AN INTRODUCTION TO THE HISTORY OF MATHEMATICS

THIRD EDITION

HOWARD EVES

UNIVERSITY OF MAINE

BARROW + Newton

HOLT, RINEHART AND WINSTON

New York Chicago San Francisco Atlanta Dallas

Montreal Toronto London Sydney

then the area of the new planar piece so formed is the same as that of the original planar piece. A similar sliding of the plane sections of a given solid yields another solid having the same volume as the original one. These results give the so-called *Cavalieri's principle*: (1) *If two planar pieces are included between a pair of parallel lines, and if the two segments cut by them on any line parallel to the including lines are equal in length, then the areas of the planar pieces are equal*; (2) *if two solids are included between a pair of parallel planes, and if the two sections cut by them on any plane parallel to the including planes are equal in area, then the volumes of the solids are equal.*

Cavalieri's principle constitutes a valuable tool in the computation of areas and volumes and can easily be rigorously established. As an illustration of the principle, consider the following application leading to the formula for the volume of a sphere. In Figure 86 we have a hemisphere of radius r on the left,

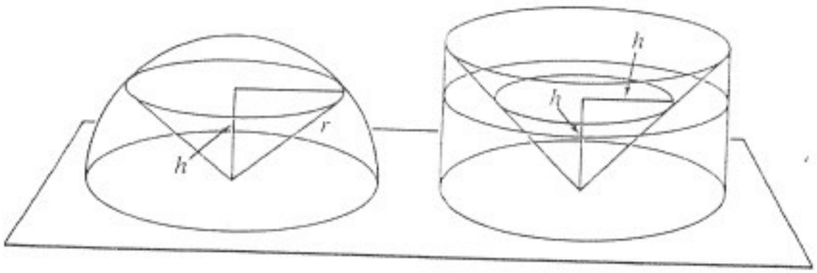

Figure 86

and on the right a cylinder of radius r and altitude r with a cone removed whose base is the upper base of the cylinder and whose vertex is the center of the lower base of the cylinder. The hemisphere and the gouged-out cylinder are resting on a common plane. We now cut both solids by a plane parallel to the base plane and at distance h from it. This plane cuts the one solid in a circular section and the other in an annular, or ring-shaped, section. By elementary geometry, we easily show that each of the two sections has an area equal to $\pi(r^2 - h^2)$. It follows, by Cavalieri's principle, that the two solids have equal volumes. Therefore the volume V of a sphere is given by

$$V = 2(\text{volume of cylinder} - \text{volume of cone})$$
$$= 2\left(\pi r^3 - \frac{\pi r^3}{3}\right) = \frac{4\pi r^3}{3}.$$

The assumption and consistent use of Cavalieri's principle simplifies the derivation of many formulas encountered in a high school course in solid geometry. This procedure has been adopted by a number of textbook writers and has been advocated on pedagogical grounds.

Cavalieri's conception of indivisibles stimulated considerable discussion, and serious criticisms were leveled by some students of the subject, particularly

by the Swiss Paul Guldin. Cavalieri recast his treatment in the vain hope of meeting these objections. The French mathematician Roberval ably handled the method and claimed to be an independent inventor of the conception. The method of indivisibles, or some process very similar to it, was effectively used by Torricelli, Fermat, Pascal, Saint-Vincent, Barrow, and others. In the course of the work of these men results were reached which are equivalent to the integration of expressions like x^n, $\sin\theta$, $\sin^2\theta$, and $\theta \sin\theta$.

11-7 THE BEGINNING OF DIFFERENTIATION

Differentiation may be said to have originated in the problem of drawing tangents to curves and in finding maximum and minimum values of functions. Although such considerations go back to the ancient Greeks, it seems fair to assert that the first really marked anticipation of the method of differentiation stems from ideas set forth by Fermat in 1629.

Kepler had observed that the increment of a function becomes vanishingly small in the neighborhood of an ordinary maximum or minimum value. Fermat translated this fact into a process for determining such a maximum or minimum. The method will be considered in brief. If $f(x)$ has an ordinary maximum or minimum at x, and if e is very small, then the value of $f(x - e)$ is almost equal to that of $f(x)$. Therefore, we tentatively set $f(x - e) = f(x)$ and then make the equality correct by letting e assume the value zero. The roots of the resulting equation then give those values of x for which $f(x)$ is a maximum or a minimum.

Let us illustrate the above procedure by considering Fermat's first example —to divide a quantity into two parts such that their product is a maximum. Fermat used Viète's notation, where constants are designated by upper case consonants and variables by upper case vowels. Following the notation to this extent, let B be the given quantity and denote the sought parts by A and $B - A$. Forming

$$(A - E)\,[B - (A - E)]$$

and equating it to $A(B - A)$ we have

$$A(B - A) = (A - E)(B - A + E)$$

or

$$2AE - BE - E^2 = 0.$$

After dividing by E, one obtains

$$2A - B - E = 0.$$

Now setting $E = 0$ we obtain $2A = B$, and thus find the required division.

Although the logic of Fermat's exposition leaves much to be desired, it is seen that his method is equivalent to setting

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = 0,$$

that is, to setting the derivative of $f(x)$ equal to zero. This is the customary method for finding ordinary maxima and minima of a function $f(x)$, and is sometimes referred to in our elementary textbooks as *Fermat's method*. Fermat, however, did not know that the vanishing of the derivative of $f(x)$ is only a necessary, but not a sufficient, condition for an ordinary maximum or minimum. Also, Fermat's method does not distinguish between a maximum and a minimum value.

Fermat also devised a general procedure for finding the tangent at a point of a curve whose Cartesian equation is given. His idea is to find the *subtangent* for the point, that is, the segment on the x-axis between the foot of the ordinate drawn to the point of contact and the intersection of the tangent line with the x-axis. The method employs the idea of a tangent as the limiting position of a secant when two of its points of intersection with the curve tend to fall together. Using modern notation the method is as follows. Let the equation of the curve (see Figure 87) be $f(x,y) = 0$, and let us seek the

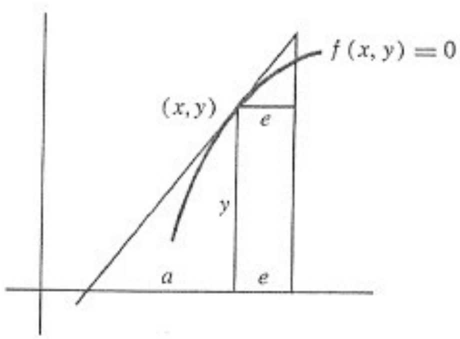

Figure 87

subtangent a of the curve for the point (x,y). By similar triangles we easily find the coordinates of a near point on the tangent to be $[x + e, y(1 + e/a)]$. This point is tentatively treated as if it were also on the curve, giving us

$$f\left[x + e, y\left(1 + \frac{e}{a}\right)\right] = 0.$$

The equality is then made correct by letting e assume the value zero. We then solve the resulting equation for the subtangent a in terms of the coordinates x and y of the point of contact. This, of course, is equivalent to setting

$$a = -y \frac{\dfrac{\partial f}{\partial y}}{\dfrac{\partial f}{\partial x}},$$

a general formula that appeared later in the work of Sluze. Fermat, in this way, found tangents to the ellipse, cycloid, cissoid, conchoid, quadratrix, and folium of Descartes. Let us illustrate the method by finding the subtangent at a general point on the folium of Descartes:

$$x^3 + y^3 = nxy.$$

Here we have

$$(x + e)^3 + y^3\left(1 + \frac{e}{a}\right)^3 - ny(x + e)\left(1 + \frac{e}{a}\right) = 0,$$

or

$$e\left(3x^2 + \frac{3y^3}{a} - \frac{nxy}{a} - ny\right) + e^2\left(3x + \frac{3y^3}{a^2} - \frac{ny}{a}\right) + e^3\left(1 + \frac{y^3}{a^3}\right) = 0.$$

Now, dividing by e and then setting $e = 0$, we find

$$a = -\frac{3y^3 - nxy}{3x^2 - ny}.$$

11-8 WALLIS AND BARROW

Isaac Newton's immediate predecessors in England were John Wallis and Isaac Barrow.

John Wallis, who was born in 1616, was one of the ablest and most original mathematicians of his day. He was a voluminous and erudite writer in a number of fields and is said to be one of the first to devise a system for teaching deaf-mutes. In 1649, he was appointed Savilian professor of geometry at Oxford, a position which he held for 54 years until his death in 1703. His work in analysis did much to prepare the way for his great contemporary, Isaac Newton.

Wallis was one of the first to discuss conics as curves of second degree rather than as sections of a cone. In 1656 appeared his *Arithmetica infinitorum* (dedicated to Oughtred), a book which, in spite of some logical blemishes, remained a standard treatise for many years. In this book, the methods of Descartes and Cavalieri are systematized and extended and a number of remarkable results are induced from particular cases. Thus the formula which we would now write as

$$\int_0^1 x^m \, dx = \frac{1}{m+1},$$

where m is a positive integer, is claimed to hold even when m is fractional or negative but different from -1. Wallis was the first to explain with any completeness the significance of zero, negative, and fractional exponents, and he introduced our present symbol (∞) for infinity.

Wallis endeavored to determine π by finding an expression for the area, $\pi/4$, of a quadrant of the circle $x^2 + y^2 = 1$. This is equivalent to evaluating $\int_0^1 (1 - x^2)^{\frac{1}{2}} \, dx$, which Wallis was unable to do directly since he was not acquainted with the general binomial theorem. He accordingly evaluated $\int_0^1 (1 - x^2)^0 \, dx$, $\int_0^1 (1 - x^2)^1 \, dx$, $\int_0^1 (1 - x^2)^2 \, dx$, and so forth, obtaining the sequence 1, 2/3, 8/15, 16/35, ... He then considered the problem of finding the law which for $n = 0, 1, 2, 3, \ldots$ would yield the above sequence. It was the interpolated value of this law for $n = 1/2$ that Wallis was seeking. By a long and complicated process, he finally arrived at his infinite product expression for $\pi/2$ given in Section 4-8. Mathematicians of his day frequently resorted to interpolation processes in order to calculate quantities that they could not evaluate directly.

Wallis accomplished other things in mathematics. He was the mathematician who came nearest to solving Pascal's challenge questions on the cycloid (see Section 9-9). It can be argued fairly that he obtained an equivalent of the formula

$$ds = \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{1}{2}} dx$$

John Wallis
Library of Congress

for the length of an element of arc of a curve. His *De algebra tractatus; historicus & practicus*, written in 1673 but published in English in 1685 and in Latin in 1693, is considered as the first serious attempt at a history of mathematics in England. It is in this work that we find the first recorded effort to give a graphical interpretation of the complex roots of a real quadratic equation. Wallis edited parts of the works of a number of the great Greek mathematicians and wrote on a wide variety of physical subjects. He was one of the founders of the Royal Society and for years he assisted the government as a cryptologist.

Whereas Wallis' chief contributions to the development of the calculus lay in the theory of integration, Isaac Barrow's most important contributions were perhaps those connected with the theory of differentiation.

Isaac Barrow was born in London in 1630. A story is told that in his early school days he was so troublesome that his father was heard to pray that should God decide to take one of his children he could best spare Isaac. Barrow completed his education at Cambridge and won renown as one of the best Greek scholars of his day. He was a man of high academic caliber, achieving recognition in mathematics, physics, astronomy, and theology. Entertaining stories are told of his physical strength, bravery, ready wit, and scrupulous conscientiousness. He was the first to occupy the Lucasian chair at Cambridge, from which he magnanimously resigned in 1669 in favor of his great pupil, Isaac Newton, whose remarkable abilities he was one of the first to recognize and acknowledge. He died in Cambridge in 1677.

Barrow's most important mathematical work is his *Lectiones opticae et geometricae*, which appeared in the year he resigned his chair at Cambridge. The preface of the treatise acknowledges indebtedness to Newton for some of the material of the book, probably the parts dealing with optics. It is in this book that we find a very near approach to the modern process of differentiation, utilizing the so-called *differential triangle* which we find in our present-day textbooks. Let it be required to find the tangent at a point P on the given curve represented in Figure 88. Let Q be a neighboring point on the curve. Then triangles PTM and PQR are very nearly similar to one another, and, Barrow argued, as the little triangle becomes indefinitely small, we have

$$\frac{RP}{QR} = \frac{MP}{TM}.$$

Let us set $QR = e$ and $RP = a$. Then if the coordinates of P are x and y, those of Q are $x - e$ and $y - a$. Substituting these values into the equation of the curve and neglecting squares and higher powers of both e and a, we find the ratio a/e. We then have

$$OT = OM - TM = OM - MP\left(\frac{QR}{RP}\right) = x - y\left(\frac{e}{a}\right),$$

and the tangent line is determined. Barrow applied this method of constructing

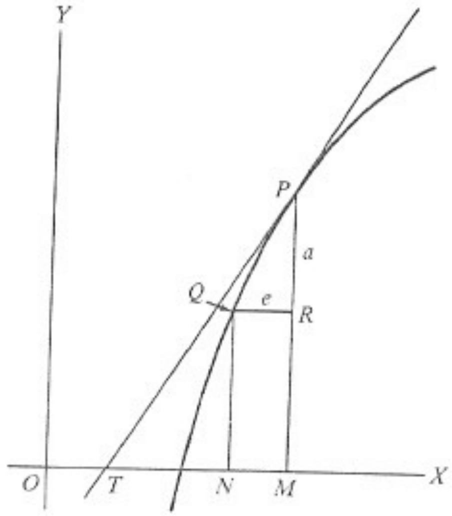

Figure 88

tangents to the curves: (a) $x^2(x^2 + y^2) = r^2y^2$ (the *kappa curve*), (b) $x^3 + y^3 = r^3$ (a special *Lamé curve*), (c) $x^3 + y^3 = rxy$ (the *folium of Descartes*, but called *la galande* by Barrow), (d) $y = (r - x) \tan \pi x/2r$ (the *quadratrix*), (e) $y = r \tan \pi x/2r$ (a *tangent curve*). As an illustration, let us apply the method to curve (b). Here we have

$$(x - e)^3 + (y - a)^3 = r^3,$$

or

$$x^3 - 3x^2e + 3xe^2 - e^3 + y^3 - 3y^2a + 3ya^2 - a^3 = r^3.$$

Neglecting the square and higher powers of e and a, and using the fact that $x^3 + y^3 = r^3$, this reduces to

$$3x^2e + 3y^2a = 0,$$

from which we obtain

$$\frac{a}{e} = -\frac{x^2}{y^2}.$$

The ratio a/e is, of course, our modern dy/dx, and Barrow's questionable procedure can easily be made rigorous by the use of the theory of limits.

In spite of tenuous evidence pointing elsewhere, Barrow is generally credited as the first to realize in full generality that differentiation and integration are inverse operations. This capital discovery is the so-called *fundamental theorem of the calculus* and appears to be stated and proved in Barrow's *Lectiones*.

Although Barrow devoted most of the latter part of his life to theology,

he did, in 1675, publish an edition (with commentary) of the first four books of Apollonius' *Conic Sections* and of the extant works of Archimedes and Theodosius.

At this stage of the development of differential and integral calculus many integrations had been performed, many cubatures, quadratures, and rectifications effected, a process of differentiation had been evolved and tangents to many curves constructed, the idea of limits had been conceived, and the fundamental theorem recognized. What more remained to be done? There still remained the creation of a general symbolism with a systematic set of formal analytical rules, and also a consistent and rigorous redevelopment of the fundamentals of the subject. It is precisely the first of these, the creation of a suitable and workable *calculus*, that was furnished by Newton and Leibniz, working independently of each other. The redevelopment of the fundamental concepts on an acceptably rigorous basis had to outwait the period of energetic application of the subject, and was the work of the great French analyst Augustin-Louis Cauchy (1789–1857) and his nineteenth-century successors.

11-9 NEWTON

Isaac Newton was born in Woolsthorpe on Christmas Day, 1642 (old style), the year in which Galileo died. His father, who died before Isaac was born, was a farmer, and it was at first planned that the son also should devote his life to farming. The youngster, however, showed great skill and delight in devising clever mechanical models and in conducting experiments. Thus, he made a toy gristmill that ground wheat to flour, with a mouse serving as motive power, and a wooden clock that worked by water. The result was that his schooling was extended, and, when 18, he was allowed to enter Trinity

Isaac Newton
David Smith Collection

College, Cambridge. It was not until this stage in his schooling that his attention came to be directed to mathematics, by a book on astrology picked up at the Stourbridge Fair. As a consequence he first read Euclid's *Elements*, which he found too obvious, and then Descartes' *La géométrie*, which he found somewhat difficult. He also read Oughtred's *Clavis*, works of Kepler and Viète, and the *Arithmetica infinitorum* by Wallis. From reading mathematics, he turned to creating it, and early in 1665, when he was 23 years old, he was in possession of the generalized binomial theorem and had created his method of fluxions, as he called what today is known as differential calculus. That year, and part of the next, the university closed because of the rampant bubonic plague, and Newton lived at home. During this period, he developed his calculus to the point where he could find the tangent and radius of curvature at an arbitrary point of a curve. He also became interested in various physical questions, performed his first experiments in optics, and formulated the basic principles of his theory of gravitation.

Newton returned to Cambridge in 1667 and for two years occupied himself chiefly with optical researches. In 1669, Barrow resigned the Lucasian professorship in favor of Newton, and the latter began his 18 years of university lecturing. His first lectures, which were on optics, were later communicated in a paper to the Royal Society and aroused considerable interest and discussion. His theory of colors and certain deductions from his optical experiments were vehemently attacked by some scientists. Newton found the ensuing argument so distasteful that he vowed never to publish anything on science again. His tremendous dislike of controversy, which seems to have bordered on the pathological, had an important bearing on the history of mathematics, for the result was that almost all of his findings remained unpublished until many years after their discovery. This postponement of publication later led to the undignified dispute with Leibniz concerning priority of discovery of the calculus. It was owing to this controversy that the English mathematicians, backing Isaac Newton as their leader, cut themselves off from continental developments, and mathematical progress in England was detrimentally retarded for practically a hundred years.

Newton continued his work in optics, and in 1675 communicated his work on the emission, or corpuscular, theory of light to the Royal Society. His reputation and his ingenious handling of the theory led to its general adoption, and it was not until many years later that the wave theory was shown to be a better hypothesis for research. Newton's university lectures from 1673 to 1683 were devoted to algebra and the theory of equations. It was in this period, in 1679, that he verified his law of gravitation[5] by using a new measurement of the earth's radius in conjunction with a study of the

[5] Any two particles in the universe attract one another with a force which is directly proportional to the product of their masses and inversely proportional to the square of the distance between them.

motion of the moon. He also established the compatibility of his law of gravitation with Kepler's laws of planetary motion, on the assumption that the sun and the planets may be regarded as heavy particles. But these important findings were not communicated to anyone until five years later, in 1684, when Halley visited Newton at Cambridge to discuss the law of force that causes the planets to move in elliptical orbits about the sun. With his interest in celestial mechanics rearoused in this way, Newton proceeded to work out many of the propositions later to become fundamental in the first book of his *Principia*. When Halley, somewhat later, saw Newton's manuscript he realized its tremendous importance, and secured the author's promise to send the results to the Royal Society. This Newton did, and at about the same time he finally solved a problem that had been bothering him for some years, namely that a spherical body whose density at any point depends only on its distance from the center of the sphere attracts an external particle as if its whole mass were concentrated at the center. This theorem completed his justification of Kepler's laws of planetary motion, for the slight departure of the sun and the planets from true sphericity is here negligible. Newton now worked in earnest on his theory and by a gigantic intellectual effort wrote the first book of the *Principia* by the summer of 1685. A year later the second book was completed and a third begun. Jealous accusations by Hooke, and the resulting unpleasantness of the matter to Newton, almost led to the abandonment of the third book, but Halley finally persuaded Newton to finish the task. The complete treatise, entitled *Philosophiae naturalis principia mathematica*, was published, at Halley's expense, in the middle of 1687 and immediately made an enormous impression throughout Europe.

In 1689, Newton represented the university in parliament. In 1692, he suffered a curious illness which lasted about two years and which involved some form of mental derangement. Most of his later life was devoted to chemistry, alchemy, and theology. As a matter of fact, even during the earlier part of his life, he probably spent about as much time on these pursuits as he did on mathematics and natural philosophy. Although his creative work in mathematics practically ceased, he did not lose his remarkable powers, for he masterfully solved numerous challenge problems that were submitted to him and which were quite beyond the powers of the other mathematicians in England. In 1696, he was appointed Warden of the Mint, and in 1699 he was promoted to be Master of the Mint. In 1703, he was elected president of the Royal Society, a position to which he was annually re-elected until his death, and in 1705 he was knighted. The last part of his life was made unhappy by the unfortunate controversy with Leibniz. He died in 1727 when 84 years old, after a lingering and painful illness, and was buried in Westminster Abbey.

As remarked above, all of Newton's important published works, except the *Principia*, appeared years after the author had discovered their contents, and almost all of them finally appeared only because of pressure from friends. The dates of these works, in order of publication, are as follows: *Principia*, 1687; *Opticks*, with two appendices on *Cubic Curves* and *Quadrature and Rectification of Curves by the Use of Infinite Series*, 1704; *Arithmetica universalis*, 1707; *Analysis per Series, Fluxiones, etc.*, and *Methodus differentialis*, 1711; *Lectiones opticae*, 1729; and *The Method of Fluxions and Infinite Series*, translated from Newton's Latin by J. Colson, 1736. One should also mention two important letters written in 1676 to H. Oldenburg, secretary of the Royal Society, in which Newton describes some of his mathematical methods.

It is in the letters to Oldenburg that Newton describes his early induction of the generalized binomial theorem, which he enunciates in the form

$$(P + PQ)^{m/n} = P^{m/n} + \frac{m}{n} AQ + \frac{m-n}{2n} BQ + \frac{m-2n}{3n} CQ + \ldots,$$

where A represents the first term, namely $P^{m/n}$, B represents the second term, namely $(m/n)AQ$, C represents the third term, and so forth. The correctness, under proper restrictions, of the binomial expansion for all complex values of the exponent was established over 150 years later by the Norwegian mathematician N. H. Abel (1802–1829).

A more important mathematical discovery made by Newton at about the same time was his method of fluxions, the essentials of which he communicated to Barrow in 1669. His *Method of Fluxions* was written in 1671, but was not published until 1736. In this work, Newton considers a curve as generated by the continuous motion of a point. Under this conception the abscissa and the ordinate of the generating point are, in general, changing quantities. A changing quantity is called a *fluent* (a flowing quantity), and its rate of change is called the *fluxion* of the fluent. If a fluent, such as the ordinate of the point generating a curve, be represented by y, then the fluxion of this fluent is represented by $\dot{y}$. In modern notation we see that this is equivalent to dy/dt, where t represents time. In spite of this introduction of time into geometry, the idea of time can be evaded by supposing that some quantity, say the abscissa of the moving point, increases constantly. This constant rate of increase of some fluent is called the *principal fluxion*, and the fluxion of any other fluent can be compared with this principal fluxion. The fluxion of $\dot{y}$ is denoted by $\ddot{y}$, and so on for higher ordered fluxions. On the other hand, the fluent of $\dot{y}$ is denoted by the symbol y with a small square drawn about it, or sometimes by y. Newton also introduces another concept, which he calls the *moment* of a fluent; it is the infinitely small amount by which a fluent such as x increases in an infinitely small interval of time o. Thus the moment of the fluent x is given by the product $\dot{x}o$. Newton remarks that we may, in any problem, neglect all terms that are multiplied by the second or higher power of o, and thus obtain an equation between the coordinates x

and y of the generating point of a curve and their fluxions $\dot{x}$ and $\dot{y}$. As an example he considers the cubic curve $x^3 - ax^2 + axy - y^3 = 0$. Replacing x by $x + \dot{x}o$ and y by $y + \dot{y}o$, we get

$$\begin{aligned} & x^3 + 3x^2(\dot{x}o) + 3x(\dot{x}o)^2 + (\dot{x}o)^3 \\ & - ax^2 - 2ax(\dot{x}o) - a(\dot{x}o)^2 \\ & + axy + ay(\dot{x}o) + a(\dot{x}o)(\dot{y}o) + ax(\dot{y}o) \\ & - y^3 - 3y^2(\dot{y}o) - 3y(\dot{y}o)^2 - (\dot{y}o)^3 = 0. \end{aligned}$$

Now, using the fact that $x^3 - ax^2 + axy - y^3 = 0$, dividing the remaining terms by o, and then rejecting all terms containing the second or higher power of o, we find

$$3x^2\dot{x} - 2ax\dot{x} + ay\dot{x} + ax\dot{y} - 3y^2\dot{y} = 0.$$

Newton considers two types of problems. In the first type, we are given a relation connecting some fluents, and we are asked to find a relation connecting these fluents and their fluxions. This is what we did above, and is, of course, equivalent to differentiation. In the second type, we are given a relation connecting some fluents and their fluxions, and we are asked to find a relation connecting the fluents alone. This is the inverse problem and its equivalent to solving a differential equation. The idea of discarding terms containing the second and higher powers of o was later justified by Newton by the use of limit notions. Newton made numerous and remarkable applications of his method of fluxions. He determined maxima and minima, tangents to curves, curvature of curves, points of inflection, and convexity and concavity of curves, and he applied his theory to numerous quadratures and to the rectification of curves. In the integration of some differential equations he showed extraordinary ability. In this work is found a method (a modification of which is now known by Newton's name) for approximating the values of the real roots of either an algebraic or a transcendental numerical equation.

The *Arithmetica universalis* contains the substance of Newton's lectures of 1673 to 1683. In it are found many important results in the theory of equations, such as the fact that imaginary roots of a real polynomial must occur in pairs, rules for finding an upper bound to the roots of a polynomial, his formulas expressing the sum of the nth powers of the roots of a polynomial in terms of the coefficients of the polynomial, an extension of Descartes' rule of signs to give limits to the number of imaginary roots of a real polynomial, and many other things.

Cubic Curves, which appeared as an appendix to the work on *Optics*, investigates the properties of cubic curves by analytic geometry. In his classification of cubic curves Newton enumerates 72 out of the possible 78 forms which a cubic may assume. Many of his theorems are stated without proof. The most attractive of these, as well as the most baffling, was his assertion that just as all conics can be obtained as central projections of a circle, so all cubics can be obtained as central projections of the curves

$$y^2 = ax^3 + bx^2 + cx + d.$$

This theorem remained a puzzle until a proof was discovered in 1731.

Of course, Newton's greatest work is his *Principia,* in which there appears for the first time a complete system of dynamics and a complete mathematical formulation of the principal terrestrial and celestial phenomena of motion. It proved to be the most influential and most admired work in the history of science. It is interesting that the theorems, although perhaps discovered by fluxional methods, are all masterfully established by classical Greek geometry aided, here and there, with some simple notions of limits. Until the development of the theory of relativity, all physics and astronomy rested on the assumption, made by Newton in this work, of a privileged frame of reference. In the *Principia* are found many results concerning higher plane curves, and proofs of such attractive geometric theorems as the two following.

(1) The locus of the centers of all conics tangent to the sides of a quadrilateral is the line (*Newton's line*) through the midpoints of its diagonals.

(2) If a point P moving along a straight line is joined to two fixed points O and O', and if lines OQ and $O'Q$ make fixed angles with OP and $O'P$, then the locus of Q is a conic.

Newton was never beaten by any of the various challenge problems that circulated among the mathematicians of his time. In one of these, proposed by Leibniz, he solved the problem of finding the orthogonal trajectories of a family of curves.

Newton was a skilled experimentalist and a superb analyst. As a mathematician, he is ranked almost universally as the greatest the world has yet produced. His insight into physical problems and his ability to treat them mathematically has probably never been excelled. One can find many testimonials by competent judges as to his greatness, such as the noble tribute paid by Leibniz, who said, "Taking mathematics from the beginning of the world to the time when Newton lived, what he did was much the better half." And there is the remark by Lagrange to the effect that Newton was the greatest genius that ever lived, and the most fortunate, for we can find only once a system of the universe to be established. His accomplishments were poetically expressed by Pope in the lines,

> Nature and Nature's laws lay hid in night;
> God said, 'Let Newton be,' and all was light.

In contrast to these eulogies is Newton's own modest estimate of his work: "I do not know what I may appear to the world; but to myself I seem to have been only like a boy playing on the seashore, and diverting myself in now and

then finding a smoother pebble or a prettier shell than ordinary, whilst the great ocean of truth lay all undiscovered before me." In generosity to his predecessors he once explained that if he had seen farther than other men, it was only because he had stood on the shoulders of giants.

It has been reported that Newton often spent 18 or 19 hours of the 24 in writing, and that he possessed remarkable powers of concentration. Amusing tales, perhaps apocryphal, are told in support of his absent-mindedness when engaged in thought.

Thus, there is the story which relates that, when giving a dinner to some friends, Newton left the table for a bottle of wine, and becoming mentally engaged he forgot his errand, went to his room, donned his surplice, and ended up in chapel.

On another occasion, Newton's friend Dr. Stukeley called on him for a chicken dinner. Newton was out for the moment, but the table was already laid with the cooked fowl in a dish under a cover. Forgetful of his dinner engagement, Newton overstayed his time, and Dr. Stukeley finally lifted the cover, removed and ate the chicken, and then replaced the bones in the covered dish. When Newton later appeared he greeted his friend and sitting down he, too, lifted the cover, only to discover the remains. "Dear me," he said, "I had forgotten that we had already dined."

And then there was the occasion when, riding home one day from Grantham, Newton dismounted from his horse to walk the animal up Spittlegate Hill just beyond the town. Unknown to Newton, on the way up the hill the horse slipped away leaving only the empty bridle in his master's hands, a fact that Newton discovered only when, at the top of the hill, he endeavored to vault into the saddle.

11-10 LEIBNIZ

Gottfried Wilhelm Leibniz, the great universal genius of the seventeenth century, and Newton's rival in the invention of the calculus, was born in Leipzig in 1646 (old style). Having taught himself to read Latin and Greek when he was a mere child, he had, before he was 20, mastered the ordinary textbook knowledge of mathematics, philosophy, theology, and law. At this young age he began to develop the first ideas of his *characteristica generalis,* which involved a universal mathematics that later blossomed into the symbolic logic of George Boole (1815–1864), and still later, in 1910, into the great *Principia mathematica* of Whitehead and Russell. When, ostensibly because of his youth, he was refused the degree of doctor of laws at the University of Leipzig, he moved to Nuremberg. There he wrote a brilliant essay on teaching law by the historical method and dedicated it to the Elector of Mainz. This led to his appointment by the Elector to a commission for the recodification of some statutes. The rest of Leibniz' life from this point on was spent in diplomatic service, first for the Elector of Mainz and then, from about 1676 until his death, for the estate of the Duke of Brunswick at Hanover.

In 1672, while in Paris on a diplomatic mission, Leibniz met Huygens, who was then residing there, and the young diplomat prevailed upon the scientist to give him lessons in mathematics. The following year Leibniz was sent on a political mission to London, where he made the acquaintance of Oldenburg and others and where he exhibited his calculating machine (see Section 9-10) to the Royal Society. Before he left Paris to take up his lucrative post as librarian for the Duke of Brunswick, Leibniz had already discovered the fundamental theorem of the calculus, developed much of his notation in this subject, and worked out a number of the elementary formulas of differentiation.

Leibniz' appointment in the Hanoverian service gave him leisure time to pursue his favorite studies, with the result that he left behind him a mountain of papers on all sorts of subjects. He was a particularly gifted linguist, winning some fame as a Sanskrit scholar, and his writings on philosophy have ranked him high in that field. He entertained various grand projects that came to nought, such as that of reuniting the Protestant and Catholic churches, and then later, just the two Protestant sects of his day. In 1682, he and Otto Mencke founded a journal called the *Acta eruditorum,* of which he became editor-in-chief. Most of his mathematical papers, which were largely written in the ten-year period from 1682 to 1692, appeared in this journal. The journal had a wide circulation in continental Europe. In 1700, Leibniz founded the Berlin Academy of Science, and endeavored to create similar academies in Dresden, Vienna, and St. Petersburg.

The closing seven years of Leibniz' life were embittered by the controversy which others had brought upon him and Newton concerning whether he

Gottfried Wilhelm Leibniz
David Smith Collection

had discovered the calculus independently of Newton. In 1714, his employer became the first German King of England, and Leibniz was left, neglected, at Hanover. It is said that two years later, in 1716, when he died, his funeral was attended only by his faithful secretary.

Leibniz' search for his *characteristica generalis* led to plans for a theory of mathematical logic and a symbolic method with formal rules that would obviate the necessity of thinking. Although this dream has only today reached a noticeable stage of realization, Leibniz had, in current terminology, stated the principal properties of logical addition, multiplication, and negation, had considered the null class and class inclusion, and had noted the similarity between some properties of the inclusion of classes and the implication of propositions (see Problem Study 11-10).

Leibniz invented his calculus sometime between 1673 and 1676. It was on October 29, 1675, that he first used the modern integral sign, as a long letter *S* derived from the first letter of the Latin word *summa* (sum), to indicate the sum of Cavalieri's indivisibles. A few weeks later he was writing differentials and derivatives as we do today, as well as integrals like $\int y\,dy$ and $\int y\,dx$. His first published paper on differential calculus did not appear until 1684. In this paper he introduces dx as an arbitrary finite interval and then defines dy by the proportion

$$dy : dx = y : \text{subtangent}.$$

Many of the elementary rules for differentiation, which a student learns early in one of our college courses in the calculus, were derived by Leibniz. The rule for finding the nth derivative of the product of two functions (see Problem Study 11-6) is still referred to as *Leibniz' rule*.

Leibniz had a remarkable feeling for mathematical form and was very sensitive to the potentialities of a well-devised symbolism. His notation in the calculus proved to be very fortunate, and is unquestionably more convenient and flexible than the fluxional notation of Newton. The English mathematicians, though, clung long to the notation of their leader. It was as late as the nineteenth century that there was formed, at Cambridge, the Analytical Society, as it was named by one of its founders, Charles Babbage (see Section 9-10), for the purpose of advocating "the principles of pure *d*-ism as opposed to the *dot*-age of the university." It should be recalled that the rationalistic philosophy *deism* was in vogue among many of the intelligentsia of the time.

The theory of determinants is usually said to have originated with Leibniz, in 1693, when he considered these forms with reference to systems of simultaneous linear equations, although a similar consideration had been made ten years earlier in Japan by Seki Kōwa. The generalization of the binomial theorem into the multinomial theorem, which concerns itself with the expansion of

$$(a + b + \ldots + n)^r,$$

is due to Leibniz. He also did much in laying the foundation of the theory of envelopes, and he defined the osculating circle and showed its importance in the study of curves.

We shall not enter here into a discussion of the unfortunate Newton-Leibniz controversy. The universal opinion today is that each discovered the calculus independently of the other. While Newton's discovery was made first, Leibniz was the earlier in publishing results. If Leibniz was not as penetrating a mathematician as Newton, he was perhaps a broader one, and while inferior to his English rival as an analyst and mathematical physicist, he probably had a keener mathematical imagination and a superior instinct for mathematical form. The controversy, which was brought upon the two principals by machinations of other parties, led to a long British neglect of European developments, much to the detriment of English mathematics.

For some time after Newton and Leibniz, the foundations of the calculus remained obscure and little heeded, for it was the remarkable applicability of the subject that attracted the early researchers. By 1700, most of our undergraduate college calculus had been founded, along with sections of more advanced fields, such as the calculus of variations. The first textbook of the subject appeared in 1696, written by the Marquis de l'Hospital (1661–1704), when, under an odd agreement, he published the lectures of his teacher, Johann Bernoulli. In this book is found the so-called *l'Hospital's rule* for finding the limiting value of a fraction whose numerator and denominator tend simultaneously to zero.

Marquis de l'Hospital
David Smith Collection

Problem Studies

11-1 The Method of Exhaustion

(a) Assuming the so-called *axiom of Archimedes*: *If we are given two magnitudes of the same kind, then we can find a multiple of the smaller which exceeds the larger*, establish the basic proposition of the method of exhaustion: *If from any magnitude there be subtracted a part not less than its half, from the remainder another part not less than its half, and so on, there will at length remain a magnitude less than any preassigned magnitude of the same kind.* (The axiom of Archimedes is implied in the fourth definition of Book V of Euclid's *Elements*, and the basic proposition of the method of exhaustion is found as Proposition 1 of Book X of the *Elements*.)

(b) Show, with the aid of the basic proposition of the method of exhaustion, that the difference in area between a circle and a circumscribed regular polygon can be made as small as desired.

11-2 The Method of Equilibrium

Figure 89 represents a parabolic segment having AC as chord. CF is tangent to the parabola at C and AF is parallel to the axis of the parabola. OPM is also parallel to the axis of the parabola. K is the midpoint of FA and $HK = KC$. Take K as a fulcrum, place OP with its center at H, and leave OM where it is. Using the geometrical fact that $OM/OP = AC/AO$ show, by Archimedes'

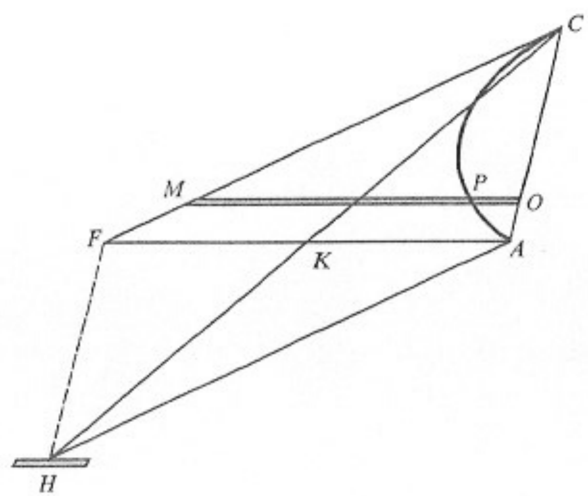

Figure 89