WARING

BOOKS BY FLORIAN CAJORI

HISTORY OF MATHEMATICS
Revised and Enlarged Edition

HISTORY OF ELEMENTARY MATHEMATICS
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HISTORY OF PHYSICS

INTRODUCTION TO THE MODERN THEORY OF EQUATIONS

A HISTORY OF MATHEMATICS

BY

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"I am sure that no subject loses more than mathematics by any attempt to dissociate it from its history."—J. W. L. GLAISHER

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a is so selected that the curve is convex toward the axis of x for the interval between a and the root. He shows that this condition is sufficient, but not necessary.[1]

In the eighteenth century proofs were given of Descartes' Rule of Signs which its discoverer had enunciated without demonstration. G. W. Leibniz had pointed out a line of proof, but did not actually give it. In 1675 *Jean Prestet* (1648–1690) published at Paris in his *Elemens des mathématiques* a proof which he afterwards acknowledged to be insufficient. In 1728 *Johann Andreas Segner* (1704–1777) published at Jena a correct proof for equations having only real roots. In 1756 he gave a general demonstration, based on the consideration that multiplying a polynomial by $(x-a)$ increases the number of variations by at least one. Other proofs were given by *Jean Paul de Gua de Malves* (1741), *Isaac Milner* (1778), *Friedrich Wilhelm Stübner*, *Abraham Gotthelf Kästner* (1745), *Edward Waring* (1782), *J. A. Grunert* (1827), *K. F. Gauss* (1828). Gauss showed that, if the number of positive roots falls short of the number of variations, it does so by an even number. E. Laguerre later extended the rule to polynomials with fractional and incommensurable exponents, and to infinite series.[2] It was established by De Gua de Malves that the absence of $2m$ successive terms indicates $2m$ imaginary roots, while the absence of $2m+1$ successive terms indicates $2m+2$ or $2m$ imaginary roots, according as the two terms between which the deficiency occurs have like or unlike signs.

Edward Waring (1734–1798) was born in Shrewsbury, studied at Magdalene College, Cambridge, was senior wrangler in 1757, and Lucasian professor of mathematics since 1760. He published *Miscellanea analytica* in 1762, *Meditationes algebraicæ* in 1770, *Proprietatis algebraicarum curvarum* in 1772, and *Meditationes analyticæ* in 1776. These works contain many new results, but are difficult of comprehension on account of his brevity and obscurity of exposition. He is said not to have lectured at Cambridge, his researches being thought unsuited for presentation in the form of lectures. He admitted that he never heard of any one in England, outside of Cambridge, who had read and understood his researches.

In his *Meditationes algebraicæ* are some new theorems on number. Foremost among these is a theorem discovered by his friend *John Wilson* (1741–1793) and universally known as "Wilson's theorem." Waring gives the theorem, known as "Waring's theorem," that every integer is either a cube or the sum of 2, 3, 4, 5, 6, 7, 8 or 9 cubes, either a fourth power or the sum of 2, 3 . . or 19 fourth powers; this has never yet been fully demonstrated. Also without proof is given the theorem that every even integer is the sum of two primes and every

[1] See F. Cajori in *Bibliotheca mathematica*, 3rd S., Vol. 11, 1911, pp. 132–137.
[2] For references to the publications of these writers, see F. Cajori in *Colorado College Publication*, General Series No. 51, 1910, pp. 186, 187.

odd integer is a prime or the sum of three primes. The part relating to even integers is generally known as "Goldbach's theorem," but was first published by Waring. Christian Goldbach communicated the theorem to L. Euler in a letter of June 30, 1742, but the letter was not published until 1843 (*Corr. math.*, P. H. Fuss).

Waring held advanced views on the convergence of series.[1] He taught that $1+\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{4^n}+\ldots$ converges when $n>1$ and diverges when $n<1$. He gave the well-known test for convergence and divergence which is often ascribed to A. L. Cauchy, in which the limit of the ratio of the $(n+1)^{th}$ to the n^{th} term is considered. As early as 1757 he had found the necessary and sufficient relations which must exist between the coefficients of a quartic and quintic equation, for two and for four imaginary roots. These criteria were obtained by a new transformation, namely the one which yields an equation whose roots are the squares of the differences of the roots of the given equation. To solve the important problem of the separation of the roots Waring transforms a numerical equation into one whose roots are reciprocals of the differences of the roots of the given equation. The reciprocal of the largest of the roots of the transformed equation is less than the smallest difference D, between any two roots of the given equation. If M is an upper limit of the roots of the given equation, then the subtraction of D, $2D$, $3D$, etc., from M will give values which separate all the real roots. In the *Meditationes algebraicæ* of 1770, Waring gives for the first time a process for the approximation to the values of imaginary roots. If x is approximately $a+ib$, substitute $x=a+a'+(b+b')i$, expand and reject higher powers of a' and b'. Equating real numbers to each other and imaginary numbers to each other, two equations are obtained which yield values of a' and b'.

Etienne Bézout (1730–1783) was a French writer of popular mathematical school-books. In his *Théorie générale des Equations Algébriques*, 1779, he gave the method of elimination by linear equations (invented also by L. Euler). This method was first published by him in a memoir of 1764, in which he uses determinants, without, however, entering upon their theory. A beautiful theorem as to the degree of the resultant goes by his name. He and L. Euler both gave the degree as in general $m \cdot n$, the product of the orders of the intersecting loci, and both proved the theorem by reducing the problem to one of elimination from an auxiliary set of linear equations. The determinant resulting from Bézout's method is what J. J. Sylvester and later writers call the Bézoutiant. Bézout fixed the degree of the eliminant also for a large number of particular cases. "One may say that he determined the number of finite intersections of algebraic loci, not only when all the intersections are finite, but also when singular

[1] M. Cantor, *op. cit.*, Vol. IV, 1908, p. 275.

in a square in such a way that in each row and column there are six officers, all of different grades as well as of different regiments. Euler thinks that no solution is obtainable when the order of the square is of the form 2 mod. 4. Arthur Cayley in 1890 reviewed what had been written; P. A. MacMahon solved it in 1915. It is called the problem of the "Latin squares," because Euler, in his notation, used "*n* lettres latines." Euler enunciated and proved a well-known theorem, giving the relation between the number of vertices, faces, and edges of certain polyhedra, which, however, was known to R. Descartes. The powers of Euler were directed also towards the fascinating subject of the theory of probability, in which he solved some difficult problems.

Of no little importance are Euler's labors in analytical mechanics. Says Whewell: "The person who did most to give to analysis the generality and symmetry which are now its pride, was also the person who made mechanics analytical; I mean Euler."[1] He worked out the theory of the rotation of a body around a fixed point, established the general equations of motion of a free body, and the general equation of hydrodynamics. He solved an immense number and variety of mechanical problems, which arose in his mind on all occasions. Thus, on reading Virgil's lines, "The anchor drops, the rushing keel is staid," he could not help inquiring what would be the ship's motion in such a case. About the same time as Daniel Bernoulli he published the *Principle of the Conservation of Areas* and defended the principle of "least action," advanced by P. Maupertius. He wrote also on tides and on sound.

Astronomy owes to Euler the method of the variation of arbitrary constants. By it he attacked the problem of perturbations, explaining, in case of two planets, the secular variations of eccentricities, nodes, etc. He was one of the first to take up with success the theory of the moon's motion by giving approximate solutions to the "problem of three bodies." He laid a sound basis for the calculation of tables of the moon. These researches on the moon's motion, which captured two prizes, were carried on while he was blind, with the assistance of his sons and two of his pupils. His *Mechanica sive motus scientia analytice exposita*, Vol. I, 1736, Vol. II, 1742, is, in the language of Lagrange, "the first great work in which analysis is applied to the science of movement."

Prophetic was his study of the movements of the earth's pole. He showed that if the axis around which the earth rotates is not coincident with the axis of figure, the axis of rotation will revolve about the axis of figure in a predictable period. On the assumption that the earth is perfectly rigid he showed that the period is 305 days. The earth is now known to be elastic. From observations taken in 1884–5,

[1] W. Whewell, *History of the Inductive Sciences*, 3rd Ed., Vol. 1, New York, 1858, p. 363.

S. C. Chandler of Harvard found the period to be 428 days.[1] For an earth of steel the time has been computed to be 441 days.

Euler in his *Introductio in analysin* (1748) had undertaken a classification of quartic curves, as had also a mathematician of Geneva, *Gabriel Cramer* (1704–1752), in his *Introduction à l'analyse des lignes courbes algébraiques*, Geneva, 1750. Both based their classifications on the behavior of the curves at infinity, obtaining thereby eight classes which were divided into a considerable number of species. Another classification was made by E. Waring, in his *Miscellanea analytica*, 1792, which yielded 12 main divisions and 84551 species. These classifications rest upon ideas hardly in harmony with the more recent projective methods, and have been abandoned. Cramer studied the quartic $y^4-x^4+ay^2+bx^2=0$ which later received the attention of F. Moigno (1840), Charles Briot and Jean Claude Bouquet, and B. A. Nievenglowski (1895), and because of its peculiar form was called by the French "courbe du diable." Cramer gave also a classification of quintic curves.

Most of Euler's memoirs are contained in the transactions of the Academy of Sciences at St. Petersburg, and in those of the Academy at Berlin. From 1728 to 1783 a large portion of the Petropolitan transactions were filled by his writings. He had engaged to furnish the Petersburg Academy with memoirs in sufficient number to enrich its acts for twenty years—a promise more than fulfilled, for down to 1818 the volumes usually contained one or more papers of his, and numerous papers are still unpublished. His mode of working was, first to concentrate his powers upon a special problem, then to solve separately all problems growing out of the first. No one excelled him in dexterity of accommodating methods to special problems. It is easy to see that mathematicians could not long continue in Euler's habit of writing and publishing. The material would soon grow to such enormous proportions as to be unmanageable. We are not surprised to see almost the opposite in J. Lagrange, his great successor. The great Frenchman delighted in the general and abstract, rather than, like Euler, in the special and concrete. His writings are condensed and give in a nutshell what Euler narrates at great length.

Jean-le-Rond D'Alembert (1717–1783) was exposed, when an infant, by his mother in a market by the church of St. Jean-le-Rond, near the Nôtre-Dame in Paris, from which he derived his Christian name. He was brought up by the wife of a poor glazier. It is said that when he began to show signs of great talent, his mother sent for him, but received the reply, "You are only my step-mother; the glazier's wife is my mother." His father provided him with a yearly income. D'Alembert entered upon the study of law, but such was his love for mathematics, that law was soon abandoned. At the age of twenty-four his reputation as a mathematician secured for him ad-

[1] For details see *Nature*, Vol. 97, 1916, p. 530.

mathematics, Smith once proposed a toast, "Pure mathematics; may it never be of any use to any one."

Ernst Eduard Kummer (1810–1893), professor in the University of Berlin, is closely identified with the theory of numbers. P. G. L. Dirichlet's work on complex numbers of the form $a+ib$, introduced by K. F. Gauss, was extended by him, by F. Eisenstein, and R. Dedekind. Instead of the equation $x^4-1=0$, the roots of which yield Gauss' units, F. Eisenstein used the equation $x^3-1=0$ and complex numbers $a+b\rho$ (ρ being a cube root of unity), the theory of which resembles that of Gauss' numbers. E. E. Kummer passed to the general case $x^n-1=0$ and got complex numbers of the form $\alpha=a_1A_1+a_2A_2+a_3A_3+\dots$, were a_i are whole real numbers, and A_i roots of the above equation. Euclid's theory of the greatest common divisor is not applicable to such complex numbers, and their prime factors cannot be defined in the same way as prime factors of common integers are defined. In the effort to overcome this difficulty, E. E. Kummer was led to introduce the conception of "ideal numbers." These ideal numbers have been applied by G. Zolotarev of St. Petersburg to the solution of a problem of the integral calculus, left unfinished by Abel.[1] J. W. R. Dedekind of Braunschweig has given in the second edition of Dirichlet's *Vorlesungen über Zahlentheorie* a new theory of complex numbers, in which he to some extent deviates from the course of E. E. Kummer, and avoids the use of ideal numbers. Dedekind has taken the roots of any irreducible equation with integral coefficients as the units for his complex numbers. F. Klein in 1893 introduced simplicity by a geometric treatment of ideal numbers.

Fermat's "Last Theorem," Waring's Theorem

E. E. Kummer's ideal numbers owe their origin to his efforts to prove the impossibility of solving in integers Fermat's equation $x^n+y^n=z^n$ for $n>2$. We premise that some progress in proving this impossibility has been made by more elementary means. For integers x, y, z not divisible by an odd prime n, the theorem has been proved by the Parisian mathematician and philosopher *Sophie Germain* (1776–1831) for $n<100$, by Legendre for $n<200$, by E. T. Maillet for $n<223$, by Dmitry Mirimanoff for $n<257$, by L. E. Dickson for $n<7000$.[2] The method used here is due to Sophie Germain and requires the determination of an odd prime p for which $x^n+y^n+z^n\equiv 0 \pmod{p}$ has no solutions, each not divisible by p, and n is not the residue modulo p of the nth power of any integer. E. E. Kummer's results rest on an advanced theory of algebraic numbers which he

[1] H. J. S. Smith, "On the Present State and Prospects of Some Branches of Pure Mathematics," *Proceed. London Math. Soc.*, Vol. 8, 1876, p. 15.

[2] See L. E. Dickson in *Annals of Mathematics*, 2. S., Vol. 18, 1917, pp. 161–187. See also L. E. Dickson in *Atti del IV. Congr. Roma*, 1908, Roma, 1909, Vol. II, p. 172.

helped to create. Once at an early period he thought that he had a complete proof. He laid it before P. G. L. Dirichlet who pointed out that, although he had proved that any number $f(\alpha)$, where α is a complex n^{th} root of unity and n is prime, was the product of indecomposable factors, he had assumed that such a factorization was unique, whereas this was not true in general.[1] After years of study, E. E. Kummer concluded that this non-uniqueness of factorization was due to $f(\alpha)$ being too small a domain of numbers to permit the presence in it of the true prime numbers. He was led to the creation of his ideal numbers, the machinery of which, says L. E. Dickson,[2] is "so delicate that an expert must handle it with the greatest care, and (is) nowadays chiefly of historical interest in view of the simpler and more general theory of R. Dedekind." By means of his ideal numbers he produced a proof of Fermat's last theorem, which is not general but excludes certain particular values of n, which values are rare among the smaller values of n; there are no values of n below 100, for which E. E. Kummer's proof does not serve. In 1857 the French Academy of Sciences awarded E. E. Kummer a prize of 3000 francs for his researches on complex integers.

The first marked advance since Kummer was made by A. Wieferich of Münster, in *Crelle's Journal*, Vol. 136, 1909, who demonstrated that if p is prime and 2^p-2 is not divisible by p^2, the equation $x^p+y^p=z^p$ cannot be solved in terms of positive integers which are not multiples of p. Waldemar Meissner of Charlottenburg found that 2^p-2 is divisible by p^2 when $p=1093$ and for no other prime p less than 2000. Recent advances toward a more general proof of Fermat's last theorem have been made by D. Mirimanoff of Geneva, G. Frobenius of Berlin, E. Hecke of Göttingen, F. Bernstein of Göttingen, Ph. Furtwängler of Bonn, S. Bohnicek and H. S. Vandiver of Philadelphia. Recent efforts along this line have been stimulated in part by a bequest of 100,000 marks made in 1908 to the Königliche Gesellschaft der Wissenschaften in Göttingen, by the mathematician F. P. Wolfskehl of Darmstadt, as a prize for a complete proof of Fermat's last theorem. Since then hundreds of erroneous proofs have been published. Post-mortems over proofs which fall still-born from the press are being held in the "Sprechsaal" of the Archiv der Mathematik und Physik.

At the beginning of the present century progress was made in proving another celebrated theorem, known as "Waring's theorem." In 1909 A. Wieferich of Münster proved the part which says that every positive integer is equal to the sum of not more than 9 positive cubes. He established also, that every positive integer is equal to the sum of not more than 37 (according to Waring, it is not more than 19) positive fourth powers, while D. Hilbert proved in 1909 that, for

[1] *Festschrift z. Feier des 100. Geburtstages Eduard Kummers*, Leipzig, 1910, p. 22.

[2] *Bull. Am. Math. Soc.*, Vol. 17, 1911, p. 371.

every integer $n>2$ (Waring had declared for every integer $n>4$), each positive integer is expressible as the sum of positive nth powers, the number of which lies within a limit dependent only upon the value of n. Actual determinations of such upper limits have been made by A. Hurwitz, E. T. Maillet, A. Fleck, and A. J. Kempner. Kempner proved in 1912 that there is an infinity of numbers which are not the sum of less than $4 \cdot 2^n$ positive 2^nth powers, $n \geqq 2$.

Other Recent Researches. Number Fields

Attracted by E. E. Kummer's investigations, his pupil, **Leopold Kronecker** (1823–1891) made researches which he applied to algebraic equations. On the other hand, efforts have been made to utilize in the theory of numbers the results of the modern higher algebra. Following up researches of Ch. Hermite, **Paul Bachmann** of Münster, now of Weimar, investigated the arithmetical formula which gives the automorphics of a ternary quadratic form.[1] Bachmann is the author of well-known texts on *Zahlentheorie*, in several volumes, which appeared in 1892, 1894, 1872, 1898, and 1905, respectively. The problem of the equivalence of two positive or definite ternary quadratic forms was solved by L. Seeber; and that of the arithmetical automorphics of such forms, by F. G. Eisenstein. The more difficult problem of the equivalence for indefinite ternary forms has been investigated by Eduard Selling of Würzburg. On quadratic forms of four or more indeterminates little has yet been done. Ch. Hermite showed that the number of non-equivalent classes of quadratic forms having integral coefficients and a given discriminant is finite, while Zolotarev and Alexander Korkine (1837–1908), both of St. Petersburg, investigated the minima of positive quadratic forms. In connection with binary quadratic forms, H. J. S. Smith established the theorem that if the joint invariant of two properly primitive forms vanishes, the determinant of either of them is represented primitively by the duplicate of the other.

The interchange of theorems between arithmetic and algebra is displayed in the recent researches of J. W. L. Glaisher (1848–) of Trinity College and J. J. Sylvester. Sylvester gave a Constructive Theory of Partitions, which received additions from his pupils, F. Franklin, now of New York city, and George Stetson Ely (?–1918), for many years examiner in the U. S. Patent Office.

By the introduction of "ideal numbers" E. E. Kummer took a first step toward a theory of fields of numbers. The consideration of super fields (Oberkörper) from which the properties of a given field of numbers may be easily derived is due mainly to R. Dedekind and to L. Kronecker. Thereby there was opened up for the theory of numbers a new and wide territory which is in close connection with

[1] H. J. S. Smith in *Proceed. London Math. Soc.*, Vol. 8, 1876, p. 13.

algebra and the theory of functions. The importance of this subject in the theory of equations is at once evident if we call to mind E. Galois' fields of rationality. The interrelation between number theory and function theory is illustrated in Riemann's researches in which the frequency of primes was made to depend upon the zero-places of a certain analytical function, and in the transcendence of e and π which is an arithmetic property of the exponential function. In 1883–1890 L. Kronecker published important results on elliptic functions which contain arithmetical theorems of great elegance. The Dedekind method of extending Kummer's results to algebraic numbers in general is based on the notion of an *ideal*. A common characteristic of Dedekind and Kronecker's procedure is the introduction of compound moduli. G. M. Mathews says[1] that, in practice it is convenient to combine the methods of L. Kronecker and R. Dedekind. Of central importance are the Galoisian or normal fields, which have been studied extensively by D. Hilbert. L. Kronecker established the theorem that all Abelian fields are cyclotomic, which was proved also by H. Weber and D. Hilbert. An important report, prepared by D. Hilbert and entitled *Theorie der algebraischen Zahlkörper*, was published in 1894.[2] D. Hilbert first develops the theory of general number-fields, then that of special fields, viz., the Galois field, the quadratic field, the circle field (Kreiskörper), the Kummer field. A report on later investigations was published by R. Fueter in 1911.[3] Chief among the workers in this subject which have not yet been mentioned are F. Bernstein, Ph. Furtwängler, H. Minkowski, Ch. Hermite, and A. Hurwitz. Accounts of the theory are given in H. Weber's *Lehrbuch der Algebra*, Vol. 2 (1899), J. Sommer's *Vorlesungen über Zahlentheorie* (1907), and Hermann Minkowski's *Diophantische Approximationen*, Leipzig (1907). H. Minkowski gives in geometric and arithmetic language both old and new results. His use of lattices serves as a geometric setting for algebraic theory and for the proof of some new results.

A new and powerful method of attacking questions on the theory of algebraic numbers was advanced by Kurt Hensel of Königsberg in his *Theorie der algebraischen Zahlen*, 1908, and in his *Zahlentheorie*, 1913. His method is analogous to that of power series in the theory of analytic functions. He employs expansions of numbers into power series in an arbitrary prime number p. This theory of p-adic numbers is generalized by him in his book of 1913 into the theory of g-adic numbers, where g is any integer.[4]

The resolution of a given large number into factors is a difficult problem which has been taken up by Paul Seelhof, François Edouard

[1] Art. "Number" in the *Encyclop. Britannica*, 11th ed., p. 857.
[2] *Jahresbericht d. d. Math. Vereinigung*, Vol. 4, pp. 177–546.
[3] *Loc. cit.*, Vol. 20, pp. 1–47.
[4] *Bull. Am. Math. Soc.*, Vol. 20, 1914, p. 259.