case is given by

$$\psi = Uz - A \sin\frac{2\pi z}{d} \exp\left\{\frac{x}{2d}\left(R - \sqrt{(R^2+16\pi^2)}\right)\right\}, \tag{65}$$

where $R = Ud/\nu$. The flow is periodic in the z-direction, and the streamlines for one period are shown for $R = 40$ in Fig. III. 4; Kovasznay suggests that this may be used to describe the flow downstream of a two-dimensional grid.

15. Examples of flows with suction

Even if the convection terms do not vanish, they will not introduce serious complications provided that they are linear in the unknown variables. Such is the case in the following problems of flows with suction.

(a) Asymptotic suction profile

A surprisingly simple solution, which is nevertheless important, describes steady flow parallel to an infinite plane surface on which the normal component of velocity takes a given non-zero value. This solution represents the steady flow far downstream of the leading edge of a semi-infinite flat plate. Without suction the boundary layer would grow indefinitely downstream so that, at any finite distance from the plate, the velocity ultimately tends to zero. But with suction this is not the case; the boundary-layer growth is eventually subdued, and we have the 'asymptotic suction profile'.

If x and z are measured along and perpendicular to the plate, the velocity components u, w, and the pressure are independent of x. Hence, from the continuity equation (4), w remains constant and equal to its value, $-W$ say, at the wall; from (12),

$$-W\,du/dz = \nu\,d^2u/dz^2 \tag{66}$$

and the pressure remains constant throughout the flow. Since $u = 0$ at $z = 0$ and tends to the main-stream value U as $z \to \infty$, the appropriate solution of (66) is

$$u = U(1 - e^{-zW/\nu}). \tag{67}$$

As $\nu \to 0$, the disturbance to the main stream becomes more and more concentrated in a 'boundary layer' at the plate, and in fact (67) is also the solution of the approximate boundary-layer equations (see Section V. 19).

(b) Circulatory flow about a rotating circular cylinder with suction

As noted in the previous section, if an infinite cylinder immersed in fluid at rest is suddenly rotated about its axis with constant angular velocity, the vorticity $\omega_z = r^{-1}\partial(rv_\theta)/\partial r$, which is initially concentrated at the surface of the cylinder, diffuses out until $\omega_z = 0$ everywhere. In this final steady state $v_\theta = \Gamma_1/2\pi r$, where Γ_1 is the circulation around the cylinder. However, if there is suction through the surface of the cylinder, the vorticity may ultimately settle down to a steady-state distribution in which the outward diffusion is balanced by the convection of vorticity towards the cylinder. Such solutions were first obtained by Hamel (1917) and have been discussed more recently by Preston (1950*a*). If r_0 is the radius of the cylinder and the suction velocity normal to the surface of the cylinder is $-V$, the radial velocity in the fluid must be given by

$$v_r = -\frac{Vr_0}{r},$$

in order to satisfy the continuity equation. The rate at which vorticity diffuses across a circle $r =$ constant is $-2\pi r\nu\, \partial\omega_z/\partial r$, and the rate of convection is $2\pi r v_r \omega_z$; therefore, a balance is achieved when

$$\frac{\partial\omega_z}{\partial r}+R\frac{\omega_z}{r} = 0,$$

where $R = r_0 V/\nu$.

The solution for ω_z is

$$\omega_z = \frac{1}{r}\frac{\partial}{\partial r}(rv_\theta) = A\left(\frac{r_0}{r}\right)^R, \tag{68}$$

where A is the value of ω_z at the cylinder. This example is of particular importance in considering the possibility of using suction to maintain different values for the circulation at the cylinder and at infinity (see Preston 1950*a*, Thwaites 1950), and it is useful, therefore, to quote the expression for the circulation $\Gamma = 2\pi r v_\theta$. We have, from (68),

$$\Gamma = \Gamma_1 - \frac{2\pi r_0^2}{R-2}A\left(\frac{r_0}{r}\right)^{R-2} \quad (R \neq 2),$$

$$\Gamma = \Gamma_1 + 2\pi r_0^2 A \log\frac{r}{r_0} \qquad (R = 2). \tag{69}$$

We see that if $R \leqslant 2$, the only solution with finite circulation at infinity is $\Gamma = \Gamma_1$, $\omega_z = 0$, $v_\theta = \Gamma_1/2\pi r$, but if $R > 2$, Γ_1 is the value of the circulation at infinity and A can be adjusted to give any circulation at the cylinder. Thus to maintain different values of the circulation at the cylinder and at infinity, it is necessary for the suction velocity V to exceed $2\nu/r_0$.

16. Similarity solutions

Apart from types of solutions described in the previous sections, in which linear equations are obtained, all exact solutions known to the present writer are similarity solutions. The dependent variables are

functions of only two coordinates and, moreover, they can be so chosen that they are functions of a *single* elementary function of the coordinates. Then the unknowns satisfy *ordinary* differential equations, and the solution of such equations (numerically, if necessary) is a simple matter compared with the solution of the original partial differential equations. If we consider equations for two variables $u(x, y)$ and $v(x, y)$, typical examples of similarity solutions are $u = x^m U(\eta)$ and $v = x^n V(\eta)$, where η is a given function of x and y. In fact the similarity solutions in this chapter all belong to the simple case where η is y itself (at least when the most suitable coordinate system is used), but the more general type arises in connexion with the approximate boundary-layer equations.

As will be illustrated below, the existence of similarity solutions is often recognized by physical considerations, and in particular from deductions as to the forms of solution which are possible dimensionally when the physical parameters in the problem do not provide both a fundamental length and a fundamental time. The emphasis is on these two units since, if such quantities appear, the unit of mass can always be derived from the density.

There is also a rather general method of testing for similarity solutions; it is most conveniently demonstrated by an example, and for the general theory reference may be made to Birkhoff (1950) and Morgan (1952). If we consider the equations for steady two-dimensional flow (equations (4) and (12) with $w = 0$), then, in applying this method, we look for a one-parameter transformation of the variables x, y, u, v, $p-p_0$ under which the equations are invariant (since only derivatives of p occur in the equations of motion, p_0 may be any constant value of the pressure). A particularly useful transformation is

$$x' = \lambda^a x, \qquad y' = \lambda^b y,$$

$$u' = \lambda^\alpha u, \qquad v' = \lambda^\beta v, \qquad p'-p_0' = \lambda^\gamma(p-p_0), \tag{70}$$

where λ is the parameter, and it is easily found that the equations are invariant provided that $\alpha = \beta = -a = -b$, $\gamma = -2a$. Then, we can say that there exists a solution in which ux, vx, $(p-p_0)x^2$ are functions of y/x, since each of these quantities is invariant in the transformation. Of course it is now clear that for this solution polar coordinates are more appropriate; then, rv_r, rv_θ, $r^2(p-p_0)$ are functions of θ alone.

The final step, having deduced a form of solution by this transformation method, is to check that the relevant boundary conditions can be satisfied. The example given here describes flow between inclined plane walls, and it is discussed in detail in the next section.

17. Two-dimensional flow between non-parallel plane walls

Although the form of solution for this problem has been found as an example of the 'transformation method', the following dimensional argument, which leads immediately to the required form, is more satisfactory. The only physical parameters involved in the problem are the kinematic viscosity ν and the volume flux Q per unit distance perpendicular to the flow plane. But Q and ν both have dimensions L^2/T. Hence, the velocity components v_r, v_θ and the distance r can appear only in the combinations rv_r and rv_θ which have dimensions L^2/T. Thus rv_r/ν and rv_θ/ν must be functions of θ and Q/ν alone. The equation of continuity then shows that $v_\theta r$ is constant, and since $v_\theta = 0$ at the walls the constant must be zero; therefore, the flow is radial. Using a similar dimensional argument to establish the form of the pressure difference $p-p_0$, we have

$$v_r = \frac{\nu F(\theta)}{r}, \qquad \frac{p-p_0}{\rho} = \frac{\nu^2 P(\theta)}{r^2}. \tag{71}$$

These expressions and the equations for F given below were first obtained and studied by Jeffery (1915) and Hamel (1917). Subsequently, many writers (Harrison 1919, Kármán 1924, Tollmien 1931, Noether 1931, Dean 1934) worked on special aspects of the problem but the most comprehensive treatments have been given by Rosenhead (1940) and Millsaps and Pohlhausen (1953); we shall refer in detail to the last two investigations.

When v_r and p given by (71) are substituted in (49), we find $P = -\frac{1}{2}(F^2+F'')$, $P' = 2F'$; hence $P = 2F+C$, where C is an arbitrary constant, and

$$F''+F^2+4F+2C = 0. \tag{72}$$

On multiplication by F', the equation for F can be integrated to

$$\tfrac{1}{2}F'^2+\tfrac{1}{3}F^3+2F^2+2CF = \text{constant}.$$

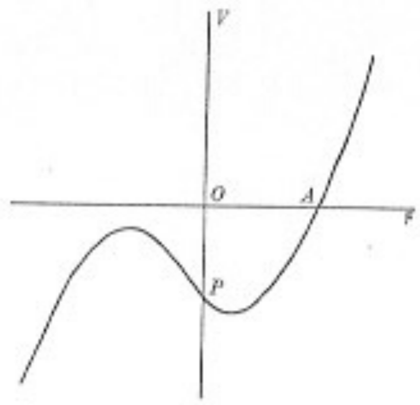
It is convenient to write

$$\tfrac{1}{2}F'^2-\tfrac{1}{3}(a-F)(F-b)(F-c) = 0, \tag{73}$$

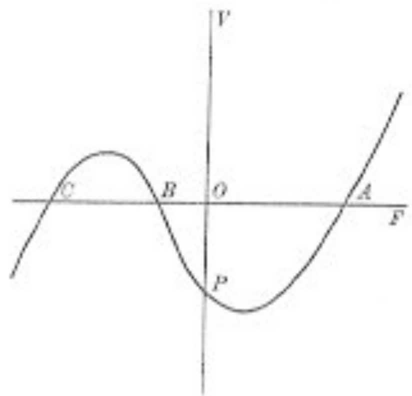
where only two of the constants a, b, c are independent since they satisfy

$$a+b+c = -6. \tag{74}$$

Now v_r must vanish at the walls, $\theta = \pm\alpha$, say, so that (73) must be solved subject to the conditions $F(\pm\alpha) = 0$. The integration can be carried out in terms of elliptic functions and the values of α and $Q/\nu = \int_{-\alpha}^{\alpha} F(\theta)\,d\theta$ for the possible range of values of a, b, and c may then be deduced. But appeal to the theory of elliptic functions can with advantage be postponed, and reduced to a minimum, by making use of a dynamical interpretation. If we consider a particle of unit mass moving along a straight line, with its displacement at time θ measured



(i)



(ii)

FIG. III. 5. Sketches of the two possible forms for the potential energy $V(F)$.

by $F(\theta)$, equation (73) is the energy equation for a motion in which the potential energy of the particle is given by

$$V(F) = -\tfrac{1}{3}(a-F)(F-b)(F-c),$$

and the total energy is zero. Since $V < 0$ in the motion and the particle starts at $F = 0$ for $\theta = -\alpha$ and returns to $F = 0$ for $\theta = \alpha$, it is clear that there are only two cases to consider. If a is real and b and c are complex conjugates, $V(F)$ must be as in Fig. III. 5 (i) with $a > 0$; if

a, b, c are all real ($a > b > c$), $V(F)$ must be as shown in Fig. III. 5 (ii) with $a > 0, b < 0, c < 0$. In case (i) the particle starts at O with finite velocity and moves to A and back to O; the representative point on the curve of $V(F)$ starts at P, moves up to A, and back to P. (The subsequent motion does not concern us since the particle cannot return to O.) For the fluid motion this represents 'pure outflow', since $F > 0$ throughout, and the flow is symmetric. But in case (ii) the fluid motion may be 'pure outflow' OAO, or 'pure inflow' OBO, or it may be composed of a number of alternate outflow and inflow regions represented by the particle oscillating between A and B; indeed, if α is too large multiple oscillations cannot be avoided. Thus, in case (ii) there are several possible values of α and Q/ν for each set of constants a, b, c. However, they can all be deduced from the appropriate combinations of the values for the two basic motions: outflow OAO and inflow OBO.

Definite integrals for α and Q are easily written down from (73) and they can be transformed into expressions involving only elliptic functions, for which tables are available. We see from the above interpretation that there may be several solutions, corresponding to different numbers of outflow and inflow regions, which lead to the same values of α and Q/ν. In fact, Rosenhead (1940) finds that there is an infinite number of solutions for each α and Q/ν and that for each *type* of solution (i.e. number of outflow and inflow regions) there is, for given α, a critical value of Q/ν above which that particular solution becomes impossible. For $\alpha < \frac{1}{2}\pi$, the critical values of Q/ν are all positive, the smallest being for pure outflow, but for $\pi > \alpha > \frac{1}{2}\pi$ pure inflow is also limited. For practical purposes, the critical curve for pure outflow and pure inflow is of particular importance and it is shown in Fig. III. 6: as $\alpha \to 0$, $(Q/\nu)_{\text{crit}} \sim 9{\cdot}424/\alpha$.

In pure outflow the flow becomes more and more concentrated in the centre of the channel as Q/ν increases, until at the critical value the velocity gradient at the wall drops to zero; for greater values of Q/ν, inflow regions must appear. But for pure inflow, as $|Q|/\nu$ increases the velocity profile becomes flatter with nearly uniform flow across the channel except for thin boundary layers at the walls. Velocity profiles for a channel with $\alpha = 5°$ are shown in Fig. III. 7, which is taken from Millsaps and Pohlhausen (1953). These authors use, in place of Q/ν, a Reynolds number based on maximum outward or inward velocity and distance r; thus, for outflow the Reynolds number is a, and for inflow it is $-b$. (This certainly simplifies the mathematical analysis, since Q is given by a somewhat complicated expression involving the Jacobian

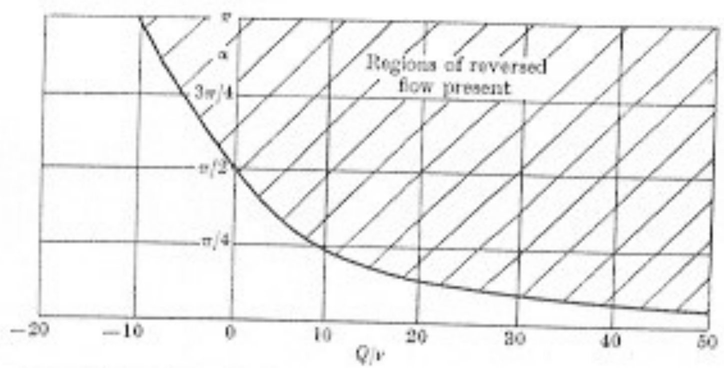


FIG. III. 6. Critical values of Q/ν for pure outflow and pure inflow.

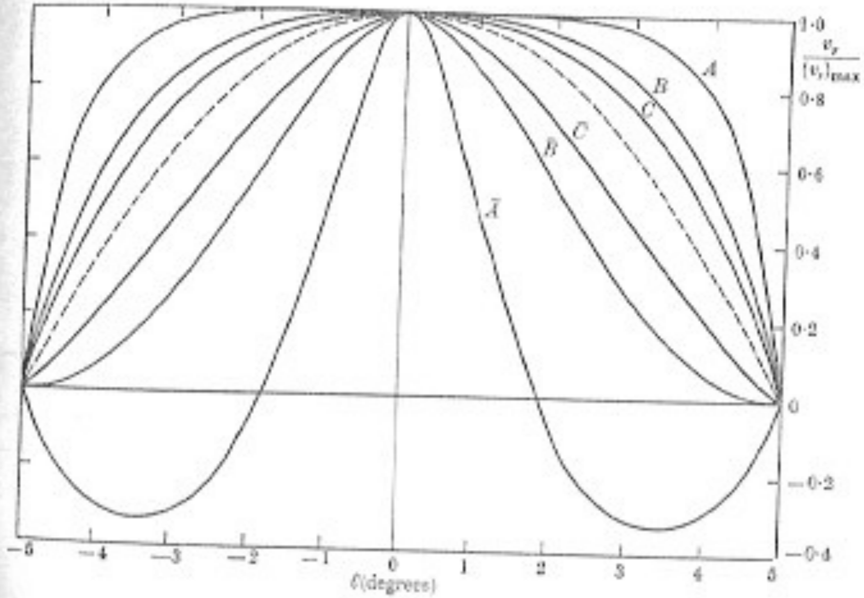


FIG. III. 7. Velocity profiles for a channel with $\alpha = 5°$. The broken line is the Poiseuille parabola. A, B, C are inflows with $|b| = 5{,}000$, $|b| = 1{,}342$, and $|b| = 684$, respectively. $\bar{A}, \bar{B}, \bar{C}$ are outflows with $a = 5{,}000$, $a = 1{,}342$, and $a = 684$, respectively, $a = 1{,}342$ being the critical value for this channel.

zeta function.) For fixed α, the variations of pure outflow with increasing a and of pure inflow with increasing $-b$ are similar to the variations with increasing $|Q|/\nu$. There is again a critical value of a above which pure outflow is impossible. The relation between the critical values of a and α is given below in equation (83); for large values of a,

$$\alpha \sim 3{\cdot}211a^{-\frac{1}{2}}. \tag{75}$$

The graph of α against a is given in Fig. III. 8.

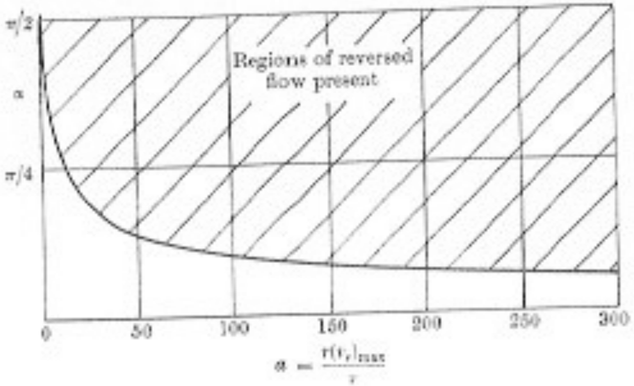


FIG. III. 8. Critical values of a for pure outflow.

On the question of which Reynolds number to use, it may be remarked that in some ways the Reynolds number $R = a\alpha$, based on maximum velocity and channel semi-width, is most significant. This is so, for example, in comparing the present results with those for flow in non-uniform channels in general. In terms of R we see from (75) that pure outflow becomes impossible when $\alpha \simeq 10{\cdot}31/R$. This is very close to the results found for the critical value of the divergence in the approximate theories for general channels of small variation. Thus Abramowitz (1949), improving on the work of Blasius (1910), finds that the critical divergence is $9{\cdot}24/R$, where R is based on maximum velocity and channel semi-width. (The value obtained by Blasius was $13{\cdot}125/R$.)

It is perhaps worth noting the main analytical results for pure outflow and inflow; the results for the more complicated flows may be deduced from the appropriate combination of these. For outflow, since $F = a$ when $\theta = 0$, we have from (73)

$$\theta = \sqrt{\tfrac{3}{2}} \int\limits_F^a \frac{dF}{\sqrt{\{(a-F)(F-b)(F-c)\}}}; \tag{76}$$

for the inflow regions, the integration is from b to F. Reducing these by the use of standard transformations (see, for example, Jeffreys and Jeffreys 1950, p. 685), we have:

(i) If a is real and b and c are complex conjugates, only pure outflow is possible and it is given by

$$F(\theta) = a - \frac{3M^2}{2}\,\frac{1-\mathrm{cn}(M\theta,\kappa)}{1+\mathrm{cn}(M\theta,\kappa)}, \tag{77}$$

$$M^2 = \tfrac{2}{3}\{(a-b)(a-c)\}^{\frac{1}{2}}, \qquad \kappa^2 = \tfrac{1}{2}+\frac{a+2}{2M^2}. \tag{78}$$

(ii) If a, b, c are real, then for outflow

$$F = a - 6k^2m^2\,\mathrm{sn}^2(m\theta, k), \tag{79}$$

and for inflow

$$F = a - 6k^2m^2\,\mathrm{sn}^2(K-m\theta, k) = b + 6k^2m^2\,\mathrm{cn}^2(K-m\theta, k), \tag{80}$$

where

$$m^2 = \tfrac{1}{6}(a-c), \qquad k^2 = \frac{a-b}{a-c}, \tag{81}$$

and $K(k^2)$ is the first complete elliptic integral.

The values of α are found by setting $F = 0$ in (76) and the corresponding integral for inflow; the values of Q are found by integrating the appropriate expressions for $F(\theta)$ from $-\alpha$ to α. The relation between α and a for the limiting condition in pure outflow may be deduced by noting that pure outflow becomes impossible when $F'(\theta)$ drops just to zero at the walls, and therefore $b = 0$. Then from (74), $c = -6-a$; hence, from (76),

$$\alpha = \sqrt{\tfrac{3}{2}} \int\limits_0^a \frac{dF}{\sqrt{\{F(a-F)(F+a+6)\}}},$$

$$= \sqrt{\frac{3}{2a}} \int\limits_0^1 \frac{dt}{\sqrt{[t(1-t)\{1+(1+6/a)t\}]}}. \tag{82}$$

This result may also be written

$$\alpha = \left(\frac{3}{3+a}\right)^{\frac{1}{2}} K\left\{\frac{1}{2}\left(\frac{a}{3+a}\right)\right\}, \tag{83}$$

and could be deduced directly from (79), since in the critical conditions $m\alpha = K(k^2)$ and

$$m^2 = \frac{3+a}{3}, \qquad k^2 = \frac{1}{2}\left(\frac{a}{3+a}\right). \tag{84}$$

For large values of a, $\alpha a^{\frac{1}{2}} \to \sqrt{3}\,K(\tfrac{1}{2}) = 3{\cdot}211$. The critical flux is

$$\frac{Q}{\nu} = 2\int\limits_0^\alpha (a-6k^2m^2\,\mathrm{sn}^2 m\theta)\,d\theta = \frac{12k^2}{\sqrt{(1-2k^2)}} \int\limits_0^K \mathrm{cn}^2\chi\,d\chi,$$

from (84). Hence

$$\frac{Q}{\nu} = \frac{12}{\sqrt{(1-2k^2)}}\{E-(1-k^2)K\},$$

where $E(k^2)$ is the second complete elliptic integral; for large values of a, $k^2 \sim \frac{1}{2}-\frac{3}{2}a^{-1}$, and therefore

$$Q/\nu \sim (a/3)^{\frac{1}{2}}\, 12\{E(\tfrac{1}{2})-\tfrac{1}{2}K(\tfrac{1}{2})\} = 2{\cdot}934a^{\frac{1}{2}}.$$

In pure inflow, as the Reynolds number, $-b$, becomes larger, the tendency for the flow to become uniform, except for boundary layers near the walls, is easily shown analytically from (80). Since α is given and m is large, we see from (80) that K must be large; hence, $k \sim 1$. But, when $k = 1$ the function $\operatorname{sn} t$ becomes $\tanh t$, and $c = b$, $a = -6-2b \sim -2b$; therefore,

$$F = b\{3\tanh^2[(-\tfrac{1}{2}b)^{\frac{1}{2}}(\alpha-\theta)+\beta]-2\}, \tag{85}$$

where $\beta = \tanh^{-1}\sqrt{(\frac{2}{3})} = 1{\cdot}146$. Thus F is approximately equal to b except in the boundary layer of thickness proportional to $(-b)^{-\frac{1}{2}}$. We see then that in this case the boundary-layer assumptions are borne out, and in fact (85) is exactly the solution which is obtained by solving the approximate boundary-layer equations (see Section V. 17).

Millsaps and Pohlhausen (1953) also obtain the temperature distributions in these flows for the case when the walls are maintained at constant temperature; for the details, reference should be made to their paper.

Hamel (1917) discussed the flow between non-parallel plane walls as a special case of flow in which the streamlines are equiangular spirals, which he showed to be the only possible form if, for a two-dimensional motion, they are to coincide with the streamlines of a potential flow without the actual motion itself being irrotational. When the more general spiral motion takes place between solid walls, the results are similar to those obtained above. Hamel's results have formed the starting-point for a number of researches by other authors (Olsson and Faxen 1927, Oseen 1927b); a general review of this work is given by Rosenblatt (1933).

18. Round jets

In axisymmetric flow there is an analogue of the plane flow described in the previous section. But it is not flow through a cone, as might have been expected; it is found to be the flow in a round jet. The reason for this becomes clear if we consider the dimensional quantities involved. For flow through a cone the volume flux has dimensions L^3T^{-1}, so that with ν, both a fundamental length and a fundamental time can be formed. Hence there is no reason why the dependence of velocities, etc., on r and θ should take a simple form. But if, instead

of a source of mass at the origin, we consider a source of momentum, the rate of transfer M of the appropriate component of momentum across a sphere of radius r gives a parameter M/ρ whose dimensions are L^4T^{-2}. Since ν has dimensions L^2T^{-1}, flows in which M/ρ and ν are the only given parameters take a simple form as in Section 17, because the problem has no fundamental length or time. Clearly the application of these flows is to jets, although more accurately we may say that they describe the flow resulting from the continuous application of a force of magnitude M at the origin.

From dimensional considerations, the flow quantities must be given, using spherical polar coordinates, by

$$v_r = \frac{\nu F(\eta)}{r}, \qquad v_\theta = \frac{\nu G(\eta)}{r},$$

$$\frac{p-p_0}{\rho} = \frac{\nu^2 P(\eta)}{r^2}, \tag{86}$$

where $\eta = \cos\theta$ and F, G, P are functions of η and $M/\rho\nu^2$ alone. (The existence of such a solution of the equations of motion may also be deduced by the transformation method described at the end of Section 16.)

The equation of continuity is satisfied by taking

$$F(\eta) = -f'(\eta), \qquad G(\eta) = -\frac{f(\eta)}{\sqrt{(1-\eta^2)}}, \tag{87}$$

corresponding to the stream function $\psi = \nu r f(\eta)$. Then the momentum equations (53) give

$$P = -\frac{f^2}{2(1-\eta^2)}-\frac{1}{2}\frac{d}{d\eta}\{ff'-(1-\eta^2)f''\},$$

$$P' = -\frac{d}{d\eta}\left\{\frac{f^2}{2(1-\eta^2)}\right\}-f''.$$

Therefore

$$P = -\frac{f^2}{2(1-\eta^2)}-f'-\tfrac{1}{2}c_1, \tag{88}$$

and

$$2ff' = 2(1-\eta^2)f''+4f+2c_1\eta+c_2, \tag{89}$$

where c_1 and c_2 are arbitrary constants of integration. Equation (89) can be integrated immediately to give

$$f^2 = 2(1-\eta^2)f'+4\eta f+\Sigma(\eta), \tag{90}$$

where $\Sigma(\eta) = c_1\eta^2+c_2\eta+c_3$, c_3 being a third constant of integration. This equation for f was first obtained by Slezkin (1934) and the solution which applies to the round jet (equation (93) below) was discussed by

Landau (1944*a*). Yatseyev (1950) has since obtained the general solution of (90) and Squire (1951) has given a detailed discussion of the round jet. Squire also obtains the temperature field for a heated jet, and makes further applications (Squire 1952, 1955) of the general solution. Morgan (1956) notes that these solutions can be interpreted as flows inside cones, if the condition of zero slip or the condition of zero normal velocity is relaxed. But the interpretation as jet flows is the fundamental one.

Before discussing the applications to jet flows, we may note in advance that $\Sigma(\eta) = K(1-\eta)^2$ is the *only* choice of $\Sigma(\eta)$ which does not lead to singularities of the flow quantities on the axis, $\eta = 1$. For, if v_θ is finite at $\eta = 1$, $f(1)$ must be zero; hence, if v_r is also finite at $\eta = 1$, with $f'(1) = -A$, say, we must have $f \sim A(1-\eta)$. But (90) then shows that $\Sigma(\eta) \sim K(1-\eta)^2$; therefore, since $\Sigma(\eta)$ is a quadratic it must be *equal* to $K(1-\eta)^2$.

Equation (90) is of the Riccati type and can be transformed into a second-order linear equation by the substitution

$$f = -2(1-\eta^2)g'/g. \tag{91}$$

Then g satisfies a form of the hypergeometric equation, but in the special case $\Sigma(\eta) = K(1-\eta)^2$ its solutions are simply

$$g = (1+\eta)^\lambda, \qquad \lambda = \tfrac{1}{2}\{1 \pm \sqrt{(1+K)}\}. \tag{92}$$

For the round jet, $\Sigma(\eta) = 0$, that is, $K = 0$, and the general solution for f becomes

$$f = \frac{2(1-\eta^2)}{a+1-\eta}, \tag{93}$$

where a is an arbitrary constant. It is found after some calculation that the rate of transfer of the x component of momentum (including the contributions of pressures and viscous stresses) across a sphere of radius r is given by

$$\frac{M}{2\pi\rho\nu^2} = \frac{32(1+a)}{3a(2+a)} + 8(1+a) - 4(1+a)^2\log\left(\frac{2+a}{a}\right), \tag{94}$$

and we see that large values of the parameter $M/\rho\nu^2$ correspond to small values of a. The streamlines of the flow are shown in Fig. III. 9 for the case $a = 10^{-2}$, $M/\rho\nu^2 = 3{,}282$. Squire (1951) has also shown that the appropriate solution of (31) for a heated jet is

$$T - T_0 = \frac{(2\sigma+1)Q}{8\pi\rho C_p \nu r}\left(\frac{a}{a+1-\eta}\right)^{2\sigma}, \tag{95}$$

where σ is the Prandtl number and Q is the strength of the heat source

at the origin. The temperature contours for the case $a = 10^{-2}$, $\sigma = 0{\cdot}72$, $(2\sigma+1)Q/8\pi\rho C_p\nu = 10$, are shown in Fig. III. 10.

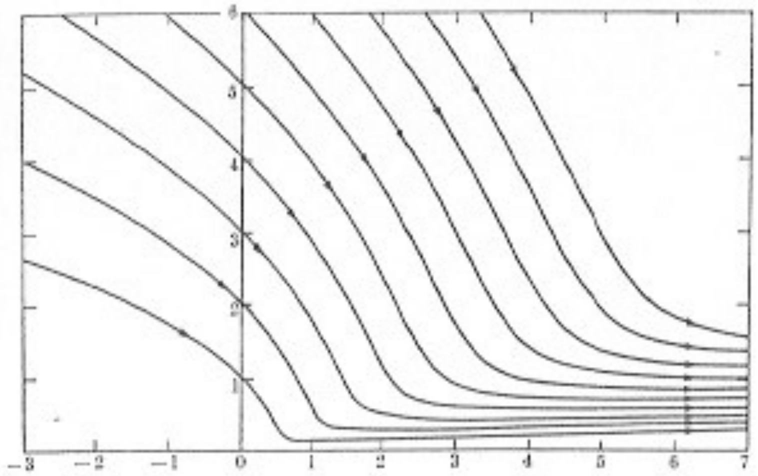


FIG. III. 9. Streamlines of round jet: $a = 10^{-2}$; $M/\rho\nu^2 = 3{,}282$.

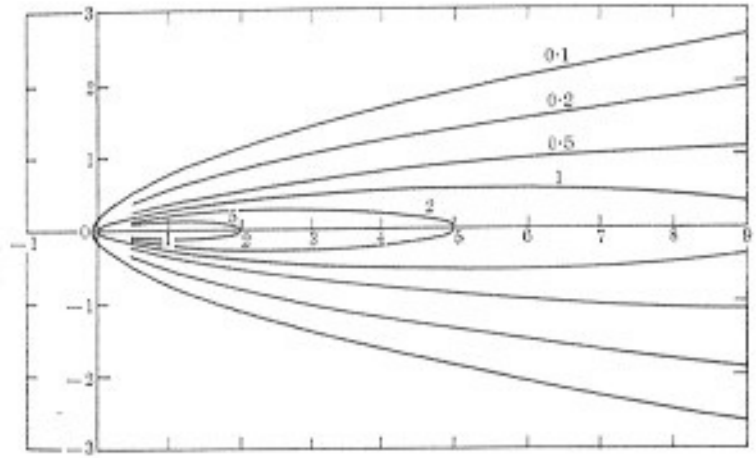


FIG. III. 10. Temperature contours of heated round jet: $a = 10^{-2}$, $\sigma = 0{\cdot}72$, $(2\sigma+1)Q/8\pi\rho C_p\nu = 10$. The numbers on the curves indicate values of $T - T_0$.

The unit of length can be chosen arbitrarily in Figs. III. 9, III. 10, and III. 11, but then the unit of temperature in Fig. III. 10 is fixed by the requirement that $(2\sigma+1)Q/8\pi\rho C_p\nu$, which has the dimensions of length multiplied by temperature, should have the value 10 in these units.

For small values of a the jet becomes concentrated near the axis and its velocity increases. In this case, $f \simeq 2(1+\eta)$ except near the axis $\eta = 1$. Thus away from the axis, that is outside the jet, there is a radial velocity $v_r = -2\nu/r$ which provides the fluid (with volume flux $8\pi\nu r$) which is entrained by the jet. Near $\eta = 1$, that is, in the jet, we may approximate f by

$$f \simeq \frac{(2/a)\theta^2}{1+\frac{1}{4}\{(2/a)\theta^2\}}.$$

From (94), $2/a \sim 3M/(16\pi\rho\nu^2)$; hence,

$$f \simeq \frac{\xi^2}{1+\frac{1}{4}\xi^2}, \qquad \xi = \frac{1}{4\nu}\left(\frac{3}{\pi}\frac{M}{\rho}\right)^{\frac{1}{2}}\theta. \tag{96}$$

This is precisely the solution obtained by Schlichting (1933*a*) directly from the approximate boundary-layer equations (see Section VIII. 17).

As a second application Squire (1952) uses (91), (92) with $K \neq 0$, to describe a jet emerging from a small hole in a plane wall. However, the boundary conditions $v_r = v_\theta = 0$ at the wall are not both satisfied: only the normal velocity component V_θ vanishes there. If K is set equal to $-(4b^2+1)$, $\lambda = \frac{1}{2} \pm ib$ in (92), and the solution for f is

$$f = (1-\eta)\left\{-1+2b\,\frac{1-c\cot[b\log(1+\eta)]}{c+\cot[b\log(1+\eta)]}\right\}, \tag{97}$$

where c is a second arbitrary constant. Since $v_\theta = 0$ for $\eta = 0$, we require $f(0) = 0$, which leads to

$$1+2bc = 0. \tag{98}$$

With this value for c,

$$f = \frac{(1+4b^2)(1-\eta)}{2b\cot[b\log(1+\eta)]-1}. \tag{99}$$

Narrow high-speed jets correspond to small values of the denominator in (99), and thus to values of b near to the root of

$$2b\cot[b\log 2] = 1.$$

The significant root is $b = 1{\cdot}8937$, and it is found that as b approaches this value, $M/\rho\nu^2 \to \infty$. The streamlines of the flow in the case $b = 1{\cdot}88$, $M/\rho\nu^2 = 3{\cdot}8\times10^3$, are shown in Fig. III. 11.

Finally, we may note that the theoretical results for the round laminar jet agree well with the experimental results for the *turbulent* jet. For the turbulent jet ν must be interpreted as an eddy viscosity, and the agreement of the results suggests that in this flow it is a good approximation to assume that the eddy viscosity is constant. Using a dimensional argument again, we can see that there is no variation of eddy viscosity in the longitudinal direction, since $M^{\frac{1}{2}}$, the only quantity of the same dimensions, is constant along the jet. But the eddy viscosity may vary with θ across the jet, and, indeed, experiments indicate some decrease near the boundary of the jet. Detailed comparison between theory and experiment is to be found in Hinze and van der Hegge Zijnen (1949). The value of a is chosen in order to give the observed rate of spread

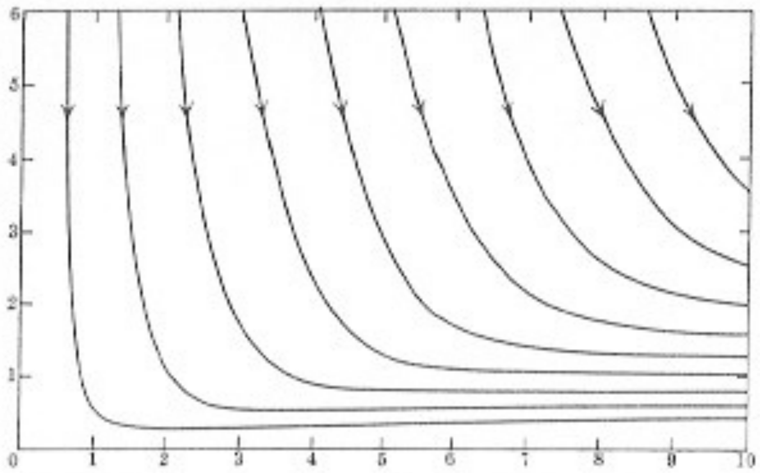


FIG. III. 11. Streamlines of jet emerging from hole in wall: $b = 1{\cdot}88$, $M/\rho\nu^2 = 3{\cdot}8\times10^3$.

of the jet and M is equated to the thrust of the jet; a value of ν can then be deduced. In applying the results, the origin in the theoretical solution is taken at a suitable distance upstream from the orifice of the jet. The calculated and experimental velocity distributions are compared and found to be in good agreement.

19. Stagnation point flows

(a) *Two-dimensional flow*

For an ideal fluid the flow against an infinite flat plate in the plane $z = 0$ is given by

$$u = kx, \qquad w = -kz, \tag{100}$$

where k is a constant. In the case of a general bluff body equation (100) applies to the flow in the neighbourhood of the stagnation point, $x = z = 0$, where x and z are small compared to the radius of curvature of the nose of the body. When viscosity is included, it must still be true that u is proportional to x, for small x and *all* z. Thus, for small x, at least, we may take $u = xF(z)$; the equation of continuity then gives $\partial w/\partial z = -F(z)$. However, it is found that the solution of this form is exact and in the case of the flat plate it applies for all x.

Introducing non-dimensional variables, we take

$$u = kxf'(\zeta), \qquad w = -(\nu k)^{\frac{1}{2}}f(\zeta), \qquad \zeta = z(k/\nu)^{\frac{1}{2}}. \tag{101}$$

These satisfy the momentum equations (12) if

$$\frac{p_0 - p}{\rho} = \tfrac{1}{2}k^2x^2 + k\nu f'(\zeta) + \tfrac{1}{2}k\nu f^2, \tag{102}$$

and the function f is a solution of

$$f''' + ff'' - f'^2 + 1 = 0. \tag{103}$$

The boundary conditions for f are

$$f(0) = f'(0) = 0, \qquad f'(\infty) = 1, \tag{104}$$

and the appropriate solution has been calculated numerically (Hiemenz 1911, Howarth 1934). The values of f, f', f'' are given later in Table V. 2.

When the viscosity is small, we see that the disturbance to the 'main stream' (100) is limited to a boundary layer whose thickness is proportional to $\sqrt{(\nu/k)}$. The boundary layer is of *constant* thickness so that the thinning of the layer due to the accelerating main stream, $u = kx$, is just sufficient to balance the thickening due to the diffusion of shear. Moreover, we may notice that the exact solution is also a solution of the approximate boundary-layer equations. For $-\rho^{-1}\,\partial p/\partial x$ takes the value k^2x which it has in the main stream and $\partial^2 u/\partial x^2 = 0$ throughout; as will be seen in detail in Chapter V, these are precisely the approximations which are made in the boundary-layer theory.

(*b*) *Axisymmetric flow*

The analogous solution for axisymmetric flow against a flat plate (Homann 1936*b*) is obtained by taking (in cylindrical polar coordinates)

$$v_r = krf'(\zeta), \qquad v_z = -2(k\nu)^{\frac{1}{2}}f(\zeta), \qquad \zeta = z(k/\nu)^{\frac{1}{2}}. \tag{105}$$

The equation for f differs from (103) only in that the term ff'' has a factor 2, and the boundary conditions are the same as (104). It is sometimes convenient to modify (105) slightly and take $v_r = krf'(\zeta)$, $v_z = -(2k\nu)^{\frac{1}{2}}f(\zeta)$, $\zeta = z(2k/\nu)^{\frac{1}{2}}$. This alternative form will be used in Chapters V and VIII. Values of the functions f, f', f'' using these modified variables are given in Table V. 3.

(*c*) *Three-dimensional flow*

The flow at a general three-dimensional stagnation point also yields an exact solution of the equations of motion: details are given in Section VIII. 21.

20. Flow due to rotating disks

The solution given by the substitutions (105) may be generalized by including a rotation of the fluid about the axis with angular velocity depending on z. It is convenient to take the velocity components as

$$v_r = Krf'(\zeta), \qquad v_\theta = \Omega r g(\zeta), \qquad v_z = -2(K\nu)^{\frac{1}{2}}f(\zeta),$$
$$\zeta = z(K/\nu)^{\frac{1}{2}}, \tag{106}$$

where K and Ω are constants having dimension T^{-1}. When the expressions are substituted in the momentum equations (49), it is found that the pressure is given by

$$(p - p_0)/\rho = \tfrac{1}{2}\lambda K^2 r^2 - 2\nu K(f^2 + f'), \tag{107}$$

where λ is an arbitrary constant, and the functions f and g satisfy

$$f'^2 - 2ff'' - g^2(\Omega/K)^2 - f''' = -\lambda, \tag{108}$$

$$2(f'g - fg') = g''. \tag{109}$$

Kármán (1921) first pointed out the existence of a solution of the form (106), and he considered its application to the flow produced by the rotation of an infinite plane disk. For this problem we take $K = \Omega$ where Ω is the angular velocity of the disk. Assuming that the disk lies in the plane $z = 0$, the appropriate boundary conditions on f and g are

$$f(0) = f'(0) = 0, \qquad g(0) = 1, \tag{110}$$

$$f' \to 0, \quad g \to 0 \quad \text{as} \quad \zeta \to \infty; \tag{111}$$

in addition, since the pressure approaches a constant value at infinity, $\lambda = 0$. These conditions determine the solution uniquely so that the value of v_z at infinity cannot be arbitrarily imposed; it is determinate in terms of Ω and ν. This is to be expected on physical grounds, for in the absence of a radial pressure gradient the fluid near the disk moves radially outwards under the influence of centrifugal force, and therefore an axial inflow at infinity is required by continuity.

An approximate solution to the problem was obtained by Kármán using momentum integral methods, and later an accurate solution was found numerically by Cochran (1934). Cochran uses expansions in powers of ζ for the flow near the disk and an expansion in powers of $e^{-c\zeta}$ for large ζ, where $c = 2f(\infty)$; by making a suitable join of these solutions at an intermediate value of ζ, the arbitrary constants in them, including the value of c, are determined. It is found that $c = 0{\cdot}886$, so that the axial inflow at infinity is

$$-0{\cdot}886(\Omega\nu)^{\frac{1}{2}}. \tag{112}$$

Graphs of the functions $2f(\zeta)$, $f'(\zeta)$, $g(\zeta)$ are shown in Fig. III. 12 and the values of these functions are given in detail in Table III. 1.

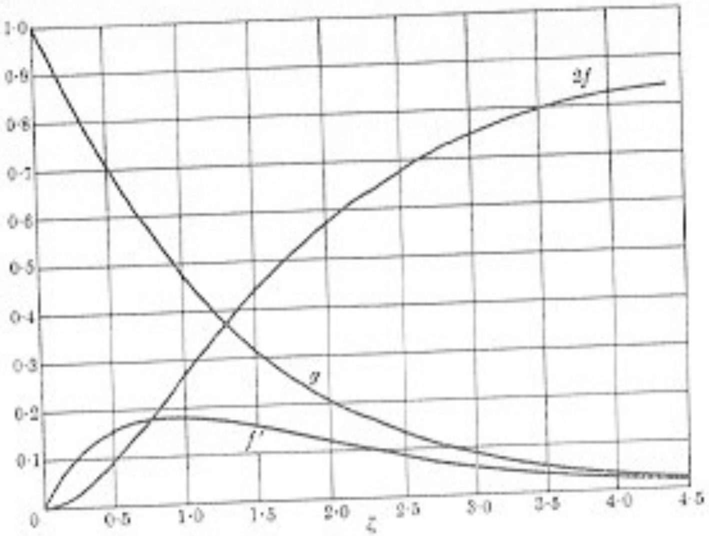


FIG. III. 12. Functions giving the velocity components in the flow produced by a rotating disk.

Again we may notice the boundary-layer character of the solution for small viscosity. The functions f, f', g are within a given percentage of their values at infinity for some finite value of $\zeta = z(\Omega/\nu)^{\frac{1}{2}}$; hence, the flow is approximately uniform except in a boundary layer of thickness $(\nu/\Omega)^{\frac{1}{2}}$. We may also note that, as in Section 19, the exact solution discussed here also satisfies the boundary-layer equations since the terms neglected in boundary-layer theory are identically zero.

Although the solution applies strictly to an infinite disk it may be used to find an approximate value for the frictional moment on a finite disk of radius a, provided that the Reynolds number $a^2\Omega/\nu$ is large. The shearing stress at the disk is

$$p_{z\theta} = \rho\nu\left(\frac{\partial v_\theta}{dz}\right)_{z=0} = \rho(\nu\Omega^3)^{\frac{1}{2}}g'(0)r,$$

so that the moment for both sides of the disk becomes

$$M = -2\int_0^a 2\pi r^2 p_{z\theta}\,dr = -\pi a^4\rho(\nu\Omega^3)^{\frac{1}{2}}g'(0).$$

TABLE III. 1

The functions $f(\zeta)$, $f'(\zeta)$, and $g(\zeta)$ for the flow due to a rotating disk (see equations (108)–(111) and Fig. III. 12)

With the exception of the two starred entries this Table is quoted directly from Cochran (1934). The starred entries incorporate minor corrections derived by smoothing differences.

ζ	f'	g	$2f$
0	0	1	0
0·1	0·046	0·939	0·005
0·2	·084	·878	·018
0·3	·114	·819	·038
0·4	·136	·762	·063
0·5	·154	·708	·092
0·6	0·166	0·656	0·124
0·7	·174	·607	·158
0·8	·179	·561	·193
0·9	·181	·517	·230
1·0	·180	·477*	·266
1·1	0·177	0·439	0·301
1·2	·173	·404	·336
1·3	·168	·371	·371
1·4	·162	·341	·404
1·5	·156	·313	·435
1·6	0·148	0·288	0·466
1·7	·141	·264	·495
1·8	·133	·242	·522
1·9	·126	·222	·548
2·0	·118	·203	·572
2·1	0·111	0·186	0·596
2·2	·104	·171	·617
2·3	·097	·156	·637
2·4	·091	·143	·656
2·5	·084	·131	·674
2·6	0·078	0·120	0·690
2·8	·068	·101	·721
3·0	·058	·085*	·746
3·2	·050	·071	·768
3·4	·042	·059	·786
3·6	0·036	0·050	0·802
3·8	·031	·042	·815
4·0	·026	·035	·826
4·2	·022	·029	·836
4·4	·018	·024	·844
∞	0	0	·886

Introducing a non-dimensional moment coefficient C_M and the Reynolds number $R = a^2\Omega/\nu$, we have

$$C_M = \frac{M}{\frac{1}{2}\rho(a\Omega)^2.\pi a^2.a} = -\frac{2g'(0)}{R^{\frac{1}{2}}} = \frac{1{\cdot}232}{R^{\frac{1}{2}}}. \tag{113}$$

For values of R less than about 10^5, this formula gives values in close agreement with experimental results, but for greater values of R, the flow becomes turbulent and the laminar theory ceases to apply.

Further applications of (106) may be made in which equations (108) and (109) have to be solved under different boundary conditions. The most straightforward of these is to include in the Kármán problem a uniform suction through the disk; this has been investigated by Stuart (1954*a*). The conditions at infinity (111) remain the same and $\lambda = 0$, but in (110) we now have

$$f(0) = \tfrac{1}{2}\alpha, \qquad f'(0) = 0, \qquad g(0) = 1 \tag{114}$$

where α is the suction parameter corresponding to the suction velocity $v_z = -(\nu\Omega)^{\frac{1}{2}}\alpha$. Stuart computed the solution by Cochran's method in the case $\alpha = 1{\cdot}0$, and obtained a series solution in descending powers of α, which is used when $\alpha > 2$. For the details reference should be made to the paper cited.

Miss D. M. Hannah (1947) considers the problem of forced flow against a rotating disk. This is a combination of the Kármán problem and the stagnation point flow described in Section 19. The boundary conditions at the disk remain as in (110), but at infinity the flow is required to approach the potential flow

$$v_r = kr, \qquad v_\theta = 0, \qquad v_z = -2kz,$$
$$\frac{p-p_0}{\rho} = -\tfrac{1}{2}k^2r^2 - 2k^2z^2. \tag{115}$$

In this case K is taken as $(k^2+\Omega^2)^{\frac{1}{2}}$, and the conditions at infinity become

$$f \sim \frac{\zeta}{(1+\mu^2)^{\frac{1}{2}}}, \qquad g \to 0, \tag{116}$$

where $\mu = \Omega/k$. From the form of the pressure at infinity, or by substitution from (116) in (108), we have

$$\lambda = -(1+\mu^2)^{-1}.$$

The special cases $\mu = 0$ and $\mu = \infty$ correspond to the Homann and Kármán flows, respectively, and Miss Hannah calculated the solution for $\mu = 0{\cdot}5$, 1, 2. Unaware of this work, Schlichting and Truckenbrodt (1952) applied Kármán's approximate method to this problem, and later Tifford and Chu (1952) solved the equations numerically for a range of values of μ.

Finally, we refer to the two related problems of the flow over a rotating disk with a prescribed rotation of the fluid at infinity and the flow

between two rotating disks. In the first problem, if the angular velocity of the disk at $z = 0$ is Ω_0 and the angular velocity of the fluid as $z \to \infty$ is Ω_∞, we may choose $\Omega = K = (\Omega_0^2+\Omega_\infty^2)^{\frac{1}{2}}$ in (106) and then equations (108) and (109) must be solved subject to the boundary conditions

$$f(0) = f'(0) = 0, \qquad g(0) = \Omega_0/\Omega,$$
$$f' \to 0, \quad g \to \Omega_\infty/\Omega \quad \text{as} \quad \zeta \to \infty; \tag{117}$$

the boundary conditions at infinity show that the constant λ in (108) takes the value $\Omega_\infty^2/(\Omega_0^2+\Omega_\infty^2)$. Bödewadt (1940) obtained a numerical solution to this problem in the case of a fixed disk ($\Omega_0 = 0$) and more accurate calculations have been carried out by Browning (unpublished) whose results are given in Schlichting (1955, p. 158). Fettis (1956) describes a new method for performing these and similar calculations and obtains results for a range of values of Ω_0/Ω_∞. The velocity components for the case $\Omega_0 = 0$ are given in Table III. 2; the oscillation in the velocities should be noted.

TABLE III. 2

The velocity distribution in the rotating flow over a fixed plane

ζ	f'	g	$-2f$
0	0	0	0
0·5	−0·343	0·382	0·190
1·0	−0·468	0·731	0·614
1·5	−0·437	1·004	1·076
2·0	−0·318	1·175	1·460
2·5	−0·171	1·246	1·704
3·0	−0·038	1·242	1·800
3·5	0·056	1·192	1·784
4·0	0·106	1·123	1·702
4·5	0·117	1·056	1·590
5·0	0·103	1·003	1·478
5·5	0·074	0·969	1·390
6·0	0·041	0·954	1·332
6·5	0·013	0·953	1·308
7·0	−0·010	0·959	1·304
7·5	−0·020	0·975	1·320
8·0	−0·023	0·990	1·340
8·5	−0·020	1·000	1·364
9·0	−0·013	1·007	1·382
9·5	−0·006	1·010	1·390
10·0	0·000	1·009	1·390
10·5	0·003	1·007	1·386
11·0	0·004	1·005	1·382
11·5	0·003	1·002	1·380
12·0	0·001	1·000	1·380
12·5	0·000	1·000	1·380
∞	0·000	1·000	1·380

In the case of two rotating disks at $z = 0$, $z = d$, say with angular velocities Ω_0 and Ω_d, respectively, if we choose $\Omega = K = (\Omega_0^2+\Omega_d^2)^{\frac{1}{2}}$, the boundary conditions are

$$\begin{aligned} f = f' = 0, \quad g = \Omega_0/\Omega \quad \text{at} \quad \zeta = 0, \\ f = f' = 0, \quad g = \Omega_d/\Omega \quad \text{at} \quad \zeta = d(\Omega/\nu)^{\frac{1}{2}}; \end{aligned} \tag{118}$$

the constant λ must be determined in the course of the solution in order to satisfy these conditions. The solution has not yet been calculated but qualitative aspects of it have been discussed by Batchelor (1951) and Stewartson (1953).

IV

FLOW AT SMALL REYNOLDS NUMBER

1. Introduction

IN Chapter III the Navier–Stokes equations of flow for a viscous incompressible fluid were introduced, and some exact solutions, valid for all Reynolds numbers, were discussed; none of them, however, deals with flow past a finite body. At present there appear to be no exact solutions of the flow past bodies of finite size, and, consequently, in order to discuss such flows, it is necessary to derive approximate solutions. These may be either numerical solutions of the exact Navier–Stokes equations, or solutions—analytical or numerical—of approximate equations. Whichever type of solution is involved it is first necessary to specify the value of the Reynolds number R, in some cases precisely, in others approximately, because the character of the flow depends critically upon it. For extreme values of R the Navier–Stokes equations can be replaced by approximate forms which are more tractable. For large R the exact equations reduce to Euler's equations of inviscid flow, except in the comparatively narrow regions in which transverse velocity gradients are large, where they reduce to Prandtl's boundary-layer equations. The study of flows obeying Prandtl's equations is the main concern of this book. For small R, by which we shall usually mean R less than 1, the exact equations are replaced by Oseen's equations, except in regions near fixed surfaces, where they reduce to Stokes's equations. This chapter is concerned with flows obeying either Oseen's or Stokes's equations, as appropriate.

In passing it may be mentioned that in the intermediate range, where R takes values between the very small and the very large, there are only a few solutions of the Navier–Stokes equations, and these are numerical. They deal, chiefly, with flow past a circular cylinder or a sphere. For example, the circular cylinder has been investigated by Thom (1933) for R (based on diameter) $= 10$ and 20, and by Allen and Southwell (1955) for $R = 0, 1, 10, 100$, and 1,000. (The last two cases, as the authors observe, are physically unrealistic since the flow becomes unstable in practice when R reaches a critical value of about 40.) Jenson (1959) has treated the sphere for $R = 5, 10, 20$, and 40. Also, in a series of papers, Kawaguti (1956, and references cited therein) has used different numerical techniques for both the circular cylinder and the sphere. His