December 29. It was ſtrangely removed, and got before, not the Eaſtern Star only of the mentioned bright *Triangle*, but alſo the moſt Northern. I think, at leaſt, in this laſt 24 Hours, it had moved 4 Degrees. The *Moon* ſhining bright, the Tail could not well be obſerved, yet ſtill it ſeemed to point directly to *Canis minor*.

II. Æquationum

II. *Æquationum Cubicarum & Biquadraticarum, tum Analytica, tum Geometrica & Mechanica, Reſolutio Univerſalis*, a J. Colſon.

§. 1. ÆQuationis Cubicæ Univerſalis

$$x^3 = \begin{array}{r} 3px^2 + 3qx + 2r, \\ -3p^2 + p^3 \\ -3pq \end{array}$$

Radices Tres ſunt,

$$x = p + \sqrt[3]{r + \sqrt{r^2 - q^3}} + \sqrt[3]{r - \sqrt{r^2 - q^3}}$$

$$x = p - \frac{1 - \sqrt{-3}}{2} \times \sqrt[3]{r + \sqrt{r^2 - q^3}} - \frac{1 + \sqrt{-3}}{2} \times \sqrt[3]{r - \sqrt{r^2 - q^3}}$$

$$x = p - \frac{1 + \sqrt{-3}}{2} \times \sqrt[3]{r + \sqrt{r^2 - q^3}} - \frac{1 - \sqrt{-3}}{2} \times \sqrt[3]{r - \sqrt{r^2 - q^3}}$$

Vel ut Calculus Arithmeticus facilior ac paratior evadat, ſi poſueris Binomii irrationalis $r + \sqrt{r^2 - q^3}$ Radicem Cubicam eſſe $m + \sqrt{n}$, erunt ejuſdem Æquationis Radices tres $x = p + 2m$, & $x = p - m \pm \sqrt{-3n}$.

Igitur data Æquatione quavis Cubica, inter ejus hujuſque Æquationis Univerſalis terminos ſingulos inſtituenda eſt comparatio, quo pacto facillime invenientur ipſæ p, q, r; & hiſce cognitis, innoteſcent Æquationis datæ Radices omnes. Hujus vero Solutionis Exempla ſint ſequentia in Numeris.

1. Æquationis Cubicæ $x^3 = 2x^2 + 3x + 4$ ſit Radix x indaganda. Erit primò juxta præſcriptum $3p = 2$,

 ſive

sive $p = \frac{2}{3}$. Secundò $3q - (3p^2) \frac{4}{3} = 3$, sive $q = \frac{13}{9}$. Tertiò $2r\,(+\overline{p^2 - 3q} \times p) - \frac{70}{27} = 4$, sive $r = \frac{89}{27}$, & $r^2 - q^3 = \frac{212}{27}$. Et propterea $x = \frac{2}{3} + \sqrt[3]{\frac{89}{27} + \sqrt{\frac{212}{27}}} + \sqrt[3]{\frac{89}{27} - \sqrt{\frac{212}{27}}}$. Reliquæ duæ Radices sunt impossibiles.

2. In Æquatione $x^3 = 12x^2 - 41x + 42$, erit primò $3p = 12$, sive $p = 4$. Secundò $3q - (3p^2)\, 48 = -41$, sive $q = \frac{7}{3}$. Tertiò $2r + (\overline{p^2 - 3q} \times p)\, 36 = 42$, sive $r = 3$; Et inde $r^2 - q^3 = -\frac{100}{27}$. At Binomii surdi $3 + \sqrt{-\frac{100}{27}}\ (= r + \sqrt{r^2 - q^3})$ Radix Cubica, per Methodos ex Arithmetica petendas extracta, est $-1 + \sqrt{-\frac{4}{3}}\ (= m + \sqrt{n},)$ & proinde Radix $x = (p + 2m = 4 - 2 =)\, 2$, vel etiam $x = (p - m \pm \sqrt{-3n} = 4 + 1 \pm (\sqrt{4})\, 2 =)\, 7$ vel 3. Vel rursus, ejusdem Binomii $3 + \sqrt{-\frac{100}{27}}$. Radix alia Cubica (tres enim agnoscit) est $\frac{3}{2} + \sqrt{-\frac{1}{12}}\ (= m + \sqrt{n},)$ & proinde Radix $x = (p + 2m = 4 + 3 =)\, 7$, & etiam $x = (p - m \pm \sqrt{-3n} = 4 - \frac{3}{2} \pm (\sqrt{\frac{1}{4}})\, \frac{1}{2} = 3$ vel 2. Vel denuo, ejusdem Binomii $3 + \sqrt{-\frac{100}{27}}$ Radix Cubica tertia est $-\frac{1}{2} - \sqrt{-\frac{25}{12}}$, $(= m + \sqrt{n},)$ & proinde Radix

$x =$

$x = (p + 2m = 4 - 1 =)\, 3$, atque etiam $x = (p - m \pm \sqrt{-3n} = 4 + \frac{1}{2} \pm (\sqrt{\frac{25}{4}})\, \frac{5}{2} =)\, 7$ vel 2.

3. In Æquatione $x^3 = -15x^2 - 84x + 100$, erit $p = -5$, $q = -3$, $r = 135$; & Binomii $135 + \sqrt{18252}$ Radix Cubica est $3 + \sqrt{12}$. Igitur Radix $x = -5 + 6 = 1$, & $x = -5 - 3 \pm \sqrt{-36} = -8 \pm \sqrt{-36}$, impossibiles.

4. In Æquatione $x^3 = 34x^2 - 310x + 1012$, erit $p = \frac{34}{3}$, $q = \frac{226}{9}$, $r = \frac{5536}{27}$; & Binomii $\frac{5536}{27} + \sqrt{\frac{707560}{27}}$ Radix Cubica est $\frac{16}{3} + \sqrt{\frac{10}{3}}$. Igitur Radix $x = \frac{34}{3} + \frac{32}{3} = 22$, & $x = \frac{34}{3} - \frac{16}{3} \pm \sqrt{-10} = 6 \pm \sqrt{-10}$, impossibiles.

5. In Æquatione $x^3 = 28x^2 + 61x - 4048$, erit $p = \frac{28}{3}$, $q = \frac{967}{9}$, $r = -\frac{25010}{27}$; & Binomii $-\frac{25010}{27} + \sqrt{-382347}$. Radix Cubica est $\frac{41}{6} + \sqrt{-\frac{243}{4}}$. Igitur $x = \frac{28}{3} + \frac{41}{3} = 23$, & $x = \frac{28}{3} - \frac{41}{6} \pm (\sqrt{\frac{729}{4}})\, \frac{27}{2} = 16$ vel -11.

6. In Æquatione $x^3 = -x^2 + 166x - 660$, erit $p = -\frac{1}{3}$, $q = \frac{499}{9}$, $r = -\frac{9658}{27}$; & Binomii $-\frac{9658}{27} + \sqrt{-\frac{1147205}{27}}$ Radix Cubica est $-\frac{22}{3} + \sqrt{-\frac{5}{3}}$. Igitur $x = -\frac{1}{3} - \frac{44}{3} = -15$, & $x = -\frac{1}{3} + \frac{22}{3} \pm \sqrt{5} = 7 \pm \sqrt{5}$, irrationales.

7. In

7. In Æquatione $x^3 = 63x^2 + 99673x + 9951705$, erit $p = 21$, $q = \frac{100996}{3}$, $r = 6031680$; & Binomii $6031680 + \sqrt{-\frac{47887175043136}{27}}$ Radix Cubica eſt $183 + \sqrt{-\frac{529}{3}}$. Igitur $x = 21 + 366 = 387$, & $x = 21 - 183 \pm (\sqrt{529})\,23 = -139$ vel 185.

Nec ſecus in cæteris procedendum: Inveſtigatur autem Theorema ad modum ſequentem. Pono Æquationis cujuſdam Cubicæ Radicem $z = a + b$, & cubicè multiplicando proveniet $z^3 = (a^3 + 3a^2b + 3ab^2 + b^3 =)\ a^3 + 3ab \times \overline{a+b} + b^3$. Jam loco ipſius $a + b$ valorem ejus z ſubſtituendo, fiet $z^3 = 3abz + a^3 + b^3$, quæ eſt Æquatio Cubica ex Radice $z = a + b$ conſtructa, cui terminus ſecundus deeſt. Ut hæc verò ad formam magis commodam magiſq; concinnam revocenter, ſumo Æquationem $z^3 = 3qz + 2r$, quæ poſthac ipſius $z^3 = 3abz + a^3 + b^3$ vices gerat. Igitur tranſmutatione hujus in illam, fiet primò $3q = 3ab$, ſive $q^3 = a^3b^3$; & ſecundò $2r = a^3 + b^3$, ſive $2ra^3 = (a^6 + a^3b^3 =)\ a^6 + q^3$. Et ſoluta hac æquatione quadratica, erit $a^3 = r + \sqrt{r^2 - q^3}$, & hinc $b^3 = (2r - a^3 =)\ r - \sqrt{r^2 - q^3}$: Atque igitur tandem $a = \sqrt[3]{r + \sqrt{r^2 - q^3}}$ & $b = \sqrt[3]{r - \sqrt{r^2 - q^3}}$. Et propterea in Æquatione Cubica $z^3 = 3qz + 2r$ erit Radix $z = (a + b =)\ \sqrt[3]{r^2 + \sqrt{r - q^3}} + \sqrt[3]{r^2 - \sqrt{r - q^3}}$

At verò hæc Radix reverâ triplex eſt, pro triplici valore quem induere poteſt $\sqrt[3]{r + \sqrt{r^2 - q^3}}$ & $\sqrt[3]{r - \sqrt{r^2 - q^3}}$. Cujuſvis enim quantitatis Radix Cubica triplex erit, & ipſius Unitatis Radix Cubica vel

eſt

eſt 1, vel $-\frac{1}{2} + \frac{1}{2}\sqrt{-3}$, vel $-\frac{1}{2} - \frac{1}{2}\sqrt{-3}$: Atque id adeo, propterea quòd harum alicujus Cubus ſit Unitas. Igitur ſi $1 \times \sqrt[3]{r + \sqrt{r^2 - q^3}}$ aut $\sqrt[3]{r + \sqrt{r^2 - q^3}}$ $(= \sqrt[3]{1 \times r + \sqrt{r^2 - q^3}} = \sqrt[3]{1} \times \sqrt[3]{r + \sqrt{r^2 - q^3}})$ Radicem aliquam [quam ſupra nominavimus $m + \sqrt{n}$, aut $1 \times \overline{m + \sqrt{n}}$,] Cubi $r + \sqrt{r^2 - q^3}$ deſignet; ipſæ $\frac{-1 + \sqrt{-3}}{2} \times \sqrt[3]{r + \sqrt{r^2 - q^3}}$ & $\frac{-1 - \sqrt{-3}}{2} \times \sqrt[3]{r + \sqrt{r^2 - q^3}}$ [i. e. $\frac{-1 + \sqrt{-3}}{2} \times \overline{m + \sqrt{n}}$ & $\frac{-1 - \sqrt{-3}}{2} \times \overline{m + \sqrt{n}}$] alias duas ejuſdem Cubi Radices deſignabunt. Similiter & $\sqrt[3]{r - \sqrt{r^2 - q^3}}$, $\frac{-1 + \sqrt{-3}}{2} \times \sqrt[3]{r - \sqrt{r^2 - q^3}}$, & $\frac{-1 - \sqrt{-3}}{2} \times \sqrt[3]{r - \sqrt{r^2 - q^3}}$, [i. e. $m - \sqrt{n}$, $\frac{-1 + \sqrt{-3}}{2} \times \overline{m - \sqrt{n}}$, $\frac{-1 - \sqrt{-3}}{2} \times \overline{m - \sqrt{n}}$,] tres Cubicæ Radices erunt Apotomes $r - \sqrt{r^2 - q^3}$. Atque has Radices debitè connectendo, fiet $z = \sqrt[3]{r + \sqrt{r^2 - q^3}} + \sqrt[3]{r - \sqrt{r^2 - q^3}}$, [i. e. $z = \overline{m + \sqrt{n}} + \overline{m - \sqrt{n}} = 2m$,] $z = \frac{-1 + \sqrt{-3}}{2} \times \sqrt[3]{r + \sqrt{r^2 - q^3}} + \frac{-1 - \sqrt{-3}}{2} \times \sqrt[3]{r - \sqrt{r^2 - q^3}}$, [i. e. $z = \frac{-1 + \sqrt{-3}}{2} \times \overline{m + \sqrt{n}} + \frac{-1 - \sqrt{-3}}{2} \times \overline{m - \sqrt{n}} = -m + \sqrt{-3n}$,] & $z = \frac{-1 - \sqrt{-3}}{2} \times \sqrt[3]{r + \sqrt{r^2 - q^3}} + \frac{-1 + \sqrt{-3}}{2} \times \sqrt[3]{r - \sqrt{r^2 - q^3}}$

[i. e.

[i. e. $z = \frac{-1-\sqrt{-3}}{2} \times \overline{m+\sqrt{n}} + \frac{-1+\sqrt{-3}}{2} \times \overline{m-\sqrt{n}} = -m - \sqrt{-3n}$,] quæ tres erunt Radices Æquationis Cubicæ $z^3 = 3qz + 2r$. Debite autem connectuntur Radices istæ ad modum præcedentem, quippe quæ sic connexæ, & more vulgari in se invicem continue ductæ, Æquationem $z^3 = 3qz + 2r$ restituunt. Denique fac $z = x - p$, & Æquatio fiet $x^3 - 3px^2 + 3p^2x - p^3 = 3qx - 3pq + 2r$, quæ universalis est, & cujus Radices evadunt ut supra fuerant exhibitæ.

Hic obiter notatu dignum est, quod Æquationis Cubicæ cujuscunque Radices omnes sint possibiles & reales, quoties Binomii membrum irrationale $\sqrt{r^2 - q^3}$ impossibilitatem in se complectitur; hoc est, quoties q est quantitas affirmativa, & simul cubus ejus major est quadrato ex latere r. At si membrum istud $\sqrt{r^2 - q^3}$ sit possibile, hoc est si q sit quantitas negativa, aut etiam si affirmativæ cubus sit minor quadrato ex latere r, tunc unicam tantùm agnoscit Æquatio Radicem possibilem & realem, reliquæque duæ erunt impossibiles.

In hoc Theoremate si fiat $p = 0$, hoc est, si desit Æquationis terminus secundus, tunc deventum erit ad casum Regularum quæ dicuntur *Cardani*, cujus solutio continetur in præcedentibus.

§. 2. Æquationis Biquadraticæ Universalis

$$x^4 = 4px^3 + \begin{array}{l} 2q \\ -4p^2 \end{array} x^2 + \begin{array}{l} 8r \\ -4pq \end{array} x + \begin{array}{l} 4s, \\ -q^2 \end{array}$$

Radices quatuor sunt $x = p - a \pm \sqrt{p^2 + q - a^2 - \frac{2r}{a}}$,

& $x = p + a \pm \sqrt{p^2 + q - a^2 + \frac{2r}{a}}$, Ubi a^2 est Radix Æquationis Cubicæ $a^6 = \begin{array}{l} p^2 \\ +q \end{array} a^4 \begin{array}{l} -2pr \\ -s \end{array} a^2 + r^2$.

Jam data Æquatione quavis Biquadratica, inter ejus hujusque Æquationis Universalis terminos singulos instituenda

enda est comparatio, quo pacto citissime invenientur ipsæ p, q, r, s; & hisce cognitis, non latebit valor ipsius a, ex Theoremate superiori inveniendus, & tum demum innotescent Æquationis datæ Radices omnes.

Huic Solutioni illustrandæ Exemplum unum aut alterum sufficiat.

1. Æquationis Biquadraticæ $x^4 = 8x^3 + 83x^2 - 162x - 936$ sint Radices extrahendæ. Erit primò juxta præscriptum $4p = 8$, sive $p = 2$. Secundò $2q - (4p^2)\ 16 = 83$, sive $q = \frac{99}{2}$. Tertiò $8r - (4pq)\ 396 = -162$, sive $r = \frac{117}{4}$. Quartò $4s - (q^2)\ \frac{9801}{4} = -936$, sive $s = \frac{6057}{16}$. Hinc $p^2 + q = \frac{107}{2}$, $2pr + s = \frac{7929}{16}$, $r^2 = \frac{13689}{19}$, & proinde $a^6 = \frac{107}{2}a^4 - \frac{7929}{16}a^2 + \frac{13689}{16}$. Jam ut Æquatio hæc aliquatenus Cubica in Radices ejus resolvatur, ad Theorema præcedens recurrendum est, in quo erit $p = \frac{107}{2}$, $q = \frac{22009}{144}$, $r = \frac{2903923}{1728}$ & $r^2 - q^3 = -\frac{11940075}{16}$. Atqui Binomii $\frac{2903923}{1728} + \sqrt{-\frac{11940075}{16}}$ Radix Cubica est $-\frac{53}{12} + \sqrt{-\frac{400}{3}}$ & propterea $a^2 = \frac{107}{6} - \frac{53}{6} = 9$, & etiam $a^2 = \frac{107}{6} + \frac{53}{12} \pm (\sqrt{400})\ 20 = \frac{169}{4}$ vel $\frac{9}{4}$: Vel quod perinde est, Æquationis præmissæ reverà Cubo-Cubicæ sex Radices sunt $a = \pm 3$, $a = \pm\frac{13}{2}$, & $a = \pm\frac{3}{2}$, quarum quævis indiscriminatim propo-

sito

fito noſtro faciet ſatis. Puta ſi in præſenti caſu fiat $a = 3$, erit juxta Theorema $x = (p - a \pm \sqrt{p^2 + q - a^2 - \frac{2r}{a}} = 2 - 3 \pm \sqrt{4 + \frac{99}{2} - 9 - \frac{39}{2}} = -1 \pm (\sqrt{25})\,5 =)\; 4$ vel -6, & $x = (p + a \pm \sqrt{p^2 + q - a^2 + \frac{2r}{a}} = 2 + 3 \pm \sqrt{4 + \frac{99}{2} - 9 + \frac{39}{2}} = 5 \pm (\sqrt{64})\,8 =)\; 13$ vel -3, quæ ſunt Æquationis datæ Radices quatuor,

2. In Æquatione $x^4 = 20x^3 + 252x^2 - 6592x + 21312$, erit $p = 5$, $q = 176$, $r = -384$, & $s = 13072$. Hinc $p^2 + q = 201$, $2pr + s = 9232$, & $r^2 = 147456$; & inde $a^6 = 201\,a^4 - 9232\,a^2 + 147456$. Jam in Theoremate pro Cubicis erit $p = 67$, $q = \frac{4235}{3}$, & $r = 65219$; eritque Binomii $65219 + \sqrt{\frac{3888930707 2}{27}}$ Radix Cubica $\frac{77}{2} + \sqrt{\frac{847}{12}}$. Igitur $a^2 = 67 + 77 = 144$, ſive $a = 12$; & proinde $x = 5 - 12 \pm \sqrt{25 + 176 - 144 + 64} = -7 \pm (\sqrt{121})\,11 = 4$ vel -18, & $x = 5 + 12 \pm \sqrt{25 + 176 - 144 - 64} = 17 \pm \sqrt{-7}$, impoſſibiles.

Hujus autem Theorematis Inventio eſt hujuſmodi, Ex duarum Æquationum Quadraticarum $z^2 + 2az - b = 0$, & $z^2 - 2az - c = 0$ in ſe invicem multiplicatione, Æquationem conficio Biquadraticam $z^4 = \overline{4a^2 + b + c} \times z^2 + \overline{2ac - 2ab} \times z - bc$, cui terminus ſecundus deeſt, quamque huic Æquationi $z^4 = ez^2 + fz + g$ ſtatuo æquipollere. Unde primo $4a^2 + b + c = e$ ſive $b = e - 4a^2 - c$. Secundò $2ac - 2ab = f$, hoc eſt, $2ac - 2ae + 8a^3 + 2ac = f$, ſive $c = \frac{f}{4a} + \frac{e}{2} - 2a^2$, &

& inde $b = (e - 4a^2 - c =) - \frac{f}{4a} + \frac{e}{2} - 2a^2$. Tertiò $-bc = g$, ſive $-\frac{f^2}{16a^2} + \frac{e^2}{4} - 2ea^2 + 4a^4 = -g$, hoc eſt, $a^6 = \frac{1}{2}ea^4 - \frac{1}{4}ga^2 - \frac{1}{16}e^2a^2 + \frac{f^2}{64}$, quæ Æquatio quaſi Cubica eſt, ex Radice a^2 & notis vel aſſumptis e, f, g conflata. Ea verò Radix per Theorema ſuperius exhiberi poteſt, & eodem Calculo innoteſcent ipſæ b & c. At Æquationum $z^2 + 2az - b = 0$ & $z^2 - 2az - c = 0$ Radices ſunt $z = -a \pm \sqrt{a^2 + b}$ & $z = a \pm \sqrt{a^2 + c}$, ſive $z = -a \pm \sqrt{\frac{1}{2}e - a^2 - \frac{f}{4a}}$, & $z = a \pm \sqrt{\frac{1}{2}e - a^2 + \frac{f}{4a}}$, quæ proinde erunt Radices Æquationis $z^4 = ez^2 + fz + g$; cognita videlicet a vel a^2 ex Æquatione $a^6 = \frac{1}{2}ea^4 - \frac{1}{4}ga^2 - \frac{1}{16}e^2a^2 + \frac{f^2}{64}$. Jam ut Æquatio iſta evadat univerſalis, & omnibus ſuis terminis inſtructa, fac. $z = x - p$, eritque $x^4 - 4px^3 + 6p^2x^2 - 4p^3x + p^4 = ex^2 - 2pex + p^2e + fx - fp + g$, item & $x = p - a \pm \sqrt{\frac{1}{2}e - a^2 - \frac{f}{4a}}$, & $x = p + a \pm \sqrt{\frac{1}{2}e - a^2 + \frac{f}{4a}}$. Tandem concinnitatis & compendii gratiâ, fac $e = 2q + 2p^2$ & $f = 8r$; tum $x^4 - 4px^3 + 4p^2x^2 = 2qx^2 - 4pqx + 2p^2q + p^4 + 8rx - 8pr + g$, $x = p - a \pm \sqrt{p^2 + q - a^2 - \frac{2r}{a}}$, $x = p + a \pm \sqrt{p^2 + q - a^2 + \frac{2r}{a}}$, & $a^6 = \overline{p^2 + q} \times a^4 - \overline{\frac{1}{4}g + \frac{1}{4}p^4 + \frac{1}{2}p^2q - \frac{1}{4}q^2} \times a^2 + r^2$. Denique fac $g = 4s - q^2 + 8pr - p^4 - 2p^2q$, & fiunt Æquationes præcedentes

$$x^4 = 4px^3 \begin{smallmatrix}+2q\\-4p^2\end{smallmatrix}x^2 \begin{smallmatrix}+8r\\-4pq\end{smallmatrix}x \begin{smallmatrix}+4s\\-q^2\end{smallmatrix} \;\&\; a^6 = \begin{smallmatrix}p^2\\+q\end{smallmatrix}a^4 \begin{smallmatrix}-2pr\\-s\end{smallmatrix}a^2 + r^2.$$

Scilicet omnia evadunt ut ſupra ſunt poſita.

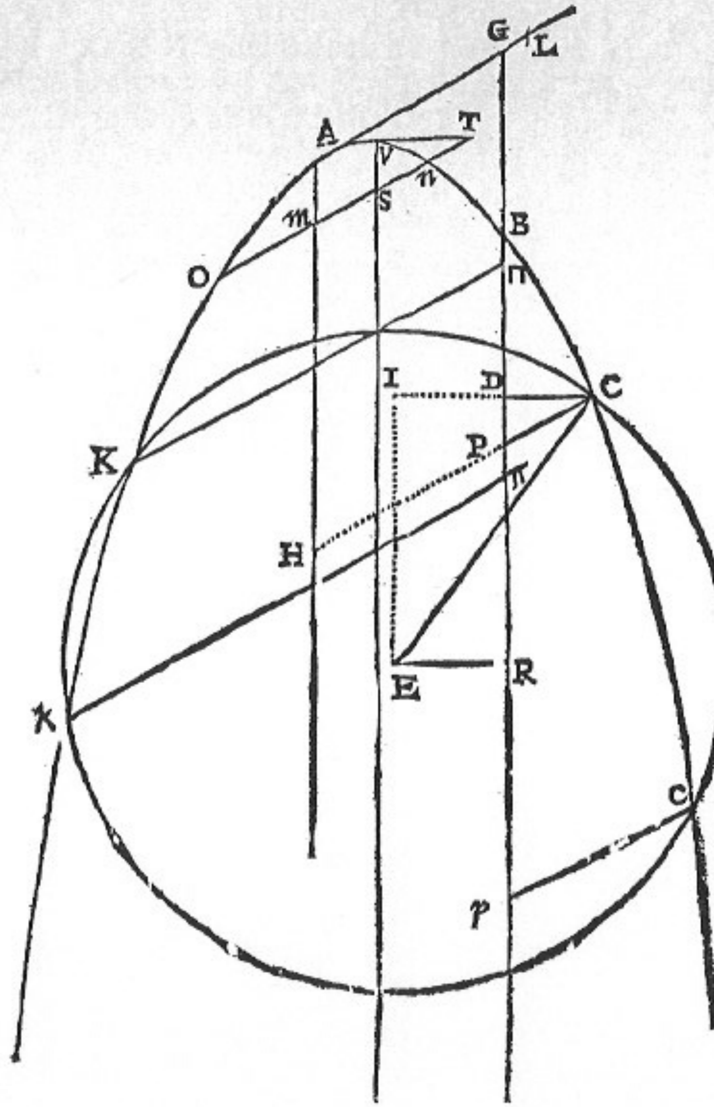
§ 3. Hacte

§ 5. Hactenus de Æquationum Cubicarum & Biquadraticarum Resolutione Analytica. Quoniam autem earundem *Effectio Geometrica* per Parabolam vulgò tradi solet, & nonnullis in pretio est, ipsam συνεπτικῶς, & quidem universalius, non pigebit hic exhibere.

Data Æquatione quavis vel Cubica vel Biquadratica, instituenda est comparatio inter terminos ejus, terminosque respondentes hujus Æquationis

$$\begin{array}{llllllll} x^4 = & \frac{2p}{q}x^3 & + & \frac{4pr}{q}x^2 & + & \frac{2p^2}{q}x & + & p^2, \\ & -4r & & -4r^2 & & -\frac{2ps}{q} & & -q^2 \\ & & & +2s & & +4rs & & -s^2 \\ & & & -1 & & -2q & & +t^2 \end{array}$$

quo pacto facile satis eruentur ipsæ p, q, r, s, t; earum interim unâ aliquâ utcunque pro lubitu assumptâ. Tum in Parabola quavis data AVB, cujus Vertex principalis V, Axis VS, & Axi



 per-

perpendicularis VT; capiatur VS = p verſus interiora Parabolæ, & in angulo SVT inſcribatur ST = q, quæ producta Parabolam ſecet in punctis binis N & O. Biſecetur ON in M, & per M agatur MA Axi parallela & Parabolæ occurrens in A. Ipſi ON parallela ducatur AL, ut ſit AL Latus rectum Parabolæ ad Diametrum AM, ſitque hæc eidem Unitas. In AL (utrinque ſi opus eſt producta) capiatur AG = r, & à puncto G ducatur GR Axi parallela, quæ Parabolam ſecet in B, à quo capiatur BR = s. A noviſſime invento puncto R ducatur RE ipſi VT parallela & æqualis, quæ ſiniſtram verſus jaceat reſpectu ipſius R ſi q ſit quantitas affirmativa, at verſus dextram ſi q ſit negativa. Atque idem de ipſis AG & BR intelligatur, quæ ad contrarias itidem partes duci debent, ſi modò valores ipſarum r & s prodeant negativi. Denique Centro E & Radio EC = t deſcribatur Circulus CKκc, qui Parabolam in totidem ſecabit punctis, quot ſunt Æquationis datæ Radices reales. Etenim à punctis iſtis C, K, &c. ducantur CP, Kπ, &c. ipſi ST parallelæ, & ad rectam GR (ſi opus eſt productam) terminatæ, eritque harum quævis x, ſeu Æquationis datæ Radix quæſita; eæ ſcilicet ad dextram jacentes erunt Radices affirmativæ, quæ verò ad ſiniſtram ſunt poſitæ erunt Radices negativæ. Punctum contactûs, ſiquod fuerit, hic ſumitur pro interſectionis punctis duobus ad invicem viciniſſimis.

Inter Æquationes Cubicas & Biquadraticas ita conſtructas hoc tantùm intercedit diſcriminis, quòd in prioribus, ob terminum ultimum in præcedente Æquatione deficientem, ſemper ſit $p^2 - q^2 - s^2 + t^2 = 0$, ſive $t = \sqrt{s^2 + q^2 - p^2}$. Igitur Centro E & Radio EB $(= \sqrt{BRq + (ERq)\,STq - VSq})$ deſcripto Circulo CKκc, Radicum una CP in priori conſtructione in nihilum abit.

Hæc autem demonſtrantur ad modum ſequentem. Manentibus jam conſtructis, & productâ CP ſi opus eſt, donec ſecat AM in H, erit CH Ordinata Parabolæ ad Diametrum

metrum AH, & proinde $CHq = AL \times AH = AH$, ob $AL = 1$. At $CH = CP + AG$, & $AH = GB + BP$, & propterea $CPq + 2\,AG \times CP + AGq = GB + BP$; ſed ob naturam Parabolæ erit $AGq = GB$, unde $CPq + 2\,AG \times CP = BP$. Jam à puncto C ad ipſam BP demittatur normalis CD, quæ occurrat etiam ipſi EI, ad BP actæ parallelæ, in puncto I. Propter ſimilia Triangula CDP & TVS, erit $DP = \frac{VS \times CP}{ST}$ & $CD = \frac{VT \times CP}{ST}$, & proinde $CPq + 2AG \times CP = BP = DP + BD = \frac{VS \times CP}{ST} + BR - IE$, ſive $CPq + 2AG \times CP - \frac{VS}{ST}CP - BR = -IE$. Aſt $IEq = CEq - CIq = CEq - CDq - VTq - 2CD \times VT = CEq - \frac{VTq \times CPq}{STq} - VTq - \frac{2VTq \times CP}{ST} =$ (ob $VTq = STq - SVq$) $CEq - CPq + \frac{SVq}{STq}CPq - STq + SVq - 2ST \times CP + \frac{2SVq}{ST}CP$, quæ igitur æqualis erit Quadrato ex Latere $CPq + 2AG \times CP - \frac{VS}{ST}CP - BR$. Atque hæc Æquatio ad terminos p, q, r, s, t revocata ipſiſſima ſit Æquatio propoſita.

Hinc liquet, quòd eadem quævis Æquatio Biquadratica innumeras per Parabolam conſtructiones ſortiri poſſit, pro indefinito valore quantitatis iſtius, quam ad arbitrium aſſumi poſſe jam diximus. Sed caſus eſt ſimpliciſſimus faciendo $VS = p = 0$, & migrat conſtructio, ſi rem ipſamſpectes, in vulgarem iſtam, in qua Radicum repræſentatrices rectæ CP, &c. ſunt ad Axem perpendiculares. Æquatio autem ſit

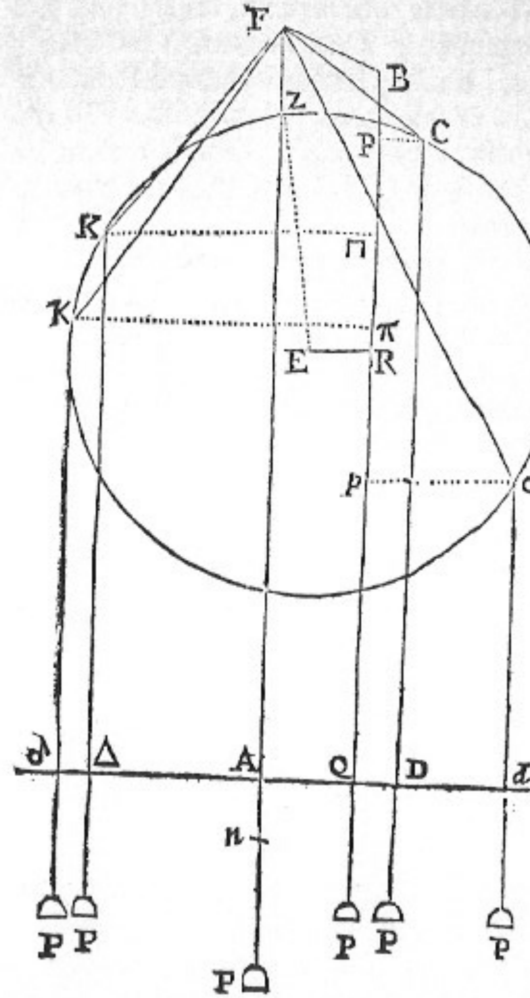
$$x^4 = \begin{array}{l} -4rx^3 \\ \\ \end{array} \begin{array}{l} -4r^2x^2 \\ +2s \\ -1 \end{array} \begin{array}{l} +4rsx \\ -2q \\ \\ \end{array} \begin{array}{l} -q^2, \\ -s^2 \\ +t^2 \end{array}$$

quæ facile conſtruitur ut ſupra.

S 4 Sed

§ 4. Sed ne Parabolæ deſcriptio Organica difficilis nimium videatur, in promptu eſt Artificium quoddam Mechanicum, ope Fili penduli pondere inſtructi peractum, cujus auxilio quam exactiſſime & facillime Æquatio noviſſima conſtrui poteſt, & proinde Æquationum quarumcunque Cubicarum & Biquadraticarum Radices inveniri; idque ſine ullo linearum ductu niſi Rectarum & Circuli. Conſtructio autem, quam appellare libet *Mechanicam*, eſt ad hunc modum.

Contra Parietem erectum, vel planum aliud quodvis Horizonti perpendiculare, ad punctum aliquod F ſuſpendatur filum tenuiſſimum & flexile FP; pondere quovis P ad extremitatem P appenſo. In hoc filo notetur punctum aliquod N, à puncto ſuſpenſionis F ſatis remotum; vel filo parvulus, ſi id mavis, innectatur Nodus N. Et ſumpta utcunque NO pro Unitate, ad punctum medium A ducatur (in plano prædicto) recta AQ Horizonti parallela, & utrinque quantum ſatis producta. Hiſce generaliter paratis, pro particulari jam applicatione fac AQ = r; ipſis q, r, s, t, ut ſæpius inculcatum, vel Arithmeticè vel Geometricè, pro datæ cujuſvis Æquationis ex-



exigentia, in Æquatione noviſſima priùs determinatis. Tunc Acu vel Stylo tenuiſſimo, aut etiam cuſpide Circini admodum gracili, flectatur filum à loco ſuo ad punctum quoddam B, ita ut punctum N cadat in noviſſime invento puncto Q. In BQ ab iſto B capiatur BR = s, & in R ad ipſam BR perpendicularis erigatur ER = q. Verùm enimverò iſtæ AQ, BR, RE ad contrarias partes ab earum initiis cadere debent, ſi fortè valores ipſarum r, s, q prodeant negativi. Denique in puncto invento E figatur Circini crus unum, &, ad diſtantiam EZ = t extentum, agatur crus alterum in orbem, ſecumque circumducat filum FZP. Hac fili circulatione pondus P nunc aſcendet nunc deſcendet motu reciproco, ut & Nodus N nunc ſupra rectam AQ extabit, nunc verò infra eandem deprimetur. Quoties autem reperietur Nodus ille N in ipſa AQ, puta in punctis D, d, Δ, δ, abſcindet is rectas DQ, dQ, ΔQ, δQ.

quæ erunt Æquationis datæ Radices omnes reales; hæ nempe ad dextram erunt Radices affirmativæ, illæ verò ad ſiniſtram Radices negativæ. Demonſtratio eſt manifeſta ex præcedentibus, habita tantùm ratione Parabolæ per puncta B, C, c, κ, K tranſeuntis. Nam poſito F foco Parobolæ, (cujus diſtantia à Vertice aſt $\frac{1}{4}$ ON,) notum eſt quod lineæ omnes ut FB + BQ, FC + CD, &c, eandem ubique conficiant ſummam.

Atque ex principiis hic poſitis proclive erit Inſtrumentum haud inconcinnum & quantumvis accuratum fabricari, cujus beneficio hujuſmodi Æquationum quarumcunque Radices nullo fere negotio inveniri poſſint, & præ oculis exhiberi. Hoc autem quilibet, ſi id Curæ ſit, variis modis pro ingenio ſuo efficere poteſt, & de his jam ſatis.

III. *Æquationum quarundam Poteſtatis tertiæ, quintæ, ſeptimæ, nonæ, & ſuperiorum, ad infinitum uſque pergendo, in terminis finitis, ad inſtar Regularum pro Cubicis quæ vocantur* Cardani, *Reſolutio Analytica.*

Per Ab. De Moivre, R. S. S.

SIt n Numerus quicunque, y quantitas incognita, ſive Æquationis Radix quæſita, ſitque a quantitas quævis omnino cognita, ſive ut vocant Homogeneum Comparationis: Atque horum inter ſe relatio exprimatur per Æquationem

$$ny + \frac{nn-1}{2\times3}ny^3 + \frac{nn-1}{2\times3}\times\frac{nn-9}{4\times5}ny^5 + \frac{nn-1}{2\times3}\times\frac{nn-9}{4\times5}\times\frac{nn-25}{6\times7}ny^7, \text{ \&c.} = a$$

Ex

Ex hujus ſeriei natura manifeſtum eſt, quod ſi n ſumatur numerus aliquis impar (integer ſcilicet, nec refert utrum ſit affirmativus vel negativus) tunc ſeries ſponte ſua terminabitur, & Æquatio fit una ex ſupra præfinitis, cujus Radix eſt

$$(1)\quad y = \frac{1}{2}\sqrt[n]{\sqrt{1+aa}+a} - \frac{\frac{1}{2}}{\sqrt[n]{\sqrt{1+aa}+a}}$$

$$\text{vel } (2)\quad y = \frac{1}{2}\sqrt[n]{\sqrt{1+aa}+a} - \frac{1}{2}\sqrt{\sqrt{1+aa}-a}$$

$$\text{vel } (3)\quad y = \frac{\frac{1}{2}}{\sqrt[n]{\sqrt{1+aa}-a}} - \frac{1}{2}\sqrt[n]{\sqrt{1+aa}-a}$$

$$\text{vel } (4)\quad y = \frac{\frac{1}{2}}{\sqrt[n]{\sqrt{1+aa}-a}} - \frac{\frac{a}{2}}{\sqrt[n]{\sqrt{1+aa}-a}}$$

Exempli gratia, ſit hujus Æquationis poteſtatis quintæ $5y + 20y^3 + 16y^5 = 4$ Radix invenienda, quo in caſu erit n = 5 & a = 4. Radix juxta formam primam erit $y = \frac{1}{2}\sqrt[5]{\sqrt{17}+4} - \frac{\frac{1}{2}}{\sqrt[5]{\sqrt{17}+4}}$, quæ in numeris vulgaribus expeditiſſime explicari poteſt ad hunc modum. Eſt $\sqrt{17}+4 = 8.1231$, cujus Logarithmus 0.9097164, & hujus pars quinta 0.1819433, huic reſpondens numerus eſt $1.5203 = \sqrt[5]{\sqrt{17}+4}$. Ipſius vero 0.1819433 Complementum Arithmeticum eſt 9.8180567. cui reſpondet numerus $0.6577 = \frac{1}{\sqrt{\sqrt{17}+4}}$ Igitur horum numerorum ſemidifferentia 0.4313 = y.

His

Hic venit Observandum quod loco Radicis generalis, non incommode sumeretur $y = \frac{1}{2}\sqrt[n]{2a} - \frac{\frac{1}{2}}{\sqrt[n]{2a}}$, si quando numerus a respectu unitatis, si satis magnus, ut si Æquatio fuerit $5y + 20y^3 + 16y^5 = 682$, erit Log. 2a = 3. 1348143, cujus pars quinta 0. 6269628, & huic respondens numerus 4. 236. Complementi autem Arithmetici 9. 3730372 numerus est 0. 236 & horum numerorum semidifferentia 2 = y.

Atqui præterea, si in Æquatione præcedenti signa alternatim sint affirmantia & negantia, vel quod eodem redit, si series obvenerit hujus modi

$$ny + \frac{1-nn}{2\times 3}ny^3 + \frac{1-nn}{2\times 3}\times\frac{9-nn}{4\times 5}ny^5 + \frac{1-nn}{2\times 3}\times\frac{9-nn}{4\times 5}\times\frac{25-nn}{6\times 7}ny^7, \&c. = a$$

erit hujus Radix

(1) $y = \frac{1}{2}\sqrt[n]{a+\sqrt{aa-1}} + \frac{\frac{1}{2}}{\sqrt[n]{a+\sqrt{aa-1}}}$

vel (2) $y = \frac{1}{2}\sqrt[n]{a+\sqrt{aa-1}} + \frac{1}{2}\sqrt[n]{a-\sqrt{aa-1}}$

vel (3) $y = \frac{\frac{1}{2}}{\sqrt[n]{a-\sqrt{aa-1}}} + \frac{1}{2}\sqrt[n]{a-\sqrt{aa-1}}$

vel (4) $y = \frac{\frac{1}{2}}{\sqrt[n]{a-\sqrt{aa-1}}} + \frac{\frac{1}{2}}{\sqrt[n]{a+\sqrt{aa-1}}}$

Hic autem Notandum, quod si $\frac{n-1}{2}$ numerus extiterit impar, Radicis inventæ signum in ei contrarium permutandum est.

Pro-

Proponatur Æquatio $5y - 20y^3 + 16y^5 = 6$, unde n = 5 & a = 6. Erit Radix $= \frac{1}{2}\sqrt[5]{6+\sqrt{35}} + \frac{\frac{1}{2}}{\sqrt[5]{6+\sqrt{35}}}$

Vel quoniam $6+\sqrt{35} = 11.916$, erit hujus logarithmus 1. 0761304 & ejus pars quinta 0. 2152561, Complementum vero Arithmeticum 9. 7847439. Horum Logarithmorum numeri sunt 1. 6415 & 0. 6091 respective, quorum semisumma 1. 1253 = y.

Verum si acciderit ut a sit minor unitate, tunc Radicis forma secunda, ut quæ proposito est magis conveniens, præ reliquis seligenda est. Sic si Æquatio fuerit $5y - 20y^3 + 16y^5 = \frac{61}{64}$, erit $y = \frac{1}{2}\sqrt[5]{\frac{61}{64}+\sqrt{\frac{-375}{4096}}} + \frac{1}{2}\sqrt[5]{\frac{61}{64}-\sqrt{\frac{-375}{4096}}}$. Et quidem si Binomialium Radix quintana ullo pacto extrahi queat, prodibit Radix proba & possibilis, etsi expressio ipsa impossibilitatem mentiatur. Binomialis vero $\frac{61}{64}+\sqrt{\frac{-375}{4096}}$ Radix quintana est $\frac{1}{4}+\frac{1}{4}\sqrt{-15}$, & Binomialis $\frac{61}{64}-\sqrt{\frac{-375}{4096}}$ Radix itidem quintana est $\frac{1}{4}-\frac{1}{4}\sqrt{-15}$, quorum Binomialium semisumma $= \frac{1}{4} = y$.

Si autem extractio ista vel non peragi posset, vel etiam difficilior videretur, res ubique confici potest per Tabulam sinuum naturalium ad modum sequentem.

Ad Radium 1 sit $a = \frac{61}{64} = 0.95112$ sinus arcus cujusdam, qui proinde erit 72°: 23′ cujus pars quinta (eo quod n = 5) est 14°: 28′; hujus sinus 0. 24981 = $\frac{1}{4}$ proxime. Nec secus procedendum in Æquationibus graduum superiorum.

 IV. Se-

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