Stokes

BOOKS BY FLORIAN CAJORI

HISTORY OF MATHEMATICS
Revised and Enlarged Edition

HISTORY OF ELEMENTARY MATHEMATICS
Revised and Enlarged Edition

HISTORY OF PHYSICS

INTRODUCTION TO THE MODERN
THEORY OF EQUATIONS

A HISTORY OF
MATHEMATICS

BY

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"I am sure that no subject loses more than mathematics by any attempt to dissociate it from its history."—J. W. L. GLAISHER

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versity in St. Louis has, since 1901, treated the subject in several articles which indicate southerly deviations. He declares that "no potential function is known that fits all parts of the earth," "that the formula of Gauss, the three formulæ of Comte de Sparre [Lyon, 1905], the formula of Professor F. R. Moulton, and my first formula, are all special cases of my general formula." [1]

Fluid Motion

The equations which constitute the foundation of the theory of fluid motion were fully laid down at the time of J. Lagrange, but the solutions actually worked out were few and mainly of the irrotational type. A powerful method of attacking problems in fluid motion is that of images, introduced in 1843 by G. G. Stokes of Pembroke College, Cambridge. It received little attention until Sir William Thomson's discovery of electrical images, whereupon the theory was extended by G. G. Stokes, W. M. Hicks, and T. C. Lewis.

George Gabriel Stokes (1819–1903) was born at Skreen, County Sligo, in Ireland. In 1837, the year of Queen Victoria's accession, he commenced residence at Cambridge, where he was to find his home, almost without intermission, for sixty-six years. At Pembroke College his mathematical abilities attracted attention and in 1841 he graduated as Senior Wrangler and first Smith's prizeman. He distinguished himself along the lines of applied mathematics. In 1845 he published a memoir on "Friction of Fluids in Motion." The general motion of a medium near any point is analyzed into three constituents—a motion of pure translation, one of pure rotation and one of pure strain. Similar results were reached by H. Helmholtz twenty-three years later. In applying his results to viscous fluids, Stokes was led to general dynamical equations, previously reached from more special hypotheses by L. M. H. Navier and S. D. Poisson. Both Stokes and G. Green were followers of the French school of applied mathematicians. Stokes applies his equations to the propagation of sound, and shows that viscosity makes the intensity of sound diminish as the time increases and the velocity less than it would otherwise be—especially for high notes. He considered the two elastic constants in the equations for an elastic solid to be independent and not reducible to one as is the case in Poisson's theory. Stokes' position was supported by Lord Kelvin and seems now generally accepted. In 1847 Stokes examined anew the theory of oscillatory waves. Another paper was on the effect of internal friction of fluids on the motion of pendulums. He assumed that the viscosity of the air was proportional to the density, which was shown later by Maxwell to be erroneous. In 1849 he treated the ether as an elastic solid in the study of diffraction. He favored Fresnel's wave theory of light as opposed to the corpuscular theory supported by David Brewster. In a report on double refraction of 1862 he correlated the work of A. L. Cauchy, J. MacCullagh, and G. Green. Assuming that the elasticity of the ether has its origin in deformation, he inferred that J. MacCullagh's theory was contrary to the laws of mechanics, but recently J. Larmor has shown that J. MacCullagh's equations may be explained on the supposition that what is resisted is not deformation, but rotation. Stokes wrote on Fourier series and the discontinuity of arbitrary constants in semi-convergent expansions over a plane. His contributions to hydrodynamics and optics are fundamental. In 1849 William Thomson (Lord Kelvin) gave the maximum and minimum theorem peculiar to hydrodynamics, which was afterwards extended to dynamical problems in general.

A new epoch in the progress of hydrodynamics was created, in 1856, by H. Helmholtz, who worked out remarkable properties of rotational motion in a homogeneous, incompressible fluid, devoid of viscosity. He showed that the vortex filaments in such a medium may possess any number of knottings and twistings, but are either endless or the ends are in the free surface of the medium; they are indivisible. These results suggested to William Thomson (Lord Kelvin) the possibility of founding on them a new form of the atomic theory, according to which every atom is a vortex ring in a non-frictional ether, and as such must be absolutely permanent in substance and duration. The vortex-atom theory was discussed by J. J. Thomson of Cambridge (born 1856) in his classical treatise on the *Motion of Vortex Rings*, to which the Adams Prize was awarded in 1882. Papers on vortex motion have been published also by Horace Lamb, Thomas Craig, Henry A. Rowland, and Charles Chree of Kew Observatory.

The subject of jets was investigated by H. Helmholtz, G. R. Kirchhoff, J. Plateau, and Lord Rayleigh; the motion of fluids in a fluid by G. G. Stokes, W. Thomson (Lord Kelvin), H. A. Köpcke, G. Greenhill, and H. Lamb; the theory of viscous fluids by H. Navier, S. D. Poisson, B. de Saint-Venant, Stokes, Oskar Emil Meyer (1834–1909) of Breslau, A. B. Stefano, C. Maxwell, R. Lipschitz, T. Craig, H. Helmholtz, and A. B. Basset. Viscous fluids present great difficulties, because the equations of motion have not the same degree of certainty as in perfect fluids, on account of a deficient theory of friction, and of the difficulty of connecting oblique pressures on a small area with the differentials of the velocities.

Waves in liquids have been a favorite subject with English mathematicians. The early inquiries of S. D. Poisson and A. L. Cauchy were directed to the investigation of waves produced by disturbing causes acting arbitrarily on a small portion of the fluid. The velocity of the long wave was given approximately by J. Lagrange in 1786 in case of a channel of rectangular cross-section, by Green in 1839 for a channel of triangular section, and by Philip Kelland (1810–1879)

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[1] See *Washington University Studies*, Vol. III, 1916, pp. 153–168.

of Edinburgh for a channel of any uniform section. Sir George B. Airy, in his treatise on *Tides and Waves*, discarded mere approximations, and gave the exact equation on which the theory of the long wave in a channel of uniform rectangular section depends. But he gave no general solutions. J. McCowan of University College at Dundee discussed this topic more fully, and arrived at exact and complete solutions for certain cases. The most important application of the theory of the long wave is to the explanation of tidal phenomena in rivers and estuaries.

The mathematical treatment of solitary waves was first taken up by S. Earnshaw in 1845, then by G. G. Stokes; but the first sound approximate theory was given by J. Boussinesq in 1871, who obtained an equation for their form, and a value for the velocity in agreement with experiment. Other methods of approximation were given by Lord Rayleigh and John McCowan. In connection with deep-water waves, Osborne Reynolds (1842–1912) of the University of Manchester gave in 1877 the dynamical explanation for the fact that a group of such waves advances with only half the rapidity of the individual waves.

The solution of the problem of the general motion of an ellipsoid in a fluid is due to the successive labors of George Green (1833), R. F. A. Clebsch (1856), and Carl Anton Bjerknes (1825–1903) of Christiania (1873). The free motion of a solid in a liquid has been investigated by W. Thomson (Lord Kelvin), G. R. Kirchhoff, and Horace Lamb. By these labors, the motion of a single solid in a fluid has come to be pretty well understood, but the case of two solids in a fluid is not developed so fully. The problem has been attacked by W. M. Hicks.

The determination of the period of oscillation of a rotating liquid spheroid has important bearings on the question of the origin of the moon. G. H. Darwin's investigations thereon, viewed in the light of G. F. B. Riemann's and H. Poincaré's researches, seem to disprove P. S. Laplace's hypothesis that the moon separated from the earth as a ring, because the angular velocity was too great for stability; G. H. Darwin finds no instability.

The explanation of the contracted vein has been a point of much controversy, but has been put in a much better light by the application of the principle of momentum, originated by W. Froude and Lord Rayleigh. Rayleigh considered also the reflection of waves, not at the surface of separation of two uniform media, where the transition is abrupt, but at the confines of two media between which the transition is gradual.

The first serious study of the circulation of winds on the earth's surface was instituted at the beginning of the second quarter of the last century by *William C. Redfield* (1789–1857), an American meteorologist and railway projector, *James Pollard Espy* (1786–1860) of Washington, through whose stimulus the present United States Weather Bureau was started and *Heinrich Wilhelm Dove* (1803–1879) of Berlin, followed by researches by *Sir William Reid* (1791–1858) a British major-general who developed his circular theory of hurricanes while in the West Indies, *Henry Piddington* (1797–1858) a British commander in the mercantile marine who accumulated data for determining the course of storms at sea and originated the term "cyclone," and *Elias Loomis* (1811–1889) of Yale University. But the deepest insight into the wonderful correlations that exist among the varied motions of the atmosphere was obtained by William Ferrel (1817–1891). He was born in Fulton County, Pa., and brought up on a farm. Though in unfavorable surroundings, a burning thirst for knowledge spurred the boy to the mastery of one branch after another. He attended Marshall College, Pa., and graduated in 1844 from Bethany College. While teaching school he became interested in meteorology and in the subject of tides. In 1856 he wrote an article on "the winds and currents of the ocean." The following year he became connected with the *Nautical Almanac*. A mathematical paper followed in 1858 on "the motion of fluids and solids relative to the earth's surface." The subject was extended afterwards so as to embrace the mathematical theory of cyclones, tornadoes, water-spouts, etc. In 1885 appeared his *Recent Advances in Meteorology*. In the opinion of *Julius Hann* of Vienna, Ferrel has "contributed more to the advance of the physics of the atmosphere than any other living physicist or meteorologist."

W. Ferrel taught that the air flows in great spirals toward the poles, both in the upper strata of the atmosphere and on the earth's surface beyond the 30th degree of latitude; while the return current blows at nearly right angles to the above spirals, in the middle strata as well as on the earth's surface, in a zone comprised between the parallels 30° N. and 30° S. The idea of three superposed currents blowing spirals was first advanced by James Thomson (1822–1892), brother of Lord Kelvin, but was published in very meagre abstract.

W. Ferrel's views have given a strong impulse to theoretical research in America, Austria, and Germany. Several objections raised against his argument have been abandoned, or have been answered by W. M. Davis of Harvard. The mathematical analysis of F. Waldo of Cambridge, Mass., and of others, has further confirmed the accuracy of the theory. The transport of Krakatoa dust and observations made on clouds point toward the existence of an upper east current on the equator, and Josef M. Pernter (1848–1908) of Vienna has mathematically deduced from Ferrel's theory the existence of such a current.

Another theory of the general circulation of the atmosphere was propounded by Werner Siemens (1816–1892) of Berlin, in which an attempt is made to apply thermodynamics to aërial currents. Important new points of view have been introduced by H. Helmholtz, who concluded that when two air currents blow one above the other in different directions, a system of air waves must arise in the same way as waves are formed on the sea. He and Anton Oberbeck (1846–1900) of Tübingen showed that when the waves on the sea attain lengths of from 16 to 33 feet, the air waves must attain lengths of from 10 to 20 miles, and proportional depths. Superposed strata would thus mix more thoroughly, and their energy would be partly dissipated. From hydrodynamical equations of rotation H. Helmholtz established the reason why the observed velocity from equatorial regions is much less in a latitude of, say, 20° or 30°, than it would be were the movements unchecked. Other important contributors to the general theory of the circulation of the atmosphere are Max Möller of Braunschweig and Luigi de Marchi of the University of Pavia. The source of the energy of atmospheric disturbances was sought by W. Ferrel and Th. Reye in the heat given off during condensation. Max Margules of the University of Vienna showed in 1905 that this heat energy contributes nothing to the kinetic energy of the winds and that the source of energy is found in the lowering of the centre of gravity of an air column when the colder air assumes the lower levels, whereby the potential energy is diminished and the kinetic energy increased.[1] Asymmetric cyclones have been studied especially by Luigi de Marchi of Pavia. Anticyclones have received attention from Henry H. Clayton of the Blue Hill Observatory, near Boston, from Julius Hann of Vienna, F. H. Bigelow of Washington, and Max Margules of Vienna.

Sound. Elasticity

About 1860 acoustics began to be studied with renewed zeal. The mathematical theory of pipes and vibrating strings had been elaborated in the eighteenth century by Daniel Bernoulli, D'Alembert, L. Euler, and J. Lagrange. In the first part of the present century P. S. Laplace corrected Newton's theory on the velocity of sound in gases; S. D. Poisson gave a mathematical discussion of torsional vibrations; S. D. Poisson, Sophie Germain, and Charles Wheatstone studied Chladni's figures; Thomas Young and the brothers Weber developed the wave-theory of sound. **Sir J. F. W. Herschel** (1792–1871) wrote on the mathematical theory of sound for the *Encyclopædia Metropolitana*, 1845. Epoch-making were H. Helmholtz's experimental and mathematical researches. In his hands and Rayleigh's, Fourier's series received due attention. H. Helmholtz gave the mathematical theory of beats, difference tones, and summation tones. **Lord Rayleigh** (John William Strutt) of Cambridge (born 1842) made extensive mathematical researches in acoustics as a part of the theory of vibration in general. Particular mention may be made of his discussion of the disturbance produced by a spherical

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[1] *Encyklopädie der Math. Wissenschaften*, Bd. VI, 1, 8, 1912, p. 216.

obstacle on the waves of sound, and of phenomena, such as sensitive flames, connected with the instability of jets of fluid. In 1877 and 1878 he published in two volumes a treatise on *The Theory of Sound*. Other mathematical researches on this subject have been made in England by William Fishburn Donkin (1814–1869) of Oxford and G. G. Stokes. An interesting point in the behavior of a Fourier's series was brought out in 1898 by J. W. Gibbs of Yale. A. A. Michelson and S. W. Stratton at the University of Chicago had shown experimentally by their harmonic analyses that the summation of 160 terms of the series $\Sigma(-1)1^{n+1}(\sin nx)/n$ revealed certain unexpected small towers in the curve for the sum, as n increased. J. W. Gibbs showed (*Nature*, Vol. 59, p. 606) by the study of the order of variation of n and x that these phenomena were not due to imperfections in the machine, but were true mathematical phenomena. They are called the "Gibbs' phenomenon," and have received further attention from Maxime Bôcher, T. H. Gronwall, H. Weyl, and H. S. Carslaw.

The theory of elasticity[1] belongs to this century. Before 1800 no attempt had been made to form general equations for the motion or equilibrium of an elastic solid. Particular problems had been solved by special hypotheses. Thus, James Bernoulli considered elastic laminæ; Daniel Bernoulli and L. Euler investigated vibrating rods; J. Lagrange and L. Euler, the equilibrium of springs and columns. The earliest investigations of this century, by Thomas Young ("Young's modulus of elasticity") in England, J. Binet in France, and G. A. A. Plana in Italy, were chiefly occupied in extending and correcting the earlier labors. Between 1820 and 1840 the broad outline of the modern theory of elasticity was established. This was accomplished almost exclusively by French writers,—Louis-Marie-Henri Navier (1785–1836), S. D. Poisson, A. L. Cauchy, Mademoiselle Sophie Germain (1776–1831), Félix Savart (1791–1841). Says H. Burkhardt: "There are two views respecting the beginnings of the theory of elasticity of solids, of which no dimension can be neglected: According to one view the deciding impulse came from Fresnel's undulatory theory of light, according to the other, everything goes back to the technical theory of rigidity (Festigkeitstheorie), the representative of which was at that time Navier. As always in such cases, the truth lies in the middle: Cauchy to whom we owe primarily the fixing of the fundamental concepts, as strain and stress, learned from Fresnel as well as from Navier."

Siméon Denis Poisson[2] (1781–1840) was born at Pithiviers. The boy was put out to a nurse, and he used to tell that when his father (a common soldier) came to see him one day, the nurse had gone out

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[1] I. Todhunter, *History of the Theory of Elasticity*, edited by Karl Pearson, Cambridge, 1886.

[2] Ch. Hermite, "Discours prononcé devant le président de la République," *Bulletin des sciences mathématiques*, XIV, Janvier, 1890.