A SOURCE BOOK IN MATHEMATICS, 1200–1800

EDITED BY D. J. STRUIK

Barron

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that □ is

$$\text{less than } \frac{3 \times 3 \times 5 \times 5 \cdots 13 \times 13}{2 \times 4 \times 4 \times 6 \cdots 12 \times 14} \sqrt{1\tfrac{1}{13}}$$

and

$$\text{greater than } \frac{3 \times 3 \times 5 \times 5 \cdots 13 \times 13}{2 \times 4 \times 4 \times 6 \cdots 12 \times 14} \sqrt{1\tfrac{1}{14}},$$

and so forth to as close an approximation as we like.[12]

14 BARROW. THE FUNDAMENTAL THEOREM OF THE CALCULUS

The so-called inverse-tangent problem consisted in finding the curve, given a law concerning the behavior of the tangent. An early example was the search for loxodromes on the sphere, which are curves intersecting the meridians at a given angle; this problem was originated by Pedro Nuñez and Simon Stevin in the sixteenth century. A later example of importance was contained in a letter to Descartes written by Florimond De Beaune in 1639, which led to the search for the curve of constant subtangent; see Descartes, *Oeuvres*, ed. C. Adam and P. Tannery, *Correspondance*, II (Paris, 1898), 510–519, and Selection V.1. The next step was the recognition that finding quadratures and solving inverse-tangent problems were identical propositions—in other words, the discovery that the integral calculus is the inverse of the differential calculus. Torricelli came to this understanding in his case of generalized parabolas and hyperbolas, satisfying the equation $x\,dy = ky\,dx$; see E. Bortolotti, *Archeion 12* (1930), 60–64. James Gregory (1638–1675), the great Scottish mathematician who died so young, seems to have been the first to see the proposition in its generality, though still in a geometric manner. This was in his *Geometriae pars universalis* (Padua, 1668); see *James Gregory tercentenary memorial volume*, ed. H. W. Turnbull (London, 1939), where Gregory's work can be enjoyed in an English paraphrase. See also M. Dehn and E. D. Hellinger, "Certain mathematical achievements of James Gregory," *American Mathematical Monthly 50* (1943), 149–163. We then find the fundamental theorem in the *Lectiones geometricae* (London, 1670) by Isaac Barrow (1630–1677), in his day a famous theologian and from 1662 to 1670 professor of mathematics at Cambridge, where he was the first to occupy the Lucasian chair. His most famous disciple was Isaac Newton, who succeeded him in his chair. See P. C. Osmond, *Isaac Barrow: his life and times* (Society for Promoting Christian Knowledge, London, 1944).

The *Lectiones geometricae* present, in 13 lectures, a curious collection of theorems, mostly concerned with the finding of tangents, areas, and lengths of arcs. Barrow himself says in the preface that he did not find the presentation very satisfactory, but instead of editing his lectures he chose rather to send them forth "in Nature's garb," just as they were born. His starting point is motion, and his early method of finding tangents is thus kinematic. He then begins to use indivisibles, but with some caution, and at the end he arrives at the method of differentiation, as used by Fermat, and at that of the characteristic (or differential) triangle (dx, dy, ds). The method is thoroughly geometrical, and this makes it not easy to recognize the importance of Barrow's results. On the (partial) translation by

[12] We now write $\frac{\pi}{2} = \lim_{n \to \infty} \frac{2 \times 2 \times 4 \times 4 \times \ldots (2n)(2n)}{3 \times 3 \times 5 \times 5 \times \ldots (2n-1)(2n+1)}$.

J. M. Child, *The geometrical lectures of Isaac Barrow* (Open Court, Chicago, London, 1916) we base certain sections of Lectures X and XI which contain theorems equivalent to $ds^2 = dx^2 + dy^2$ (rectification) and $(d/dx)\int_0^x y\,dx = y$ (the fundamental theorem). The notation is slightly modernized; see footnote 11.

LECTURE X

1. Let AEG [Fig. 1] be any curve whatever, and AFI another curve so related to it that, if any straight line EF is drawn parallel to a straight line given in

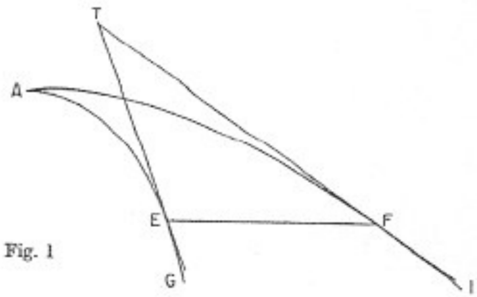

Fig. 1

position (which cuts AEG in E and AFI in F), EF is always equal to the arc AE of the curve AEG, measured from A; also let the straight line ET touch the curve AEG at E, and let ET be equal to the arc AE; join TF; then TF touches the curve AFI.

The proof follows.

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2. Moreover, if the straight line EF always bears the same ratio to the arc AE, in just the same way FT can be shown to touch the curve AFI.[1] . . .

3. Let AGE [Fig. 2] be any curve, D a fixed point, and AIF another curve such that, if any straight line DEF is drawn through D, the intercept EF is always equal to the arc AE; and let the straight line ET touch the curve AGE; make TE equal to the arc AE; let TKF be a curve such that, if any straight

[1] This is one of the many theorems by which Barrow passes from the knowledge of the tangent of one curve to that of another by means of methods which originally are based on motion (EF is moving parallel to itself), but eventually can be interpreted purely geometrically. If $E(x, y)$ and the y-axis are in the EF direction, then $EF = y + s$ (s = arc AE), and $F(x, y + s)$. Hence the slope of FT is

$$\frac{1 + \sin\varphi}{\cos\varphi} = \tan\varphi + \sec\varphi = \frac{dy}{dx} + \frac{ds}{dx}, \quad \text{if } \frac{ds}{dx} = \sec\varphi,$$

hence $\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$. Here we have taken $\frac{dy}{dx} = \tan\varphi$, hence the X-axis is $\perp AF$.

line DHK is drawn through D, cutting the curve TKF in K and the straight line TE in H, $HK = HT$; then let FS be drawn to touch TKF at F; FS will touch the curve AIF also.[2]

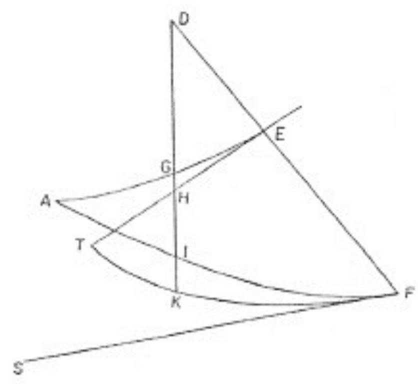

Fig. 2

4. Moreover, if the straight line EF always bears the same ratio to the arc AE, the tangent to it can easily be found from the above and Lect. VIII, §8.

A number of similar theorems follow, and applications to some special curves (from a straight line to a hyperbola, from a circle to a quadratix). Then Barrow says: "I add one or two theorems, which it will be seen are of great generality, and not lightly to be passed over." Here they are:

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11. Let ZGE [Fig. 3] be any curve of which the axis is VD and let there be perpendicular ordinates to this axis (VZ, PG, DE) continually increasing from the initial ordinate VZ; also let VIF be a line such that, if any straight line EDF is drawn perpendicular to VD, cutting the curves in the points E, F, and VD in D, the rectangle contained by DF and a given length R is equal to the intercepted space $VDEZ$; also let $DE : DF = R : DT$, and join [T and F]. Then TF will touch the curve VIF.[3] For, if any point I is taken in the line VIF (first on the side of F towards V), and if through it IG is drawn parallel to VZ, and IL is parallel to VD, cutting the given lines as shown in the figure; then $LF : LK = DF : DT = DE : R$, or $R \times LF = LK \times DE$.

[2] This is similar to Art 1, but now in polar form.

[3] If the curve ZGE is given by $y = f(x)$ and curve AIF by $z = g(x)$, then $Rz = \int_0^x y\,dx$, and $y : z = R : DT$. The theorem that DT is tangent to the curve AIF gives $y : z = R : z\dfrac{dx}{dz}$, hence $y = R\dfrac{dz}{dx}$ or $y = \dfrac{d}{dx}\int_0^x y\,dx$. This therefore is, in geometrical form, the fundamental theorem of the calculus. Figure 4 gives the text in facsimile.

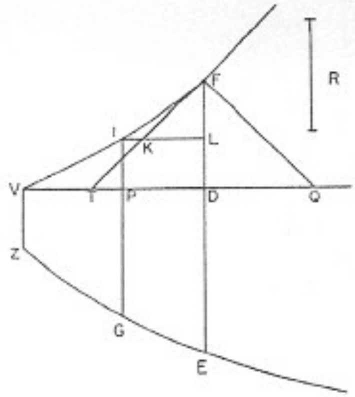


Fig. 3

But, from the stated nature of the lines DF, LK, we have $R \times LF =$ area $PDEG$; therefore $LK \times DE =$ area $PDEG < DP \times DE$; hence $LK < DP < LI$.

Again, if the point I is taken on the other side of F, and the same construction is made as before, plainly it can be easily shown that $LK > DP > LI$.

From which it is quite clear that the whole of the line TKF lies within or below the curve $VIFI$.

Other things remaining the same, if the ordinates, VZ, PG, DE, continually decrease, the same conclusion is attained by similar argument; only one distinction occurs, namely, in this case, contrary to the other, the curve VIF is concave to the axis VD.

Corollary. It should be noted that $DE \times DT = R \times DF =$ area $VDEZ$.

12. From the preceding we can deduce the following theorem.

Let ZGE, VIE be any two lines so related that, if any straight line EDF is applied to a common axis VD, the square on DF is always equal to twice the space $VDEZ$; also take DQ, along VD produced, equal to DE, and join FQ; then FQ is perpendicular to the curve VIF.[4]

I will also add the following kindred theorems.

13. Let $AGEZ$ be any curve [Fig. 5], and D a certain fixed point such that the radii, DA, DG, DE, drawn from D, decrease continually from the initial radius DA; then let DKE be another curve intersecting the first in E and such that, if any straight line DKG is drawn through D, cutting the curve AEZ in G and the curve DKE in K, the rectangle contained by DK and a given length R is equal to the area ADG; also let DT be drawn perpendicular to DE, so that $DT = 2R$; join TE. Then TE touches the curve DKE.

. . . .

Moreover, if any point, K say, is taken in the curve DKE, and through it DKG is drawn, and $DG : DK = R : P$; then, if DT is taken equal to $2P$ and TG is joined, and also KS is drawn parallel to GT; KS will touch the curve DKE.

[4] This shows how to construct the normal to the figure of Art. 11. Arts. 13 and 14 show how the argument runs in terms of what we would call polar coordinates.

78 **LECT. X.**

Hujusmodi plura quædam cogitaram hic inserere; verùm hæc existimo sufficere subindicando modo, juxta quem, citra *Calculi molestiam*, *curvarum tangentes* exquirere licet, unaque constructiones demonstrare. Subjiciam tamen unum aut alterum non aspernanda, ut videtur *Theoremata* perquam generalia.

Fig. 109. XI. Sit linea quæpiam Z G E, cujus axis V D; ad quam imprimis applicatæ perpendiculares (V Z, P G, D E) ab initio V Z continuè utcunque crescant; sit item linea V I F talis, ut ductâ quâcunq; rectâ E D F ad V D perpendiculari (quæ *curvas* secet punctis F, E, ipsam V D in D) sit semper *rectangulum* ex D F, & designatâ quâdam R æquale *spatio* respectivè *intercepto* V D E Z; fiat autem D E. D F :: R. D T; & connectatur recta T F; hæc curvam V I F continget.

Fig. 110. Sumatur enim in linea V I F punctum quodpiam I (illud primò supra punctum F, versus initium V) & per hoc ducantur rectæ I G ad V Z, ac K L ad V D parallelæ (quæ lineas expositas secent, ut vides) estque tum L F. L K :: (D F. D T ::) D E. R; adeóque L F × R = L K × D E. Est autem (ex præstituta linearum istarum natura) L F × R æquale spatio P D E G; ergo L K × D E = P D E G ⊐ D P × D E. Unde est L K ⊐ D P; vel L K ⊐ L I.

Rursus accipiatur quodvis punctum I, infra punctum F, reliquaq; fiant, uti priùs; similique jam planè discursu constabit fore L K × D E = P D E G ⊏ D P × D E, unde jam erit L K ⊏ D P, vel L I. E quibus liquidò patet totam rectam T K F K intra (seu extra) curvam V I F I existere.

Iisdem quoad cætera positis, si *ordinatæ* V Z, P G, D E, *&c.* continuè decrescant, eadem conclusio simili ratiocinio colligetur; unicum obvenit *Discrimen*, quod in hoc casu (contra quam in priore) linea V I F concavas suas axi V D obvertat.

Corol. Notetur D E × D T æquari spatio V D E Z.

Fig. 111. XII. Exindè deducitur hoc *Theorema*: Sint duæ lineæ quævis Z G E, V K F ita relatæ, ut ad communem ipsarum axem V D applicatâ quâvis rectâ E D F, sit semper quadratum ex D E æquale *duplo spatio* V D E Z; sumatur autem D Q = D E, & connectatur F Q; hæc curvæ V K F perpendicularis erit.

Concipiatur enim linea V I F, per F transiens, talis qualem mox attigimus (cujus scilicet ad V D applicatæ se habeant ut spatia VDEZ; hoc est ut quadrata ex applicatis à curva V K F in præsente hypothesi) lineamque

Fig. 4

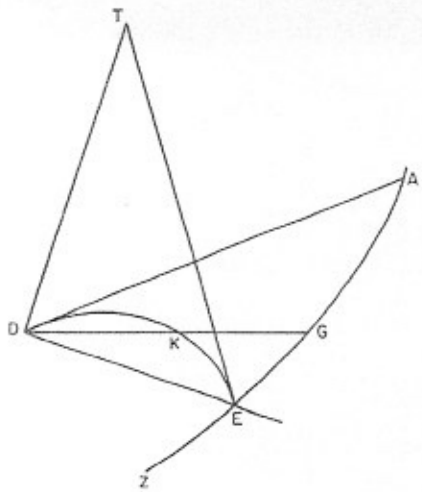


Fig. 5

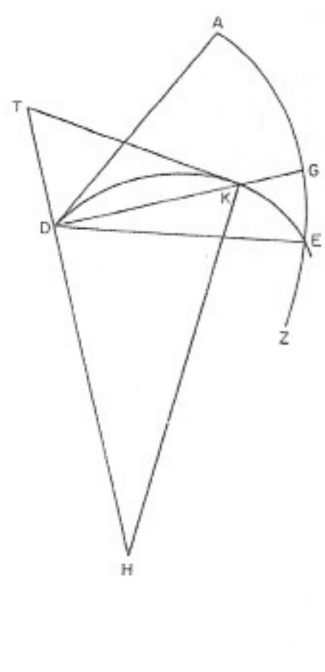
Observe that Square on DG : Square on $DK = 2R : DS$.

Now, the above theorem is true, and can be proved in a similar way, even if the radii drawn from D, DA, DG, DE, are equal (in which case the curve $AGEZ$ is a circle and the curve DKE is the Spiral of Archimedes), or if they continually increase from A.

14. From this we may easily deduce the following theorem.

Let AGE, DKE [Fig. 6] be two curves so related that, if straight lines DA, DG are drawn from some fixed point D in the curve DKE (of which the latter cuts

Fig. 6



the curve DKE in K), the square on DK is equal to four times the area ADG; then, if DH is drawn perpendicular to DG, and $DK : DG = DG : DH$; and HK is joined; then HK is perpendicular to the curve DKE.

We have now finished in some fashion the first part, as we declared, of our subject. Supplementary to this we add, in the form of appendices, a method for finding tangents by calculation frequently used by us. Although I hardly know, after so many well-known and well-worn methods of the kind above, whether there is any advantage in doing so. Yet I do so on the *advice of a friend*;[5] and all the more willingly, because it seems to be more profitable and general than those which I have discussed.

Let AP, PM be two straight lines given in position [Fig. 7] of which PM cuts a given curve in M, and let MT be supposed to touch the curve at M, and to cut the straight line at T.

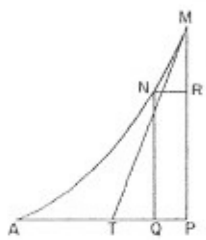


Fig. 7

In order to find the length of the straight line PT, I set off an indefinitely small arc, MN, of the curve; then I draw NQ, NR parallel to MP, AP; I call $MP = m$, $PT = t$, $MR = a$, $NR = e$, and other straight lines, determined by the special nature of the curve, useful for the matter in hand, I also designate by name; also I compare MR, NR (and through them, MP, PT) with one another by means of an equation obtained by calculation; meantime observing the following rules.[6]

Rule 1. In the calculation, I omit all terms containing a power of a or e, or products of these (for these terms have no value).[7]

Rule 2. After the equation has been formed, I reject all terms consisting of letters denoting known or determined quantities or terms which do not contain a or e (for these terms, brought over to one side of the equation, will always be equal to zero).

Rule 3. I substitute m (or MP) for a, and t (or PT) for e. Hence at length the quantity of PT is found.

Moreover, if any indefinitely small arc of the curve enters the calculation, an indefinitely small part of the tangent, or of any straight line equivalent to it (on

[5] This friend probably is Newton, to whom Barrow refers by name in the preface, saying that Newton has helped him in preparing the book, adding some things from his own work.

[6] This introduces the "characteristic triangle" (NR, RM, NM) or (dx, dy, ds), on the advice, it seems, of Newton.

[7] This neglecting of terms of higher order reminds us of Fermat (Selections IV.7, 8) and also of Newton's fluxion theory (Selection V.7).

account of the indefinitely small size of the arc) is substituted for the arc. But these points will be made clearer by the following examples.

Barrow gives five examples of this method of the characteristic triangle. Two of them are the folium of Descartes $x^3 + y^3 = axy$ (written by Barrow AP cub + PM cub = $AX \times AP \times PM$; he calls the curve La Galande) and the quadratrix; the others are the curves $x^3 + y^3 = a^3$, $r = a \tan \theta$, and $y = a \tan x$. The result of the differentiation of $y = a \tan x$ is shown to be (in our notation, of course), $dy/dx = a \sec^2 x$.

The next lecture deals with integration.

LECTURE XI

1. If VH [Fig. 8] is a curve whose axis is VD, and HD is an ordinate perpendicular to VD, and $\varphi Z\psi$ is a line such that, if from any point chosen at random on the curve, say E, a straight line EP is drawn normal to the curve, and a straight line EAZ perpendicular to the axis, AZ is equal to the intercept AP; then the area $VD\psi\varphi$ will be equal to half the square on the line DH.

For if the angle HDO is half a right angle, and the straight line VD is divided into an infinite number of equal parts at A, B, C, and if through these points

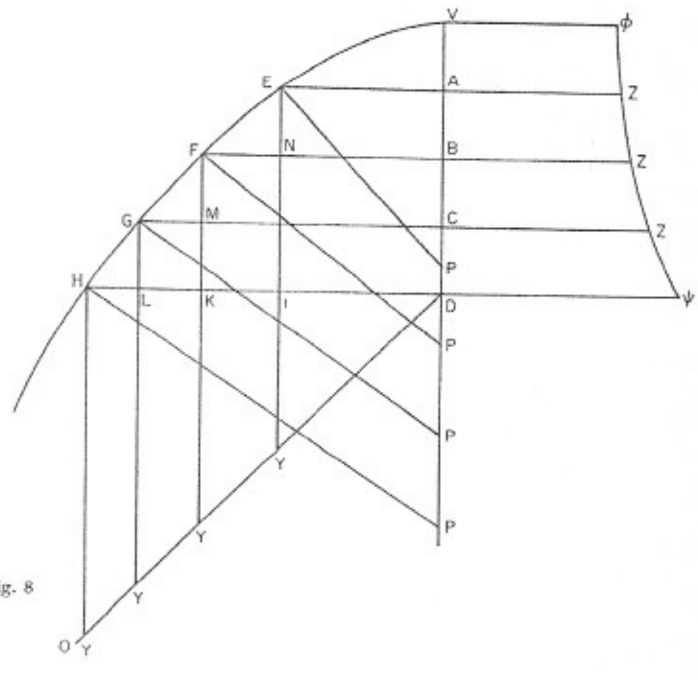


Fig. 8

straight lines EAZ, FBZ, GCZ, are drawn parallel to HD, meeting the curve in E, F, G; and if from these points are drawn straight lines EIY, FKY, GLY, parallel to VD (or HO); and if also EP, FP, GP, HP are normals to the curve, the lines intersecting as in the figure; then the triangle HLG is similar to the triangle PDH (for, on account of the infinite section, the small arc HG can be considered as a straight line).

Hence, $HL:LG = PD:DH$, or $HL \times DH = LG \times PD$; that is,

$$HL \times HO = DC \times D\psi.$$

By similar reasoning it may be shown that, since the triangle GMF is similar to the triangle PCG, $LK \times LY = CB \times CZ$; and in the same way,

$$KI \times KY = BA \times BZ, \quad ID \times IY = AV \times AZ.$$

Hence it follows that the triangle DHO (which differs in the slightest degree only from the sum of the rectangles $HL \times HO + LK \times LY + KI \times KY + ID \times IY$) is equal to the space $VD\psi\varphi$ (which similarly differs in the least degree only from the sum of the rectangles $DC \times D\psi + CB \times CZ + BA \times BZ + AV \times AZ$); that is,

$$DH^2/2 = \text{area } VD\psi\varphi.$$

A lengthier indirect argument may be used; but what advantage is there?[8]

2. With the same data and construction as before, the sum of the rectangles $AZ \times AE$, $BZ \times BF$, $CZ \times CG$, etc., is equal to one-third of the cube on the base DH.[9]

For, since $HL:LG = PD:DH = PD \times DH:DH^2$; therefore $HL \times DH^2 = LG \times PD \times DH$ or $LH \times HO^2 = DC \times D\psi \times DH$; and, similarly, $LK \times LY^2 = CB \times CZ \times CG$, $KI \times KY^2 = BA \times BZ \times BF$, etc.

But the sum $HL \times HO^2 + LK \times LY^2 + KI \times KY^2 +$ etc. $= DH^3/3$; and the proposition follows at once.

3. By similar reasoning, it follows that

$$\text{the sum of } AZ \times AE^2, BZ \times BF^2, CZ \times CG^2, \text{etc.} = DH^4/4;$$
$$\text{the sum of } AZ \times AE^3, BZ \times BF^3, CZ \times CG^3, \text{etc.} = DH^5/5;$$

and so on.

4. Hence we may deduce the following important theorems.

Let $VD\psi\varphi$ be any space of which the axis VD is equally divided [as in Fig. 7];

[8] If we measure x along VP and y in the direction of EA, then $AP = AZ = y\,dy/dx$ and the theorem states that

$$\int_0^{x_0} y\frac{dy}{dx}\,dx = \int_0^{y_0} y\,dy = \frac{y^2}{2}\Big]_0^{y_0} = \frac{y_0^2}{2},$$

when D has the coordinates (x_0, y_0). This is a form of change of independent variable from x to y.

[9] That is, $\int_0^{x_0} y^2\frac{dy}{dx}\,dx = \int_0^{y_0} y^2\,dy = \frac{y_0^3}{3}$.

then if we imagine that each of the spaces $VAZ\varphi$, $VBZ\varphi$, $VCZ\varphi$, etc., is multiplied by its own ordinate AZ, BZ, CZ, etc., respectively, the sum which is produced will be equal to half the square of the space $VD\psi\varphi$.[10]

Several more examples are given, concerning the area of a quadrant of a circle and of a parabolic segment and the volume of a surface of rotation, after which comes the following theorem:

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10. Again, if VH [Fig. 9] is a curve whose axis is VD and base DH, and DZZ is a curve such that, if any point such as E is taken on the curve VH and ET is drawn to touch the curve, and a straight line EIZ is drawn parallel to the axis, then IZ is always equal to AT; in that case, I say, the space DHO is equal to the space VHD.

This extremely useful theorem is due to that most learned man, Gregory of Aberdeen: we will add some deductions from it . . .[11]

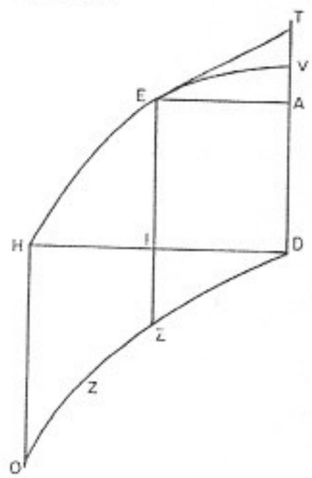

Fig. 9

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[10] That is,

$$\int_0^{y_0} y\,dy \int_0^x y\frac{dy}{dx}\,dx = \int_0^{y_0} \tfrac{1}{2}\,y^3\,dy = \frac{y_0^4}{8} = \tfrac{1}{2}\left(\frac{y_0^2}{2}\right)^2.$$

Art. 5 shows that

$$\int_0^{y_0} y^{1/2}\,dy = \tfrac{2}{3}y_0^{3/2}.$$

[11] When D is taken as origin, $DI = x$, $DA = y$, then $AT = IZ = x\,dy/dx$, and if we write $H(x_0, 0)$, $V(0, y_0)$, then

$$\int_0^{y_0} x\,dy = \int_0^{x_0} y\,dx.$$

Barrow refers to James Gregory, who, in 1668, had arrived independently at the fundamental theorem.

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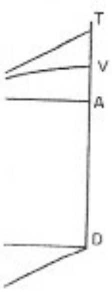

$= x\,dy/dx$, and if we

ndently at the funda-

Barrow goes on to give more examples and Art. 19 arrives again at the fundamental theorem, now in the form converse to that given in Lecture X, Art. 11: if the curve AMB is given by $z = f(x)$, and the curve KZL by $y = f_1(x)$, $z\,dx/dz : z = R : y$, then $\int y\,dx = R \int dz$, or $\int y\,dx = Rz$.

We list here some of Barrow's notations which we have modified: $A \sqsubset B$, A is greater than B; $A \sqsupset B$, A is less than B; $A\,.\,B :: C\,.\,D$, $A : B = C : D$; Aq, the square of A, for instance, in Lecture X, Art. 13, square on DG is written DGq; Ac or A cub, the cube of A; $DHqq$ the fourth power of DH. We have kept his symbol of multiplication, $A \times B$.

We end with a word of caution. Despite the fact that, in order to understand these seventeenth-century mathematicians, we are inclined to translate their reasoning into the notation and language with which we are familiar, we must constantly be aware that our point of view is not equivalent to theirs. They saw geometric theorems in the sense of Euclid, where we see operations and calculating processes. At the same time, just because these mathematicians applied their geometric notions in an attempt to transcend the static character of classical mathematics, their geometric thought has a richness that may easily escape observation in the modern transcription. If we were to rewrite Euclid in the notation of analytic geometry we would obtain a body of knowledge with a character different from that of Euclid and, despite all the advantages that the algebraic computations would bring, we would lose some of the more subtle and esthetic qualities of Euclid.

15 HUYGENS. EVOLUTES AND INVOLUTES

The search for reliable clocks, a necessity for scientific navigation and geography as well as for theoretical astronomy, led Christiaan Huygens (1629–1695), a Dutch patrician and a founding member of the French Academy of Sciences (1666), to the invention of the pendulum clock (the idea of which seems to have already occurred to Galilei). Huygens described this invention in the *Horologium oscillatorium* (Paris, 1673; reprinted, with French translation, in *Oeuvres complètes de Christiaan Huygens*, XVIII, 68–368). This book, in its five parts, contains a number of important discoveries in mechanics and mathematics, so that, with the books of Cavalieri and Wallis (see Selections IV.5, 6, 13), it is a landmark on the path that led to the invention of the calculus.

After describing his pendulum clock in Part I, Huygens deals in Part II with "The fall of heavy bodies and their cycloidal movement." Here we find a theory of the cycloid and, based on it, the following theorem on a heavy point moving on a cycloid in a field of gravity:

> *Proposition* XXV. On a cycloid with a vertical axis whose vertex is below, the times of descent in which a mobile point, starting from rest at an arbitrary point of the curve, reaches the lowest point, are all equal, and have to the times of the vertical fall along the total axis of the cycloid a ratio equal to that of the semicircumference of a circle to that of the diameter [in our terms, as $\pi : 2$].

In other words, the cycloid is a *tautochrone*. From this theorem Huygens obtains the tautochronic pendulum, which has a period independent of its amplitude. This property of