MATHEMATICAL ANALYSIS

S. C. Malik
Savita Arora
Department of Mathematics
S.G.T.B. Khalsa College
University of Delhi
Delhi, India

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5. STOKES' THEOREM (First generalization of Green's Theorem)

We recall that Green's Theorem expresses a relation between a double integral over a plane region and a line integral taken round its plane boundary. There are two ways to generalise this in R^3. One of these extensions, known as *Stokes' Theorem*, relates a surface integral taken over a surface to a line integral taken around the boundary curve of the surface. This generalisation is due to an English mathematician, *George Gabriel Stokes* (1819-1903).

A second generalisation arises when the double integral is replaced by a triple integral, and the line integral by a surface integral. This generalisation is named *Gauss's Theorem* and will be taken up later.

Stokes' Theorem *If S is a smooth oriented surface bounded by a curve* **C** *oriented in the same sense, and f, g, h are three functions which along with their first order partial derivatives are continuous in a three dimensional domain containing S, then*

$$\int_{\mathbf{C}} (f\,dx + g\,dy + h\,dz) = \iint_S \left[\left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right) dy\,dz + \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\right) dz\,dx + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx\,dy\right]$$

Let the oriented surface be represented as

$$x = x(u, v), \quad y = y(u, v), \quad z = z(u, v), \quad (u, v) \in D$$

where D is an oriented surface in uv-plane. Also, let its boundary be an oriented curve Γ represented by

$$u = u(t), \quad v = v(t), \quad a \leqslant t \leqslant b$$

The proof of the theorem involves the following steps:

1. The line integral along **C** is expressed as an ordinary integral,
2. The ordinary integral is expressed as a line integral along Γ,
3. The line integral along Γ is then expressed, by Green's Theorem, as double integral over D, and finally
4. The double integral along D is expressed as a surface integral over S.

Now

$$\int_{\mathbf{C}} (f\,dx + g\,dy + h\,dz)$$

$$= \int_a^b \left[f\left(\frac{\partial x}{\partial u}\frac{du}{dt} + \frac{\partial x}{\partial v}\frac{dv}{dt}\right) dt + g\left(\frac{\partial y}{\partial u}\frac{du}{dt} + \frac{\partial y}{\partial v}\frac{dv}{dt}\right) dt + h\left(\frac{\partial z}{\partial u}\frac{du}{dt} + \frac{\partial z}{\partial v}\frac{dv}{dt}\right) dt\right]$$

$$= \int_{\Gamma} \left(f \frac{\partial x}{\partial u} + g \frac{\partial y}{\partial u} + h \frac{\partial z}{\partial u} \right) du + \left(f \frac{\partial x}{\partial v} + g \frac{\partial y}{\partial v} + h \frac{\partial z}{\partial v} \right) dv$$

$$= \iint_D \left[\frac{\partial}{\partial u} \left(f \frac{\partial x}{\partial v} + g \frac{\partial y}{\partial v} + h \frac{\partial z}{\partial v} \right) - \frac{\partial}{\partial v} \left(f \frac{\partial x}{\partial u} + g \frac{\partial y}{\partial u} + h \frac{\partial z}{\partial u} \right) \right] du\, dv \qquad (1)$$

But

$$\frac{\partial}{\partial u} \left(f \frac{\partial x}{\partial v} \right) = \left(\frac{\partial f}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial u} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial u} \right) \frac{\partial x}{\partial v} + f \frac{\partial^2 x}{\partial u \partial v}$$

Writing down similar expressions for the other terms of the integrand and rearranging, the double integral on the right hand side of (1) becomes

$$\iint_D \left[\left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z} \right) \frac{\partial(y, z)}{\partial(u, v)} + \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x} \right) \frac{\partial(z, x)}{\partial(u, v)} + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) \frac{\partial(x, y)}{\partial(u, v)} \right] du\, dv$$

$$= \iint_S \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z} \right) dy\, dz + \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x} \right) dz\, dx + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dx\, dy \qquad (2)$$

Hence the proof.

Also by the definition of surface integral, relation (2) is equivalent to

$$\int_C (f\, dx + g\, dy + h\, dz) = \iint_S \left[\left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z} \right) \cos \alpha + \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x} \right) \cos \beta + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) \cos \gamma \right] dS \qquad (3)$$

where $\cos \alpha$, $\cos \beta$, $\cos \gamma$ are the direction cosines of the normal at any point to the surface.

Note 1 (*Vectorial formulation*). Let

$$\mathbf{r} = \mathbf{i}x + \mathbf{j}y + \mathbf{k}z$$

be the position vector of any point on the surface S, and

$$\mathbf{F}(x, y, z) = \mathbf{i}P(x, y, z) + \mathbf{j}Q(x, y, z) + \mathbf{k}R(x, y, z)$$

be a vector function defined on S.

Let $\mathbf{n}$ denote the unit normal at any point of the surface under consideration, so that

$$\mathbf{n} = \mathbf{i} \cos \alpha + \mathbf{j} \cos \beta + \mathbf{k} \cos \gamma$$

$$\therefore \quad \text{curl } \mathbf{F} \cdot \mathbf{n} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \cos \alpha + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \cos \beta + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \cos \gamma$$

and

$$\mathbf{F}\cdot d\mathbf{r} = P\,dx + Q\,dy + R\,dz$$

so that by (3) Stokes' theorem can be written as

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \text{curl}\,\mathbf{F}\cdot\mathbf{n}\,dS$$

2. If the surface S is a piece of a plane parallel to the xy-plane then $dz = 0$ and we get Green's Theorem as special case of Stokes' Theorem.

5.1 Deductions from Stokes' Theorem

Stokes' theorem has various applications in mathematical analysis. Here we are going to establish only one such deduction: the conditions for a line integral to be independent of the path of integration. These conditions, in fact, generalise the results obtained from Green's theorem (§ 4.1 Ch. 17) concerning the question of path independence of an integral over a plane curve. With that view, we introduce the following concept.

Definition. A three-dimensional domain V is said to be ***simply connected*** if, for any closed contour belonging to V there exists a surface, with the contour as its boundary, entirely lying in V.

A sphere (ball), the whole space, the domain lying between two concentric spheres are examples of a *simply connected* space. An example of a domain which is not simply connected (referred to as *multiply connected*) is a ball with a cylindrical tunnel passing through it.

Now we proceed to establish the following result analogous to § 4.1, Chapter 17.

THEOREM 2. *If three functions $f(x, y, z)$, $g(x, y, z)$, and $h(x, y, z)$, defined in a bounded closed simply connected domain V, are continuous along with their first order partial derivatives in the domain, then the following four assumptions are equivalent to each other.*

1. *The line integral $\int f\,dx + g\,dy + h\,dz$ taken along any closed contour lying inside V is equal to zero.*

2. *The line integral $\int_{AB} f\,dx + g\,dy + h\,dz$ is independent of the path of integration connecting two arbitrary points A and B.*

3. *The expression $f\,dx + g\,dy + h\,dz$ is the total differential of a single valued function defined in V.*

4. *The conditions*

$$\frac{\partial g}{\partial x}=\frac{\partial f}{\partial y},\ \frac{\partial h}{\partial y}=\frac{\partial g}{\partial z},\ \frac{\partial f}{\partial z}=\frac{\partial h}{\partial x}$$

are fulfilled at each point of the domain V.

The theorem is proved according to the scheme $1\to 2\to 3\to 4\to 1$ which we followed while proving § 4.1 Ch. 16. We leave the proof to the reader with the only hint that to deduce condition 1 from condition 4, one must take an arbitrary closed contour Γ lying within V and consider a surface entirely lying in V whose boundary is Γ; such a surface existing because of the condition that V is a simply connected domain. Then the application of Stoke's theorem to the line integral along Γ shows that condition 4 implies the relation $\int_\Gamma f\,dx + g\,dy + h\,dz = 0$.

Example 15. Use Stokes' theorem to find the line integral

$$\int_C x^2y^3\,dx + dy + z\,dz$$

where C is the circle $x^2+y^2=a^2,\ z=0$.

Now, by Stokes' theorem

$$\int_C x^2y^3\,dx+dy+z\,dz=\iint_S\left(\frac{\partial z}{\partial y}-\frac{\partial .1}{\partial z}\right)dy\,dz+\left(\frac{\partial x^2y^3}{\partial z}-\frac{\partial z}{\partial x}\right)dz\,dx$$

$$+\left(\frac{\partial .1}{\partial x}-\frac{\partial x^2y^3}{\partial y}\right)dx\,dy$$

$$=-\iint_S 3x^2y^2\,dx\,dy$$

where S is the circle $x^2+y^2=a^2$ in the xy-plane. Changing to polars,

$$=-3\int_{-\pi}^{\pi}\int_0^a r^4\cos^2\theta\sin^2\theta\; r\,dr\,d\theta$$

$$=-\frac{3a^6}{6}\int_{-\pi}^{\pi}\cos^2\theta\sin^2\theta\,d\theta=-\frac{\pi a^6}{8}$$

Example 16. Show that

$$\iint_S (y-z)\,dy\,dz+(z-x)\,dz\,dx+(x-y)\,dx\,dy=a^3\pi$$

where S is the portion of the surface $x^2+y^2-2ax+az=0,\ z\geq 0$.

By Stokes' theorem

$$\iint_S (y-z)\,dy\,dz+(z-x)\,dz\,dx+(x-y)\,dx\,dy$$

$$= \frac{1}{2}\int_C (y^2+z^2)\,dx + (z^2+x^2)\,dy + (x^2+y^2)\,dz$$

where C is the curve

$$(x-a)^2 + y^2 = a^2, \quad z = 0$$

On putting $x = a + a\cos\theta = a(1+\cos\theta)$, $y = a\sin\theta$, the line integral becomes

$$\frac{1}{2}\int_{-\pi}^{\pi} [a^2\sin^2\theta(-a\sin\theta) + a^2(1+\cos\theta)^2\, a\cos\theta]\,d\theta$$

$$= \frac{a^3}{2}\int_{-\pi}^{\pi} (-\sin^3\theta + \cos\theta + 2\cos^2\theta + \cos^3\theta)\,d\theta$$

$$= a^3\int_0^{\pi} (2\cos^2\theta + \cos^3\theta)\,d\theta$$

$$= a^3\int_0^{\pi/2} (2\cos^2\theta + \cos^3\theta)\,d\theta + a^3\int_0^{\pi/2} (2\sin^2\theta - \sin^3\theta)\,d\theta$$

$$= \pi a^3$$

Note. The method employed to convert a surface integral into a line integral is not general.

EXERCISES

1. Using Stokes' theorem, show that

$$\int_C y\,dx + z\,dy + x\,dz = -\iint_S (\cos\alpha + \cos\beta + \cos\gamma)\,dS$$

2. Show, using Stokes' theorem, that

$$\int_\Gamma (y+z)\,dx + (z+x)\,dy + (x+y)\,dz = 0$$

where Γ is the circle $x^2+y^2+z^2 = a^2$, $x+y+z = 0$.

3. Using Stokes' theorem, prove that

$$\int_\Gamma y\,dx + z\,dy + x\,dz = -2\pi a^2\sqrt{2}$$

where Γ is the curve $x^2+y^2+z^2-2ax-2ay = 0$, $x+y = 2a$.

4. Apply Stokes' theorem to transform the integral

$$\int_C (y^2+z^2)\,dx + (x^2+z^2)\,dy + (x^2+y^2)\,dz$$

taken along a smooth curve C to a certain integral over a smooth oriented surface with C as its boundary.

5. Verify Stokes' theorem for the integral

$$\int_C x^2\,dx + yx\,dy$$

where C is a square in the $z = 0$ plane with sides along the lines, $x = 0,\ y = 0,\ x = a,\ y = a.$

6. Verify Stokes' theorem in each case

(*i*) $\mathbf{F} = z\mathbf{i} + x\mathbf{j} + y\mathbf{k}$

S is the part of the paraboloid $z = 1 - x^2 - y^2$ for which $z \geqslant 0$, $\mathbf{n}\cdot\mathbf{k} > 0$.

(*ii*) $\mathbf{F} = y^2\mathbf{i} + xy\mathbf{j} - 2yz\mathbf{k}$

S is the hemisphere $x^2 + y^2 + z^2 = a^2$, $z \geqslant 0$ with $n\cdot\mathbf{k} > 0$.

6. THE VOLUME OF A CYLINDRICAL SOLID BY DOUBLE INTEGRALS

We have shown earlier that the volume of a cylindrical solid can be found with the help of double integrals.

Let a cylindrical solid be bounded above by a surface $z = \psi(x, y)$, below by a plane region D (on the xy-plane) and on the sides by lines parallel to z-axis. Its volume V is given by

$$V = \iint_D \psi(x, y)\,dx\,dy, \text{ in cartesian coordinates}$$

or

$$= \iint_D \psi(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta, \text{ in polar coordinates}$$

or

$$= \iint_S z\cos\gamma\,dS, \text{ as a surface integral}$$

where S is the surface of the solid.

If the equation of the surface is given in the form

$$x = \theta(y, z), \text{ or } y = \phi(z, x)$$

then the corresponding formulas for calculating the volumes are of the form

$$V = \iint_{D_1} \theta(y, z)\,dy\,dz \quad\text{or}\quad \iint_{S_1} x\cos\alpha\,dS$$

$$V = \iint_{D_2} \phi(z, x)\,dz\,dx \quad\text{or}\quad \iint_{S_2} y\cos\beta\,dS$$

where D_1, D_2 are the domains in the yz-plane, and zx-plane in which the given surface is projected.

Note 1. Clearl
surface $z = \psi(x$

2. The functio
the surface is m

3. If the functi
two parts. (*i*) th
where $\psi \leqslant 0$. Tl
the volume of th
will be negative
lying below the
difference betwee
values of the two
solid.

4. *Volume by ite*
ted geometrically.

$V = \iint_D \psi$

$= \int_a^b S(x)$

where

$S(x) = \int_{\phi_1(x}^{\phi_2(x}$
= area
paral

$\therefore\ V =$ vol