

DISSERTATION

Between the Finite and the Infinite

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To my parents.

Between the Finite and the Infinite

by

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Abstract

Various results in Proof Theory, Set Theory, and Recursion Theory are proved.

We introduce the *reflection order*, an ordering on patterns of iterated Σ^1_1- and Π^1_1-reflection for ordinals. Extending Aanderaa's theorem that $\pi^1_1 < \sigma^1_1$, we prove that σ^1_1 has rank ω in the reflection order and $\sigma^1_1(\sigma^1_1)$ has rank ω^ω. We prove that the reflection order is a prewellordering of length $\geq \varepsilon_0$ and that its conjunction-free fragment has length exactly ω^ω. As part of the proof, we extend classical results of Aczel, Richter, and Gostanian.

We study the determinacy of games of transfinite length, mostly of length ω^2. We prove that clopen determinacy of length ω^2 is equivalent to σ-projective determinacy of length ω, to the existence of a certain uncountable sequence of canonical inner models with finitely many Woodin cardinals, and to the cut elimination theorem for Takeuti's Determinate Logic for languages with no atomic formulae of infinite arity. We obtain further characterizations of this in joint work with S. Müller and P. Schlicht.

We compute the consistency strength of determinacy for games of length ω^2 and payoff in other pointclasses, including the open sets, the F_σ sets, the Borel sets, the projective sets, and the σ-projective sets (the latter two jointly with S. Müller). We give new upper bounds for the consistency strength of clopen determinacy for games with length of the form $\omega^2 \cdot \beta$ and show that if there are ω^2 Woodin cardinals below a measurable cardinal, then there is a model of Kripke-Platek set theory satisfying $\mathsf{AD}_{\mathbb{R}}$.

We study proofs in finitary and infinitary systems with weakened eigenvariable conditions. Jointly with M. Baaz, we prove a speed-up theorem for cut-free proofs and characterize the sequents provable in intuitionistic ε-calculus. In the infinitary case, we characterize cut elimination theorems for these systems in terms of determinacy assertions, following Takeuti.

Jointly with D. Fernández-Duque, we prove that provability logic is strongly complete with respect to its topological semantics, and, in particular, with respect to the ordinal $\omega^\omega + 1$. More generally, we introduce a generalization of Icard spaces and show that provability logic is individually strongly complete with respect to every Icard space of large enough rank. Finally, we prove a completeness theorem for transfinite provability logic with respect to finite polytopologies based on Icard spaces.

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1

Introduction

This thesis is centered around two recurring themes, each corresponding to one possible sense that the preposition in the title can have. It is composed of four parts, which are related in ways that might not be obvious at first sight.

This work mostly fits within the scope of the program in which one attempts to isolate criteria to determine the truth or falsity of general mathematical statements. By Gödel's incompleteness theorems, not all mathematical problems can be solved by resorting exclusively to deductive reasoning, and this occurs in particular when traversing the consistency strength hierarchy in the direction of inconsistency.

Two natural ways to climb the consistency strength hierarchy are provided by consistency statements and by large-cardinal axioms. Gödel's hope was that large-cardinal axioms would settle otherwise independent mathematical questions such as the Continuum Hypothesis. Unfortunately, Lévy and Solovay's [LS67] contributions to the technique of forcing showed that this was not the case. This brought upon the need to isolate other key mathematical axioms which are not large-cardinal assumptions. The results in parts II and III of this thesis can be arguably interpreted as set-theoretic and proof-theoretic evidence towards the view, ubiquitous within the author's scientific circle, that these axioms must include (strong) determinacy hypotheses.

Although arguing for axioms of definable determinacy might seem pointless to the reader already convinced of the existence of large cardinals, we believe that there is a subtle point to be raised: the position that axioms of definable determinacy must hold, if taken to the extreme, must necessarily be accompanied by a rejection of forcing axioms:

Theorem 1.0.1 (Larson [Lar04])**.** *Suppose that Martin's Maximum holds. Then, there is a definable game of length ω_1 that is not determined.*

This theorem is not vacuous, in that Woodin has shown (from suitable assumptions) that it is consistent with ZFC that all definable games of length ω_1 are determined. The question remains as to the extent to which determinacy of games of length ω_1 is *provable*, perhaps with the use of large-cardinal or other (consistent) combinatorial statements. We mention the following result, which builds heavily on work of Neeman and is not proved in this thesis:

Theorem 1.0.2 (with D. Blue [ABb])**.** *Suppose that there is a countably iterable model of* ZFC *with a measurable Woodin cardinal and that the Continuum Hypothesis holds. Let $X \subset 2^{\omega_1}$ be provably Δ_1 definable from $\mathbb{R}$. Then X is determined.*

The assumption on provability in the above statement can be weakened to Ω-provability, under suitable assumptions. The proof, however, depends crucially on the Continuum Hypothesis, although we do not know if it is necessary for the conclusion. Larson's theorem suggests that this may well be the case. If it were, this fact would make a very strong case for the truth of the Continuum Hypothesis, provided countably iterable models of ZFC with measurable Woodin cardinals exist, regardless of whether this is provable from large cardinals alone.[1] For more details on the connection between the Continuum Hypothesis and the determinacy of definable games, we refer the reader to Woodin [Woo02] or Woodin [Woo16].

In this dissertation, we study—and argue for—axioms of definable determinacy for games of countable length. Unlike the set of possibilities, the size of this dissertation must be necessarily finite, and so we will mosltly focus on games of length ω^2. Our expectation is that this lead to further studies of axioms of definable determinacy for games of countable and uncountable lengths, which may in turn eventually provide an answer to questions such as the Continuum Hypothesis.

Our first (and main) argument is based on the realization that there is striking similarity between the regions in the consistency strength hierarchy linking PA and ZFC on the one hand, and the theories

$$\mathsf{ZFC} + \{\text{“there are } n \text{ Woodin cardinals”: } n \in \mathbb{N}\}$$

and

$$\mathsf{ZFC} + \text{“there are infinitely many Woodin cardinals,”}$$

[1] This is not known to be the case, albeit it is widely conjectured.

on the other. Starting from the assumption that the theories PA and ZFC are both true, this suggests an approach to isolating a collection of true theories extending ZFC and increasing in consistency strength; namely, if PA and ZFC are true, then there must be a sequence of intermediate true theories linking them. If the two regions of the consistency strength hierarchy specified above indeed resemble each other, then the theories below each will have stronger counterparts, and one may argue for their truth on the basis of this counterparthood; this is what we do.

There is one potential issue: the choice of intermediate theories. There are three canonical ways of extending PA and obtaining approximations to ZFC that have been thoroughly studied; these are:

1. axioms of comprehension, asserting the existence of initial segments of the constructible hierarchy L;
2. axioms of definable determinacy for (Borel) games of length ω,
3. axioms asserting the closure of $\mathbb{R}$ under various jump operators.

A possible fourth alternative, one which goes back to Turing's thesis (see also Feferman [Fef62]), would be the extension of PA by iterated consistency assertions in some canonical way. We believe that more research is needed in this area before it can be incorporated in an argument of the present sort, so we shall make no further comment on this. The reason why is this mentioned at all is that it is in principle possible, albeit unlikely, that advances in this area be in conflict with the argument we present here.

These three ways of extending PA have analogues which yield higher consistency strength:

1. axioms of choiceless determinacy, asserting the determinacy of initial segments of the relativized constructible hierarchy $L(\mathbb{R})$;
2. axioms of definable determinacy for (Borel) games of length ω^2,
3. axioms asserting the closure of $\mathbb{R}$ under various inner-model-theoretic operators.

Earlier work in Reverse Mathematics shows that the three paths from PA to ZFC that we mentioned coincide (see Figure 1.1). The results in Part II show that their respective analogues also coincide (see Figure 2.1 on page 21). This means not only that the two aforementioned regions of the consistency hierarchy resemble each other in a very precise way, but also that the earlier potential issue was in fact vacuous. It seems extremely plausible that higher

Base	Comprehension	Determinacy	Operators	
RCA_0	$L_1[x]$-comprehension for all $x \in \mathbb{R}$, i.e., ACA_0	$\mathbf{\Sigma}^0_1$-determinacy for games of length $n \in \mathbb{N}$	$x^{(n)}$ exists for all $x \in \mathbb{R}$	Post, Simpson
RCA_0	$L_\alpha[x]$-comprehension for all $x \in \mathbb{R}$ and all $\alpha < \omega_1$, i.e., ATR_0	$\mathbf{\Delta}^0_1$-determinacy	$x^{(\alpha)}$ exists for all $x \in \mathbb{R}$ and all $\alpha < \omega_1$	Steel [Ste77]
KP	$L_{\omega_1^{ck}}$-comprehension, i.e., there is an admissible set	Σ^0_1-determinacy	$\mathcal{O}$ exists	Solovay
KP	$L_{\sigma^1_1}$-comprehension, i.e., there is a Σ^1_1-reflecting ordinal	Σ^0_2-determinacy		Solovay
KP + Sep	for all $x \in \mathbb{R}$ and all $\alpha < \omega_1$, there is some β^x_α such that $L_{\beta^x_\alpha}[x] \models$ "V_α exists"	$\mathbf{\Delta}^1_1$-determinacy	the theory of $L_{\beta^x_\alpha}[x]$ exists for all $x \in \mathbb{R}$ and all $\alpha < \omega_1$	Martin
ZFC		Projective Determinacy	$M^\sharp_n$ exists for all $n \in \mathbb{N}$	Neeman [Nee04], Woodin

Figure 1.1: Theories between PA and ZFC + Projective Determinacy in consistency strength. Theories in the same row are equivalent. (The indicated base theories are not meant to be optimal).

regions of the consistency hierarchy also resemble the gap between PA and ZFC, and this is evidenced by work of Neeman [Nee04], Sargsyan [Sar09], Steel, Trang [Tra13], Woodin, and others.

The analogy between Figures 1.1 and 2.1 does extend beyond the results presented in this thesis. For example, the σ-projective hierarchy has a natural analogue which can be defined by paralleling the definition of simple clopen games (see Definition 5.2.2 on p. 140). Determinacy for games at various levels of this hierarchy can be shown to be equivalent to the existence of sequences of iterations of the Turing jump of an appropriate length. Another example is given by the difference hierarchy between Π^0_2 and Δ^0_3. Although we shall not prove this here, determinacy at these levels corresponds, on the one hand, to the existence of iteratively Σ^1_1-reflecting ordinals (MedSalem and Tanaka [MT08]) and, on the other, to the determinacy of $L(\mathbb{R})$ up to iteratively Π^+_1-reflecting ordinals.

The study of the reflecting structure of L (or of $L(\mathbb{R})$) at the level of iteratively Σ^1_1- and Π^1_1-reflecting ordinals is foundationally very interesting, because of its essential role in the proof-theoretic analyses of theories in the vicinity of Π^1_2-comprehension (see Rathjen [Rat05b, Rat05a]). It seems plausible that extensions of weak theories by iterated consistency assertions provide a fourth bridge between PA and ZFC. Our position seems to predict that these theories will have higher analogues which coincide with the theories of determinacy for definable games of length ω^2, although it is hard to make precise claims before further progress in the proof-theoretic analysis of theories beyond Π^1_2-comprehension is made.

This region of the constructible hierarchy is also very interesting mathematically because of its connections to generalized recursion theory. Part I of the thesis is devoted to this study. Its goal is to determine the order in which various patterns of iterated reflection first occur within the constructible hierarchy. Patterns of reflection are finite objects, but their order can only be determined by analyzing various transfinite ordinals α in terms of their degrees of stability, the lengths of α-recursive wellorderings, and their reflecting properties across different (possibly nonstandard) models of (fragments of) set theory. We have chosen to lead the thesis with our study of the reflection order because it is the most self-contained part, although the computations involved might seem easier to the reader familiar with the examples and/or results in Part IV.

Our second argument for axioms of definable determinacy as extensions of ZFC is proof-theoretical. It is based on cut-elimination theorems that arise as natural extensions of proof systems with locally unsound rules. The initial goal was to consider examples of generalized notions of proof in which correctness is determined by analyzing a proof in its entirely, and without the restriction that every initial segment of a proof be also a proof. The example we give at the beginning of Part III is the following argument, which has a true conclusion and is admissible when formalized within the framework we develop:

1. That Kurt Gödel is Austrian entails that Kurt Gödel is Austrian.

2. Hence, that Kurt Gödel is Austrian entails that everyone is Austrian.

3. That is, if Kurt Gödel is Austrian, then all people are Austrian.

4. Therefore, there exists a person such that, if that person is Austrian, then all people are Austrian.

This generalized notion of proof results from extending the permissible circumstances in which one may quantify terms. This type of reasoning is not novel—it originates from the way terms are treated in Hilbert's ε-calculus. A different, but similar, situation occurs in reasoning by contradiction, where a proof cannot be prematurely terminated after a contradictory assumption has been made, but before the contradiction has been reached. Our proof systems have natural infinitary extensions by allowing countable conjunctions and quantifier chains. The general rules for quantification can be applied not only to quantifier strings of the form

$$\forall x_0, \forall x_1, \forall x_2, \dots$$

but also of the form

$$\forall x_0, \exists x_1, \forall x_2, \dots .$$

It is natural to ask whether the cut-elimination theorem extends to this proof system. This is a purely syntactical question, independent of any possible interpretation of the symbols and, in particular, of quantifier chains. It turned out that this question was already studied by Takeuti, in the case that the language contains atomic formulae of infinite arity and only quantifier strings of length ω; he showed that the cut-elimination theorem was equivalent to the Axiom of Determinacy. We study the question in the setting in which all atomic formulae have finite arity and show that the cut-elimination theorem is equivalent to σ-projective determinacy, if only quantifier strings of length ω are allowed, or to projective determinacy for games of countable length, if quantifier strings of all countable lengths are allowed. Thus, we arrive again at axioms of definable determinacy, this time by adopting extensions of the cut-elimination theorem to infinitary logic.

It was in fact these two cut-elimination theorems that led to our studies of σ-projective determinacy, on the one hand, and to games of length ω^2, on the other, before we knew of the relation between the two determinacy hypotheses (via clopen games).

The final part of this thesis stems from the study of arithmetical theories by means of iterated consistency assertions and/or reflection principles (see Beklemisev [Bek04]). This area combines proof-theoretic techniques with the representation of provability predicates by modal operators, allowing the use of general algebraic methods to manipulate them. Part IV concerns the topological counterparts of these algebras and builds on work of Beklemisev and Gabelaia [BG13] and Fernández-Duque [FD14]. The key idea is to exploit the arithmetical structure of the ordinals to derive results about scattered spaces.

Structure of the thesis

Other than this introductory chapter and the forthcoming summary of results, this thesis contains thirteen chapters, four of which are joint work with other researchers. Chapter 4 was written jointly with S. Müller and P. Schlicht [AMS]; Chapter 5, with S. Müller [AM]; Chapter 12, with M. Baaz [ABa]; Chapter 14, with D. Fernández-Duque [AFD17]. All chapters, except for the one on open games, are intended to be submitted individually for publication and thus are formatted as individual articles. Because of this, their respective introductory and preliminary sections might overlap. The ones not mentioned already correspond (in order) to the articles [Ague, Agug, Aguf, Agud, Agub, Aguc, Agua, Aguh].

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In the fall of 2015, I met Rachid Atmai, to whom, after having heard of these-things-called-measurable-cardinals during John Steel's talk at the Fifth European Set Theory Society meeting in Cambridge that I, an exclusively-proof-theorist, attended as an excuse to escape the unsurmountable Viennese summer, I professed my intrigue concerning these-things-called-core-models. Skeptical, however, I expressed unease linked to the apparent great deal of preliminary reading needed before one could begin doing research in that area. He promptly assured me this was not the case. This dissertation would be extremely different had that lie of blatancy previously unbeknownst to man not been spoken. I would like to thank him for this, and for his patience with me during the times of Devlin and and the times of Mitchell.

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2

Summary of Results

Here, we briefly summarize the main results proved in this thesis.

The Order of Reflection

In Part I, we study the *reflection order*, a binary relation on *reflection patterns*. These are defined as follows: σ^1_1 and π^1_1 are reflection patterns. If s and t are reflection patterns, then so are $\sigma^1_1(s)$, $\pi^1_1(s)$, and $s \wedge t$. Let X be a class of ordinals. An ordinal α is Σ^1_1*-reflecting on* X if

$$L_\alpha \models \phi \implies \exists \beta \in \alpha \cap X \, (L_\beta \models \phi),$$

whenever ϕ is a (set-theoretic) Σ^1_1-sentence, possibly with parameters in L_α. An ordinal being Π^1_1-reflecting on X is defined similarly.

We say α is σ^1_1-reflecting if it is Σ^1_1-reflecting on Ord and we say it is π^1_1-reflecting if it is Π^1_1-reflecting on Ord. Inductively, we say α is $\sigma^1_1(s)$-reflecting if it is Σ^1_1-reflecting on s-reflecting ordinals and we say it is $\pi^1_1(s)$-reflecting if it is Π^1_1-reflecting on s-reflecting ordinals. We say it is $s \wedge t$-reflecting if it is both s-reflecting and t-reflecting. Given two reflection patterns s and t, we define

$$s < t \tag{2.1}$$

if the least s-reflecting ordinal is smaller than the least t-reflecting ordinal. An analysis of the reflection order involves answering questions such as (2.1). The reflection order is essentially an order on finite figures defined by making reference to much larger objects. Moreover, there can be no way of defining it which is much more simple, as we will see that it is not equivalent to any linear re-ordering of $\mathbb{N}$ which is wellfounded provably in PA.

The starting point is a result of Aanderaa by which

$$\pi^1_1 < \sigma^1_1.$$

A generalization of Aanderaa's theorem is that

$$\pi^1_1 < \pi^1_1(\pi^1_1) < \pi^1_1(\pi^1_1(\pi^1_1)) < \cdots < \sigma^1_1;$$

in fact, σ^1_1 has rank ω in the reflection order. We prove:

Theorem 2.0.1. *$\sigma^1_1(\sigma^1_1)$ has rank ω^ω in the reflection order.*

In proving this theorem, we prove various facts about particular well-known ordinals. For instance, we prove that σ^1_1 is stable precisely to the supremum of lengths of σ^1_1-recursive wellorderings, but that $\sigma^1_1(\sigma^1_1)$ is strictly more stable. We also prove various relationships between reflection patterns which we believe to be of interest and unintuitive, such as

$$\sigma^1_1(\pi^1_1(\sigma^1_1)) < \sigma^1_1(\sigma^1_1)$$

or

$$\pi^1_1(\sigma^1_1) \equiv \pi^1_1(\sigma^1_1) \wedge \pi^1_1(\pi^1_1).$$

Our main result concerns the length of the reflection order. Clearly it is a prewellordering; we show:

Theorem 2.0.2. *The length of the reflection order is at least ε_0.*

The proof of Theorem 2.0.2 shows that the rank of $(\sigma^1_1)^n$ in the reflection order is at least the result of iterating the ω-based exponential n-many times.

When discussing the reflection order with D. Schrittesser, he asked what the length of the conjunction-free fragment of the reflection order was. The answer is an easy consequence of the methods developed along the way to Theorem 2.0.1:

Theorem 2.0.3. *The length of the conjunction-free fragment of the reflection order is ω^ω.*

Another easy consequence of the proof of Theorem 2.0.1 is that the conjunction-free fragment of the reflection order is cofinal in the full reflection order. In fact,

Theorem 2.0.4. *The sequence of reflection patterns of the form $(\sigma^1_1)^n$ is cofinal in the reflection order.*

We do not know of any recursive upper bound for the length of the reflection order, although we show that for every reflection pattern s, the least s-reflecting ordinal appears below the least α such that

$$L_\alpha \prec_1 L_{\alpha^+ + 1},$$

where α^+ denotes the least admissible ordinal greater than α. (This is optimal in the sense that if one replaces $\alpha^+ + 1$ by α^+ in the equation above, the ordinal α is equal to π^1_1, the least of the reflecting ordinals.) Presumably, all the relationships between reflection patterns are provable in any theory which proves the existence of the corresponding ordinals, so this suggests that the length of the reflection order is smaller than the proof-theoretic ordinal of (lightface) Π^1_2-comprehension. We do not have a proof of this, but conjecture that it is in fact much shorter.

The computations in this chapter make use of the hyperexponential notation $e^n\beta$. The reader who desires a more careful treatment of hyperexponentials and logarithms, or examples, may consult Part IV.

At the end of this summarizing chapter, we include diagrams describing the order between some ordinals given by the reflection order, together with comments, including the ranks of some ordinals in the reflection order.

Games of Transfinite Length

Part II is concerned with two-player, perfect-information games of transfinite length. The games we consider are defined as follows: fix a countable wellorder α (the length of the game), and fix a set $A \subset \mathbb{N}^\alpha$ of α-sequences of natural numbers (the payoff set). Two players, Player I and Player II, alternate turns playing natural numbers $x(\iota)$, producing a sequence:

I	$x(0)$		$x(2)$		$\dots$	$x(\omega)$		$\dots$
II		$x(1)$		$x(3)$	$\dots$		$x(\omega+1)$	$\dots$

After α-many numbers have been played, the game ends. Let x be the sequence thus produced. Player I wins if, and only if, $x \in A$; otherwise, Player II wins. Since the space $\mathbb{N}^\alpha$ is homeomorphic to the irrationals, we refer to plays x as *reals*.

The first half of Part II deals with *σ-projective* sets of real numbers; these are the smallest collection of subsets of $\mathbb{R}$ closed under complements, countable unions, and continuous images. Specifically, we are concerned with infinite games on $\mathbb{N}$ whose payoff is a σ-projective set. We consider

the question of computing the consistency strength of the assertion that all such games are determined. Because it is too coarse, this question cannot be answered in terms of the usual large-cardinal hierarchy, but only in terms of the existence of *canonical inner models*. It is an easy consequence of known methods that the consistency strength of σ-projective determinacy follows from the existence of infinitely many Woodin cardinals and thus is strictly weaker (since it only speaks about objects below the least inaccessible cardinal). Moreover, a theorem of Woodin states that projective determinacy implies the consistency of ZFC with any finite amount of Woodin cardinals; hence, the consistency strength of σ-projective determinacy lies strictly between that of ZFC with all finite amounts of Woodin cardinals, and that of ZFC with infinitely many Woodin cardinals.

Given a countable ordinal $\alpha < \omega_1$, we introduce the notion of a model of *class* S_α; these are countably iterable fine-structural extender models constructed relative to a real parameter. The existence of models of each class S_α can be directly proved from the existence of infinitely many Woodin cardinals, and thus seen to be weaker in consistency strength. We prove:

Theorem 2.0.5. *The following are equivalent over* ZFC*:*

1. *σ-projective determinacy;*
2. *For each $\alpha < \omega_1$, there is a model of set theory of class S_α.*

Afterwards, we include joint work with S. Müller and P. Schlicht in which we study the relationship between games of length ω with a complicated payoff, and longer games with a simpler payoff. The methods used to prove the main results are perhaps more important than the results themselves: they generalize in very useful ways, as we will see later. We introduce the notion of *simple* games of a given length $\alpha + \omega^2$ and show, among other things:

Theorem 2.0.6. *(with S. Müller and P. Schlicht) The following are equivalent over* ZFC*:*

1. *σ-projective determinacy;*
2. *Determinacy for simple clopen games of length ω^2;*
3. *Determinacy for simple σ-projective games of length ω^2;*

The second half of Part II studies games of fixed countable length greater than ω. The starting point is Neeman's theorem that if there is an active,

countable, iterable extender model with θ Woodin cardinals, then all analytic games of length $\omega \cdot \theta$ are determined. We study lower bounds for the consistency strength of this assertion. Since analytic games of length $\omega \cdot n$ can be simulated by projective games of length ω, the first place at which something new occurs is at length ω^2; thus, most of our work in this section focuses on games of length ω^2.

Theorem 2.0.7. *(with S. Müller) The following two schemata are equiconsistent over* ZFC*:*

1. *Determinacy for projective games of length ω^2;*
2. *{"there are $\omega + n$ Woodin cardinals": $n \in \mathbb{N}$}.*

Our proof yields local lower bounds and extends to compute the strength of σ-projective determinacy and of $\Im^{\mathbb{R}}\mathbf{\Pi}^1_1$-determinacy for games of length ω^2.

After studying projective determinacy, we move on to simpler pointclasses. Like σ-projective determinacy of length ω, determinacy assumptions for games of length ω^2 with payoffs in pointclasses between the clopen sets and the Borel sets have consistency strengths that lie within ZFC plus all finite amounts of Woodin cardinals and ZFC with infinitely many Woodin cardinals.

Theorem 2.0.8. *The following are equivalent over* ZFC*:*

1. *Clopen determinacy for games of length ω^2;*
2. *σ-projective determinacy.*

Together with the results from the first chapter of Part II, we obtain a computation of the consistency strength of clopen determinacy for games of length ω^2.

A classical theorem of Steel states that, over Recursive Comprehension, clopen determinacy is equivalent to open determinacy. This seems to suggest that clopen determinacy for games of length ω^2 is equivalent to open determinacy for games of length ω^2. However, this is not the case. It is not hard to see that the σ-projective sets of reals are precisely the sets of reals in $L_{\omega_1}(\mathbb{R})$. In contrast,

Theorem 2.0.9. *The following are equivalent over* ZFC*:*

1. *Open determinacy for games of length ω^2;*

2. *There is a transitive model of* $\mathsf{KP} + \mathsf{AD}$ *containing* $\mathbb{R}$.

As for Borel determinacy, we obtain a triple equivalence:

Theorem 2.0.10. *The following are equivalent over* ZFC*:*

1. *Borel determinacy for games of length* ω^2*;*

2. *For each* $\alpha < \omega_1$*, there is a transitive model of* $\mathsf{KP} + \mathsf{AD}$ + *"*V_α *exists" containing* $\mathbb{R}$.

3. *For each* $\alpha < \omega_1$*, there is a countably iterable extender model of* Z + *"there is a limit of Woodin cardinals* λ *and* $V_{\lambda+\alpha}$ *exists."*

Borel determinacy of length ω^2 is the first theorem that we know of whose proof over ZFC requires models with large cardinals over a theory weaker than ZFC.

The techniques employed adapt to prove results on the determinacy sets in a specific level of the Borel hierarchy. Since we have considered the first one, we consider the second one. For this, we need a definition. Here, A^+ denotes the intersection of all admissible sets containing A.

Definition 2.0.11. A set A is Π^+_1-reflecting if for all Π_1 statements ϕ, if $A^+ \models \phi(A)$, then there is $B \in A$ such that $B^+ \models \phi(B)$.

Theorem 2.0.12. *The following are equivalent over* ZFC*:*

1. F_σ*-determinacy for games of length* ω^2*;*

2. *There is a transitive,* Π^+_1*-reflecting model of* $\mathsf{KP} + \mathsf{AD}$ *containing* $\mathbb{R}$.

Our results on games of length ω^2 can be summarized by Figure 2.1 (compare with Figure 1.1).

After our study of games of length ω^2, we move on to longer games. We generalize our work on clopen games to compute upper bounds for the strength of clopen determinacy of a various lengths. These upper bounds are probably optimal, but we do not prove this here. The proof is based on the methods of the first half of Part II and "shortening" theorems such as the ones for games of length ω^2. We mention the following result:

Theorem 2.0.13. *The following are equivalent over* ZFC*:*

1. *Clopen determinacy for games of length* ω^3*;*

Base	$L(\mathbb{R})$-determinacy	Games of length ω^2	Operators	
ZFC	$L_1(\mathbb{R}) \models \mathsf{AD}$	$\mathbf{\Sigma}^1_1$-determinacy for games of length $\omega \cdot n$	$M_n^\sharp$ exists for all $n \in \mathbb{N}$	Neeman [Nee04], Woodin
ZFC	$L_{\omega_1}(\mathbb{R}) \models \mathsf{AD}$, i.e., $L_\alpha(\mathbb{R})$-determinacy for all $\alpha < \omega_1$	$\mathbf{\Delta}^0_1$-determinacy	$N^\sharp_\alpha(x)$ exists for all $\alpha < \omega_1$ and some $x \in \mathbb{R}$	
ZFC	$L_{\kappa^\mathbb{R}}(\mathbb{R}) \models \mathsf{AD}$, i.e., there is an admissible set containing $\mathbb{R}$ and satisfying AD	$\mathbf{\Sigma}^0_1$-determinacy		
ZFC	$L_{\sigma_\mathbb{R}}(\mathbb{R}) \models \mathsf{AD}$, i.e., there is a Π^+_1-reflecting set containing $\mathbb{R}$ and satisfying AD	$\mathbf{\Sigma}^0_2$-determinacy		
ZFC	$L_\vartheta(\mathbb{R}) \models \mathsf{AD}$, i.e., for all $\alpha < \omega_1$, there is some β such that $L_\beta(\mathbb{R}) \models \mathsf{AD}$ + "V_α exists"	$\mathbf{\Delta}^1_1$-determinacy	for all $\alpha < \omega_1$, there is an iterable extender model of Zermelo set theory with a limit of Woodin cardinals λ satisfying "$V_{\lambda+\alpha}$ exists"	
ZFC	AD + {"there are n Woodin cardinals": $n \in \mathbb{N}$}*	Projective Determinacy	{"there are $\omega + n$ Woodin cardinals": $n \in \mathbb{N}$}*	with S. Müller

Figure 2.1: Games of length ω^2. For the theories marked with *, we include only a proof of equiconsistency.

2. *Determinacy of all games of length ω^2 with payoff in the smallest σ-algebra containing the open sets and closed under the real-game quantifier.*

Since clopen determinacy for games of length ω^3 is false in $L(\mathbb{R})$, the above result cannot be improved by replacing the second statement with a determinacy hypothesis for games of length ω.

To finish the chapter, we use long games to prove a theorem on the strength of $\mathsf{AD}_{\mathbb{R}}$. By theorems of Steel and Woodin, the following are equiconsistent:

1. $\mathsf{ZFC}+$ "there is a cardinal λ which is a limit of Woodin cardinals and a limit of $<\lambda$-strong cardinals";
2. $\mathsf{ZF} + \mathsf{AD}_{\mathbb{R}}$.

In particular, over ZF, $\mathsf{AD}_{\mathbb{R}}$ is strictly stronger than the existence of a proper class of Woodin cardinals.

We have seen that the theory $\mathsf{KP} + \mathsf{AD}$ has consistency strength almost as high as $\mathsf{ZF} + \mathsf{AD}$ (it implies open determinacy for games of length ω^2, which implies σ-projective determinacy). We see, however, that the theory $\mathsf{KP} + \mathsf{AD}_{\mathbb{R}}$ is much weaker than $\mathsf{ZF} + \mathsf{AD}_{\mathbb{R}}$.

Theorem 2.0.14. *Suppose that there are ω^2 Woodin cardinals and a measurable cardinal above them. Then, there is a transitive model of $\mathsf{KP} + \mathsf{AD}_{\mathbb{R}}$ containing $\mathbb{R}$.*

Proofs in Systems with Unsound Rules

In Part III, we study proofs of sequents in formal systems with locally unsound rules. We define the systems LK^{++} and LK^{+}, obtained from Gentzen's LK by weakening its characteristic-variable condition in a certain way. More specifically, we ask only that each strong quantifier inference satisfy the following three properties, abstracted from proofs in Hilbert's ε-calculus. Here, for free variables a, b, we write $a <_\pi b$ if the proof π contains a strong-quantifier inference of the form:

$$\frac{\Gamma \vdash \Delta, A(a, b, \vec{c})}{\Gamma \vdash \Delta, \forall x\, A(x, b, \vec{c})}$$

or of the form:

$$\frac{A(a, b, \vec{c}), \Gamma \vdash \Delta}{\exists x\, A(x, b, \vec{c}), \Gamma \vdash \Delta}$$

The conditions are:

- (substitutability) the characteristic variable does not appear in the conclusion of π.
- (side-variable condition) the relation $<_\pi$ is acyclic.
- (weak regularity) the characteristic variable is not the characteristic variable of another strong-quantifier inference in π.

We show that every LK^{++}- and LK^{+}-provable sequent is already LK-provable. The main result is:

Theorem 2.0.15. *(with M. Baaz) There is no elementary function bounding the length of the shortest cut-free* LK*-proof in terms of the shortest cut-free* LK^{++}*- or the shortest cut-free* LK^{+}*-proof.*

We also obtain results on the complexity of Herbrand sequents in LK^{++} and LK^{+}, as well as the corresponding intuitionistic analogues, LJ^{+} and LJ^{++}. Unlike their classical counterparts, LJ^{+} and LJ^{++} are not conservative over intuitionistic logic and do not admit cut elimination. As a consequence of our study of proofs in unsound system, we obtain a characterization of the collection of sequents provable in intuitionistic ε-calculus. Below, by a *quantifier shift*, we mean an instance of a sequent expressing distributivity of a quantifier with respect to a Boolean connective, such as

$$\forall x\,(A \vee B(x)) \vdash A \vee \forall x\,B(x).$$

Theorem 2.0.16. *(with M. Baaz) Let S be a sequent in first-order logic. The following are equivalent:*

1. *S is provable in intuitionistic logic with all quantifier shifts added as axioms;*
2. *S is provable in intuitionistic ε-calculus;*
3. *S is provable in LJ^{+};*
4. *S is provable in LJ^{++}.*

The characteristic variable conditions of LK^{++} are closely related to those in Takeuti's system for *Determinate Logic*. Roughly, Takeuti's conditions are the natural generalizations of the ones for LK^{++} and LK^{+} to infinitary logic. Thus, Takeuti's system for Determinate Logic is the natural extension of LK^{++} and LK^{+} to infinitary logic.

Theorem 2.0.17 (Takeuti)**.** *The following are equivalent for every transitive model M of $\mathsf{ZF}+\mathsf{DC}$ containing $\mathbb{R}$:*

1. $M \models$ *"* DL *satisfies the cut-elimination theorem,"*
2. $M \models \mathsf{AD}$.

The proof of Takeuti's theorem is unusual in that it uses the Axiom of Choice to prove that a statement inconsistent with it holds in an inner model. Using a simple forcing argument, Takeuti's theorem can be rephrased without making reference to inner models.

Corollary 2.0.18. *Assume $\mathsf{ZF}+\mathsf{DC}$. The following are equivalent:*

1. DL *satisfies the cut-elimination theorem,*
2. AD.

In the second part of Chapter III, we prove variants of Takeuti's theorem by modifying his proof. We mention two. The langauge of Takeuti's system DL is uncountable. Its formulae are countable, but sequents need not be. This language involves a restriction with regard to the length of strings of quantifiers one is allowed to form. Removing this restriction gives rise to a proof system we call $\mathsf{DL}_{\mathbb{R}}$.

Theorem 2.0.19. *Assume $\mathsf{ZF}+\mathsf{DC}$. Then, the following are equivalent:*

1. $\mathsf{DL}_{\mathbb{R}}$ *satisfies the cut-elimination theorem.*
2. $\mathsf{AD}_{\mathbb{R}}$.

Let "weak cut elimination" stand for the assertion that every sequent having a proof with no atomic formulae of infinite arity has a cut-free proof. Although full cut elimination contradicts the Axiom of Choice, weak cut elimination does not:

Theorem 2.0.20. *Assume $\mathsf{ZF}+\mathsf{DC}$. The following are equivalent:*

1. DL *satisfies weak cut elimination.*
2. *σ-projective determinacy.*

Putting everything so far together, we obtain the following theorem relating proof-theoretic, set-theoretic, and game-theoretic hypotheses:

Theorem 2.0.21. *The following are equivalent over ZFC:*

1. *all σ-projective games of length ω are determined;*
2. *all simple σ-projective games of length ω^2 are determined;*
3. *all simple clopen games of length ω^2 are determined;*
4. *all clopen games of length ω^2 are determined;*
5. *for every $\alpha < \omega_1$, there is a model of class S_α;*
6. *for every $\alpha < \omega_1$, for almost every $x \in \mathbb{R}$, there is a model of class S_α containing x;*
7. *Takeuti's* DL *satisfies cut elimination for languages with no atomic formulae of infinite arity.*

Topological Interpretations of Provability

Our last chapter concerns the topological interpretations of provability logic. The provability logic GL is the modal logic whose axioms are

$$\Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$$

and

$$\Box(\Box\phi \to \phi) \to \Box\phi$$

and whose rules of inference are modus ponens and necessitation (from ϕ deduce $\Box\phi$). The intent is that formulae ϕ be iterpreted as formulae in the language of some formal system, such as ZFC, with formulae of the form $\Box\phi$ read as "ϕ is provable."

Solovay proved that GL is complete for its provability semantics, and Segerberg proved that it is complete for its relational semantics. Esakia proved that it is complete with respect to its topological semantics, in which $\Diamond$ (the dual of $\Box$) is interpreted as the limit-point operator. A strengthening of this, due to Abazhidze and Blass, is that GL is complete with respect to a single topological space, the ordinal ω^ω with the order topology.

In Part IV, we introduce *generalized Icard topologies*, obtained from an arbitrary scattered space X by adding the sets of all points with (Cantor-Bendixson) rank in some (α, β). This process can be iterated transfinitely to generate the Icard topologies

$$\{X_{\uparrow\lambda} : \lambda \in \mathsf{Ord}\}.$$

The simplest example of an Icard topology is an ordinal with the order topology, obtained when $\lambda = 1$ and X is the same ordinal with the scattered topology given by initial segments. The following generalization of Abashizde and Blass' theorem was proved in the author's Bachelor's thesis, written under the supervision of D. Fernández-Duque. It employs hyperexponential notation $e^\lambda\alpha$; roughly, this is the result of iterating the function

$$\alpha \to -1 + \omega^\alpha$$

λ-many times.

Theorem 2.0.22. *Let λ be nonzero and X be a scattered space of rank at least $e^\lambda\omega$. Then, GL is complete with respect to $X_{\uparrow\lambda}$.*

Here, we prove the following strengthening to *strong completeness*:

Theorem 2.0.23. *(with D. Fernández-Duque) Let λ be nonzero and X be a scattered space of rank at least $e^\lambda\omega + 1$. Then, GL is strongly complete with respect to $X_{\uparrow\lambda}$.*

This is in contrast to the well-known fact that GL is not strongly complete with respect to its relational semantics. On the way to proving the theorem, we prove strong completeness with respect to a class of Kripke-like structures we call *ω-bouquets*. Some particular cases are:

Corollary 2.0.24. *Let α be an ordinal. Let $\mathcal{I}_1$ be the order topology and τ_c be the topology generated by closed, cofinal sets on ordinals of uncountable cofinality.*

1. *Suppose $\omega^\omega + 1 \leq \alpha$; then GL is strongly complete with respect to $(\alpha, \mathcal{I}_1)$.*
2. *Suppose $\omega_{\omega^\omega} \leq \alpha$; then GL is complete with respect to $(\alpha, (\tau_c)_{\uparrow 1})$.*
3. *Suppose $\omega_{\omega^\omega+1} \leq \alpha$, then GL is strongly complete with respect to $(\alpha, (\tau_c)_{\uparrow 1})$.*

In the second part of Chapter IV, we prove a completeness theorem for the polymodal provability logic GLP. This logic contains a sequence of modalities

$$\{\langle\xi\rangle : \xi \in \mathsf{Ord}\}$$

interpreted as varying degrees of consistency. It is known to be complete with respect to various arithmetical interpretations and to be incomplete

with respect to relational semantics. There is a *canonical* topological interpretation of GLP given in terms of generalized stationarity on sufficiently indescribable cardinals, but it is open whether GL is complete with respect to each individual topology, though we observe in Chapter IV that it is not strongly complete.

We introduce the notion of a $\vec{\vartheta}$-polytopology, where $\vec{\vartheta}$ is a finite sequence of ordinals and prove completeness of GLP for $\vec{\vartheta}$-polytopologies. Roughly, we show that given a scattered space X of large-enough rank and a formula ϕ consistent with GLP, and letting $\vec{\vartheta}$ be the sequence of modalities appearing in ϕ, there is a polytopological space

$$\big(X, \tau_{\vartheta_i} : i < \textsc{lth}(\vec{\vartheta})\big)$$

satisfying ϕ and such that, moreover, each τ_i is a rank-preserving extension of the Icard topology $X_{\uparrow\vartheta_i}$.

Theorem 2.0.25. *Let $\vec{\vartheta}$ be an increasing sequence of nonzero ordinals. Denote by GLP$\restriction\vec{\vartheta}$ the fragment of GLP whose only modalities appear in $\vec{\vartheta}$.*

1. *All theorems of GLP$\restriction\vec{\vartheta}$ hold in every $\vec{\vartheta}$-polytopology.*
2. *Let Λ be a limit ordinal greater than each element in $\vec{\vartheta}$ and (X, τ) be a scattered space of rank $\geq e^{\Lambda}1$. Suppose ϕ only contains modalities in $\vec{\vartheta}$ and is consistent with GLP. Then, there is an open subset of X such that ϕ is satisfied on a $\vec{\vartheta}$-polytopology over $(U, \tau_{\uparrow 1})$.*

This strengthens a result from the author's Bachelor's thesis in which $\vec{\vartheta}$ was a pair of the form $(1, \Omega)$.

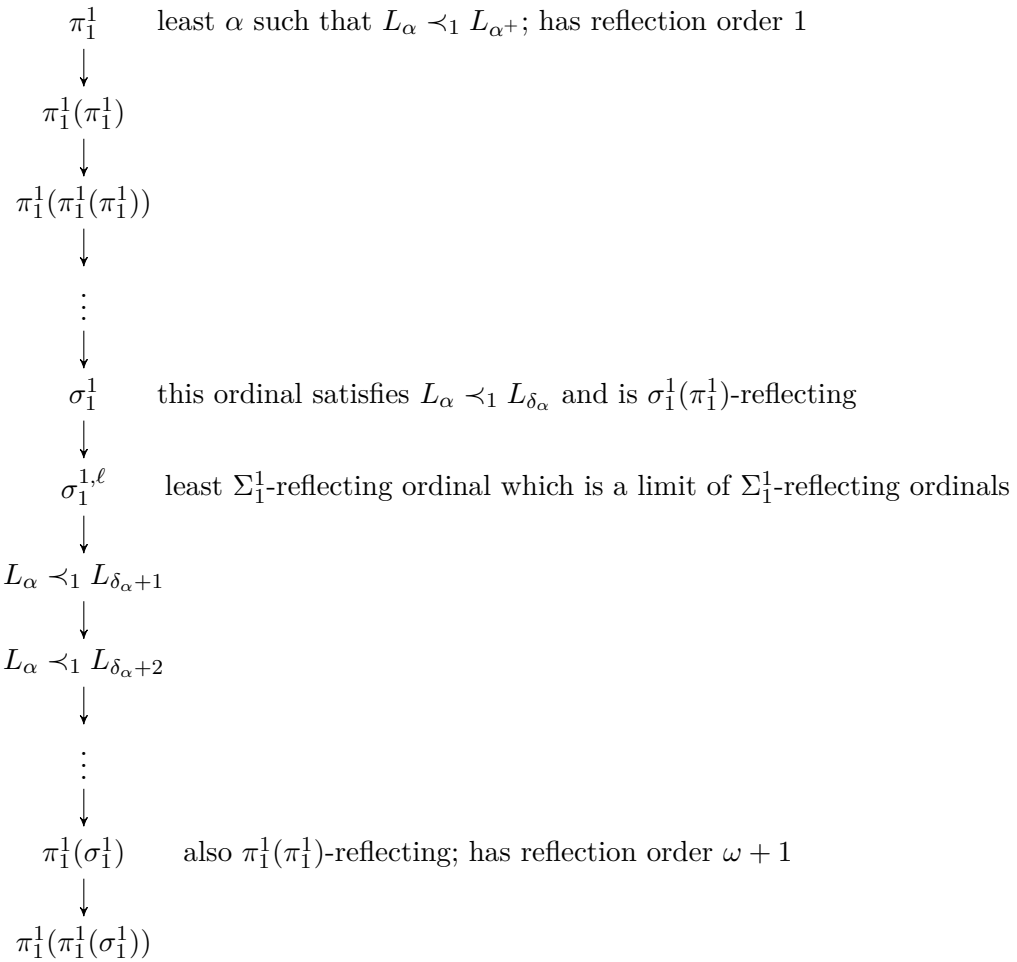


Figure 2.2: Reflecting ordinals, I. The topmost ordinal is the smallest and arrows represent strict inequalities. Here δ_α is the supremum of lengths of α-recursive wellorderings.

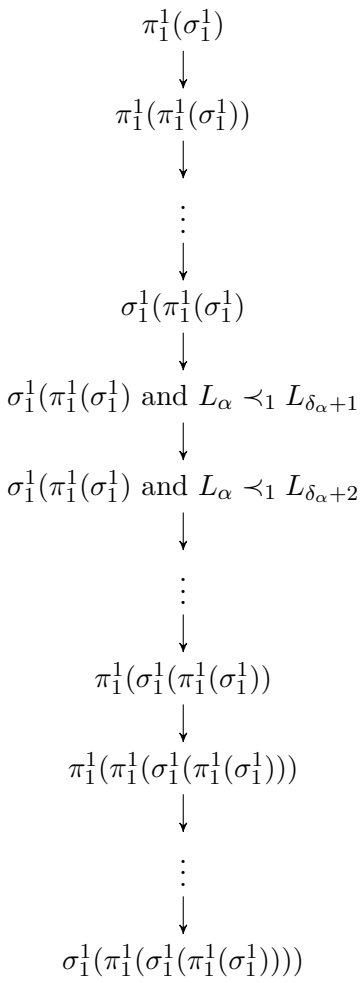


Figure 2.3: Reflecting ordinals, II.

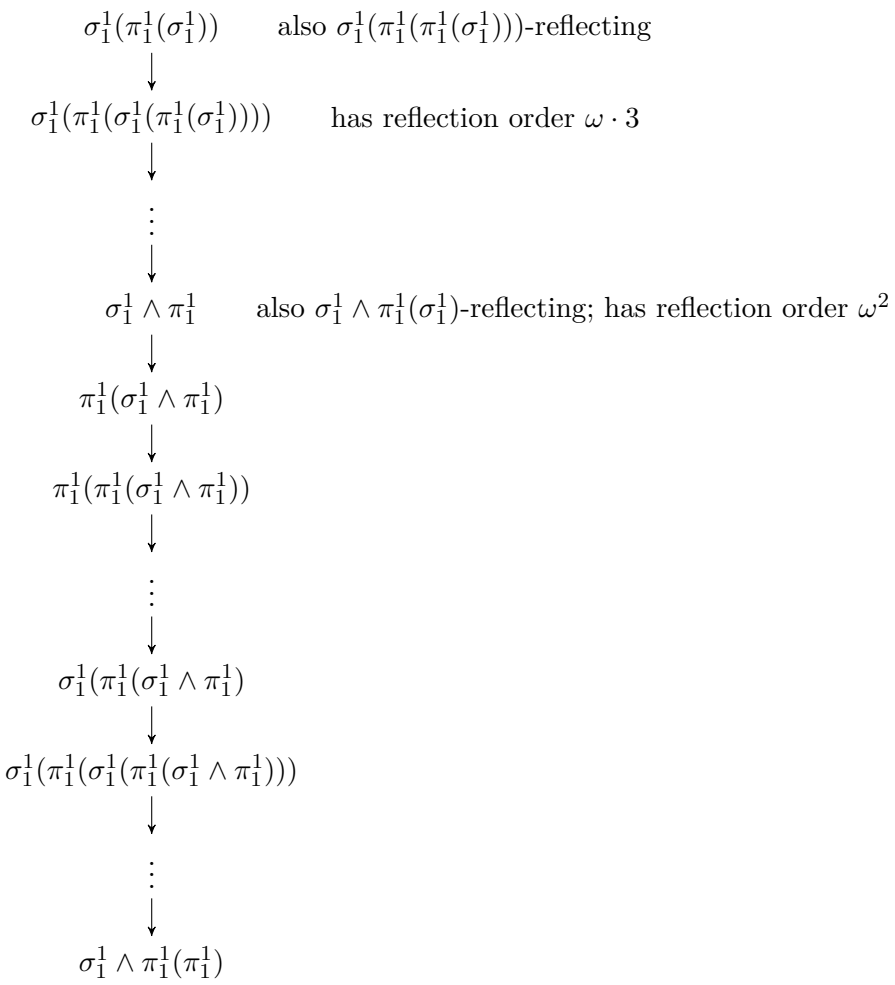


Figure 2.4: Reflecting ordinals, III.

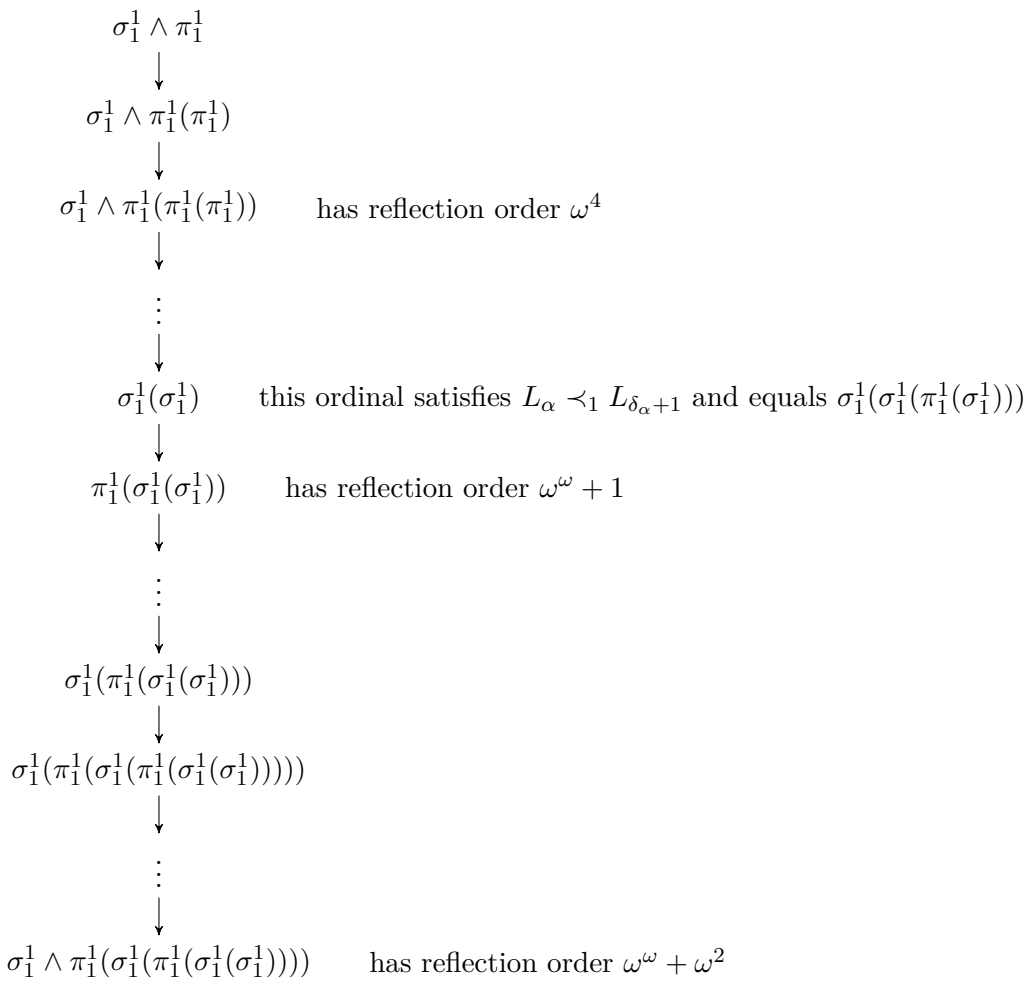


Figure 2.5: Reflecting ordinals, IV.

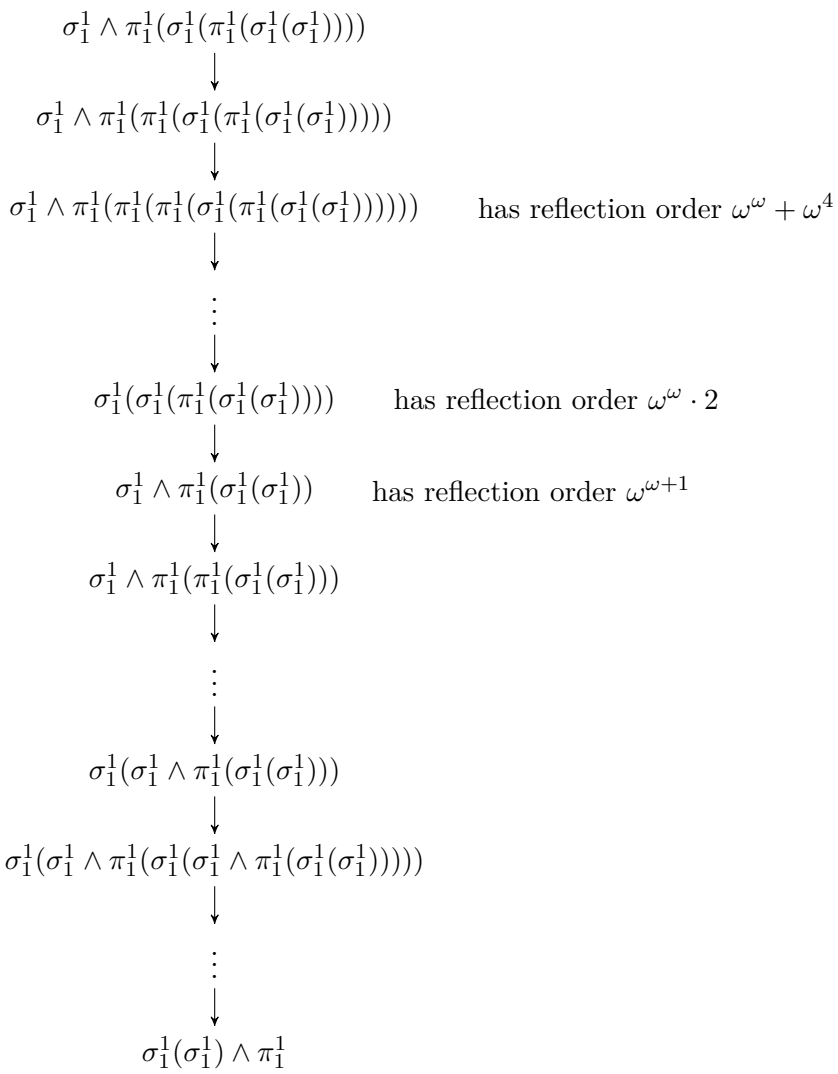


Figure 2.6: Reflecting ordinals, V.

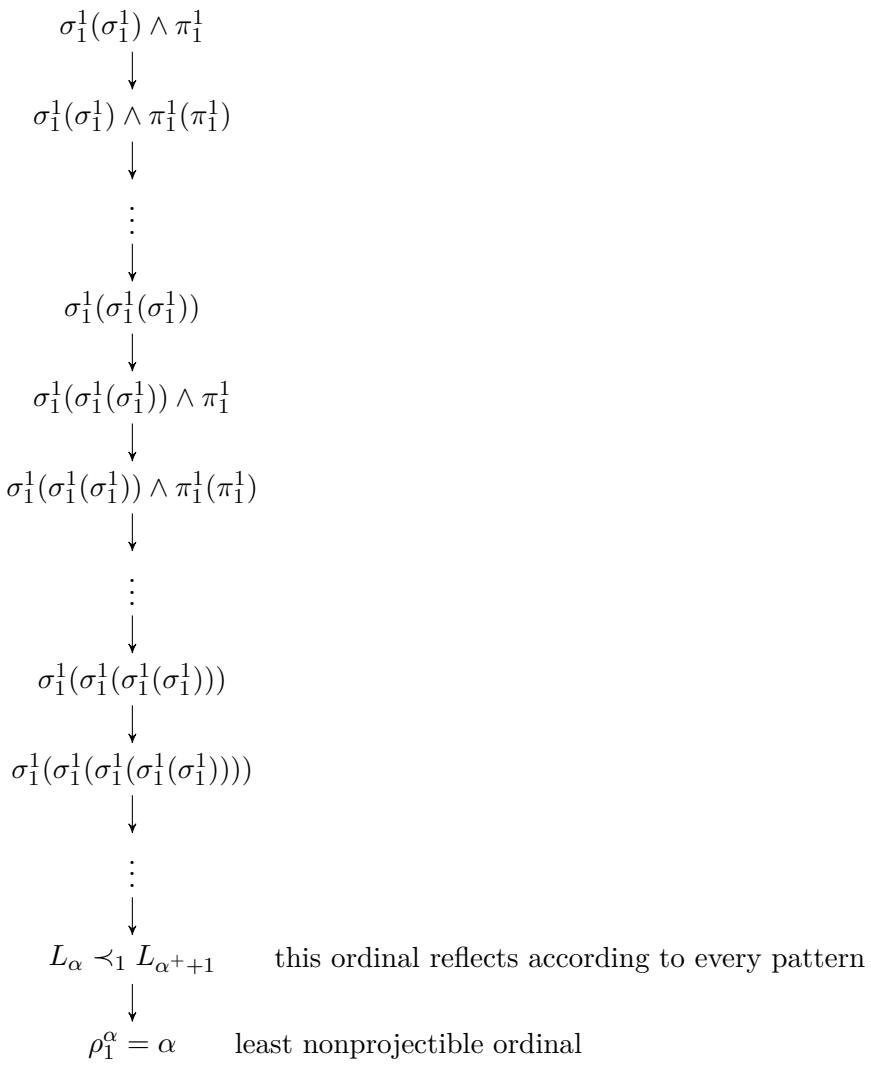


Figure 2.7: Reflecting ordinals, VI.

Part I

The Order of Reflection

3

The Order of Reflection

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3.1 Introduction

Let L_α denote the αth level of Gödel's constructible hierarchy, given by $L_0 = \varnothing$, $L_{\alpha+1} =$ all sets definable over L_α with parameters, and $L_\eta = \bigcup_{\alpha<\eta} L_\alpha$ at limit stages. In α-recursion theory, one lifts the usual notion of "computation" over the natural numbers (or, equivalently, over L_ω) to L_α, for sufficiently closed α. As became evident from early work by Kreisel, Kripke, Platek, Sacks, Takeuti, and others (see e.g., Simpson [Sim78]), facts about recursion on L_α can be translated into facts about recursion on L_ω in various ways. In particular, the termination of simple inductive definitions of sets of natural numbers is deeply connected with the reflecting structure of L (see e.g., Cenzer [Cen74] or Aczel and Richter [AR74]). The purpose of this chapter is to study the order in which various reflecting properties given

in terms of iterated Σ^1_1- and Π^1_1-reflection first occur in the constructible hierarchy.

A formula in the language of set theory is Σ^1_1 if it contains only existential second-order quantifiers (i.e., ranging over classes) followed by arbitrary first-order quantifiers. An ordinal α is said to be Σ^1_1-*reflecting* if whenever ϕ is a Σ^1_1 formula in the language of set theory and $a_1, \ldots, a_n$ are finitely many elements of L_α, then

$$L_\alpha \models \phi(a_1, \ldots, a_n) \text{ implies } \exists \beta < \alpha \Big(a_1, \ldots, a_n \in L_\beta \wedge L_\beta \models \phi(a_1, \ldots, a_n) \Big).$$

Given a class of ordinals X, an ordinal α is said to be Σ^1_1-reflecting *on* X if one can additionally demand that the ordinal β above belong to X. The least Σ^1_1-reflecting ordinal is denoted by σ^1_1, and π^1_1 is defined dually.

An ordinal α is said to be β-*stable* if

$$L_\alpha \prec_1 L_\beta.$$

Given an ordinal α, write α^+ for the smallest admissible ordinal greater than α. Aczel and Richter [AR74] showed that $\pi^1_1 \neq \sigma^1_1$ and that a countable ordinal α is Π^1_1-reflecting if, and only if, it is α^+-stable. Afterwards, Aanderaa [Aan74] showed that $\pi^1_1 < \sigma^1_1$. Gostanian [Gos79] showed that σ^1_1 is smaller than the least α which is $(\alpha^+ + 1)$-reflecting; in fact, he showed that any α which is both $(\alpha^+ + 1)$-stable and locally countable is also Σ^1_1-reflecting. Later Gostanian and Hrbacek [GH79] employed Gostanian's method to give a new proof of Aanderaa's theorem. A third, apparently folklore proof appears in Simpson [Sim78].

Let us now generalize the definitions of σ^1_1 and π^1_1 as follows:

Definition 3.1.1. The notion of a *reflection pattern* is given inductively: the empty set is a reflection pattern; if s and t are reflection patterns, then so too are $s \wedge t$, $\sigma^1_1(s)$, and $\pi^1_1(s)$.

We write σ^1_1 for $\sigma^1_1(\varnothing)$ and π^1_1 for $\pi^1_1(\varnothing)$.

Definition 3.1.2. A reflection pattern is *linear* if it contains no conjunctions, and *non-linear* otherwise.

Definition 3.1.3. An ordinal is said to be $\sigma^1_1(\varnothing)$-reflecting if it is Σ^1_1-reflecting; it is said to be $\pi^1_1(\varnothing)$-reflecting if it is Π^1_1-reflecting. Let s and t be reflection patterns. Inductively, an ordinal α is said to be $\sigma^1_1(s)$-reflecting if it reflects Σ^1_1 statements onto s-reflecting ordinals; it is said to be $\pi^1_1(s)$-reflecting if it reflects Π^1_1 statements onto s-reflecting ordinals; it is said to be $s \wedge t$-reflecting if it is both s-reflecting and t-reflecting.

We may alternate between uppercase Σ and Π and lowercase σ and π in speaking about patterns of reflection.

The *ordering problem* is: given two reflection patterns s and t, determine whether the least s-reflecting ordinal is smaller than the least t-reflecting ordinal. We will identify a pattern s with the least s-reflecting ordinal. Thus, instances of the ordering problem are e.g., determining whether

$$\sigma^1_1(\sigma^1_1) < \sigma^1_1(\pi^1_1(\sigma^1_1 \wedge \pi^1_1))$$

or whether

$$\pi^1_1(\sigma^1_1) < \pi^1_1(\sigma^1_1) \wedge \pi^1_1(\pi^1_1).$$

Other related problems emerge. For instance, one may ask whether $\sigma^1_1(\sigma^1_1)$ is the least Σ^1_1-reflecting ordinal which is also a limit of Σ^1_1-reflecting ordinals. (Incidentally, the answer to all three questions is "no.")

Let us mention the main results of this chapter. First, we solve the ordering problem for linear patterns and in particular show:

Theorem 3.1.4. *The linear order of reflection is a prewellordering of length ω^ω.*

Although Theorem 3.1.4 makes no reference to the The proof requires analyzing the structure full reflection order, its proof does require an analysis thereof, to a certain extent. We shall see that all reflection patterns are witnessed for the first time by ordinals between the least α which is α^+-stable and the least α which is $(\alpha^+ + 1)$-stable. In addition, we show:

Theorem 3.1.5. *The linear reflection order is cofinal in the full reflection order.*

This raises the question of whether the full reflection order also has length ω^ω. This turns out to be false:

Theorem 3.1.6. *The reflection order has length at least ε_0. Moreover, the pattern $(\sigma^1_1)^n$ has rank at least the result of iterating the ω-base exponential n-many times.*

We also compute the rank of specific reflection patterns in the reflection order. By Aanderaa's theorem, the pattern π^1_1 has rank 1 in the reflection order. To name a few further results:

Theorem 3.1.7. *The pattern σ^1_1 has rank ω in the reflection order.*

Theorem 3.1.8. *The pattern $\pi^1_1 \wedge \sigma^1_1$ has rank ω^2 in the reflection order.*

Theorem 3.1.9. *The pattern $\sigma^1_1(\sigma^1_1)$ has rank ω^ω in the reflection order.*

These results easily extend beyond what is explicitly written here. For instance:

Theorem 3.1.10. *The pattern $\sigma^1_1 \wedge \pi^1_1(\sigma^1_1(\sigma^1_1)))$ has rank $\omega^{\omega+1}$ in the reflection order.*

Theorem 3.1.11. *The pattern $\sigma^1_1(\sigma^1_1 \wedge \pi^1_1(\sigma^1_1(\sigma^1_1)))$ has rank $\omega^{\omega+\omega}$ in the reflection order.*

Convention Every ordinal in this chapter is assumed to be both countable and locally countable (i.e., for all $\beta < \alpha$, there is a surjection from ω to β in L_α. These are the hypotheses for the theorems of Gostanian and Aczel-Richter mentioned above, respectively.

3.2 Stability and Gandy Ordinals

For an admissible ordinal α, write

$$\delta_\alpha = \sup\{\delta : \delta \text{ is the length of an } \alpha\text{-recursive wellordering of a subset of } \alpha\},$$

where a subset of α is said to be *α-recursive* if it is Δ_1-definable over L_α with parameters. The value of δ_α remains unchanged if one replaces "α-recursive" by "α-r.e." in the definition. For every admissible α, δ_α is easily seen to be a limit and e.g., additively indecomposable. We always have $\delta_\alpha \leq \alpha^+$; an ordinal α is *Gandy* if $\delta_\alpha = \alpha^+$. Gostanian [Gos79] showed that σ^1_1 is the smallest ordinal which is not Gandy. In fact, he showed that a locally countable ordinal is not Gandy if, and only if, it is Σ^1_1-reflecting. Abramson and Sacks [AS76] showed that $(\aleph^L_\omega)^+$ is Gandy.

Since we know what the degree of stability of π^1_1 is (viz. $(\pi^1_1)^+$), a possible first question is that of the degree of stability of σ^1_1.

Proposition 3.2.1. *σ^1_1 is not $(\delta_{\sigma^1_1} + 1)$-stable.*

Proof. Let $\delta = \delta_{\sigma^1_1}$. Since $\delta < (\sigma^1_1)^+$, it is not admissible. As we observed before, δ is a limit ordinal; thus, the failure of admissibility must be due to an instance of collection. Choose some Δ_0 formula ψ such that for some $\vec{a} \in L_\delta$, $L_\delta \not\models \psi(\vec{a})$-collection. To see that σ^1_1 is not $(\delta+1)$-stable, consider the formula ϕ in the language of set theory asserting that there are sets A, B such that:

1. A and B are transitive sets satisfying V=L, A is admissible, $A \in B$, and there is $\vec{a} \in B$ such that B does not satisfy $\psi(\vec{a})$-collection;

2. for each $\mathsf{Ord} \cap A$-recursive linear ordering $R \in B$, either there is an infinite descending sequence b through R with $b \in A$, or there is an ordinal $\beta \in B$ and an isomorphism $f \in B$ from R to β;

3. for each $\beta \in B$, there is an $(\mathsf{Ord} \cap A)$-recursive linear ordering $R \in B$ and an isomorphism $f \in B$ from R to β.

Notice that ϕ is a Σ_1 formula, since the only unbounded quantifier is the one on B. Moreover, it does not hold in $L_{\sigma_1^1}$, for the sets A and B would need to be of the form L_α and L_β, with $\alpha < \beta < \sigma_1^1$. Conditions (2) and (3) together imply that $\beta = \delta_\alpha$, but Gostanian's characterization of σ_1^1 then implies $\beta = \alpha^+$, contradicting condition (1). Finally, it does hold in $L_{\delta+1}$, as witnessed by $A = L_{\sigma_1^1}$ and $B = L_\delta$. To see that (2) holds, recall a theorem of Gostanian [Gos79, Theorem 3.2] by which if α is Σ_1^1-reflecting, then every α-recursive linear ordering which is not a wellordering has an infinite descending sequence in L_α. Thus, every σ_1^1-recursive linear ordering R either has an infinite descending sequence in $L_{\sigma_1^1}$, or else is isomorphic to some ordinal $\beta < \delta$. One can construct an isomorphism witnessing this by transfinite recursion: at stage $\gamma < \beta$, one has defined $f \restriction \gamma$ and sets $f(\gamma)$ equal to the R-least element not in the range of $f \restriction \gamma$. Since this process takes β-many stages and $R \in L_{\sigma_1^1+1}$, such an ismorphism belongs to $L_{\sigma_1^1+\delta}$. Since δ is additively indecomposable, it belongs to L_δ. The proof that (3) holds is similar. □

The proof of Proposition 3.2.1 shows:

Corollary 3.2.2. *Suppose σ is Σ_1^1-reflecting and $(\delta_\sigma + 1)$-stable. Then, it is a limit of Σ_1^1-reflecting ordinals.*

One cannot improve the conclusion of Proposition 3.2.1—every Σ_1^1-reflecting ordinal is stable to the supremum of its recursive wellorderings:

Proposition 3.2.3. *Suppose σ is Σ_1^1-reflecting. Then σ is δ_σ-stable.*

Proof. Since δ is a limit ordinal, it suffices to consider arbitrary $\gamma < \delta$ and show that

$$L_\sigma \prec_1 L_\gamma.$$

Let $a \in L_\sigma$ and ϕ be a Σ_1-formula such that $L_\gamma \models \phi(a)$. Without loss of generality, assume that a is an ordinal. Let R be a σ-recursive wellordering

of length γ. In particular, R is σ-r.e., so there is a Σ_1 formula ψ, such that for all $x, y \in L_\sigma$,

$$xRy \leftrightarrow L_\sigma \models \psi(x, y).$$

Let us assume for notational simplicity that ψ is defined without parameters. Given an ordinal σ', let $R_{\sigma'}$ be the binary relation given by

$$xR_{\sigma'}y \leftrightarrow L_{\sigma'} \models \psi(x, y).$$

Since ψ is Σ_1, we have $R_{\sigma'} \subset R_\tau$ whenever $\sigma' \leq \tau \leq \sigma$. In particular, $R_{\sigma'}$ is wellfounded for all $\sigma' < \sigma$.

Because $L_\gamma \models \phi(a)$, there is a subset A of L_σ such that

1. A codes a model (M, E) of $\mathsf{KP} + V = L$;
2. M has a largest admissible ordinal τ and (τ, E) is isomorphic to $(\sigma, \in)$;
3. there is an ordinal β of M and a function $f \in M$ which is an isomorphism between R_τ^M (i.e., R_τ computed within M) and β, and $L_\beta^M \models \phi(a)$.

The existence of such an A can be expressed by a set-theoretic Σ_1^1 formula over L_σ with parameter a (as well as any other parameters involved in the definition of R).

By Σ_1^1-reflection, there is some $\sigma' < \sigma$ and some $A_{\sigma'} \subset L_{\sigma'}$ such that $a \in L_{\sigma'}$ and

1. $A_{\sigma'}$ codes a model (N, F) of $\mathsf{KP} + V = L$;
2. N has a largest admissible ordinal η and (η, F) is isomorphic to $(\sigma', \in)$;
3. there is an ordinal b of N and a function $g \in N$ which is an isomorphism between R_η^N and b, and $L_b^N \models \phi(a)$.

Here and for the rest of our lives, let us identify the wellfounded part of N with its transitive collapse. Condition (2) implies that $L_{\sigma'} \in N$. By (3), there is an ordinal b of N and an isomorphism $g \in N$ from $R_{\sigma'}$ to b. Because $R_{\sigma'} \subset R$, it is wellfounded, and so b really is an ordinal. Now, $L_b \models \phi(a)$, and N has no admissible ordinals above σ', so $b < (\sigma')^+ < \sigma$. Since ϕ is Σ_1, we conclude that $L_\sigma \models \phi(a)$, as was to be shown. □

We have shown that σ_1^1 is $\delta_{\sigma_1^1}$-stable and not $(\delta_{\sigma_1^1} + 1)$-stable. Note the following consequence of the preceding proposition:

Proposition 3.2.4. *There is a Σ^1_1-sentence ϕ such that for every locally countable, admissible σ, $L_\sigma \models \phi$ if, and only if, σ is Σ^1_1-reflecting or Π^1_1-reflecting.*

Proof. Let ϕ be the sentence that asserts the existence of some $A \subset L_\sigma$ coding a model (M, E) of $\mathsf{KP} + V = L$ containing σ and such that

$$M \models L_\sigma \prec_1 L_{\delta_\sigma}.$$

Clearly, every Π^1_1-reflecting ordinal satisfies this sentence, as does every Σ^1_1-reflecting ordinal, by Proposition 3.2.3.

Suppose that $L_\sigma \models \phi$, as witnessed by (M, E). Suppose moreover that σ is not Σ^1_1-reflecting, so that $\delta_\sigma = \sigma^+$ by Gostanian's characterization. Since $\sigma \in M$, a well-known theorem of F. Ville (see e.g., Barwise [Bar75] for a proof) implies that $L_{\sigma^+} \subset M$. Given an arbitrary $\beta < \sigma^+$, we then have $\beta \in M$ and $\beta < \delta^M_\sigma$, for otherwise $\delta^M_\sigma < \sigma^+$, which is impossible, since any σ-recursive wellordering of a subset of σ of length δ^M_σ would belong to M. By choice of M,

$$M \models L_\sigma \prec_1 L_{\delta_\sigma},$$

and so $M \models L_\sigma \prec_1 L_\beta$. However, being Σ_1-elementary is absolute, so we really do have $L_\sigma \prec_1 L_\beta$ and, since β was arbitrary, we have $L_\sigma \prec_1 L_{\sigma^+}$, so σ is Π^1_1-reflecting. □

An immediate consequence is Aanderaa's classical result:

Corollary 3.2.5 (Aanderaa)**.** $\pi^1_1 < \sigma^1_1$.

Corollary 3.2.5 holds in a strong form:

Corollary 3.2.6. *σ^1_1 reflects Σ^1_1 sentences on Π^1_1-reflecting ordinals.*

Proof. Let ϕ be the sentence from Proposition 3.2.4. Then, if ψ is another Σ^1_1 sentence, so is the conjunction $\phi \wedge \psi$. □

Corollary 3.2.6 is not new; it also follows from the proof of Corollary 3.2.5 written down in Simpson [Sim78]. Our method for analyzing the reflection order is to prove results akin to Corollary 3.2.6. Now that we know the degree of stability of σ^1_1, it is natural to ask what the least ordinal α which is $(\delta_\alpha + 1)$-stable is. We shall eventually see that it is rather small and in fact smaller than the successor of σ^1_1 in the reflection order. We finish this section with some related results.

Proposition 3.2.7. *Suppose σ is locally countable and Σ^1_1-reflecting on Σ^1_1-reflecting ordinals. Then σ is $(\delta_\sigma + 1)$-stable.*

Proof. This is similar to the proof of Proposition 3.2.3. Again, it is easy to see that δ is a limit. Let $\gamma = \delta_\sigma + 1$ and $a \in L_\sigma$ be such that $L_\gamma \models \phi(a)$, for some Σ_1 formula ϕ. Let ψ be the Σ^1_1 formula expressing that σ is locally countable and there is a set $A \subset L_\sigma$ coding a model (M, E) of $\mathsf{KP} + V = L$ with $\sigma \in M$ and such that

$$M \models \text{“}L_{\delta_\sigma+1} \models \phi(a).\text{”}$$

Then $L_\sigma \models \psi$. By hypothesis, there is a Σ^1_1-reflecting $\tau < \sigma$ such that $L_\tau \models \psi$. Thus, τ is locally countable and there is a model (M, E) of $\mathsf{KP} + V = L$ with $\tau \in M$ and such that

$$M \models \text{“}L_{\delta_\tau+1} \models \phi(a).\text{”}$$

By Ville's theorem, $L_{\tau^+} \subset M$ and, since τ is Σ^1_1-reflecting, $\delta_\tau < \tau^+$. Hence, M computes δ_τ and $\delta_\tau + 1$ correctly and so we really have $L_{\delta_\tau+1} \models \phi(a)$. Since ϕ is Σ_1, we conclude $L_\sigma \models \phi(a)$, as desired. □

The preceding proof shows that if σ is as in Proposition 3.2.7, then σ is $(\delta_\sigma + 2)$-stable, $(\delta_\sigma)^\omega$-stable, etc. It shows that if f is a function on ordinals which is uniformly Σ_1-definable (with parameters in L_σ) on say, multiplicatively indecomposable levels of L, then σ is $f(\delta_\sigma)$-stable.

Definition 3.2.8. We denote by $\sigma^{1,\ell}_1$ the least Σ^1_1-reflecting ordinal which is a limit of Σ^1_1-reflecting ordinals.

Proposition 3.2.9. *$\sigma^{1,\ell}_1$ is smaller than the least α which is $(\delta_\alpha + 1)$-stable.*

Proof. Let α be as in the statement. We claim that α is Σ^1_1-reflecting. Otherwise, $\delta_\alpha = \alpha^+$, and so α is $(\alpha^+ + 1)$-stable. But surely α is locally countable, and thus Σ^1_1-reflecting, by Gostanian's result mentioned in the introduction. Thus, α is Σ^1_1-reflecting. By Corollary 3.2.2, α is a limit of Σ^1_1-reflecting ordinals. However, this is expressible in L_α; thus, the proof of Proposition 3.2.1 shows that α is a limit of ordinals which are both Σ^1_1-reflecting and limits of Σ^1_1-reflecting ordinals. □

As a consequence, we obtain a negative answer to one of the questions posed in the introduction.

Corollary 3.2.10. *$\sigma^{1,\ell}_1$ does not reflect Σ^1_1 statements onto Σ^1_1-reflecting ordinals.*

We state without proof a result implying that $\sigma^{1,\ell}_1 < \pi^1_1(\sigma^1_1)$. Its proof is similar to that of the much more powerful Theorem 3.3.9 below.

$$L_\alpha \prec_1 L_{\alpha^+} \to \sigma_1^1 \to \sigma_1^{1,\ell} \to L_\alpha \prec_1 L_{\delta_\alpha+1} \to L_\alpha \prec_1 L_{\delta_\alpha+2} \to \cdots \to \pi_1^1(\sigma_1^1)$$

Figure 3.1: Ordinals below $\pi_1^1(\sigma_1^1)$

Proposition 3.2.11. *For every $\alpha < \pi_1^1(\sigma_1^1)$, there is some $\sigma < \pi_1^1(\sigma_1^1)$ which is both Σ_1^1-reflecting and $(\delta_\sigma + \alpha)$-stable.*

We end this section with Figure 3.1, which summarizes the relationships between the ordinals considered so far. We shall also see that

$$\pi_1^1(\sigma_1^1) < \sigma_1^1(\sigma_1^1).$$

3.3 Reflection Transfer Theorems

In this section, we will present some results on the transfer of reflection properties, i.e., results of the form

$$\text{if } \sigma \text{ is } s\text{-reflecting, then it is } t\text{-reflecting,}$$

where s and t are reflection patterns. Recall our convention that every ordinal considered is countable and locally countable. The first reflection transfer results we present are rather elementary:

Lemma 3.3.1. *Let s be a reflection pattern and σ be an ordinal.*

1. *If σ is $\Sigma_1^1(\sigma_1^1(s))$-reflecting, then σ is $\Sigma_1^1(s)$-reflecting.*
2. *If σ is $\Pi_1^1(\pi_1^1(s))$-reflecting, then σ is $\Pi_1^1(s)$-reflecting.*

Proof. If σ is $\Sigma_1^1(\sigma_1^1(s))$-reflecting and L_σ satisfies a Σ_1^1 sentence ϕ, then, by definition, there is a $\Sigma_1^1(s)$-reflecting $\tau < \sigma$ such that $L_\tau \models \phi$. By $\Sigma_1^1(s)$-reflection, there is an s-reflecting $\eta < \tau$ such that $L_\tau \models \phi$. Hence, σ is $\Sigma_1^1(s)$-reflecting. The argument for $\Pi_1^1(\pi_1^1(s))$-reflection is similar. □

Lemma 3.3.2. *Let s be a reflection pattern and σ be an ordinal.*

1. *If σ is $\sigma_1^1(s) \wedge \pi_1^1(t)$-reflecting, then it is $\pi_1^1(\sigma_1^1(s) \wedge t)$-reflecting.*
2. *If σ is $\sigma_1^1(s) \wedge \pi_1^1(t)$-reflecting, then it is $\sigma_1^1(s \wedge \pi_1^1(t))$-reflecting.*
3. *If σ is $\sigma_1^1(s)$-reflecting, then it is $\sigma_1^1(s \wedge \pi_1^1)$-reflecting.*

Proof. Recall that if an ordinal σ is s-reflecting, for any nontrivial reflection pattern s, then it is recursively inaccessible and, in fact, a limit of recursively inaccessible ordinals. (1) then follows from the simple observation that being $\Sigma^1_1(s)$-reflecting is expressible by a Π^1_1 sentence ψ. Thus, if σ is $\sigma^1_1(s) \wedge \pi^1_1(t)$-reflecting and satisfies some Π^1_1-sentence ϕ, then the conjunction $\phi \wedge \psi$ is also Π^1_1, and any ordinal satisfying it must be $\Sigma^1_1(s)$-reflecting.

(2) is similar. For (3), there are two cases: if σ is Π^1_1-reflecting, then the result follows from (2). If σ is not Π^1_1-reflecting, it is not σ^+-stable. Hence, there is a least $\gamma < \sigma^+$ such that σ is not γ-stable, i.e., there is a Σ_1-formula ψ and some parameter $\alpha < \sigma$ such that $L_\gamma \models \psi(\alpha)$, but $L_\sigma \not\models \psi(\alpha)$. The remainder of the proof is an adaptation of the proof of Corollary 3.2.5 presented in Simpson [Sim78]:

Let ϕ be the Σ^1_1-statement expressing that there is a model (M, E) of $\mathsf{KP}+V = L$ end-extending $L_{\sigma+1}$ such that for some $\gamma' \in M$, $L^M_{\gamma'} \models \psi(\alpha)$ and, moreover, if γ' is least such, then $M \models$"σ is $(<\gamma')$-stable." Then $L_\sigma \models \phi$. By choice of σ, there is an s-reflecting ordinal $\tau < \sigma$ such that $L_\tau \models \phi$. This means that there is a model (M, E) of $\mathsf{KP} + V = L$ end-extending L_τ such that for some $\gamma' \in M$, $L^M_{\gamma'} \models \psi(\alpha)$ and, for the least such γ', we have $M \models$"τ is $(<\gamma')$-stable.". Since ψ is Σ_1 and $L_\sigma \not\models \psi(\alpha)$, γ' must belong to the illfounded part of M. So τ is s-reflecting and, as in Proposition 3.2.4, τ is Π^1_1-reflecting. By taking conjunctions as before, one sees that every Σ^1_1 satisfied by L_σ is satisfied by some $(s \wedge \pi^1_1)$-reflecting $\tau < \sigma$, as was to be shown. □

Example 3.3.3. In the introduction, we claimed that

$$\sigma^1_1(\pi^1_1(\sigma^1_1)) < \sigma^1_1(\sigma^1_1).$$

To see this, notice that Lemma 3.3.2(1) and 3.3.2(2) imply that

$$\sigma^1_1 \wedge \pi^1_1 = \sigma^1_1 \wedge \pi^1_1(\sigma^1_1) = \sigma^1_1(\pi^1_1(\sigma^1_1)) \wedge \pi^1_1 = \sigma^1_1 \wedge \pi^1_1(\sigma^1_1(\pi^1_1(\sigma^1_1)))$$

and thus that

$$\sigma^1_1(\pi^1_1(\sigma^1_1)) < \sigma^1_1 \wedge \pi^1_1.$$

On the other hand, Lemma 3.3.2(3) implies that

$$\sigma^1_1(\sigma^1_1) = \sigma^1_1(\sigma^1_1 \wedge \pi^1_1),$$

and so

$$\sigma^1_1(\pi^1_1(\sigma^1_1)) < \sigma^1_1 \wedge \pi^1_1 < \sigma^1_1(\sigma^1_1),$$

as claimed. □

A natural question is whether one can strengthen π^1_1 in the statement of Lemma 3.3.2(3) and, in particular, whether σ^1_1 is $\sigma^1_1(\pi^1_1(\pi^1_1))$-reflecting. By generalizing the proof of Lemma 3.3.2(3), we see that the answer is "yes."

Definition 3.3.4. Let s be a reflection pattern. An ordinal α is *β-stable on s* if whenever L_β satisfies a Σ_1 sentence $\phi(L_\alpha)$ with additional parameters in L_α, there is an s-reflecting $\gamma < \alpha$ such that $L_{\gamma^+} \models \phi(L_\gamma)$.

We caution the reader that an ordinal α being β-stable on $\varnothing$ is not the same as it being β-stable, for the first definition allows L_α as a parameter. However, the definition coincides at sufficiently strong closure points:

Lemma 3.3.5. *Let s be a reflection pattern. The following are equivalent:*

1. *α is α^+-stable on s;*
2. *α is $\Pi^1_1(s)$-reflecting.*

We omit the proof of Lemma 3.3.5, which is a simple consequence of Aczel and Richter's characterization of Π^1_1-reflection.

Theorem 3.3.6. *Let s be a reflection pattern. Suppose σ is $\sigma^1_1(s)$-reflecting. Then, it is $\sigma^1_1(s \wedge \pi^1_1(s))$-reflecting.*

Proof. The conclusion of the theorem follows from Lemma 3.3.2 if σ is $\pi^1_1(s)$-reflecting, so we may assume that it is not.

Since σ is not $\pi^1_1(s)$-reflecting, it is not σ^+-stable on s, so there is a least $\beta < \sigma^+$ and a Σ_1-formula $\exists x\, \phi(y, x)$ such that $L_\beta \models \exists x\, \phi(L_\sigma, x)$ and whenever $\gamma < \sigma$ and γ is s-reflecting, then $L_{\gamma^+} \not\models \exists x\, \phi(L_\gamma, x)$. Let ψ be the formula expressing that there is a model M of $\mathsf{KP} + V = L$ such that

1. M contains σ.
2. $M \models$ "$\exists x\, \phi(L_\sigma, x)$ and, letting γ be least such that $M \models \phi(L_\sigma, a)$ for some $a \in L_\gamma$, σ is $<\gamma$-stable on s."

By reflection, there is $\tau < \sigma$ with $L_\tau \models \psi$, as witnessed by some model N which end-extends L_{τ^+}. Since σ is $\sigma^1_1(s)$-reflecting, we may assume that τ is s-reflecting. Let γ be N-least such that $N \models \exists x \in L_\gamma\, \phi(L_\tau, x)$. Then, we cannot have $\gamma < \tau^+$, for otherwise τ is an s-reflecting ordinal such that $L_{\tau^+} \models \exists x\, \phi(L_\tau, x)$, contradicting the choice of ϕ. Thus, γ belongs to the illfounded part of N and, in N, τ is $<\gamma$-stable on s. Since τ is recursively inaccessible (this can be assumed also if $s = \varnothing$), N is correct about s-reflection below τ, so an argument as before shows that τ is τ^+-stable on s and thus $\Pi^1_1(s)$-reflecting. □

Example 3.3.7. By repeatedly applying Theorem 3.3.6, we obtain

$$\sigma_1^1 = \sigma_1^1(\pi_1^1) = \sigma_1^1(\pi_1^1(\pi_1^1)) = \sigma_1^1(\pi_1^1(\pi_1^1(\pi_1^1))) = \dots.$$

This implies the sequence of inequalities

$$\pi_1^1 < \pi_1^1(\pi_1^1) < \pi_1^1(\pi_1^1(\pi_1^1)) < \cdots < \sigma_1^1,$$

which strengthens Corollary 3.2.5.

The following strengthening of Proposition 3.2.3 is proved similarly:

Lemma 3.3.8. *Suppose σ is $\Sigma_1^1(s)$-reflecting. Then, it is δ_σ-stable on s.*

Proof. Let $\theta(L_\sigma)$ be a Σ_1 sentence with parameters in L_σ, say, of the form $\exists x\, \theta_0(x, L_\sigma)$. Let $\eta < \delta_\sigma$ and $b \in L_\eta$ be such that $L_{\delta_\sigma} \models \theta_0(b, L_\sigma)$. Since $\eta < \delta_\sigma$, there is a σ-recursive wellorder R of length η. Let ψ be the sentence asserting the existence of a model M of KPi such that

1. M end-extends $L_{\sigma+1}$;
2. $M \models$"σ^+ exists";
3. in M, R is isomorphic to an ordinal η_0 and there is $b_0 \in L_{\eta_0}^M$ such that $L_{\eta_0}^M \models \theta_0(b_0, L_\sigma)$.

Then $L_\sigma \models \psi$. Moreover, ψ is Σ_1^1 so, by reflection, there is an s-reflecting $\tau < \sigma$ such that $L_\tau \models \psi$, as witnessed by some model N which end-extends L_{τ^+}. Now, in N, $L_{\tau^+}^N \models \theta_0(b_0, L_\tau)$ for some $b_0 \in L_{\eta_0}^N$, where η_0 is some N-ordinal isomorphic to R_τ. However, $R_\tau \subset R$, since R is σ-recursive, and θ_0 is Σ_0, so we really have $L_{\tau^+} \models \theta(L_\tau)$, as desired. □

The following theorem, although perhaps odd-looking at first, is crucial for our analysis of the reflection order.

Theorem 3.3.9. *Let s be a reflection pattern. Suppose σ is $\Pi_1^1(\sigma_1^1(s))$-reflecting but not Σ_1^1-reflecting. Then σ is $\Pi_1^1(s)$-reflecting.*

Proof. Suppose σ is Π_1^1-reflecting on $\Sigma_1^1(s)$-reflecting ordinals but not Σ_1^1-reflecting. Let ψ be the statement expressing that whenever (M, E) is an end-extension of $L_{\sigma+1}$ satisfying KPi, then $M \models$"σ is not $\Sigma_1^1(s)$-reflecting." This sentence is Π_1^1 and thus cannot be satisfied by L_σ, for otherwise it would be reflected to a $\Sigma_1^1(s)$-reflecting ordinal. But clearly L_α cannot satisfy ψ if α is $\Sigma_1^1(s)$-reflecting.

Thus, $L_\sigma \not\models \psi$, so there is a model M of KPi end-extending $L_{\sigma+1}$ such that

$$M \models \text{“}\sigma \text{ is } \Sigma^1_1(s)\text{-reflecting.”}$$

For ordinals $\tau < \sigma$, whether τ is t-reflecting is computed correctly by, say, τ^{++}, and thus too by M, for any reflection pattern t. By Lemma 3.3.8 applied within M,

$$M \models \text{“}\sigma \text{ is } \delta_\sigma\text{-stable on } s\text{.”}$$

Let ϕ be a Π^1_1 statement, and $a \in L_\sigma$ be a parameter such that $L_\sigma \models \phi(a)$. By Barwise-Gandy-Moschovakis [BGM71], there is a Σ_1 formula ϕ^* such that for all admissible α with $a \in L_\alpha$, $L_\alpha \models \phi(a)$ if, and only if, $L_{\alpha^+} \models \phi^*(a, L_\alpha)$; thus, $L_{\sigma^+} \models \phi^*(a, L_\sigma)$. Let $b \in L_{\sigma^+}$ be a witness for ϕ^* and let $\gamma < \sigma^+$ be large enough so that $b \in L_\gamma$. Since σ is not Σ^1_1-reflecting (in the real world), $\delta_\sigma = \sigma^+$, and thus

$$\gamma < \delta^M_\sigma.$$

Since ϕ^* is Σ_1,

$$M \models \text{“}L_{\delta_\sigma} \models \phi^*(a, L_\sigma),\text{”}$$

so by the δ_σ-stability of σ on s within M, there is an s-reflecting $\tau < \sigma$ such that

$$M \models \text{“}L_{\tau^+} \models \phi^*(a, L_\tau)\text{.”}$$

Since $\tau^+ < \sigma$, we really do have

$$L_{\tau^+} \models \phi^*(a, L_\tau),$$

and so $L_\tau \models \phi(a)$. This completes the proof of the theorem. □

Remark 3.3.10. The assumption that σ is not Σ^1_1-reflecting cannot be removed from the statement of Theorem 3.3.9. To see this, let s be the empty word. By Lemma 3.3.2(1), $\sigma^1_1 \wedge \pi^1_1$ is $\Pi^1_1(\sigma^1_1)$-reflecting. However, $\sigma^1_1 \wedge \pi^1_1$ is not $\Pi^1_1(\pi^1_1)$-reflecting, for being Σ^1_1-reflecting is expressible by a Π^1_1-formula, and thus every ordinal which is $(\sigma^1_1 \wedge \pi^1_1(\pi^1_1))$-reflecting is also $\pi^1_1(\sigma^1_1 \wedge \pi^1_1)$-reflecting and, in particular, a limit of $(\sigma^1_1 \wedge \pi^1_1)$-reflecting ordinals. □

Remark 3.3.11. One cannot improve the statement of Theorem 3.3.9 to conclude that σ is $\Pi^1_1(\sigma^1_1(s)\wedge s)$-reflecting, for let $s = \pi^1_1$. Then, $\pi^1_1(\sigma^1_1(\pi^1_1)) = \pi^1_1(\sigma^1_1)$ by Lemma 3.3.2(3). However, as in Example 3.3.3,

$$\pi^1_1(\sigma^1_1) < \sigma^1_1(\pi^1_1(\sigma^1_1)) < \sigma^1_1 \wedge \pi^1_1.$$

This should be contrasted with Theorem 3.3.6 below. □

We conclude this section with two examples.

Example 3.3.12. By combining Theorems 3.3.9 and 3.3.6, one sees that

$$\pi^1_1(\sigma^1_1) = \pi^1_1(\sigma^1_1) \wedge \pi^1_1(\pi^1_1) = \pi^1_1(\sigma^1_1) \wedge \pi^1_1(\pi^1_1(\pi^1_1)) = \dots$$

Since we have seen that these reflection patterns are all smaller than $\sigma^1_1(\sigma^1_1)$ and $\sigma^1_1 \wedge \pi^1_1$, it follows that σ^1_1 and $\pi^1_1(\sigma^1_1)$ have order-types ω and $\omega + 1$ in the reflection order, respectively. □

Example 3.3.13. Let us present a proof of the inequality

$$\pi^1_1\Big(\sigma^1_1 \wedge \pi^1_1(\sigma^1_1 \wedge \pi^1_1)\Big) < \sigma^1_1(\sigma^1_1).$$

First, apply Theorem 3.3.6 to see that

$$\sigma^1_1(\sigma^1_1) = \sigma^1_1(\sigma^1_1 \wedge \pi^1_1(\pi^1_1)).$$

Then, apply Lemma 3.3.2 to see that

$$\sigma^1_1 \wedge \pi^1_1(\pi^1_1) = \sigma^1_1 \wedge \pi^1_1(\sigma^1_1 \wedge \pi^1_1),$$

so that

$$\sigma^1_1\Big(\sigma^1_1 \wedge \pi^1_1(\pi^1_1)\Big) = \sigma^1_1\Big(\sigma^1_1 \wedge \pi^1_1(\sigma^1_1 \wedge \pi^1_1)\Big).$$

Finally, by Theorem 3.3.6, every $\sigma^1_1(\sigma^1_1 \wedge \pi^1_1(\sigma^1_1 \wedge \pi^1_1))$-reflecting ordinal is also $\sigma^1_1(\pi^1_1(\sigma^1_1 \wedge \pi^1_1(\sigma^1_1 \wedge \pi^1_1)))$-reflecting, so that $\sigma^1_1(\sigma^1_1)$ is a limit of $\pi^1_1(\sigma^1_1 \wedge \pi^1_1(\sigma^1_1 \wedge \pi^1_1))$-reflecting ordinals.

3.4 Patterns Below $\sigma^1_1(\sigma^1_1)$

In this section, we describe the reflection order below $\sigma^1_1(\sigma^1_1)$. To ease notation, we shall omit subscripts and superscripts (in this section) and simply write σ for σ^1_1 and π for π^1_1. We shall also sometimes omit parentheses; thus, e.g., we will write

$$\sigma \wedge \pi\sigma$$

instead of

$$\sigma^1_1 \wedge \pi^1_1(\sigma^1_1).$$

We will also express concatenation by direct juxtaposition, so that e.g., if $s = \sigma \wedge \pi$, then

$$ss = \sigma \wedge \pi(\sigma \wedge \pi).$$

We remind the reader one last time of our convention on all ordinals being countable and locally countable. The following notation will be useful:

Definition 3.4.1. Let s and t be reflection patterns. We write $s \equiv t$ if for every ordinal α, α is s-reflecting if, and only if, it is t-reflecting.

Definition 3.4.2. Let $k \in \mathbb{N}$ and s be a reflection pattern. We write $c^k_0 s = s$; inductively,

$$c^k_{n+1} s = \sigma \wedge \pi^k \sigma\pi c^k_n s.$$

We write c^k_n for $c^k_n s$, where s is the empty pattern.

We remark that, in particular, $c^0_n = (\sigma\pi)^n$.

Lemma 3.4.3. *For every $n, k \in \mathbb{N}$ and every reflection pattern s, every $(\sigma \wedge \pi^{k+1} s)$-reflecting ordinal is $(\pi c^k_n s)$-reflecting.*

Proof. We first show by induction on n that

$$\sigma \wedge \pi^{k+1} \equiv \sigma \wedge \pi c^k_n.$$

Suppose that every $(\sigma \wedge \pi^{k+1})$-reflecting ordinal is (πc^k_n)-reflecting, i.e., that

$$\sigma \wedge \pi^{k+1} \equiv \sigma \wedge \pi c^k_n \equiv \sigma \wedge \pi(c^k_n \wedge \pi^k).$$

After some applications of Lemma 3.3.2, we have

$$\begin{aligned}
\sigma \wedge \pi c^k_n &\equiv \sigma\pi c^k_n \wedge \pi c^k_n \\
&\equiv \sigma\pi c^k_n \wedge \pi^{k+1} \\
&\equiv \sigma\pi c^k_n \wedge \pi^{k+1}\sigma\pi c^k_n \\
&\equiv \sigma \wedge \pi^{k+1}\sigma\pi c^k_n \\
&\equiv \sigma \wedge \pi(\pi^k \sigma\pi c^k_n) \\
&\equiv \sigma \wedge \pi(\sigma \wedge \pi^k \sigma\pi c^k_n) \\
&\equiv \sigma \wedge \pi c^k_{n+1},
\end{aligned}$$

as desired. The argument given also shows that every every $(\sigma \wedge \pi^{k+1} s)$-reflecting ordinal is $(\sigma \wedge \pi c^k_n s)$-reflecting, although it does not prove the converse (which, incidentally, is not true). □

Corollary 3.4.4. *For every $n, k \in \mathbb{N}$, every $l \leq k$, and every reflection pattern s, every $(\sigma \wedge \pi^{k+1}s)$-reflecting ordinal is $(\pi^{k+1}\sigma\pi c_n^l s)$-reflecting.*

Proof. Suppose α is $(\sigma \wedge \pi^{k+1}s)$-reflecting. By Lemma 3.4.3,

$$\alpha \text{ is } \pi c_{n+1}^k s\text{-reflecting.}$$

By definition,

$$\pi c_{n+1}^k s \equiv \pi(\sigma \wedge \pi^k \sigma\pi c_n^k s);$$

in particular,

$$\alpha \text{ is } \pi^{k+1}\sigma\pi c_n^k s\text{-reflecting.}$$

We may thus apply Lemma 3.3.1 n times to see that every $c_n^k s$-reflecting ordinal is also c_n^l-reflecting, from which the result follows. □

Lemma 3.4.5. *For every $k \in \mathbb{N}$, every $n_0, \ldots, n_k \in \mathbb{N}$, and every reflection pattern s, every $(\sigma \wedge \pi^{k+1}s)$-reflecting ordinal is $(\pi^{k+1}\sigma\pi c_{n_0}^0 c_{n_1}^1 \ldots c_{n_k}^k s)$-reflecting.*

Proof. This follows from applying Corollary 3.4.4 repeatedly. □

Definition 3.4.6. A reflection pattern is in 2-*normal form* if it is of the form

$$\pi^n c_{n_0}^0 c_{n_1}^1 \ldots c_{n_k}^k,$$

for some natural numbers k, $n_0, n_1, \ldots, n_k$. If w is the reflection pattern above, we define

$$o(w) = \omega^{k+1} \cdot n_k + \omega^k \cdot n_{k-1} + \cdots + \omega \cdot n_0 + n.$$

For now, we shall simply refer to patterns in 2-normal form as being in *normal form*. We shall see that patterns in normal form have very nice properties.

Lemma 3.4.7. *Suppose s is a reflection pattern in normal form. Then, every $\sigma\sigma$-reflecting ordinal is $\sigma(s)$-reflecting.*

Proof. By Theorem 3.3.6, any such ordinal is $\sigma(\sigma \wedge \pi^k)$-reflecting for any k. The lemma now follows from Corollary 3.4.4 and Lemma 3.3.1, and Lemma 3.3.2. □

We also have the following "contraction" lemma, which will be crucial:

Lemma 3.4.8. *Suppose s is a reflection pattern. Then, every $\sigma\pi\sigma\pi s$-reflecting ordinal is $\sigma\pi s$-reflecting.*

Proof. Suppose α is $\sigma\pi\sigma\pi s$-reflecting. By an argument as in Lemma 3.3.1, applying Lemma 3.3.2, for every Σ_1^1-sentence ϕ satisfied by L_α, one can find some $\pi\sigma\pi s$-reflecting $\beta < \alpha$ such that $L_\beta \models \phi$ and β is not Σ_1^1-reflecting. By Theorem 3.3.9, β is $\pi\pi s$-reflecting. By Lemma 3.3.1, β is πs-reflecting, as desired. □

Lemma 3.4.9. *Suppose πt and πs are reflection patterns in normal form such that $o(\pi t) \leq o(\pi s)$. Then, every (πs)-reflecting ordinal is either (πt)-reflecting or σ-reflecting.*

Proof. Let

$$\pi t = \pi^n c^0_{n_0} c^1_{n_1} \dots c^k_{n_k},$$

and

$$\pi s = \pi^m c^0_{m_0} c^1_{m_1} \dots c^l_{m_l},$$

where n and m are nonzero. Without loss of generality, we assume that n_k and m_l are also nonzero. It follows that $k \leq l$. Suppose that $o(\pi t) < o(\pi s)$. It will be convenient, for illustrative purposes, to consider the case that $k < l$ first. If so, it suffices to show that every $\pi(\sigma \wedge \pi^l)$-reflecting ordinal which is not σ-reflecting is (πt)-reflecting, for then the result follows from Theorem 3.3.9. By Lemma 3.4.5, every $\pi(\sigma \wedge \pi^l)$-reflecting ordinal is

$$\pi(\sigma \wedge \pi^l \sigma\pi c^0_{n_0} c^1_{n_1} \dots c^k_{n_k})\text{-reflecting.}$$

By Lemma 3.3.2, every such ordinal is

$$\pi(\sigma\pi^l c^0_{n_0} c^1_{n_1} \dots c^k_{n_k})\text{-reflecting,}$$

and, by Theorem 3.3.6 or Lemma 3.3.1, according as $n \leq l$ or $l \leq n$, it is

$$\pi(\sigma\pi^n c^0_{n_0} c^1_{n_1} \dots c^k_{n_k})\text{-reflecting,}$$

so that, if it is not σ-reflecting, then it is

$$\pi(\pi^n c^0_{n_0} c^1_{n_1} \dots c^k_{n_k})\text{-reflecting,}$$

by Theorem 3.3.9, i.e., $\pi(t)$-reflecting.

The general case is similar: let $i \leq k$ be greatest such that $n_i < m_i$ and notice that

$$c^i_{m_i} = c^i_{m_i - n_i} c^i_{n_i}.$$

Thus,

$$\pi t = \pi^n c^0_{n_0} c^1_{n_1} \dots c^{i-1}_{n_{i-1}} t',$$

where $t' = c^i_{n_i} \dots c^k_{n_k}$; and

$$\pi s = \pi^m c^0_{m_0} c^1_{m_1} \dots c^i_{m_i - n_i} t'.$$

It suffices to show that every $\pi(\sigma \wedge \pi^i \sigma \pi t')$-reflecting ordinal which is not σ-reflecting is (πt)-reflecting, for then the result follows from Theorem 3.3.9. Lemma 3.4.5 (with $\sigma \pi t'$ being the s in the statement) shows that every such ordinal is

$$\pi(\sigma \wedge \pi^i \sigma \pi c^0_{n_0} c^1_{n_1} \dots c^{i-1}_{n_{i-1}} \sigma \pi t')\text{-reflecting.}$$

As before, by Lemma 3.3.2, every such ordinal is

$$\pi(\sigma \pi^i \sigma \pi c^0_{n_0} c^1_{n_1} \dots c^{i-1}_{n_{i-1}} \sigma \pi t')\text{-reflecting.}$$

By Theorem 3.3.6 and Lemma 3.3.1, every such ordinal is

$$\pi(\sigma \pi^n \sigma \pi c^0_{n_0} c^1_{n_1} \dots c^{i-1}_{n_{i-1}} \sigma \pi t')\text{-reflecting.}$$

By contraction (Lemma 3.4.8), it is

$$\pi(\sigma \pi^n \sigma \pi c^0_{n_0} c^1_{n_1} \dots c^{i-1}_{n_{i-1}} t')\text{-reflecting,}$$

and by an argument like the one of Lemma 3.4.8, it is

$$\pi(\sigma \pi^n c^0_{n_0} c^1_{n_1} \dots c^{i-1}_{n_{i-1}} t')\text{-reflecting,}$$

so that if it is not σ-reflecting, then it is

$$\pi(\pi^n c^0_{n_0} c^1_{n_1} \dots c^{i-1}_{n_{i-1}} t')\text{-reflecting,}$$

by Theorem 3.3.9, as desired. □

Lemma 3.4.10. *Suppose t is a reflection pattern in normal form. Then $\sigma \wedge t$ is equivalent to a reflection pattern in normal form.*

Proof. Put $t = \pi^k c^0_{m_0} c^1_{m_1} \dots c^l_{m_l}$. The lemma is immediate unless $k \neq 0$ and there is some least $i \leq l$ such that $m_i \neq 0$. Thus,

$$\begin{aligned} t &= \pi^k c^i_{m_i} c^1_{m_1} \dots c^l_{m_l} \\ &= \pi^k (\sigma \wedge \pi^i \sigma \pi c^i_{m_i - 1} c^1_{m_1} \dots c^l_{m_l}). \end{aligned}$$

Let

$$s = c^{k+i}_{m_{k+i}} c^{k+i+1}_{m_{k+i+1}} \dots c^l_{m_l},$$

so that

$$t = \pi^k(\sigma \wedge \pi^i \sigma\pi c^i_{m_i-1} c^1_{m_1} \dots c^{k+i-1}_{m_{k+i-1}} s).$$

If all m_j are empty for $j \neq 0$, then the result follows easily; otherwise, by Lemma 3.3.2 and Corollary 3.4.4,

$$t = \pi^k(\sigma \wedge \pi^i \sigma\pi c^i_{m_i-1} c^1_{m_1} \dots c^{k+i-1}_{m_{k+i-1}} \sigma\pi s).$$

By Lemma 3.3.2,

$$\begin{aligned}\sigma \wedge t &= \sigma \wedge \pi^k(\sigma \wedge \pi^i \sigma\pi c^i_{m_i-1} c^1_{m_1} \dots c^{k+i-1}_{m_{k+i-1}} \sigma\pi s) \\ &= \sigma \wedge \pi^k(\pi^i \sigma\pi c^i_{m_i-1} c^1_{m_1} \dots c^{k+i-1}_{m_{k+i-1}} \sigma\pi s) \\ &= \sigma \wedge \pi^{k+i} \sigma\pi c^i_{m_i-1} c^1_{m_1} \dots c^{k+i-1}_{m_{k+i-1}} \sigma\pi s)\end{aligned}$$

By Lemma 3.4.5 on the one hand and Lemma 3.3.1 and Corollary 3.4.4 on the other,

$$\sigma \wedge \pi^{k+i} \sigma\pi s \equiv \sigma \wedge t,$$

but the reflection pattern on the left-hand side is readily seen to be equivalent to one in normal form. □

Theorem 3.4.11. *Let s be a reflection pattern in which the string $\sigma\sigma$ does not occur. Then, it is equivalent to a reflection pattern in 2-normal form.*

Proof. This is proved by induction on the construction of s. Clearly, if s is equivalent to a reflection pattern in normal form, then so too is πs. We only need consider the string σs in the case that s is of the form

$$\pi^m c^0_{m_0} c^1_{m_1} \dots c^l_{m_l},$$

where m is nonzero. Then, it is easy to see that

$$\begin{aligned}\sigma\pi^m c^0_{m_0} c^1_{m_1} \dots c^l_{m_l} &= \sigma\pi c^0_{m_0} c^1_{m_1} \dots c^l_{m_l} \\ &= c^0_{m_0+1} c^1_{m_1} \dots c^l_{m_l}.\end{aligned}$$

Now, let s be as above, and let

$$t = \pi^n c^0_{n_0} c^1_{n_1} \dots c^l_{n_l}.$$

We need to show that $s \wedge t$ is equivalent to a reflection pattern in normal form. If both n and m are nonzero, then the result follows from Lemma 3.4.9; so suppose that one of m and n is zero, so that

$$s \wedge t = \sigma \wedge s \wedge t.$$

Write $n = n_{-1}$ and $m = m_{-1}$ and let i and j be least such that n_i and m_j are nonzero, respectively. There are four cases to consider. The first one is that in which both i and j are equal to 0. Then, there are reflection patterns u and v, both in normal form, such that

$$t = \sigma\pi u$$

and

$$s = \sigma\pi v.$$

Suppose without loss of generality that $o(\pi u) < o(\pi v)$. By Lemma 3.4.9, every (πv)-reflecting ordinal which is not σ-reflecting is (πu)-reflecting. Thus, every s-reflecting ordinal is either t-reflecting or $\sigma\sigma$-reflecting, in which case it is also t-reflecting by Lemma 3.4.7.

The second case is that in which $i = 0$ but $j \neq 0$. Then, there are reflection patterns u and v, both in normal form, such that

$$t = \sigma\pi u$$

and

$$\sigma \wedge s = \sigma \wedge \pi^{m_j}\sigma\pi v.$$

By direct computation,

$$\begin{aligned} s \wedge t &\equiv \sigma\pi u \wedge \pi^{m_j}\sigma\pi v \\ &\equiv \sigma(\pi u \wedge \pi^{m_j}\sigma\pi v) \wedge \pi^{m_j}\sigma\pi v \\ &\equiv \sigma(\pi u \wedge \pi^{m_j}\sigma\pi v) \wedge \pi^{m_j}(\sigma(\pi u \wedge \pi^{m_j}\sigma\pi v) \wedge \sigma\pi v) \\ &\equiv \sigma \wedge \pi^{m_j}(\sigma(\pi u \wedge \pi^{m_j}\sigma\pi v) \wedge \sigma\pi v). \end{aligned}$$

Now, it is easily seen that

$$\begin{aligned} \pi^{m_j}\sigma(\pi u \wedge \pi^{m_j}\sigma\pi v) &\equiv \pi^{m_j}\sigma(\pi u \wedge \pi^{m_j}\sigma\pi v \wedge \pi\sigma\pi v) \\ &\equiv \pi^{m_j}\Big(\sigma(\pi u \wedge \pi^{m_j}\sigma\pi v) \wedge \sigma\pi\sigma\pi v\Big) \\ &\equiv \pi^{m_j}\Big(\sigma(\pi u \wedge \pi^{m_j}\sigma\pi v) \wedge \sigma\pi v\Big), \end{aligned}$$

where the last equivalence follows from Lemma 3.4.8, and so

$$s \wedge t \equiv \sigma \wedge \pi^{m_j}\sigma(\pi u \wedge \pi^{m_j}\sigma\pi v).$$

Since each of πu and $\pi^{m_j}\sigma\pi v$ is a reflection pattern in normal form, Lemma 3.4.9 implies that their conjunction is equal to one of πu and $\pi^{m_j}\sigma\pi v$. Let us denote this conjunct by w. Then,

$$s \wedge t \equiv \sigma \wedge \pi^{m_j}\sigma w.$$

Since w is in normal form, so too is $\pi^{m_j}\sigma w$, so the result follows from Lemma 3.4.10. The case in which $i \neq 0$ and $j = 0$ is analogous.

The remaining case is that in which both i and j are nonzero. Suppose without loss of generality that $n_i < m_j$ By replacing n_i and m_j by larger numbers if necessary (this might need to be done in the case $i = -1$ and $m_0 = 0$—a situation similar to the one in Lemma 3.4.10) we may assume that there are reflection patterns u and v such that

$$\sigma \wedge t = \sigma \wedge \pi^{n_i}\sigma\pi u$$

and

$$\sigma \wedge s = \sigma \wedge \pi^{m_j}\sigma\pi v.$$

Then, we have

$$\begin{aligned}\sigma \wedge s \wedge t &\equiv \sigma \wedge \pi^{n_i}\sigma\pi u \wedge \pi^{m_j}\sigma\pi v \\ &\equiv \sigma\big(\pi^{n_i}\sigma\pi u \wedge \pi^{m_j}\sigma\pi v\big) \wedge \pi^{n_i}\sigma\pi u \wedge \pi^{m_j}\sigma\pi v \\ &\equiv \sigma \wedge \pi^{n_i}\sigma\pi u \wedge \pi^{m_j}\Big(\sigma\big(\pi^{n_i}\sigma\pi u \wedge \pi^{m_j}\sigma\pi v\big) \wedge \sigma\pi v\Big).\end{aligned}$$

By contraction,

$$\begin{aligned}\sigma \wedge \pi^{m_j}\sigma\pi^{n_i}\sigma\pi u &\equiv \sigma \wedge \pi^{m_j}\sigma\pi\sigma\pi u \\ &\equiv \sigma \wedge \pi^{m_j}\sigma\pi u.\end{aligned}$$

Since $n_i < m_j$, it follows by contraction that every ordinal which is

$$(\sigma \wedge \pi^{m_j}\sigma\pi^{n_i}\sigma\pi u)\text{-reflecting}$$

is also

$$(\sigma \wedge \pi^{n_i}\sigma\pi u)\text{-reflecting.}$$

Similarly, we have

$$\begin{aligned}\sigma \wedge \pi^{m_j}\sigma\pi^{m_j}\sigma\pi v &\equiv \sigma \wedge \pi^{m_j}\Big(\sigma\pi^{m_j}\sigma\pi v \wedge \sigma\pi\sigma\pi v\Big) \\ &\equiv \sigma \wedge \pi^{m_j}\Big(\sigma\pi^{m_j}\sigma\pi v \wedge \sigma\pi v\Big)\end{aligned}$$

and so every ordinal which is $(\sigma \wedge \pi^{m_j}\sigma\pi^{m_j}\sigma\pi v)$-reflecting is also

$$\Big(\sigma \wedge \pi^{m_j}\big(\sigma\pi^{m_j}\sigma\pi v \wedge \sigma\pi v\big)\Big)\text{-reflecting.}$$

It follows that

$$\begin{aligned}\sigma \wedge s \wedge t &\equiv \sigma \wedge \pi^{n_i}\sigma\pi u \wedge \pi^{m_j}\Big(\sigma\big(\pi^{n_i}\sigma\pi u \wedge \pi^{m_j}\sigma\pi v\big) \wedge \sigma\pi v\Big)\\ &\equiv \sigma \wedge \pi^{m_j}\sigma\big(\pi^{n_i}\sigma\pi u \wedge \pi^{m_j}\sigma\pi v\big).\end{aligned}$$

Since both m_j and n_i are nonzero, Lemma 3.4.9 implies that the conjunction

$$\pi^{n_i}\sigma\pi u \wedge \pi^{m_j}\sigma\pi v$$

is equivalent to one of $\pi^{n_i}\sigma\pi u$ or $\pi^{m_j}\sigma\pi v$. Denote this conjunct by w. Then, w is in normal form. Since w begins with the symbol π, $\pi^{m_j}\sigma w$ is also equivalent to a pattern in normal form. The result then follows from Lemma 3.4.10. This proves the theorem. □

Theorem 3.4.12. *$\sigma\sigma$ has order-type ω^ω in the reflection order.*

Proof. By Theorem 3.4.11, every reflection pattern in which the string $\sigma\sigma$ does not occur is equivalent to one in $\sigma\sigma$-normal form. By Lemma 3.4.7, this implies that $\sigma\sigma$ is strictly bigger than each reflection pattern in which the string $\sigma\sigma$ does not occur. Conversely, if a reflection pattern does contain the string $\sigma\sigma$, then naturally, it cannot be strictly smaller than $\sigma\sigma$. An easy induction shows that, for reflection patterns u and v in 2-normal form, $u < v$ if, and only if, $o(u) < o(v)$, so the result follows. □

3.5 Linear Patterns

Our first result in this section concerns the length of the linear reflection order; its proof induces a simple algorithm for comparing two arbitrary linear reflection patterns.

Theorem 3.5.1. *The length of the linear fragment of the reflection order is ω^ω.*

Proof. Let us employ the simplified notation from the previous section. Recursively, we assign ordinals to reflection patterns without conjunction: we assign the ordinal ω^n to the pattern

$$\sigma^n.$$

In particular, the ordinal 1 is assigned to the empty pattern. If s and t are patterns to which ordinals $o(s)$ and $o(t)$ have been assigned, we assign the ordinal

$$o(s) + o(t)$$

to the pattern

$$t\pi s.$$

Note that if $n < m$, then, on the one hand,

$$\alpha + \omega^n + \omega^m = \alpha + \omega^m,$$

while, on the other,

$$\begin{aligned}
\sigma^m\pi s &\equiv \sigma^{m-n}\sigma^n\pi s \\
&\equiv \sigma^{m-n}(\sigma^n\pi s \wedge \pi) && \text{by Lemma 3.3.2(3)} \\
&\equiv \sigma^{m-n}(\sigma^n\pi s \wedge \pi\sigma^n\pi s) && \text{by Lemma 3.3.2(1)} \\
&\equiv \sigma^{m-n}\Big(\sigma^n(\pi s \wedge \pi\sigma^n\pi s) \wedge \pi\sigma^n\pi s\Big) && \text{by Lemma 3.3.2(2)} \\
&\equiv \sigma^m(\pi s \wedge \pi\sigma^n\pi s) \\
&\equiv \sigma^m\pi s \wedge \sigma^m\pi\sigma^n\pi s.
\end{aligned}$$

Thus, every $\sigma^m\pi s$-reflecting ordinal is also $\sigma^m\pi\sigma^n\pi s$-reflecting when $n < m$; the converse is also true, by Lemma 3.3.1 and Theorem 3.3.9 (cf. the argument of Lemma 3.4.8 on p. 52). It follows that this assignment of ordinals is well-defined. By Lemma 3.3.1 and Theorem 3.3.6, we have

$$\sigma\pi^n\sigma \equiv \sigma\pi^m\sigma$$

for any pair of nonzero numbers n and m, so every conjunction-free reflection pattern is equivalent to one to which an ordinal has been assigned. It should be clear by now that, for conjunction-free reflection patterns, $s < t$ if, and only if, $o(s) < o(t)$, which completes the proof. □

The second result of this section is that the linear patterns are cofinal in the reflection order.

Theorem 3.5.2. *The sequence $\{(\sigma^1_1)^n : n \in \mathbb{N}\}$ is cofinal in the reflection order.*

Proof. To prove the theorem, we shall prove by induction on the construction of a reflection pattern s that if s contains no occurrence of σ^{n+1}, then every $\sigma(\sigma^n \wedge t)$-reflecting ordinal is also $\sigma(\sigma^n \wedge t \wedge s)$-reflecting, for every reflection

pattern t (cf. Lemma 3.4.7 on p. 52). We may assume that $n \neq 0$, for otherwise the conclusion follows from Theorem 3.3.6. Note that the case that s is a conjunction is immediate from the induction hypothesis and the case that s is of the form $\pi s'$ is also immediate from the induction hypothesis and Theorem 3.3.6, thus, we suppose that s is of the form $\sigma s'$. The pattern s' might be a conjunction, say,

$$s' = s_0 \wedge s_1 \wedge \cdots \wedge s_k \wedge r_0 \wedge r_1 \wedge \cdots \wedge r_l,$$

where each s_i is of the form $\sigma^{m_i} s_i'$ for some $m_i < n$ and some s_i' which is a conjunction of patterns of the form πs^*, and each r_i is of the form $\pi r_i'$. Instead of proving that every $\sigma(\sigma^n \wedge t)$-reflecting ordinal is $\sigma(\sigma^n \wedge t \wedge \sigma s')$-reflecting, we shall prove the stronger fact that it is

$$\sigma\Big(\sigma^n \wedge t \wedge \sigma(\sigma^{n-1}(s_0' \wedge s_1' \wedge \cdots \wedge s_k') \wedge r_0 \wedge r_1 \wedge \cdots \wedge r_l)\Big)\text{-reflecting.}$$

By the induction hypothesis applied to s', and the fact that each of s_i' and

$$r := r_0 \wedge r_1 \wedge \cdots \wedge r_l$$

is a conjunction of patterns of the form πs^*, we obtain:

$$\begin{aligned}
\sigma\Big(\sigma^n \wedge t\Big) &\equiv \sigma\Big(\sigma^n \wedge t \wedge s_0' \wedge \cdots \wedge s_k' \wedge r\Big) \\
&\equiv \sigma\Big(\sigma^n \wedge t \wedge \sigma^n \wedge s_0' \wedge \cdots \wedge s_k' \wedge r\Big) \\
&\equiv \sigma\Big(\sigma^n \wedge t \wedge \sigma(\sigma^{n-1} \wedge s_0' \wedge \cdots \wedge s_k' \wedge r)\Big) \\
&\equiv \sigma\Big(\sigma^n \wedge t \wedge \sigma(\sigma^{n-1}(s_0' \wedge \cdots \wedge s_k') \wedge r)\Big),
\end{aligned}$$

where the last two equivalences follow from Lemma 3.3.2. This completes the proof of the theorem. □

3.6 Co-stability

Let us temporarily go back to our original notation with parentheses and subscripts. By Lemma 3.4.7, if $s < \sigma_1^1(\sigma_1^1)$, then we additionally obtain the stronger fact that $\sigma_1^1(\sigma_1^1)$ is $\sigma_1^1(s)$-reflecting. One may wonder if this phenomenon obtains in general. We will see that it does not. In particular, $\sigma_1^1(\pi_1^1(\sigma_1^1(\sigma_1^1)))$ is not $\sigma_1^1(\sigma_1^1)$-reflecting, and, consequently,

$$\sigma_1^1(\sigma_1^1) < \sigma_1^1(\pi_1^1(\sigma_1^1(\sigma_1^1))) < \sigma_1^1(\pi_1^1(\sigma_1^1(\sigma_1^1))) \wedge \sigma_1^1(\sigma_1^1).$$

The proof, however, will lead to a third reflection transfer theorem which has the unintuitive consequence that

$$\sigma_1^1(\pi_1^1(\sigma_1^1(\sigma_1^1))) \wedge \sigma_1^1(\sigma_1^1) = \sigma_1^1(\sigma_1^1(\pi_1^1(\sigma_1^1(\sigma_1^1))))$$

The central notion for this is that of *co-stability*.

Definition 3.6.1. An ordinal σ is *δ_σ-costable* if for every Π_1 formula ϕ with parameters in L_σ, if $L_{\delta_\sigma} \models \phi(L_\sigma)$, then there is $\alpha < \sigma$ such that $L_{\alpha^+} \models \phi(L_\alpha)$. Let s be a reflection pattern. We say σ is *δ_σ-costable on s* if moreover α can be chosen to be s-reflecting.

The same way that π_1^1 can be characterized as the least σ which is δ_σ-stable, σ_1^1 can be characterized as the least σ which is δ_σ-costable. More generally:

Proposition 3.6.2. *Let s be a reflection pattern and σ be an ordinal. Then,*

1. *If σ is δ_σ-costable on s, then it is $\Sigma_1^1(s)$-reflecting.*
2. *If s is a (possibly empty) conjunction of reflection patterns of the form $\pi_1^1(t)$ and σ is $\Sigma_1^1(s)$-reflecting, then it is δ_σ-costable on s.*

Proof. The first item is immediate from the definition. For the second, suppose σ is $\Sigma_1^1(s)$-reflecting. Let ϕ be a Π_1 formula with parameters in L_σ such that $L_{\delta_\sigma} \models \phi(L_\sigma)$. Let ψ be the sentence asserting the existence of a model M of $\mathsf{KP} + V = L$ such that

1. M contains σ;
2. $L_{\delta_\sigma}^M \models \phi(L_\sigma)$.

By reflection, let $\sigma_0 < \sigma$ be an s-reflecting ordinal such that $L_{\sigma_0} \models \psi$. If σ_0 is Σ_1^1-reflecting, then it is $(\sigma_1^1 \wedge s)$-reflecting and thus $\sigma_1^1(s)$-reflecting, by Lemma 3.3.2 and the assumption on s. Choose an s-reflecting $\sigma_1 < \sigma_0$ such that $L_{\sigma_1} \models \psi$. After finitely many steps, one arrives at some s-reflecting σ_n which is not Σ_1^1-reflecting and such that $L_{\sigma_n} \models \psi$, as witnessed by some model N. Since σ_n is not Σ_1^1-reflecting, $\delta_{\sigma_n} = \sigma_n^+$, so $\delta_{\sigma_n}^N$ is greater than every $\beta < \sigma_n^+$. Since ϕ was chosen to be Π_1, it follows that $L_{\sigma_n^+} \models \phi(L_\sigma)$, as desired. □

If one could remove the restriction on s in the statement of Proposition 3.6.2, then $\sigma_1^1(\sigma_1^1)$ would be the least ordinal σ which is δ_σ-costable on Σ_1^1-reflecting ordinals. Proposition 3.6.3 below shows that this is false.

Proposition 3.6.3. *$\sigma^1_1(\sigma^1_1)$ is not the least ordinal α which is δ_α-costable on Σ^1_1-reflecting ordinals.*

Proof. The main point is that being δ_σ-costable on s is expressible over L_{δ_σ} using L_σ as a parameter (see the proof of Proposition 3.2.1). Let $\sigma = \sigma^1_1(\sigma^1_1)$ and suppose towards a contradiction that σ is indeed the least ordinal α which is δ_α-costable on Σ^1_1-reflecting ordinals; then $L_{\delta_\sigma+1}$ satisfies the statement expressing that there is an admissible α such that δ_α exists and is not admissible and α is δ_α-costable on Σ^1_1-reflecting ordinals. By Proposition 3.2.7, σ is $(\delta_\sigma+1)$-stable and thus a limit of ordinals α which are δ_α-costable on Σ^1_1-reflecting ordinals—a contradiction. □

Proposition 3.6.4. *Let s be a reflection pattern of the form $\pi^1_1(t)$. Then, the least $\sigma^1_1(s)$-reflecting ordinal is not $\sigma^1_1(\sigma^1_1)$-reflecting.*

We shall not prove Proposition 3.6.4, as it is a consequence of Theorem 3.6.6 below.

Example 3.6.5. The inequality stated at the beginning of the section now follows: clearly $\sigma^1_1(\pi^1_1(\sigma^1_1(\sigma^1_1)))$ is a limit of $\sigma^1_1(\sigma^1_1)$-reflecting ordinals and thus

$$\sigma^1_1(\sigma^1_1) < \sigma^1_1(\pi^1_1(\sigma^1_1(\sigma^1_1))).$$

By Proposition 3.6.4 applied to $t = \pi^1_1(\sigma^1_1(\sigma^1_1))$, the ordinal $\sigma^1_1(\pi^1_1(\sigma^1_1(\sigma^1_1)))$ is not $\sigma^1_1(\sigma^1_1)$-reflecting, and thus

$$\sigma^1_1(\pi^1_1(\sigma^1_1(\sigma^1_1))) < \sigma^1_1(\pi^1_1(\sigma^1_1(\sigma^1_1))) \wedge \sigma^1_1(\sigma^1_1).$$

In particular, we have

$$\sigma^1_1(\pi^1_1(\sigma^1_1(\sigma^1_1))) \neq \sigma^1_1(\pi^1_1(\sigma^1_1(\sigma^1_1))) \wedge \sigma^1_1(\sigma^1_1).$$

This is in contrast to the fact that

$$\sigma^1_1(\pi^1_1(\sigma^1_1(\sigma^1_1))) = \sigma^1_1(\pi^1_1(\sigma^1_1(\pi^1_1(\sigma^1_1(\sigma^1_1))))) \wedge \sigma^1_1(\pi^1_1(\sigma^1_1(\sigma^1_1))).$$

by Lemma 3.4.8.

We now state and prove the third reflection transfer theorem:

Theorem 3.6.6. *Let s be a reflection pattern. Suppose σ is $\sigma^1_1(\pi^1_1(s))$-reflecting and $\sigma^1_1(\sigma^1_1 \wedge t)$-reflecting. Then, it is*

$$\sigma^1_1\big(\sigma^1_1(\pi^1_1(s)) \wedge t\big)\text{-reflecting.}$$

Proof. Put $r = \pi^1_1(s)$. Let σ be an ordinal which is $\sigma^1_1(\pi^1_1(s))$-reflecting and also $\sigma^1_1(\sigma^1_1 \wedge t)$-reflecting. By Proposition 3.6.2, σ is δ_σ-costable on r. Let ψ be the sentence asserting the existence of a model M of $\mathsf{KP} + V = L$ such that

1. M contains σ;

2. In M, σ is δ_σ-costable on t.

By $\sigma^1_1(\sigma^1_1 \wedge t)$-reflection, there is a $(\sigma^1_1 \wedge t)$-reflecting ordinal $\tau < \sigma$ such that $L_\tau \models \psi$, as witnessed by some model N. Thus, in N, τ is δ_τ-costable on r. However, since τ is Σ^1_1-reflecting, $\delta_\tau < \tau^+$. It follows that $\delta_\tau \in N$, so N is correct about the δ_τ-costability on r of τ. Hence, τ really is δ_τ-costable on r. By Proposition 3.6.2, it is $\Sigma^1_1(r)$-reflecting. Therefore, τ is $\Sigma^1_1(r)$-reflecting and t-reflecting, so σ is as desired. □

Repeated applications of Theorem 3.6.6 yield:

Corollary 3.6.7. *Let s be a reflection pattern. Suppose σ is $\sigma^1_1(\pi^1_1(s))$-reflecting and $\sigma^1_1((\sigma^1_1)^n \wedge t)$-reflecting. Then, it is*

$$\sigma^1_1((\sigma^1_1)^n(\pi^1_1(s)) \wedge t)\text{-reflecting.}$$

We finish this section with a final reflection transfer theorem stated as a (very useful) dichotomy. It is a strengthening of Theorem 3.3.9 which clarifies the hypothesis on σ not being Σ^1_1-reflecting.

Theorem 3.6.8. *Suppose σ is $\pi^1_1(\sigma^1_1(t \wedge \pi^1_1(s)))$-reflecting. Then, one of the following holds:*

1. *σ is $\pi^1_1(s)$-reflecting; or*

2. *σ is $\sigma^1_1(t \wedge \pi^1_1(s))$-reflecting.*

Proof. Suppose σ is $\pi^1_1(\sigma^1_1(t \wedge \pi^1_1(s)))$-reflecting but not $\pi^1_1(s)$-reflecting. Let θ be a Σ^1_1 sentence with parameters in L_σ such that

$$L_\sigma \models \theta;$$

we need to find a $(t \wedge \pi^1_1(s))$-reflecting $\tau < \sigma$ such that

$$L_\tau \models \theta.$$

By Barwise-Gandy-Moschovakis [BGM71], there is a Π_1 formula $\theta^*(a)$ such that for every admissible α containing the parameters of θ, $L_\alpha \models \theta$ if, and only if, $L_{\alpha^+} \models \theta^*(L_\alpha)$. In particular,

$$L_{\sigma^+} \models \theta^*(L_\sigma).$$

Since σ is not $\pi^1_1(s)$-reflecting, there is a least $\beta < \sigma^+$ such that σ is not β-stable on $\pi^1_1(s)$. Because θ^* is Π_1,

$$L_\beta \models \theta^*(L_\sigma).$$

Let ϕ be the sentence asserting the non-existence of a model M of $\mathsf{KPi}+V = L$ end-extending $L_{\sigma+1}$ in which σ is $\sigma^1_1(t \wedge \pi^1_1(s))$-reflecting. This is a Π^1_1 sentence and thus cannot be satisfied by any $\pi^1_1(\sigma^1_1(t \wedge \pi^1_1(s)))$-reflecting ordinal and, in particular, by σ. Thus, there is a model M of $\mathsf{KPi} + V = L$ end-extending $L_{\sigma+1}$ and such that

$$M \models \text{“}\sigma \text{ is } \sigma^1_1(t \wedge \pi^1_1(s))\text{-reflecting.”}$$

Let χ be the sentence asserting the existence of a model N of $\mathsf{KP} + V = L$ such that

1. N contains σ;
2. in N, letting β be least such that σ is not β-stable on $\pi^1_1(s)$, we have

$$N \models \text{“}L_\beta \models \theta^*(L_\sigma).''$$

Since $\beta < \sigma^+$ and M must end-extend L_{σ^+}, $\beta \in M$ and M is correct about β being the least ordinal at which σ fails to be stable on $\pi^1_1(s)$. Thus, we have

$$M \models \text{“}L_\sigma \models \chi,''$$

as witnessed, say, by $L^M_{\sigma^+}$. Within M, σ is $\sigma^1_1(t \wedge \pi^1_1(s))$-reflecting and thus there is some $\tau < \sigma$ such that,

$$M \models \text{“}L_\tau \models \chi,''$$

and so we really do have

$$L_\tau \models \chi.$$

Moreover, M is correct about reflection below σ, so we may assume that τ is $(t \wedge \pi^1_1(s))$-reflecting. By the definition of χ, there is a model N of $\mathsf{KP} + V = L$ such that

1. N contains τ;
2. in N, letting γ be least such that σ is not γ-stable on $\pi^1_1(s)$, we have

$$N \models \text{“}L_\beta \models \theta^*(L_\tau).''$$

Since τ is $\pi^1_1(s)$-reflecting, it is τ^+-stable on $\pi^1_1(s)$, and thus γ cannot be a true ordinal smaller than τ^+. By Ville's Theorem, N must end-extend L_{τ^+}. Because θ^* is Π_1, it follows that

$$L_{\tau^+} \models \theta^*(L_\tau),$$

and thus, that

$$L_\tau \models \theta,$$

as was to be shown. □

3.7 A Chain of Length ε_0

In this section, we extend the analysis of the reflection order beyond $\sigma^1_1(\sigma^1_1)$. We will go back to our simplified notation in which we omit parentheses, subscripts, and superscripts. We will show that the reflection pattern σ^n has order-type at least

$$\underbrace{\omega^{\omega^{\cdot^{\cdot^{\omega}}}}}_{n \text{ times}}$$

and thus that the reflection order has order-type at least ε_0. We will again assign an ordinal $o(s)$ to each reflection pattern s in some normal form, prove that the ordering induced by this assignment coincides with the reflection order on patterns on which it is defined.

The computations below will be a bit shorter if we employ hyperexponential notation (see Fernández-Duque and Joosten [FDJ13a]). Below, we write

$$-1 + \alpha = \begin{cases} \alpha - 1 & \text{if } \alpha \text{ is finite and nonzero,} \\ 0 & \text{if } \alpha = 0, \\ \alpha & \text{if } \alpha \text{ is infinite.} \end{cases}$$

Define:

1. $e^0\alpha = \alpha$,

2. $e^1\alpha = e\alpha := -1 + \omega^\alpha$, and
3. $e^{n+1}\alpha = ee^n\alpha$.

The left-inverse of the exponential e is the logarithm ℓ given by $\ell 0 = 0$ and

$$\ell(\alpha + \omega^\beta) = \beta.$$

Inductively,

$$\ell^{n+1}\alpha = \ell\ell^n\alpha.$$

In this construction, the main patterns we will work with are those of the form

$$\sigma^n \wedge \pi^{k+1};$$

for example, $\sigma\sigma\sigma$ will not figure in it.

As before, we will be able to combine them to form more complicated patterns. This can be done through direct juxtaposition, or with the use of a "linking" pattern; these will be those of the form

$$\sigma(\sigma^h \wedge \pi).$$

Note that this includes the pattern $\sigma\pi$ from before.

We will need the notion of the *right side* of an ordinal, which we define below. The *Hyperexponential Normal Form Theorem* [AFD17] states that every ordinal ξ can be written in the form $e^\alpha\beta$, where β is either 1 or additively decomposable. In the case $\xi < \varepsilon_0$, we additionally have $\alpha < \omega$, so this expresion is covered by our earlier definition of e^α.

If $\xi = e^\alpha\beta$, where $\beta \leq \omega$, then we say that the *right side* of ξ is ξ itself. Inductively, if $\xi = e^\alpha\beta$, where β is additively decomposable and greater than ω, then we say that the *right side* of ξ is $e^\alpha\beta'$, where β' is the right side of β.

Example 3.7.1. The right side of

$$e^3(e^2 2 + e(e1 + 2)) = \omega^{\omega^{\omega^{\omega^2}} + \omega^{\omega+2}}$$

is

$$e^3(e(2)) = e^4 2 = \omega^{\omega^{\omega^{\omega^2}}}.$$

Definition 3.7.2. We assign ordinals to some patterns:

1. If s is a pattern below $\sigma\sigma$, then $o(s)$ is the order-type of s in the reflection order.

2. Let $n, k \in \mathbb{N}$. $o(\sigma^{n+1} \wedge \pi^{k+1}) = e^{n+1}(k+1)$.

3. Let $h \in \mathbb{N}$. Suppose s is a pattern and $o(s)$ is defined. Then,

   (a) $o(\pi^k s) = o(s) + k$;

   (b) if $0 < \ell^{h+1} o(s)$ (i.e., if $\ell^h o(s)$ is a limit ordinal), then $o(\sigma(\sigma^h \wedge \pi s)) = e^h(\ell^h o(s) + \omega)$;

   (c) if $0 < \ell^{h+1} o(s)$ (i.e., if $\ell^h o(s)$ is a limit ordinal) and $k \in \mathbb{N}$, then $o(\sigma^h \wedge \pi^{k+1} s) = e^h(\ell^h o(s) + k + 1)$.

4. Let $h \in \mathbb{N}$. Suppose s and t are patterns such that both $o(s)$ and $o(t)$ are defined. Suppose moreover that $1 < \ell^{h+1} o(s)$ (i.e., that $\ell^h o(s)$ is a limit and the last term in its Cantor normal form has exponent at least 2) and that $o(t)$ is of the form $e^{h+1}\alpha$, with α equal to its right side, with $\omega^2 \leq \ell^h o(t) \leq \ell^h o(s)$. Under these conditions,

$$o(t \wedge \sigma(\sigma^h \wedge \pi s)) = e^h(\ell^h o(s) + \ell^h o(t)).$$

Thus, for patterns with ordinal above ω^ω, we have "initial" patterns of the form

$$\sigma^{n+1} \wedge \pi^{n+1}$$

and defined four constructions to generate more complicated patterns. When one considers the corresponding ordinals, what these constructions do is modify the right side in various ways. Let us illustrate this with an example:

Example 3.7.3. The pattern $\sigma\sigma\sigma\sigma \wedge \pi$ has ordinal

$$e^4 1 = \omega^{\omega^{\omega^\omega}}$$

while the pattern $\sigma\sigma\sigma \wedge \pi$ has ordinal

$$e^3 1 = \omega^{\omega^\omega}.$$

Note that $e^3 1$ is equal to its right side. Using various linking patterns of the form $\sigma(\sigma^h \wedge \pi)$, these two patterns can be combined.

1. When $h = 0$: $o\big(\sigma\sigma\sigma \wedge \pi \wedge \sigma\pi(\sigma\sigma\sigma\sigma \wedge \pi)\big) = e^4 1 + e^3 1 = \omega^{\omega^{\omega^\omega}} + \omega^{\omega^\omega}$.

2. When $h = 1$: $o\Big(\sigma\sigma\sigma \wedge \pi \wedge \sigma\big(\sigma \wedge \pi(\sigma\sigma\sigma\sigma \wedge \pi)\big)\Big) = e(e^3 1 + e^2 1) = \omega^{\omega^{\omega^\omega} + \omega^\omega}$.

3. When $h = 2$, this is no longer possible, as $\ell^2 o(\sigma\sigma\sigma \wedge \pi) = \ell^2 e^3 1 = \omega < \omega^2$. Instead, we may consider the pattern

$$\sigma(\sigma\sigma \wedge \pi(\sigma\sigma\sigma\sigma \wedge \pi)$$

to which the ordinal

$$e^2(e^2 1 + \omega) = \omega^{\omega^{\omega^\omega + \omega}}$$

has been assigned.

4. For $h = 3$, the pattern

$$\sigma(\sigma\sigma\sigma \wedge \pi(\sigma\sigma\sigma\sigma \wedge \pi)$$

has the ordinal

$$e^3(e^1 1 + \omega) = \omega^{\omega^{\omega^{\omega+\omega}}}$$

assigned.

5. An ordinal that has not been considered so far is

$$e^2(e^2 1 + e^2 1) = \omega^{\omega^{\omega^\omega + \omega^\omega}};$$

this corresponds to the result of linking two copies of $\sigma\sigma\sigma\sigma \wedge \pi$ via the pattern $\sigma(\sigma\sigma \wedge \pi)$, i.e., to the pattern

$$\sigma\sigma\sigma\sigma \wedge \pi \wedge \sigma\big(\sigma\sigma \wedge \pi(\sigma\sigma\sigma\sigma \wedge \pi)\big).$$

Definition 3.7.4. Let $\alpha < \varepsilon_0$. We denote by c^α the unique reflection pattern s such that $o(s)$ is defined and equal to α.

An easy induction shows that c^α is well defined for all $\alpha < \varepsilon_0$.

Lemma 3.7.5. *Let $\alpha < \varepsilon_0$ and let s be a reflection pattern. Then, every $c^\alpha s$-reflecting ordinal is either πs-reflecting or $\sigma\pi s$-reflecting.*

Proof. This follows by simply observing the definition of c^α and contracting with Lemma 3.3.1 and Lemma 3.4.8. □

Lemma 3.7.6. *Suppose that $\beta < \alpha < \varepsilon_0$. Then, one of the following holds:*

1. *every c^α-reflecting ordinal is πc^β-reflecting; or*
2. *every c^α-reflecting ordinal is $\sigma\pi c^\beta$-reflecting.*

Proof. This is proved by induction on α. Let ξ be a c^α-reflecting ordinal. It suffices to show that ξ is $\sigma\pi c^\beta$-reflecting or $\pi\sigma\pi c^\beta$-reflecting, for in the latter case either it is σ-reflecting, thus $\sigma\pi\sigma\pi c^\beta$-reflecting, and so $\sigma\pi c^\beta$-reflecting by Lemma 3.4.8; or it is not, and thus it is πc^β-reflecting by Theorem 3.3.9.

If α is a successor ordinal $\gamma + 1$, then $c^\alpha = \pi c^\gamma$, so ξ is πc^γ-reflecting. Let $\beta < \gamma$ be arbitrary. By induction hypothesis, either every c^γ-reflecting ordinal is πc^β-reflecting, whence ξ is $\pi\pi c^\beta$-reflecting and thus πc^β-reflecting; or every c^γ-reflecting ordinal is $\sigma\pi c^\beta$-reflecting, so that ξ is $\pi\sigma\pi c^\beta$-reflecting.

The more interesting case is when α is a limit ordinal. If α is of the form $e^{n+1}1$, then $c^\alpha = \sigma^{n+1} \wedge \pi$. By Theorem 3.3.6,

$$\sigma^{n+1} = \sigma(\sigma^n \wedge \pi^k)$$

for every $k \in \mathbb{N}$. By definition,

$$o(\sigma^n \wedge \pi^{k+1}) = e^n(k+1)$$

for all k and clearly

$$\lim_{k\in\mathbb{N}} e^n(k+1) = e^{n+1}1.$$

So that if $\beta < \alpha$, then there is $k \in \mathbb{N}$ such that—by induction hypothesis—one of the following holds:

1. $\sigma^n \wedge \pi^k \equiv \sigma^n \wedge \pi^k \wedge \pi c^\beta$; or
2. $\sigma^n \wedge \pi^k \equiv \sigma^n \wedge \pi^k \wedge \sigma\pi c^\beta$.

Both cases easily imply that

$$\sigma^n \wedge \pi^k \equiv \sigma^n \wedge \pi^k \wedge \sigma\pi c^\beta$$

and hence

$$\sigma^{n+1} \equiv \sigma\Big(\sigma^n \wedge \pi^k \wedge \sigma\pi c^\beta\Big)$$

so, in particular, σ^{n+1} is $\sigma\pi c^\beta$-reflecting, as desired. The proof in this case illustrates how the induction hypothesis is used. The remaining cases are similar, but the computations are more involved.

By the previous observations, we may reduce to the case that α is a limit ordinal not of the form $e^{n+1}1$. We shall proceed by cases according to the way c^α was constructed. There are four cases:

1. $c^\alpha = \sigma^{h+1} \wedge \pi^{k+1}$, where $k \neq 0$, so that $e^{n+1}(k+1)$ and α is equal to its right side.

2. $c^\alpha = \sigma^{h+1} \wedge \pi^{k+1} c^\gamma$ for some $\gamma < \alpha$ with $0 < \ell^{h+2}\gamma$, so that the right side of α is equal to $e^{h+1}(k+1)$.

3. $c^\alpha = \sigma(\sigma^{h+1} \wedge \pi c^\gamma)$ for some $\gamma < \alpha$ with $0 < \ell^{h+2}\gamma$, so that the right side of α is equal to $e^{h+1}\omega$.

4. $c^\alpha = c^\beta \wedge \sigma(\sigma^h \wedge \pi c^\gamma)$ for some $\beta, \gamma < \alpha$ such that $1 < \ell^{h+1}\gamma$, $\beta = e^h\beta'$ with $\omega^2 \leq \ell^h\beta \leq \gamma$ and β equal to its right side. If so, the right side of α is then β.

Case (1) is treated very similarly to case (2) and so we omit the proof. We deal with case (2) now. To ease notation we will assume that γ is equal to its right side and replace it by $e^{h+1}\gamma$, although the general proof is similar. Thus, we assume $\alpha = e^{h+1}(\gamma + k + 1)$. First note that

$$\begin{aligned} e^{h+1}(\gamma + k + 1) &= e^h e(\gamma + k + 1) \\ &= e^h \omega^{\gamma+k+1} \\ &= e^h \omega^{\gamma+k} \cdot \omega \\ &= \lim_{m\in\mathbb{N}} e^h\Big(\omega^{\gamma+k} \cdot m\Big) \\ &= \lim_{m\in\mathbb{N}} e^h\Big(e(\gamma+k) \cdot m\Big). \end{aligned}$$

Let

$$\alpha_m = e^h\big(e(\gamma+k) \cdot (m+1)\big).$$

By definition,

$$c^{\alpha_0} = c^{e^{h+1}(\gamma+k)} = \sigma^{h+1} \wedge \pi^k c^{e^{h+1}\gamma}.$$

Recall that the "linking" pattern for addition within e^h is $\sigma(\sigma^h \wedge \pi)$, so that

$$c^{\alpha_{m+1}} = \sigma^{h+1} \wedge \pi^k c^{e^{h+1}\gamma} \wedge \sigma\Big(\sigma^h \wedge \pi c^{\alpha_m}\Big).$$

We will prove by induction on m that

$$c^\alpha \equiv c^\alpha \wedge \pi c^{\alpha_m}.$$

Clearly,

$$\begin{aligned} c^\alpha \equiv \sigma^{h+1} \wedge \pi^{k+1} c^{e^{h+1}\gamma} &\equiv \sigma^{h+1} \wedge \pi\Big(\sigma^{h+1} \wedge \pi^k c^{e^{h+1}\gamma}\Big) \\ &\equiv \sigma^{h+1} \wedge \pi^{k+1} \wedge \pi c^{\alpha_0} \end{aligned}$$

Inductively,

$$\begin{aligned}
c^{\alpha} &\equiv \sigma^{h+1} \wedge \pi^{k+1} \wedge \pi c^{\alpha_m} \\
&\equiv \sigma\Big(\sigma^h \wedge \pi c^{\alpha_m}\Big) \wedge \pi^{k+1} c^{e^{h+1}\gamma} \\
&\equiv \sigma^{h+1} \wedge \pi\Big(\pi^k c^{e^{h+1}\gamma} \wedge \sigma\Big(\sigma^h \wedge \pi c^{\alpha_m}\Big)\Big) \\
&\equiv \sigma^{h+1} \wedge \pi^{h+1} c^{e^{h+1}\gamma} \wedge \pi\Big(\pi^k c^{e^{h+1}\gamma} \wedge \sigma\Big(\sigma^h \wedge \pi c^{\alpha_m}\Big)\Big) \\
&\equiv \sigma^{h+1} \wedge \pi^{h+1} c^{e^{h+1}\gamma} \wedge \pi\Big(\sigma^{h+1} \wedge \pi^k c^{e^{h+1}\gamma} \wedge \sigma\Big(\sigma^h \wedge \pi c^{\alpha_m}\Big)\Big) \\
&\equiv \sigma^{h+1} \wedge \pi^{h+1} c^{e^{h+1}\gamma} \wedge \pi c^{\alpha_{m+1}}.
\end{aligned}$$

Because

$$\lim_{m\in\mathbb{N}} \alpha_m = \alpha,$$

an argument as before with the induction hypothesis finishes the proof in this case.

We may now proceed to case (3). Again assuming for notational simplicity that γ is equal to its right side and replacing γ by $e^{h+1}\gamma$, we have

$$c^{\alpha} = \sigma(\sigma^{h+1} \wedge \pi c^{e^{h+1}\gamma}).$$

Notice that

$$\alpha = e^{h+1}(\gamma + \omega) = \lim_{m\in\mathbb{N}} e^{h+1}(\gamma + m).$$

We had computed before that

$$c^{e^{h+1}(\gamma+m+1)} = \sigma^{h+1} \wedge \pi^{m+1} c^{e^{h+1}\gamma},$$

so, by an argument as before, it would suffice to verify that

$$c^{\alpha} \equiv c^{\alpha} \wedge \sigma\big(\sigma^{h+1} \wedge \pi^{m+1} c^{e^{h+1}\gamma}\big)$$

for all m. But this is immediate from Theorem 3.3.6.

Finally, for case (4), we assume $c^{\alpha} = c^{\beta} \wedge \sigma(\sigma^h \wedge \pi c^{\gamma})$ for some $\beta, \gamma < \alpha$ such that $1 < \ell^{h+1}\gamma$, $\beta = e^h\beta'$ with $\omega^2 \leq \ell^h\beta \leq \gamma$ and β equal to its right side. If so, the right side of α is then β. Again for notational simplicity we assume that γ is equal to its right side, and that α is of the form

$$\alpha = e^h(\gamma + \beta'),$$

with $\omega^2 \leq \beta'$. We need to distinguish two subcases:

1. β' is of the form $e^{l+1}1$ for some $l \in \mathbb{N}$ (necessarily nonzero, by the assumption $\omega^2 \leq \beta'$);

2. β' is of the form $e^{l+1}(k+1)$, for some $l \in \mathbb{N}$ and some $k \neq 0$.

By our assumption that β is equal to its right side, these are the only possibilities. In the second subcase, we have

$$\begin{aligned}\alpha = e^h(\gamma + \beta') &= e^h(\gamma + e^{l+1}(k+1)) \\ &= e^h(\gamma + e^l\omega^{k+1}) \\ &= \lim_{m\in\mathbb{N}} e^h(\gamma + e^l(\omega^k \cdot m)).\end{aligned}$$

Put

$$\alpha_m = e^h(\gamma + e^l(ek \cdot (m+1))).$$

Thus,

$$\begin{aligned}c^{\alpha_0} &= c^{e^h e^l(ek)} \wedge \sigma(\sigma^h \wedge \pi c^{e^h\gamma}) \\ &= c^{e^{h+l+1}k} \wedge \sigma(\sigma^h \wedge \pi c^{e^h\gamma}) \\ &= \sigma^{h+l+1} \wedge \pi^k \wedge \sigma(\sigma^h \wedge \pi c^{e^h\gamma})\end{aligned}$$

and, for each m,

$$\begin{aligned}c^{\alpha_{m+1}} &= c^{e^h e^l(ek)} \wedge \sigma(\sigma^{h+l} \wedge \pi c^{\alpha_m}) \\ &= c^{e^{h+l+1}k} \wedge \sigma(\sigma^{h+l} \wedge \pi c^{\alpha_m}) \\ &= \sigma^{h+l+1} \wedge \pi^k \wedge \sigma(\sigma^{h+l} \wedge \pi c^{\alpha_m}).\end{aligned}$$

Now, we have

$$\begin{aligned}c^\alpha &\equiv c^\beta \wedge \sigma(\sigma^h \wedge \pi c^{e^h\gamma}) \\ &\equiv c^{e^{h+l+1}(k+1)} \wedge \sigma(\sigma^h \wedge \pi c^{e^h\gamma}) \\ &\equiv \sigma^{h+l+1} \wedge \pi^{k+1} \wedge \sigma(\sigma^h \wedge \pi c^{e^h\gamma}) \\ &\equiv \sigma^{h+l+1} \wedge \pi\Big(\sigma^{h+l+1} \wedge \pi^k \wedge \sigma(\sigma^h \wedge \pi c^{e^h\gamma})\Big) \wedge \sigma(\sigma^h \wedge \pi c^{e^h\gamma}) \\ &\equiv \sigma^{h+l+1} \wedge \pi c^{\alpha_0} \wedge \sigma(\sigma^h \wedge \pi c^{e^h\gamma}).\end{aligned}$$

Inductively,

$$\begin{aligned}
c^{\alpha} &\equiv \sigma^{h+l+1} \wedge \pi^{k+1} \wedge \pi c^{\alpha_m} \\
&\equiv \sigma\Big(\sigma^{h+l} \wedge \pi c^{\alpha_m}\Big) \wedge \pi^{k+1} \wedge \pi c^{\alpha_m} \\
&\equiv \sigma^{h+l+1} \wedge \pi\Big(\pi^{k} \wedge \sigma\Big(\sigma^{h+l} \wedge \pi c^{\alpha_m}\Big)\Big) \wedge \pi c^{\alpha_m} \\
&\equiv \sigma^{h+l+1} \wedge \pi\Big(\sigma^{h+l+1} \wedge \pi^{k} \wedge \sigma\Big(\sigma^{h+l} \wedge \pi c^{\alpha_m}\Big)\Big) \wedge \pi c^{\alpha_m} \\
&\equiv \sigma^{h+l+1} \wedge \pi^{k+1} \wedge \pi\Big(\sigma^{h+l+1} \wedge \pi^{k} \wedge \sigma\Big(\sigma^{h+l} \wedge \pi c^{\alpha_m}\Big)\Big) \\
&\equiv \sigma^{h+l+1} \wedge \pi^{k+1} \wedge \pi c^{c_{m+1}},
\end{aligned}$$

so an argument as before completes the proof. Finally, in the first subcase, where β' is of the form $e^{l+1}1$, we have

$$\begin{aligned}
\alpha = e^h(\gamma + \beta') &= e^h(\gamma + e^{l+1}1) \\
&= e^h(\gamma + e^l\omega) \\
&= \lim_{m\in\mathbb{N}} e^h(\gamma + e^l m).
\end{aligned}$$

Put

$$\alpha_m = e^h(\gamma + e^l m).$$

Thus,

$$\begin{aligned}
c^{\alpha_m} &= c^{e^h e^l m} \wedge \sigma(\sigma^h \wedge \pi c^{e^h\gamma}) \\
&= c^{e^{h+l} m} \wedge \sigma(\sigma^h \wedge \pi c^{e^h\gamma}) \\
&= \sigma^{h+l} \wedge \pi^m \wedge \sigma(\sigma^h \wedge \pi c^{e^h\gamma}).
\end{aligned}$$

Now, we have

$$\begin{aligned}
c^{\alpha} &\equiv c^{\beta} \wedge \sigma(\sigma^h \wedge \pi c^{e^h\gamma}) \\
&\equiv c^{e^{h+l+1}1} \wedge \sigma(\sigma^h \wedge \pi c^{e^h\gamma}) \\
&\equiv \sigma^{h+l+1} \wedge \pi \wedge \sigma(\sigma^h \wedge \pi c^{e^h\gamma}) \\
&\equiv \sigma^{h+l+1} \wedge \pi\sigma(\sigma^h \wedge \pi c^{e^h\gamma}) \wedge \sigma(\sigma^h \wedge \pi c^{e^h\gamma}) \\
&\equiv \sigma\Big(\sigma^{h+l} \wedge \pi\sigma(\sigma^h \wedge \pi c^{e^h\gamma})\Big) \wedge \pi \wedge \sigma(\sigma^h \wedge \pi c^{e^h\gamma}).
\end{aligned}$$

Suppose an ordinal ξ is

$$\pi\sigma(\sigma^h \wedge \pi c^{e^h\gamma})\text{-reflecting}.$$

By Theorem 3.6.8, one of the following holds:

1. ξ is $\sigma(\sigma^h \wedge \pi c^{e^h\gamma})$-reflecting; or
2. ξ is $\pi c^{e^h\gamma}$-reflecting.

Thus, if ξ is

$$\sigma^{h+l} \wedge \pi\sigma(\sigma^h \wedge \pi c^{e^h\gamma})\text{-reflecting},$$

then, one of the following holds:

1. ξ is $\sigma^{h+l} \wedge \pi \wedge \sigma(\sigma^h \wedge \pi c^{e^h\gamma})$-reflecting; or
2. ξ is $(\sigma^{h+l} \wedge \pi c^{e^h\gamma})$-reflecting.

However, note that

$$\begin{aligned}\sigma^{h+l} \wedge \pi c^{e^h\gamma} &\equiv \sigma^{h+l} \wedge \sigma^{l+h} \wedge \pi c^{e^h\gamma}\\ &\equiv \sigma^{h+l} \wedge \sigma^l\Big(\sigma^h \wedge \pi c^{e^h\gamma}\Big) \wedge \pi c^{e^h\gamma}\\ &\equiv \sigma^{h+l} \wedge \sigma\Big(\sigma^h \wedge \pi c^{e^h\gamma}\Big) \wedge \pi c^{e^h\gamma}.\end{aligned}$$

(The last equivalence is possible because l is nonzero.) We have shown that every ordinal which is

$$\sigma^{h+l} \wedge \pi\sigma(\sigma^h \wedge \pi c^{e^h\gamma})\text{-reflecting}$$

is also

$$\sigma^{h+l} \wedge \pi \wedge \sigma\Big(\sigma^h \wedge \pi c^{e^h\gamma}\Big)\text{-reflecting}.$$

Therefore, an application of Theorem 3.3.6 shows:

$$\begin{aligned}c^\alpha &\equiv \sigma\Big(\sigma^{h+l} \wedge \pi\sigma(\sigma^h \wedge \pi c^{e^h\gamma})\Big) \wedge \pi \wedge \sigma(\sigma^h \wedge \pi c^{e^h\gamma})\\ &\equiv \sigma\Big(\sigma^{h+l} \wedge \pi \wedge \sigma\Big(\sigma^h \wedge \pi c^{e^h\gamma}\Big)\Big) \wedge \pi \wedge \sigma(\sigma^h \wedge \pi c^{e^h\gamma})\\ &\equiv \sigma\Big(\sigma^{h+l} \wedge \pi^i \wedge \sigma\Big(\sigma^h \wedge \pi c^{e^h\gamma}\Big)\Big) \wedge \pi \wedge \sigma(\sigma^h \wedge \pi c^{e^h\gamma})\end{aligned}$$

for every $i \in \mathbb{N}$. But

$$\sigma\Big(\sigma^{h+l} \wedge \pi^i \wedge \sigma\Big(\sigma^h \wedge \pi c^{e^h\gamma}\Big)\Big) = \sigma c^{\alpha_m},$$

so an appeal to the induction hypothesis—as before—finishes the proof of the lemma. □

Lemma 3.7.6 immediately implies that if $\beta < \alpha < \varepsilon_0$, then c^β is strictly smaller than c^α in the reflection order. We have shown:

Theorem 3.7.7. *The reflection order has length at least ε_0.*

We also mention the following consequence of the proof of Theorem 3.7.7:

Theorem 3.7.8. *The reflection pattern σ^n has rank at least $e^n 1$ in the reflection order.*

3.8 Concluding remarks

We have shown that the reflection order is a prewellordering of length at least ε_0 and, in fact, that the reflection pattern σ^n has rank at least $e^n 1$ in the reflection order.

We have not given any upper bounds on the length of the reflection order. Let us say something about this:

Proposition 3.8.1. *Let s be a reflection pattern and suppose α is a countable, locally countable ordinal such that*

$$L_\alpha \prec_1 L_{\alpha^+ + 1}.$$

Then, α is s-reflecting.

Proof. Since

$$L_\alpha \prec_1 L_{\alpha^+},$$

it follows that α is π^1_1-reflecting. By Gostanian's theorem [Gos79] mentioned in the introduction, α is σ^1_1-reflecting. Inductively, suppose it is

$$(\sigma^1_1)^n\text{-reflecting}$$

and let ψ be a Σ^1_1 sentence such that

$$L_\alpha \models \psi.$$

Choose a Π_1 sentence ψ^* such that for all admissible β containing all relevent parameters,

$$L_\beta \models \psi$$

if, and only if,

$$L_{\beta^+} \models \psi^*(L_\beta),$$

so that, in particular,

$$L_{\alpha^+} \models \psi^*(L_\alpha).$$

Then, from the point of view of $L_{\alpha^+ +1}$, there are admissible sets L_α and L_{α^+} such that

1. α is $(\sigma^1_1)^n$-reflecting, i.e., for every Π_1 sentence ϕ^*, if $L_{\alpha^+} \models \phi^*(L_\alpha)$, then there is a $(\sigma^1_1)^{n-1}$-reflecting $\beta < \alpha$ such that $L_{\beta^+} \models \phi^*(L_\beta)$. (The quantification over ϕ^* is bounded.)

2. $L_{\alpha^+} \models \psi^*(L_\alpha)$.

Thus, by stability, there are admissible sets L_β and L_{β^+} in L_α such that β is $(\sigma^1_1)^n$-reflecting and $L_\beta \models \psi^*(L_\beta)$. Hence, α is $(\sigma^1_1)^{n+1}$-reflecting. Therefore, a simple induction on the construction of a reflection pattern s, using Theorem 3.3.9 shows that α is s-reflecting. □

The length of the reflection order is thus at most the least ordinal α such that

$$L_\alpha \prec_1 L_{\alpha+1}.$$

However, we believe that it is shorter and, in fact, recursive. The reason is that each inequality between reflection patterns is surely provable in any theory that proves the existence of the corresponding ordinals. Together with Proposition 3.8.1, this suggests that the length of the reflection order is smaller than the proof-theoretic ordinal of (lightface) Π^1_2-CA_0 and thus smaller than the ordinal described in Rathjen [Rat05a].

Part II

Games of Transfinite Length

4

σ-Projective Determinacy

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4.1 Introduction

We consider infinite (Gale-Stewart) games on the natural numbers. These games can be assumed to have the following normal form: given some fixed $A \subset \mathbb{R}$, players I and II alternate turns playing digits $x(n) \in \{0, 1, \ldots, 9\}$. After infinitely many rounds have been played, Player I wins if, and only if,

$$x := \sum_{n=0}^{\infty} \frac{x(n)}{10^{n+1}}$$

belongs to A; otherwise Player II wins.

Definition 4.1.1. A set of reals is *σ-projective* if it belongs to the smallest σ-algebra on $\mathbb{R}$ containing the open sets and closed under continuous images.

The question of *σ-projective determinacy*—whether σ-projective games are determined—cannot be decided within ZFC, Zermelo-Fraenkel Set Theory with the Axiom of Choice, but it is decidable in natural extensions thereof. It can be shown, using the proof of a theorem of Martin and Steel [MS89], that if there are infinitely many Woodin cardinals, then σ-projective games are determined.[1] By a theorem of Woodin (cf. [MSW16]), if projective games are determined, then there are models of ZFC with any finite number of Woodin cardinals (a set $A \subset \mathbb{R}$ is projective if it can be obtained from open sets in finitely many stages by applying complements and continuous images). Thus, the hypothesis that corresponds to σ-projective determinacy lies somewhere strictly between infinitely many Woodin cardinals, and all finite amounts of Woodin cardinals.[2]

This hypothesis is given in terms of certain *smallness classes* and can be described as follows. A model M of ZFC is an *initial segment* of $N = L_\gamma[A]$ if $M = L_\beta[A]$ for some $\beta \leq \gamma$. For the purposes of this introduction, "active" can mean "with a measurable cardinal."

Definition 4.1.2. Let M be a countable, ω_1-iterable extender model of some fragment of ZFC.

1. M is of class S_0 above δ if it has an initial segment which is active above δ;

2. M is of class $S_{\alpha+1}$ above δ if it has an initial segment N of class S_α above some $\delta_0 > \delta$ which is Woodin in N;

3. M is of class S_λ above δ if $\lambda < \omega_1^M$ and it has an active initial segment in all classes S_α above δ, for all $\alpha < \lambda$;

4. M is of class S_α if it is of class S_α above 0.

(See Figure 4.1 below for an example.)

If n is finite, then an extender model of class S_n is usually called "not n-small." If α is infinite, then a model of class S_α will have only a finite amount of Woodin cardinals, but it will also have many ordinals that are Woodin cardinals in some of its initial segments. The existence of models

[1] This was first pointed out to us by G. Sargsyan.

[2] As we will see, it is much closer to the latter.

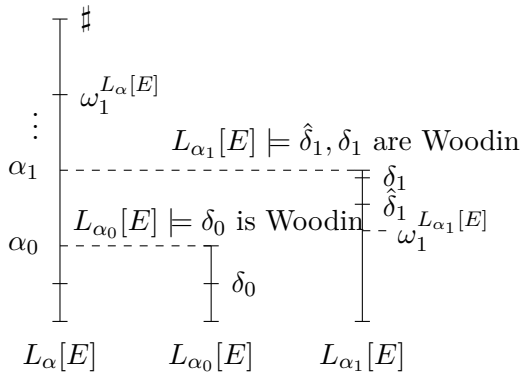


Figure 4.1: A model $L_\alpha[E]$ of class S_ω

of any class S_α does not imply the existence of models of set theory with infinitely many Woodin cardinals.

The definition of the classes S_λ, for limit λ limit, has been chosen for convenience; we will comment on this below. The main theorem is:

Theorem 4.1.3. *The following are equivalent:*

1. *All σ-projective games are determined.*

2. *The classes S_α are all non-empty, for $\alpha < \omega_1$.*

We begin in Section 4.2 by fixing conventions and proving basic facts about the σ-projective hierarchy and minimal models in the classes S_α, $\alpha < \omega_1$. Roughly, we verify that the σ-projective hierarchy behaves much like the projective hierarchy and that inner models of class S_α behave much like the familiar model $M_n^\sharp$. In Section 4.3, σ-projective determinacy is proved from the large cardinal assumption. The proof is short, granted the contents of Section 4.2.

The idea for the converse is to adapt the proof in Müller-Schindler-Woodin [MSW16] for the projective levels to our context. The proof of the existence of $M_n^\sharp$ in [MSW16] involves different arguments for odd and even n, because of periodicity in the projective hierarchy. Since we do not attempt to prove either direction of Theorem 12.2.6 from locally optimal hypotheses, we do not need to do this. We have chosen to adapt the proof for even n here, which is essentially the successor case of a core model induction. Our definition of the clases S_λ for limit λ ensures that the limit case of the induction is immediate (cf. Lemma 4.2.16); nonetheless, this definition

adds consistency strength at later stages in a subtle way. This is because there are many equivalent formulations of S_λ giving rise to non-equivalent formulations of $S_{\lambda+1}$ (this is exploited in the proof of Lemma 4.3.4).

A recurrent obstacle we shall face is that various arguments in the context of models with finite amounts of Woodin cardinals involve $L[E]$-constructions à la Mitchell-Steel [MS94]. These arguments do not work here, because many of the extenders witnessing that a given model is of some class S_α (for $\omega \leq \alpha$) are not total and so cannot be used for this type of backgrounding. These obstacles can be overcome by appealing to the theory of operator premice.

The proof of Theorem 12.2.6 requires verifying that some notions of Steel [Ste95] carry over to the σ-projective hierarchy; this is done in Section 4.4. The remainder of the proof takes place in the last two sections, inductively. The main goal in Section 4.5 is to show that the ω_2 of a suitable inner model is inaccessible in its HOD. Once this is done, the proof of [MSW16, Theorem 3.20] carries over smoothly. We include a sketch in Section 4.6 in which we highlight the (albeit superficial) differences, but refer the reader to [MSW16] for some of the arguments.

Large-cardinal assumptions of this form have previously been considered by Rudominer [Rud97], who showed that minimal extender models satisfying them contain precisely the reals ordinal definable in various initial segments of $L(\mathbb{R})$.

4.2 Preliminaries

We write $\mathbb{R}$ for the space of all infinite sequences of natural numbers, with the product topology. Rigorously speaking, this space is not $\mathbb{R}$, but rather (homeomorphic to) $\mathbb{R} \setminus \mathbb{Q}$. Since rational numbers play no role in matters of determinacy, we abuse notation. With this convention, $\mathbb{R}$ is homeomorphic to $\mathbb{R}^n$ and to $\mathbb{R}^{\mathbb{N}}$, so we sometimes identify reals with tuples of reals. A *pointset* is a collection of points in $\mathbb{R}$, or any of the spaces we identify with $\mathbb{R}$. A *pointclass* is a collection of pointsets, in practice specified in terms of definability (e.g., $\mathbf{\Delta}^1_1$, the pointclass of all Borel sets).

Our notation is—to the best of our knowledge—standard. If f is a function and $A \subset \mathrm{dom}(f)$, we write $f[A]$ for

$$\{f(a) : a \in A\}.$$

If f is a function and A is a set, we say that A is closed under f if $f(a) \in A$ for all $a \in A \cap \mathrm{dom}(f)$. An ordinal is even if it is of the form $\omega \cdot \alpha + 2k$,

where $k \in \mathbb{N}$.

An ordinal α is *recursive* (in $x \in \mathbb{R}$) if there is a recursive (in x) wellordering of a subset of $\mathbb{N}$ having order-type α. If $x \in \mathbb{R}$, we write ω_1^x for the least ordinal which is not recursive in x. Thus, ω_1^0 is the well-known Church-Kleene ordinal. If $x \in \mathbb{R}$, the *Turing cone* generated by x is given by

$$\{y \in \mathbb{R} : \ x \text{ is recursive in } y\}.$$

The Turing filter is the filter generated by Turing cones. We sometimes say "almost all $x \in \mathbb{R}$" to mean "for all $x \in \mathbb{R}$ in some set in the Turing filter." If x and y are real numbers, we write $x \oplus y$ for the recursive join of x and y. If

$$(M, \vec{P})$$

is a structure consisting of a set with added predicates, we might abuse notation by identifying it with M, or perhaps sometimes write $|M|$ to refer to its universe.

If κ and λ are ordinals, the *collapse algebra* for κ and λ, $\mathrm{Coll}(\kappa, \lambda)$ is the partial order of all partial injective functions of size $< \kappa$ from κ to λ, ordered by reverse inclusion. If M is a transitive model of set theory containing κ and λ and $g \subset \mathrm{Coll}(\kappa, \lambda)$ is M-generic, then $M[g] \models |\kappa| = |\lambda|$.

4.2.1 σ-projective sets

If $A \subset \mathbb{R}$ is a continuous image of some $B \subset \mathbb{R}$, then there is a real x_f coding the values of some continuous function f such that $f[B] = A$ on a countable dense subset of $\mathbb{R}$. Conversely, projections from $\mathbb{R}^{n+1}$ to $\mathbb{R}^n$ are continuous. Thus, σ-projective sets are the smallest σ-algebra on $\mathbb{R}$ containing the open sets and closed under projections. One can also check that the σ-projective sets are exactly the sets of reals in $L_{\omega_1}(\mathbb{R})$ (cf. [AMS] for a proof); thus, σ-projective determinacy is the statement

$$L_{\omega_1}(\mathbb{R}) \models \mathsf{AD}$$

(which is not a schema). We will make no use of this fact, though.

Definition 4.2.1. The lightface σ-projective (or σ-analytical) hierarchy is given as follows.

1. A set is $\Sigma^1_0(x)$ if it is a union of basic open sets which is recursive in x.

2. A set is $\Delta^1_\alpha(x)$ if it is $\Sigma^1_\alpha(x)$ and $\Pi^1_\alpha(x)$.

3. A set is $\Pi^1_\alpha(x)$ if it is the complement of a $\Sigma^1_\alpha(x)$ set.

4. A set is $\Sigma^1_{\alpha+1}(x)$ if it is the projection of a $\Pi^1_\alpha(x)$ set.

5. A set A is $\Sigma^1_\lambda(x)$, for a countable limit λ, if it is a countable union, recursive in x, of sets $A_k \in \Sigma^1_{\lambda_k}(x)$, with $\lambda_k < \lambda$.

6. A set A is $\Pi^1_\lambda(x)$, for a countable limit λ, if it is a countable intersection, recursive in x, of sets $A_k \in \Pi^1_{\lambda_k}(x)$, with $\lambda_k < \lambda$.

Lemma 4.2.2 below follows by an easy application of the Axiom of Choice for countable sets:

Lemma 4.2.2. *A set of reals is σ-projective if, and only if, it belongs to $\Sigma^1_\lambda(x)$ for some $\lambda < \omega_1$ and some $x \in \mathbb{R}$.*

Let us mention the following elementary fact:

Lemma 4.2.3. *The class $\Sigma^1_\lambda(x)$ is closed under real existential quantification.*

Proof. This is clear if λ is a successor ordinal. Otherwise, inductively, consider a set $A = \bigcup_{k\in\mathbb{N}} A_k$ in $\Sigma^1_\lambda(x) \cap \mathbb{R}^2$. We have

$$\begin{aligned} a \in \exists^{\mathbb{R}} A \text{ if, and only if, } & \exists y \in \mathbb{R}\, (a,y) \in A \\ \text{if, and only if, } & \exists y \in \mathbb{R}\, \exists k \in \mathbb{N}\, (a,y) \in A_k \\ \text{if, and only if, } & \exists k \in \mathbb{N}\, \exists y \in \mathbb{R}\, (a,y) \in A_k \\ \text{if, and only if, } & a \in \bigcup_{k\in\mathbb{N}} \exists^{\mathbb{R}} A_k \end{aligned}$$

The result follows. □

The boldface versions $\mathbf{\Sigma}^1_\alpha$ and $\mathbf{\Pi}^1_\alpha$ of the classes above are defined in the natural way. The σ-projective hierarchy can be recast in terms of infinitary logic:

Definition 4.2.4. Let ϕ be a formula in the language of $\mathcal{L}_{\omega_1,\omega}$.

1. ϕ is $\Sigma^1_0(x)$ if it is a disjunction of quantifier-free formulae with no infinite connectives recursive in x.

2. ϕ is $\Delta^1_\alpha(x)$ if it is $\Sigma^1_\alpha(x)$ and $\Pi^1_\alpha(x)$.

3. ϕ is $\Pi^1_\alpha(x)$ if it is $\mathcal{L}_{\omega_1,\omega}$-equivalent to the negation of a $\Sigma^1_\alpha(x)$ formula.

4. ϕ is $\Sigma^1_{\alpha+1}(x)$ if it is of the form $\exists x\, \psi$, where ψ is $\Pi^1_\alpha(x)$.

5. ϕ is $\Sigma^1_\lambda(x)$, for a countable limit λ, if $\phi = \bigvee_{k\in\mathbb{N}} \phi_k$, where $\phi_k \in \Sigma^1_{\lambda_k}(x)$ with $\lambda_k < \lambda$, and the disjunction is recursive in x

6. ϕ is $\Pi^1_\lambda(x)$, for a countable limit λ, if $\phi = \bigwedge_{k\in\mathbb{N}} \phi_k$, where $\phi_k \in \Pi^1_{\lambda_k}(x)$ with $\lambda_k < \lambda$, and the conjunction is recursive in x

Assuming that we have fixed a reasonable arithmetization of formulae in $\mathcal{L}_{\omega_1,\omega}$, we will identify them with elements of $\mathbb{R}$.

The following lemma can easily be proved by induction.

Lemma 4.2.5. *A set of reals is $\Sigma^1_\alpha(x)$ if, and only if, it is definable over $\mathbb{R}$ by a formula in $\Sigma^1_\alpha(x)$ with x as a parameter.*

We refer to an $\mathcal{L}_{\omega_1,\omega}$-formula defining a $\Sigma^1_\alpha(x)$ set A from x as a *definition* of A. If A is $\Sigma^1_\alpha(x)$, then it has a definition that is recursive in x. The σ-projective hierarchy is of course not absolute, neither in its length nor in the contents of each level.

4.2.2 Uniformization

Let $A \subset \mathbb{R}\times\mathbb{R}$. A function f is said to *uniformize* A if

$$\mathrm{dom}(f) = \{x \in \mathbb{R} : \exists y\, (x,y) \in A\}$$

and $(x, f(x)) \in A$ for all $x \in \mathrm{dom}(f)$. A pointclass Γ is said to have the *uniformization property* if every set in Γ has a uniformizing function in Γ.

Let $A \subset \mathbb{R}$. A *norm* on A is a function

$$\rho : A \to \mathsf{Ord}.$$

Norms can be identified with prewellorderings of the reals, i.e., subsets of $\mathbb{R}\times\mathbb{R}$ that are reflexive, transitive, and wellfounded. A pointclass Γ has the *prewellordering property* if every set in Γ has a norm in Γ.

A *scale* on A is a sequence $\{\rho_i : i \in \mathbb{N}\}$ of norms satisfying the following semicontinuity property: suppose that $\{x_k : k \in \mathbb{N}\}$ is a sequence of reals such that

$$\lim_{k\in\mathbb{N}} x_k = x.$$

Suppose, additionally, that for each $i \in \mathbb{N}$, the sequence of ordinals

$$\{\rho_i(x_k) : k \in \mathbb{N}\}$$

is eventually constant, say with value λ_i. Then the following hold:

1. $x \in A$, and
2. $\rho_i(x) \leq \lambda_i$ for all $i \in \mathbb{N}$.

A pointclass Γ is said to have the *scale property* if every set in Γ has a scale which is uniformly in Γ. A pointclass is *adequate* if it contains all recursive subsets of $\mathbb{R}$ and is closed under finite unions and intersections and recursive substitutions.

Theorem 4.2.6 (Moschovakis [Mos71])**.** *Let Γ be an adequate pointclass closed under real universal quantification and with the scale property. Then Γ has the uniformization property.*

If Γ is a (lightface) pointclass, we write $\mathbf{\Gamma}$ for the associated boldface pointclass, and

$$\mathbf{\Delta}_\Gamma = \{A : \{A, \neg A\} \subset \mathbf{\Gamma}\}.$$

This is the Second Periodicity Theorem:

Theorem 4.2.7 (Moschovakis [Mos71])**.** *Suppose Γ is adequate.*

1. *If $A \in \Gamma$ admits a scale in Γ, then $\exists^{\mathbb{R}} A$ admits a scale in $\exists^{\mathbb{R}} \forall^{\mathbb{R}} \Gamma$.*
2. *Suppose that sets in $\mathbf{\Delta}_\Gamma$ are determined. If $A \in \Gamma$ admits a scale in Γ, then $\forall^{\mathbb{R}} A$ admits a scale in $\forall^{\mathbb{R}} \exists^{\mathbb{R}} \Gamma$.*

A consequence of the Second Periodicity Theorem is the following:

Lemma 4.2.8. *Let α be even, nonzero, and recursive in some $x \in \mathbb{R}$.*

1. *Suppose that sets in $\mathbf{\Delta}^1_\alpha$ are determined. Then the classes*

$$\Pi^1_{\alpha+1}(x), \Sigma^1_{\alpha+2}(x), \mathbf{\Pi}^1_{\alpha+1}, \text{ and } \mathbf{\Sigma}^1_{\alpha+2},$$

   have the scale property.

2. *Suppose that sets in $\mathbf{\Delta}^1_\beta$ are determined for all even $\beta < \alpha$. Then the classes $\Sigma^1_\alpha(x)$ and $\mathbf{\Sigma}^1_\alpha$ have the scale property.*

Proof. This is proved by induction. The case where α is finite is well known and due to Moschovakis; it follows from Theorem 4.2.7. In fact, Theorem 4.2.7 yields the result whenever α is a successor ordinal. Thus, it suffices to show that if α is a countable limit ordinal, then the pointclasses

$$\Sigma^1_\alpha(x) \text{ and } \mathbf{\Sigma}^1_\alpha$$

have the scale property. The result for the lightface class can be proved directly and the one for the boldface class follows by relativization. Let $A \in \Sigma^1_\alpha(x)$ and write A as a countable union

$$A = \bigcup_{k\in\mathbb{N}} A_k, \text{ where } A_k \in \Sigma^1_{\alpha_k}(x),\ \alpha_k < \alpha.$$

Since the union is recursive in x, we shall assume for simplicity that the sequence $\{\alpha_k : k \in \mathbb{N}\}$ is non-decreasing. In fact, we can replace multiple sets of the same complexity by their union and assume that the sequence is increasing.

Granted this abuse of notation (or replacing the finite union below by an infinite one), we have

$$A_k \setminus \bigcup_{n<k} A_n \in \Sigma^1_{\alpha_k}(x),$$

so we may as well assume that all the sets A_k are disjoint. By induction hypothesis, each set A_k admits a scale in $\Sigma^1_{\alpha_k+1}(x)$ with associated prewellorderings, say, $\{\preceq^k_i : i \in \mathbb{N}\}$. Define a scale on A by setting $x \preceq_i y$ if, and only if, $x \in A_k$ and $y \in A_{k'}$ and

1. $k < k'$; or
2. $k = k'$, and $x \preceq^k_i y$.

Each $\preceq_i$ is clearly a prewellordering of A. Moreover, each family $\{\preceq^k_i : i \in \mathbb{N}\}$ is uniformly in $\Sigma^1_{\alpha_k+1}(x)$, so it follows that $\{\preceq_i : i \in \mathbb{N}\}$ is uniformly in $\Sigma^1_\alpha(x)$, provided that for each (some) $i \in \mathbb{N}$, the sequence of norms

$$\{\preceq^k_i : k \in \mathbb{N}\}$$

is recursive in x. For this, it suffices to assume without loss of generality that each scale $\{\preceq^k_i : i \in \mathbb{N}\}$ had been obtained by recursively applying the proof of Theorem 4.2.7, together with this argument.

It remains to verify that $\{\preceq_i : i \in \mathbb{N}\}$ is a scale. Suppose that $\{x_k : k \in \mathbb{N}\}$ is a sequence of reals converging to x and that for each $i \in \mathbb{N}$, the sequence of ranks

$$\{|x_k|_{\preceq_i} : k \in \mathbb{N}\}$$

is eventually constant. This can only happen if all but finitely many x_k belong to the same A_n for some $n \in \mathbb{N}$. Then, for $i \in \mathbb{N}$ the sequences of ranks

$$\{|x_k|_{\preceq^n_i} : k \in \mathbb{N}\}$$

are all eventually constant. It follows that $x \in A_n$ and $x \preceq_i^n x_k$ for all i and sufficiently large k, and so $x \in A$ and $x \preceq_i x_k$ for all $i \in \mathbb{N}$ and sufficiently large k, as desired. □

We note an immediate application of Lemma 4.2.8 which we will use later on.

Lemma 4.2.9. *Let α be an even ordinal and suppose that sets $\mathbf{\Delta}^1_\alpha$ are determined. Let $B \in \Pi^1_{\alpha+1}(x_0)$ for some $x_0 \in \mathbb{R}$ such that $\alpha < \omega_1^x$ and suppose that*

$$(x, y) \in A \text{ if, and only if, } \exists z \in \Delta^1_{\alpha+1}(y, x_0)\, (x, y, z) \in B.$$

Then $A \in \Pi^1_{\alpha+1}(x_0)$.

Proof. A theorem of Kleene (cf. [Mos09, Theorem 4D.3]) states that if e.g., Γ is a lightface pointclass with the prewellordering property, closed under recursion and quantification over $\mathbb{N}$, and closed under either universal or existential quantification over $\mathbb{R}$, and if B is in Γ and

$$x \in A \text{ if, and only if, } \exists z \in \Delta_\Gamma(y)\, (x, y, z) \in B,$$

then A is in Γ. The result then follows from Lemma 4.2.8. □

4.2.3 Inner Model Theory

In this section we review some inner-model-theoretic results we will use and fix some conventions. We will then focus in the next section on models in the classes S_α, $\alpha < \omega_1$.

We assume familiarity with e.g., Steel [Ste10] or Mitchell-Steel [MS94], specifically, we assume familiarity with extender models (premice) and their fine structure, iteration strategies, and iteration trees. By a "premouse," we sometimes mean "x-premouse, for some $x \in \mathbb{R}$," or even "a-premouse, for some reasonably chosen set a." Throughout this paper, all extender models are assumed to be tame (i.e., they do not have extenders on the sequence overlapping local Woodin cardinals) and defined using Mitchell-Steel indexing. If M is a (sound) premouse, we write $\rho(M)$ or $\rho_\omega(M)$ for the projectum of M. If $\rho(M) = \delta$, we say that M *projects to* δ. If $\rho(M) < \delta$, we say that M projects *across* δ. We might also say it projects strictly across δ, to avoid confusion. In this paper, we will only need to consider iterations of premice

along *normal* iteration trees.[3] Thus, every iteration tree is henceforth assumed to be normal, and δ-iterability refers to the existence of an iteration strategy whose domain is the set of all normal trees of length $< \delta$.

We recall two versions of the comparison theorem:

Theorem 4.2.10 (cf. Theorem 3.11 of [Ste10])**.** *Let M and N be countable k-sound, ω_1+1-iterable extender models. Then, there exist iterates M' and N' of M and N such that one of the following holds:*

1. *M' is a non-dropping iterate of M and M' is an initial segment of N'; or*
2. *N' is a non-dropping iterate of N and N' is an initial segment of M'.*

Theorem 4.2.11 (Lemma 1.11 of Steel [Ste95])**.** *Let M and N be countable, ω_1+1-iterable extender models such that $\rho_{k_0+1}(M) \leq \delta$ and $\rho_{k_1+1}(N) \leq \delta$, where δ is a cutpoint of M and N. Suppose M is k_0-sound above δ and N is k_1-sound above δ. Then, one of the following holds:*

1. *$M \trianglelefteq N$; or*
2. *$N \trianglelefteq M$.*

If T is an iteration tree, we denote by $\delta(T)$ the supremum of the lengths of all extenders used in T and by $M(T)$ the unique passive premouse M such that $\mathsf{Ord} \cap M = \delta(T)$ and such that for all $\alpha < \mathrm{LTH}(T)$,

$$M|\mathrm{LTH}(E^T_\alpha) = M^T_\alpha|\mathrm{LTH}(E^T_\alpha).$$

Definition 4.2.12. Let T be an iteration tree on M of limit length. Q is a *Q-structure* for T if

1. it extends $M(T)$ (perhaps non-properly),
2. it has $\delta(T)$ as a cutpoint,
3. if $\delta(T) \in Q$, then $\delta(T)$ is Woodin in Q,
4. either Q projects strictly across $\delta(T)$ or there is a subset A of $\delta(T)$, definable over Q, such that the extender sequence of Q does not witness any ordinal to be $< \delta(T)$-A-strong.

We normally will require Q-structures to be sufficiently iterable above $\delta(T)$. In this case, there will be a unique one, denoted $Q(T)$.

[3] This means ω-normal.

Definition 4.2.13. Let T be an iteration tree on M of limit length and b be a cofinal wellfounded branch through T. We define $Q(b,T) = J_\alpha^{M_b^T}$, where α is largest such that either $\alpha = \delta(T)$ holds or $\delta(T) < \beta \leq \mathsf{Ord} \cap M_b^T$ and $\delta(T)$ is Woodin in $J_\alpha^{M_b^T}$.

Recall that if T is an iteration tree and b and c are distinct branches such that $Q(b,T)$ and $Q(c,T)$ both exist, then neither is an initial segment of the other; in fact, the extenders in $J_{\delta(T)}^{Q(b,T)}$ witness that $\delta(T)$ is Woodin with respect to functions in $M_b^T \cap M_c^T$. If $Q(b,T)$ exists, then it is a Q-structure for T. It follows that there is at most one cofinal wellfounded branch of T such that

$$Q(T) = Q(b,T).$$

For a premouse M, the (possibly partial) Q-structure iteration strategy Σ^t is defined as follows: given an iteration tree T of limit length, $\Sigma^t(T)$ is the unique cofinal wellfounded branch b such that

$$Q(T) = Q(b,T),$$

and $Q(b,T)$ is sufficiently iterable, if such a branch exists. By a theorem due "perhaps to Mitchell and Steel" (cf. Steel-Woodin [SW16, Theorem 2.3]), if M is a tame premouse projecting to ω, then it has at most one ω_1+1-iteration strategy, namely Σ^t.

By a theorem of Woodin, arbitrary reals can be absorbed by generic extensions of iterates of premice with Woodin cardinals. Neeman has shown that this can be achieved by only considering generic extensions by collapse algebras.

Theorem 4.2.14 (Neeman)**.** *Suppose that M is a countable, ω_1-iterable model of ZFC with a Woodin cardinal δ. Then, for every $x \in \mathbb{R}$, there is an iterate N of M and an N-generic $g \subset Coll(\omega, j(\delta))$ such that $x \in N[g]$, where j is the embedding.*

4.2.4 Models of class S_α

We will consider canonical extender models of each class S_α. Because will always want α to be countable therein, the models will be defined at most on a Turing cone. The first such premice we consider will be the analogs of $M_n^\sharp(x)$, for $x \in \mathbb{R}$. Since "$M_\alpha^\sharp(x)$" seems to suggest an increasing sequence of Woodin cardinals of length α, we have chosen completely different notation.

$N^\sharp_\alpha(x)$

Definition 4.2.15. $N^\sharp_\alpha(x)$ is the unique ω_1-iterable sound x-premouse of class S_α, all of whose proper initial segments are not of class S_α, if it exists.

(If there is more than one such premouse, then $N^\sharp_\alpha(x)$ is undefined.) In our context, we will usually not have anything stronger than ω_1-iterability; this will work, however, as we will be dealing with boldface-type assumptions, which will imply stronger forms of iterability in sufficiently closed countable inner models. We will make this precise and show that if for each $x \in \mathbb{R}$ there is some model satisfying Definition 4.2.15, then there is at most one for each $x \in \mathbb{R}$.

By minimality, if $N^\sharp_\alpha(x)$ exists, then every element of $N^\sharp_\alpha(x)$ is definable in $N^\sharp_\alpha(x)$ from x. Hence, $\rho(N^\sharp_\alpha(x)) = \omega$. If $N^\sharp_\alpha(x)$ exists, then

$$\alpha < \omega_1^{N^\sharp_\alpha(x)}.$$

(This follows from the definition of the class S_α.)

The existence of models of class S_λ carries no additional consistency strength over the existence of models of class S_α, for all $\alpha < \lambda$. This is done on purpose, as this ensures that the limit case of the inductive proof of Theorem 12.2.6 will be trivial:

Lemma 4.2.16. *Suppose that $\lambda < \omega_1^x$ is a countable limit ordinal and $N^\sharp_\alpha(y)$ exists for all $\alpha < \lambda$ and almost all $y \in \mathbb{R}$. Then $N^\sharp_\lambda(x)$ exists.*

Proof. Suppose that λ and x are as in the statement of the lemma. Letting

$$M = \bigcup_{\alpha<\lambda} N^\sharp_\alpha(x),$$

the choice of x implies that Ord^M is countable in $L(M)$. Since $y^\sharp$ exists for all $y \in \mathbb{R}$, there is an elemetary embedding

$$j : L(M) \to L(M),$$

so the active premouse N induced by j is as desired. (Iterability follows from the fact that the top extender of N is its only total extender; uniqueness follows from the uniqueness of $N^\sharp_\alpha(x)$, for $\alpha < \lambda$). □

We could denote by $N_\alpha(x)$ the set-like part of the class-sized iterated ultrapower of $N^\sharp_\alpha(x)$ by its topmost measure and its iterates, although we

will have no reason to. The operator $a \mapsto N^\sharp_\alpha(a)$ can be extended to act on uncountable a; in this case, we demand that (i) $L(a)$ be a model of ZFC (although this is not necessary, we will not deal with other types of a here), (ii) a contain a real x in which α is recursive, and (iii) countable elementary substructures of $N^\sharp_\alpha(a)$ be ω_1-iterable.

Our goal for the remainder of this section is to prove Lemma 4.2.21 below. To do so in what we think is the most cost-effective way, we briefly recall some notions from the theory of operator premice. Although the theory is very general, we will mostly be concerned with operators of the form

$$a \mapsto N^\sharp_\alpha(a).$$

We also refer the reader to Schindler-Steel [SS14, Chapter 1] or Wilson [Wil12, Chapter 2] for more details, and to Schlutzenberg-Trang [ST16] for a more general treatment.

Let N be a transitive set such that

$$L(N) \models \mathsf{ZFC}$$

and write $Lp(N)$ for the union of all sound premice over N, projecting to or across N, all of whose countable elementary substructures have ω_1-iterable transitive collapses. If α is recursive, or if there is some $x \in N$ such that $\alpha < \omega_1^x$, then we denote by

$$N^\sharp_\alpha(N)$$

the least initial segment of $Lp(N)$ which is an N-premouse of class S_α above N, if it exists. If $\alpha < \omega_1^x$, then being an (active/passive) x-premouse of class S_α is expressible by an rQ formula (with the exception of ω_1-iterability),[4] and so the function

$$N \mapsto N^\sharp_\alpha(N)$$

is a *mouse operator with parameter x on HC,* in the sense of [Wil12]. By [Wil12, Lemma 2.1.12], it is a *model operator with parameter x on HC* that condenses well, in the sense of [SS14].

[4]Recall that rQ formulae are of the form

$$\forall x\, \forall \theta \leq \mathrm{crit}(E_M)^+\, \exists y\, \exists \nu\, (x \subset y \wedge (\theta \leq \nu < \mathrm{crit}(E_M)^+) \wedge \phi(y, \nu, \bar{u}))$$

for active premice, and of the form

$$\forall x\, \exists y\, (x \subset y \wedge \phi(y, \bar{u}))$$

for passive premice. Here E_M denotes the top extender and $\phi(\cdot, \cdot, \bar{u})$ is $r\Sigma_1$ with parameters and does not have x or θ free.

In the context of σ-projective determinacy, we will only be concerned with countable extender models. However, we will frequently need to consider them inside other models of set theory in which they are not countable. The reason why this does not make the theory more complicated is the following lemma, which says that Σ^t can be extended to act on uncountable models on whose countable elementary substructures Σ^t is total, provided the complexity of Q-structures can be bounded:

Lemma 4.2.17 (cf. Lemma 1.6.12 of [SS14])**.** *Let M be tame and Z be a transitive and rudimentary-closed set containing HC. Let F be a mouse operator defined on all $M(T)$ such that $T \in Z$ is by Σ^t_M and suppose that whenever P is countable and elementarily embeddable into M, Σ^t_P is total and for all T by Σ^t_P, $\Sigma^t_P(T)$ is the unique b such that*

$$Q(b,T) \trianglelefteq F(M(T)).$$

Then Σ^t_M is total and for all $T \in Z$ by Σ^t_M, $\Sigma^t_M(T)$ is the unique b such that

$$Q(b,T) \trianglelefteq F(M(T)).$$

Suppose that $N^\sharp_\alpha(P)$ exists for some domain of sets P, all containing some given $x \in \mathbb{R}$. Let us temporarily write

$$F(P) = N^\sharp_\alpha(P),$$

if the right-hand side is defined. Under these assumptions and notation, if $x \leq_T z$ and $a = z$ is a real or a is a z-premouse, one can then define the certified construction $K^{c,F}(a)$ and the relativized core model $K^F(a)$ as in [SS14] (perhaps making use of Jensen-Steel [JS13]). Recall that $K^{c,F}(a)$ is the inductive construction of an a-(F-)premouse in which at each successor stage one cores down and either adds a certified (e.g., countably certified) extender to the extender sequence E or applies F (and not just the rudimentary-closure operator), and that $K^F(a)$ is the core model defined within $K^{c,F}(a)$ as K is defined within K^c, cf. e.g., Steel [Ste96], Jensen [Jec03b], Mitchell-Schindler [MS04] or Jensen-Steel [JS13] for the latter construction.

Structures to which we have given rise in this way are not premice in the usual sense, but F-premice (or relativized a-F-premice). By construction, they come equipped with an extender sequence E, but may have many more (partial) extenders, arising from applications of F. Fine structural notions, as well as that of an iteration tree, carry over to F-premice. Two important points that should be made are that (i) the definition of an iteration tree in this context requires that each node be an F-premouse (as opposed to merely

a premouse) and that (ii) iterability for F-premice refers only to extenders in the sequence E. The reader should note that the distinction between F-premice and premice (for F as has been defined) is not very consequential, as every F-premouse can be repackaged into an ordinary premouse without much trouble.

Recall also that $M_1^F(a)$ is the core of the least stage in the $K^{c,F}(a)$-construction with a Woodin cardinal (above $\mathsf{Ord} \cap a$), if it exists. By minimality, this least stage is not obtained by adding an extender in the $K^{c,F}(a)$-construction, but by applying F. Thus, $M_1^F(a)$ is $N^\sharp_{\alpha+1}(a)$, modulo repackaging it into a structure in the appropriate language and verifying iterability for extenders on the sequence of the latter, but not on that of the former.

Lemma 4.2.18 below is standard.

Lemma 4.2.18. *Suppose that $N^\sharp_\alpha(y)$ exists and is ω_1+1-iterable for all $y \in \mathbb{R}$ and that $N^\sharp_{\alpha+1}(x)$ exists and is ω_1+1-iterable by a strategy Σ. Then, if T is a normal tree on $N^\sharp_{\alpha+1}(x)$, one of the following holds:*

1. *$Q(T)$ exists and is a proper initial segment of $N^\sharp_\alpha(M(T))$, and $\Sigma(T)$ is the unique branch b through T such that $Q(T) = Q(b,T)$; or*
2. *T is non-dropping in model, $\delta(T) = j(\delta)$, where j is the embedding of T and δ is the smallest Woodin cardinal of $N^\sharp_{\alpha+1}(x)$, and $M^T_{\Sigma(T)} = N^\sharp_\alpha(M(T))$.*

Proof. This is simply the Mitchell-Steel theorem on the uniqueness of iteration strategies for tame mice stated explicitly for $N^\sharp_{\alpha+1}(x)$. The only difference is that now we bound the complexity of the Q-structures we allow. If T is a tree for which no proper initial segment of $N^\sharp_\alpha(M(T))$ provides a Q-structure, then, since $N^\sharp_{\alpha+1}(x)$ satisfies "I am of class S_α above δ and no proper initial segment of me is of class S_α above δ," for some Woodin cardinal δ, we must have

1. T is non-dropping in model,
2. $\delta(T) = j(\delta)$, where j is the embedding of T,
3. $M^T_{\Sigma(T)}$ satisfies "I am of class S_α above $j(\delta)$ and no proper initial segment of me is of class S_α above $j(\delta)$."

The last clause implies $\rho(M^T_{\Sigma(T)}) \leq j(\delta)$. By applying the comparison theorem to $M^T_{\Sigma(T)}$ and $N^\sharp_\alpha(M(T))$ (both models are ω_1+1-iterable above $\delta(T)$), we see that they must be equal. $\square$

A consequence of the preceding lemma is a simple description of the unique iteration strategy for $N^\sharp_{\alpha+1}(x)$, assuming ω_1+1-iterablity of all relevant structures: given a normal tree T, one first attempts to find a Q-structure within the proper initial segments of $N^\sharp_\alpha(M(T))$. If this is not possible, then one searches for branches b such that $M^T_b = N^\sharp_\alpha(M(T))$. If so, then further iterates of this model take place entirely above $M(T)$, so one proceeds similarly, this time in search of even simpler Q-structures. We say that such an iteration strategy is *guided by Q-structures* which are not of class S_α. The point is that in the boldface scenario, iteration strategies can also be shown to be of this form if one only assumes ω_1-iterability. The next lemma verifies that in the context of models of the form $N^\sharp_\alpha(x)$.

Lemma 4.2.19. *Let α be a countable ordinal and suppose $N^\sharp_\alpha(x)$ exists for almost all $x \in \mathbb{R}$.*

1. *Let N be a premouse of class $S_{\alpha+1}$ above some cutpoint δ and let $a \in N|\delta$. Suppose that $L(a) \models \mathsf{ZFC}$ and $tc(a)$ contains some real in which α is recursive. Then $N^\sharp_\alpha(a) \in N$.*

2. *If $\alpha < \omega_1^x$, M is an x-premouse none of whose proper initial segments are of class S_α, and either (i) $\rho_k(M) = \omega$ for some k such that M is k-sound, or (ii) M has no Woodin cardinals, then the unique ω_1-iteration strategy of M is guided by Q-structures that are not of class S_β for some $\beta < \alpha$.*

3. *If M and N are two ω_1-iterable x-premice none of whose proper initial segments are of class S_α, then the conclusion of the comparison theorem holds for M and N.*

Proof. The case that α is finite is well known. Thus, we assume that α is infinite and proceed by induction. Clauses 2 and 3 for limit α are easy to show. To prove the lemma, we will show that 1 holds for α (i.e., N in the statement will be of class $S_{\alpha+1}$), after which we will show that 2 holds for $\alpha+1$. Finally, we prove 3 for $\alpha+1$. For convenience, we restate 2 and 3 for $\alpha+1$:

2. Suppose $N^\sharp_{\alpha+1}(x)$ exists for almost all $x \in \mathbb{R}$. If $\alpha < \omega_1^x$, M is an x-premouse none of whose proper initial segments are of class $S_{\alpha+1}$, and either (i) $\rho_k(M) = \omega$ for some k such that M is k-sound, or (ii) M has no Woodin cardinals, then the unique ω_1-iteration strategy of M is guided by Q-structures that are not of class S_α.

3. Suppose $N^\sharp_{\alpha+1}(x)$ exists for almost all $x \in \mathbb{R}$. If M and N are two ω_1-iterable x-premice none of whose proper initial segments are of class $S_{\alpha+1}$, then the conclusion of the comparison theorem holds for M and N.

We begin with 1. Let N be a premouse of class $S_{\alpha+1}$ above some cutpoint δ and suppose that $a \in N|\delta$ and $\mathrm{tc}(a)$ contains some real in which α is recursive and $L(a) \models \mathsf{ZFC}$. By an auxiliary induction, we show that $N^\sharp_\nu(a) \in N$ for all such a and all $\nu \leq \alpha$; this follows at once from Lemma 4.2.16 if ν is a limit ordinal, so we may as well assume that ν is a successor, say, $\hat{\alpha}+1$ and (for notational simplicity) that $\nu = \alpha$.

Without loss of generality, assume that N has a least Woodin cardinal δ_0 above δ and that N is of class S_α above δ_0 (this holds of an initial segment of N, by definition). Let β be the smallest N-cardinal greater than δ_0 such that

$$|N|\beta| \models \mathsf{ZFC} + \text{“there is some measurable } \kappa_0 \text{ above } \delta_0\text{.”}$$

Such a β exists because N has a Woodin cardinal greater than δ_0. (The measurable is here only to allow us to talk of core models as in [Ste96].) By minimality, N is of class S_α above β, so, by the induction hypothesis applied to 1, $N|\beta$ is closed under

$$b \mapsto N^\sharp_{\hat{\alpha}}(b).$$

Moreover, all such $N^\sharp_{\hat{\alpha}}(b)$ are countable and ω_1-iterable in V and (by the induction hypothesis applied to 2) their iteration strategies are given by Q-structures which are not of class S_ξ, for some $\xi < \hat{\alpha}$. By induction hypothesis (applied to 1) these Q-structures belong to $N|\beta$. A consequence of this is:

Claim 4.2.20. *Let $N^\sharp_{\hat{\alpha}}(b)$ be as above. Then $N^\sharp_{\hat{\alpha}}(b)$ is κ_0-iterable in $N|\beta$.*

Proof. We show, by induction on $\alpha' \leq \hat{\alpha}$, that the restriction of the iteration strategy Σ of $N^\sharp_{\alpha'}(b)$ in V to trees of length κ_0 or less in $N|\beta$ is in $N|\beta$. The limit step is easy, so as before let us only consider the case $\alpha' = \hat{\alpha} = \xi + 1$.

Let $T \in N|\beta$ be a tree by this strategy. If $Q(T)$ exists and is a proper initial segment of $N^\sharp_\xi(M(T))$, then $\Sigma(T)$ is the unique b such that $Q(T) = Q(b,T)$. Let $g \subset \mathrm{Coll}(\omega, \delta(T))$ be $N|\beta$-generic. By Mostowski's Absoluteness Theorem,

$$(N|\beta)[g] \models \text{“there is a unique } b \text{ such that } Q(T) = Q(b,T)\text{.”}$$

(The collapse is needed because b might not be countable in $N|\beta$.) Uniqueness follows from the induction hypothesis applied to 3 and the argument of [Ste10, Theorem 6.12]. By homogeneity, $b \in N|\beta$.

The other case is that in which no proper initial segment of $N^\sharp_\xi(M(T))$ provides a Q-structure for T. In this case, Σ selects a branch b such that $M^T_b = N^\sharp_\xi(M(T))$. Again by the induction hypothesis applied to 3 and the argument of [Ste10, Theorem 6.12], such a branch must be unique. By the induction hypothesis applied to 1, $N^\sharp_\xi(M(T)) \in N|\beta$, so the same argument as above shows that $b \in N|\beta$. If so, the argument for the remainder of the iteration is immediate from the induction hypothesis (applied to the claim). This proves the claim. □

Let F be the operator

$$b \mapsto N^\sharp_{\hat{\alpha}}(b)$$

and consider the output of a $K^{c,F}$-construction above a and up to κ_0, as computed in $N|\beta$. Since $N|\kappa_0$ is closed under F and has a Woodin cardinal (namely, δ_0), it has enough certifiable extenders to ensure that the $K^{c,F}$-construction is closed under F and has a Woodin cardinal, i.e., it reaches $M^F_1(a)$. As remarked before, $M^F_1(a)$ is $N^\sharp_\alpha(a)$, provided we can show iterability for extenders not on the sequence of $M^F_1(a)$. We show that it is ω_1-iterable in V; a similar argument shows that it is countably iterable in N.

Let M be the result of repackaging $M^F_1(a)$ as a structure in the language of a-premice. Let E be the extender sequence of M and $E' \subset E$ be the sequence from $M^F_1(a)$. We view M as a structure in the language of set theory with three additional predicate symbols, corresponding to E, E', and to its topmost extender. Ultrapowers of structures in this language are defined in the natural way. For the remainder of this proof, if T is an iteration tree on M, we write

$$(M^T_\xi, E_\xi, E'_\xi, E^{\mathrm{top}}_\xi)$$

for the ξth element of T. By the argument of [Ste96], $M^F_1(a)$ inherits the ω_1-iterablility of N in V, so M is ω_1-iterable with respect to extenders on E'. Since all its proper initial segments are not of class S_α, its iteration strategy Σ^t is given by Q-structures which are not of class $S_{\hat{\alpha}}$, by the induction hypothesis applied to 2.

Let T be a normal iteration tree on M, in the usual sense. If T uses only extenders from E' and its images, then we can use the iteration strategy Σ^t to pick branches at limit stages. If

$$(M^T_\xi, E_\xi, E'_\xi, E^{\mathrm{top}}_\xi)$$

is an iterate of M arising this way, the fact that Σ^t has been used guarantees that $(M_\xi^T, E'_\xi, E_\xi^{\mathrm{top}})$ is an a-F-premouse. Suppose that ξ is least such that F_ξ, the ξth extender used in T, does not come from E'_ξ. Let $\kappa = \mathrm{CRIT}(F_\xi)$ and $\nu = \nu(F_\xi)$. By the remark just given, there is some $\zeta < \kappa$ such that

1. F_ξ belongs to the sequence of $N^\sharp_{\hat{\alpha}}(M_\xi^T|\zeta)$,
2. $N^\sharp_{\hat{\alpha}}(M_\xi^T|\zeta) \trianglelefteq M_\xi^T$, and
3. $N^\sharp_{\hat{\alpha}}(M_\xi^T|\zeta)$ projects to or across ζ.

Since T is normal, $\mathrm{LTH}(F_\eta) < \mathrm{LTH}(F_\xi)$ if $\eta < \xi$, and, since F_ξ is least such that $F_\xi \notin E'_\xi$, we must also have $\mathrm{LTH}(F_\eta) < \kappa$. It follows that ξ is the least η such that $\kappa < \nu(F_\eta)$, so F_ξ must be applied to an initial segment of M_ξ^T. Since F_ξ is not a total extender in M_ξ^T, it is applied to a strict initial segment of M_ξ^T. In fact, it must be applied to an initial segment of

$$N^\sharp_{\hat{\alpha}}(M_\xi^T|\zeta)$$

This means that, starting from stage ξ, the iteration tree is a tree on $N^\sharp_{\hat{\alpha}}(M_\xi^T|\zeta)$ above $M_\xi^T|\zeta$, and, since $N^\sharp_{\hat{\alpha}}(M_\xi^T|\zeta)$ is ω_1-iterable above $M_\xi^T|\zeta$, we are done. This proves 1.

The argument for 2 is similar to that of Lemma 4.2.18 and uses the induction hypothesis for 3. The difference here is that we do not assume M is of the form $N^\sharp_{\alpha+1}(x)$, but (i) if $\rho_k(M) = \omega$ for some k such that M is k-sound, then every normal iteration tree on M has to drop in model or degree. Suppose T is an iteration tree on M, Σ is an iteration strategy, and $\Sigma(T) = b$. Then $\Sigma^t(T) = b$; otherwise, let $c = \Sigma^t(T)$. Since there are drops in b, $\rho_m(M_b^T) < \delta(T)$ for some m, so $Q(b, T)$ exists. Since $Q(c, T)$ also exists, the argument of [Ste10, Lemma 6.12] and the induction hypothesis for 3 yield a contradiction.

If (ii) M has no Woodin cardinals and Σ is an iteration strategy for M, then one can always find Q-structures for normal trees T on M such that

$$\delta(T) < \mathsf{Ord} \cap M^T_{\Sigma(T)},$$

so a similar argument shows $\Sigma = \Sigma^t$. This proves 2.

It remains to prove 3, but this is essentially immediate from 1 and 2, for if M and N are as in the statement, then one can find some $x \in \mathbb{R}$ coding M and N and an x-premouse P of class $S_{\alpha+1}$ and apply the comparison theorem within P. By 2, the iteration strategies of M and N are guided by Q-structures, and P contains Q-structures for all its iteration trees by 1. This completes the proof of the lemma. □

We can finally prove the main result of this section—$N^\sharp_\alpha(x)$ is well defined:

Lemma 4.2.21. *If for almost all $x \in \mathbb{R}$, there is an x-premouse of class S_α, then there is at most one x-premouse satisfying the definition of $N^\sharp_\alpha(x)$ for each $x \in \mathbb{R}$*

Proof. This follows from Lemma 4.2.19.3, for any two models satisfying the definition must project to ω and can be compared. Thus, they must be equal. □

We mention another application of comparison (from ω_1-iterability):

Lemma 4.2.22. *Suppose that $N^\sharp_\alpha(x)$ exists for almost all $x \in \mathbb{R}$ and that $N^\sharp_{\alpha+1}(x)$ exists. Let N be an iterate of $N^\sharp_{\alpha+1}(x)$ obtained by applying Lemma 4.2.14 and $g \subset Coll(\omega, j(\delta))$ be N-generic, where δ is the least Woodin cardinal of $N^\sharp_{\alpha+1}(x)$ and j is the embedding. Then, there is a real $x_g \in \mathbb{R}$ such that*

$$|N[g]| = |N^\sharp_\alpha(x_g)|.$$

Proof. First note that

$$N^\sharp_{\alpha+1}(x) = N^\sharp_\alpha(N^\sharp_{\alpha+1}(x)|\delta).$$

This can be shown by a comparison argument. N is an iterate of $N^\sharp_{\alpha+1}(x)$ by a tree of length $\omega + 1$, non-dropping in model, below δ, so a similar comparison argument shows that

$$N = N^\sharp_\alpha(N|j(\delta)),$$

where j is the iteration embedding. Pick any $x_g \in \mathbb{R} \cap N[g]|(j(\delta)+\omega)$ coding $N|j(\delta)$ and g. If ξ is any x_g-indiscernible, then

$$\{N|j(\delta), g\} \in L_\xi[x_g].$$

It follows that the extender sequences of $N^\sharp_\alpha(x_g)$ and $N[g]$ agree above δ, so the lemma follows. □

$Lp^\alpha(x)$ and co.

We will also consider a different—but closely related—collection of models: the *lower part* models $Lp^\alpha(x)$. These will mainly be studied in Sections 4.5 and 4.6. The main reason for us considering them is that they have very strong closure properties. We will simply state the definitions and a representation lemma for successor ordinals and prove the main properties later on, when they are needed.

Definition 4.2.23. Let α be a successor ordinal and $x \in \mathbb{R}$ be such that $\alpha < \omega_1^x$. We recursively define

$$\begin{aligned}
Lp^\alpha_0(x) &= x,\\
o_\beta &= \mathsf{Ord} \cap Lp^\alpha_\beta(x)\\
Lp^\alpha_{\beta+1}(x) &= \bigcup\{M : M \text{ is an } x\text{-premouse not of class } S_\alpha \text{ above } o_\beta\\
&\qquad \text{that extends } Lp^\alpha_\beta(x),\ \rho(M) \leq o_\beta,\ o_\beta \text{ is a cutpoint of } M,\\
&\qquad M \text{ is sound above } o_\beta \text{ and } \omega_1\text{-iterable above } o_\beta\},\\
Lp^\alpha_\lambda(x) &= \bigcup_{\beta<\lambda} Lp^\alpha_\beta(x), \text{ for } \lambda \in \mathsf{Lim} \cap \omega_1,\\
Lp^\alpha(x) &= \bigcup_{\beta<\omega_1} Lp^\alpha_\beta(x).
\end{aligned}$$

$Lp^\alpha(x)$ can be thought of as a structure in the language of premice—its extender sequence is the union of all extender sequences of the premice that gave rise to it. In fact, $Lp^\alpha(x)$ is indeed a premouse and it is rather easy to describe.

Lemma 4.2.24. *Suppose $\alpha = \beta + 1 < \omega_1^x$. Let M be the ω_1th iterated ultrapower of $N^\sharp_\alpha(x)$ by a normal measure on its smallest measurable cardinal. Then*

$$Lp^\alpha(x) = M|\omega_1.$$

In particular

$$\mathbb{R} \cap Lp^\alpha(x) = \mathbb{R} \cap N^\sharp_\alpha(x).$$

Lemma 4.2.24 can be proved by following the argument of Müller-Schindler-Woodin [MSW16, Lemma 3.23].

We now define $Lp^\alpha(x)$ for limit ordinals. For these, we again briefly resort to mouse operators.

Definition 4.2.25. Let α be a limit ordinal recursive in a real $x \in \mathbb{R}$ and let F be the operator

$$a \mapsto \bigcup_{\beta<\alpha} N^\sharp_\beta(a).$$

The constructible hierarchy relative to F and x, $L^F(x)$, is given by

$$\begin{aligned} L^F_0(x) &= \mathrm{tc}(\{x\}), \\ L^F_{\xi+1}(x) &= F(L^F_\xi(x)), \\ L^F_\lambda(x) &= \bigcup_{\xi<\lambda} L^F_\xi(x), \qquad \text{for } \lambda \in \mathsf{Lim}, \\ L^F(x) &= \bigcup_{\xi\in\mathsf{Ord}} L^F_\xi(x), \end{aligned}$$

provided the object on the right-hand side exists. The lower part model for limit α is given by

$$Lp^\alpha(x) = L^F_{\omega_1}(x).$$

It will be useful to slice $Lp^\alpha(x)$ into a finer hierarchy:

Definition 4.2.26. Suppose $\alpha < \omega_1^x$ is a limit ordinal. Define

$$\begin{aligned} Lp^\alpha_{0,0}(x) &= x, \\ Lp^\alpha_{\beta,\gamma+1}(x) &= N^\sharp_{\gamma+1}(Lp^\alpha_{\beta,\gamma}(x)), \\ Lp^\alpha_{\beta,\lambda}(x) &= \bigcup_{\gamma<\lambda} Lp^\alpha_{\beta,\gamma}(x), \text{ for } \lambda \in \mathsf{Lim} \cap \alpha, \\ Lp^\alpha_{\beta,0}(x) &= \bigcup_{\eta<\beta,\gamma<\alpha} Lp^\alpha_{\eta,\gamma}(x), \text{ for } \beta < \omega_1, \\ Lp^\alpha(x) &= \bigcup_{\beta<\omega_1} Lp^\alpha_{\beta,0}(x). \end{aligned}$$

We may write $Lp^\alpha_\beta(x)$ for $Lp^\alpha_{\beta,0}(x)$.

The definition of $Lp^\alpha(x)$ can be naturally extended beyond ω_1. We may also write $Lp^\alpha(x)$ for $Lp^\alpha_{\mathsf{Ord}}(x)$, hopefully causing no confusion.

We conclude with a simple remark on relativized constructible hierarchies. We state it for F as in Definition 4.2.25.

Lemma 4.2.27. *Let α be a limit ordinal and suppose that the operator F given by*

$$a \mapsto \bigcup_{\beta<\alpha} N^\sharp_\beta(a)$$

is defined almost everywhere. Suppose x and y are real numbers that compute α and such that $x \in L^F(y)$ and $y \in L^F(x)$. Then

$$L^F(x) = L^F(y).$$

Proof. By hypothesis, $N^\sharp_\alpha(z)$ exists for almost all $z \in \mathbb{R}$. By Lemma 4.2.19, $L^F(x)$ and $L^F(y)$ are closed under F. An inductive argument, using Lemma 4.2.19 again, shows that if $z \in \mathbb{R} \cap L^F(x)$ and $\beta < \alpha$, then $N^\sharp_\beta(z)$ is iterable in $L^F(x)$. Since $y \in L^F(x)$, $L^F(x)$ can carry out the definition of $L^F(y)$ and, in fact,

$$L^F(y)^{L^F(x)} = L^F(y).$$

Similarly,

$$L^F(x)^{L^F(y)} = L^F(x),$$

so it follows that $L^F(x) = L^F(y)$. □

4.3 A Proof of σ-Projective Determinacy

The main result in this section will be:

Theorem 4.3.1. *Suppose that $N^\sharp_{\alpha+1}(x)$ exists for almost all $x \in \mathbb{R}$. Then, all sets in $\mathbf{\Sigma}^1_\alpha$ are determined.*

Theorem 4.3.1 is not optimal, but it is easier to prove as stated. In particular, we give a proof that works for even and odd α. We will need a general determinacy result due to Neeman, a version of which is the following:

Theorem 4.3.2 (Neeman [Nee04])**.** *Let M be an ω_1-iterable proper class transitive model with a Woodin cardinal δ which is countable in V. Let $\dot{A}$ be a Coll(ω, δ)-name for a set of reals. Then there is a strategy*

$$\sigma : \bigcup_{n \in \mathbb{N}} \mathbb{N}^n \to \mathbb{N}$$

such that one of the following holds:

1. *Whenever $x \in \mathbb{R}$ is a play of a Gale-Stewart game in which Player I plays by σ, there is a countable iterate N of M and an N-generic $g \subset$ Coll(ω, δ) such that $x \in N[g]$ and $N[g] \models x \in \dot{A}[g]$.*

2. *Whenever $x \in \mathbb{R}$ is a play of a Gale-Stewart game in which Player II plays by σ, there is a countable iterate N of M and an N-generic $g \subset$ Coll(ω, δ) such that $x \in N[g]$ and $N[g] \models x \notin \dot{A}[g]$.*

Neeman proved Theorem 4.3.2 for games of any (fixed) countable length (starting with a sufficient amount of Woodin cardinals). This yields Corollary **??**, granted Theorem 12.2.6 holds.

Definition 4.3.3. Let Γ be a class of formulae, perhaps with parameters.

1. A transitive set M is said to be Γ-*correct* if for all formulae $\phi \in \Gamma \cap M$,
$$\phi \text{ if, and only if, } M \models \phi.$$
2. A transitive set M is said to *compute truth in* Γ if there is a mapping
$$\phi \mapsto \phi^*$$
in M such that for all formulae $\phi \in \Gamma \cap M$,
$$\phi \text{ if, and only if, } M \models \phi^*.$$

For example, if $\Gamma = \mathbf{\Sigma}^1_2$, then L is Γ-correct, by the Shoenfield Absoluteness Theorem. By a theorem of Woodin, M_1 can compute $\mathbf{\Sigma}^1_3$-truth, but it is not $\mathbf{\Sigma}^1_3$-correct.

Lemma 4.3.4. *Suppose $N^\sharp_{\alpha+1}(x)$ exists for almost all $x \in \mathbb{R}$. Then $N^\sharp_{\alpha+1}(x)$ is $\mathbf{\Sigma}^1_{\alpha+1}$-correct.*

Proof. The case where α is finite is well known (cf. e.g. Müller-Schindler-Woodin [MSW16, Lemma 1.17] for a proof). We begin with the limit case. Suppose that α is a limit ordinal and $N^\sharp_{\alpha+1}(x)$ exists for almost all $x \in \mathbb{R}$ and let $x \in \mathbb{R}$ have large-enough Turing degree. By Lemma 4.2.19, $N^\sharp_{\alpha+1}(x)$ is closed under
$$a \mapsto N^\sharp_\alpha(a),$$
for $a \in \mathbb{R} \cap N^\sharp_{\alpha+1}(x)$, so it is $\mathbf{\Pi}^1_\alpha$-correct—and hence upwards $\mathbf{\Sigma}^1_{\alpha+1}$-correct—by induction hypothesis. For the converse we need a slightly more involved argument.

We begin by noting that simpler models than $N^\sharp_{\alpha+1}(x)$ can compute (boldface) $\mathbf{\Pi}^1_\alpha$-truth. For instance, let F be the operator
$$a \mapsto \bigcup_{\beta<\alpha} N^\sharp_\beta(a).$$
Thus, if $x \in \mathrm{dom}(F)$, then $L^F_0(x) = \{x\} \cup \omega$ and
$$L^F_1(x) = F(x) = \bigcup_{\beta<\alpha} N^\sharp_\beta(x).$$

Claim 4.3.5. *Suppose α ($\in \mathsf{Lim}$) is recursive in y. Then $L_1^F(y)$ computes truth in $\mathbf{\Pi}^1_\alpha$.*

Proof. It would not be very useful to claim that $L_1^F(y)$ is $\mathbf{\Pi}^1_\alpha$-correct, since $\mathbf{\Pi}^1_\alpha$ subsets of $\mathbb{R}^{L_1^F(y)}$ do not in general belong to $L_1^F(y)$, although they do belong to, say, $L_{\alpha+1}(L_1^F(y))$.

The main point is that the sequence

$$\{N^\sharp_\beta(y) : \beta < \alpha\}$$

is definable over $L_1^F(y)$. By induction hypothesis, $N^\sharp_{\beta+1}(y)$ is $\mathbf{\Pi}^1_\beta$-correct, so given a set $A \in \Pi^1_\beta(a)$, where $a \in \mathbb{R} \cap L_1^F(y)$, $L_1^F(y)$ can compute whether $z \in A$ by asking whether

$$N^\sharp_{\gamma+1}(y) \models z \in A,$$

where γ is arbitrary such that

1. $\beta < \gamma$, and
2. $z, a \in N^\sharp_{\gamma+1}(y)$.

For the remaining case, if $A \in \Pi^1_\alpha(a)$ is a set, say,

$$A = \bigcap_{n\in\mathbb{N}} A_n,$$

where $A_n \in \Pi^1_{\alpha_n}(a)$, $\alpha_n < \alpha$, and $a \in \mathbb{R} \cap L_1^F(y)$, then the sequence

$$\{\alpha_n : n \in \mathbb{N}\}$$

is recursive in a and thus belongs to $L_1^F(y)$, so $L_1^F(y)$ can compute whether $z \in A$ by asking whether for all $n \in \mathbb{N}$ and all sufficiently large γ_n we have

$$N^\sharp_{\gamma_n+1}(y) \models z \in A_n.$$

This proves the claim. □

We now prove downwards $\mathbf{\Sigma}^1_{\alpha+1}$-correctness of $N^\sharp_{\alpha+1}(x)$. By Claim 4.3.5, $L_1^F(y)$ computes truth in $\mathbf{\Pi}^1_\alpha$ correctly for all y in which x is recursive. Moreover, this computation is uniform, i.e., there is a fixed mapping

$$\phi \mapsto \phi^*$$

witnessing that Definition 4.3.3 holds of every such $L_1^F(y)$. (This mapping is obtained by following the proof of Claim 4.3.5.) Let ϕ be a formula in $\Sigma^1_{\alpha+1}(a,x)$ for some $a \in N^\sharp_{\alpha+1}(x)$, say, of the form $\exists y\, \phi(a,x,y)$ and suppose that $y \in \mathbb{R}^V$ is such that $\phi(a,x,y)$ holds.

Let δ_0 be the Woodin cardinal witnessing that $N^\sharp_{\alpha+1}(x)$ is of class $S_{\alpha+1}$ and let

$$\kappa = \sup\{\mathsf{Ord} \cap N^\sharp_\beta(N^\sharp_{\alpha+1}(x)|\delta_0) : \beta < \alpha\}$$
$$(= \mathsf{Ord} \cap L_1^F(N^\sharp_{\alpha+1}(x)|\delta_0)\,).$$

Thus, $N^\sharp_{\alpha+1}(x)|\kappa$ is of class S_β above δ_0 for all $\beta < \alpha$. (It is not of class S_α above δ_0!)

By the Löwenheim-Skolem Theorem applied to $N^\sharp_{\alpha+1}(x)|\kappa$ within $N^\sharp_{\alpha+1}(x)$ and the Condensation Lemma, there is an initial segment P of $N^\sharp_{\alpha+1}(x)$ such that

$$P \models \text{“for all } \beta < \alpha\text{, I am of class } S_\beta \text{ above some fixed Woodin cardinal } \delta\text{”}$$

and such that P is countable in $N^\sharp_{\alpha+1}(x)$. Since P is an initial segment of $N^\sharp_{\alpha+1}(x)$, it is ω_1-iterable in V. Let N be an iterate of P such that $y \in N[g]$ for some N-generic $g \subset \mathrm{Coll}(\omega, j(\delta))$, where j is the iteration embedding. By the argument of Lemma 4.2.22, $N[g]$ is essentially a premouse of the form

$$L_1^F(x_g) = \bigcup_{\beta<\alpha} N^\sharp_\beta(x_g),$$

so it computes truth in $\mathbf{\Pi}^1_\alpha$, by the claim. The proof of Lemma 4.2.22 shows a way of defining x_g from g and x, so, by homogeneity,

$$N \models \text{“}\Vdash_{\mathrm{Coll}(\omega,j(\delta))} \exists z\, \phi^*(a,x,z),\text{”}$$

so, by elementarity and the fact that the mapping $\phi \mapsto \phi^*$ is not moved by the iteration embedding (because α is countable in P),

$$P \models \text{“}\Vdash_{\mathrm{Coll}(\omega,\delta)} \exists z\, \phi^*(a,x,z)\text{”};$$

but P is countable in $N^\sharp_{\alpha+1}(x)$, so $N^\sharp_{\alpha+1}(x)$ can construct generic filters for partial orders in P. Thus,

$N^\sharp_{\alpha+1}(x) \models$ “there is a P-generic h and a $z' \in P[h]$ such that $P[h] \models \phi^*(a,x,z')$”.

Since we already know that $N^\sharp_{\alpha+1}(x)$ is $\mathbf{\Pi}^1_\alpha$-correct, Claim 4.3.5 shows that $P[h]$ computes truth in $\mathbf{\Pi}^1_\alpha$ also from the point of view of $N^\sharp_{\alpha+1}(x)$, and this is witnessed also by the mapping $\phi \mapsto \phi^*$. It follows that

$$N^\sharp_{\alpha+1}(x) \models \phi(a, x, z'),$$

as desired.

For the successor case, note that $N^\sharp_{\alpha+2}(x)$ is closed under

$$a \mapsto N^\sharp_{\alpha+1}(a),$$

for $a \in \mathbb{R} \cap N^\sharp_{\alpha+2}(x)$ by Lemma 4.2.19, so it is $\mathbf{\Sigma}^1_{\alpha+1}$-correct. An argument as before shows that it is $\mathbf{\Sigma}^1_{\alpha+2}$-correct. □

We can finally proceed to the:

Proof of Theorem 4.3.1. The case for finite n is well known. It follows from the results of Neeman [Nee04, Section 2].

Let us first consider the case of double successor ordinals, i.e., let α be an infinite ordinal and suppose that $N^\sharp_{\alpha+2}(x)$ exists for almost all $x \in \mathbb{R}$. Let A be a set in $\Sigma^1_{\alpha+1}(x)$, for some $x \in \mathbb{R}$, so A is definable from x by a formula ϕ in $\Sigma^1_{\alpha+1}(x)$, say

$$\phi(y, x) = \exists z\, \phi_0(z, y, x).$$

By hypothesis, $N^\sharp_{\alpha+2}(x)$ exists. Apply Neeman's theorem to $N^\sharp_{\alpha+2}(x)$, using its least Woodin cardinal δ and setting $\dot{A}$ equal to the canonical $\mathrm{Coll}(\omega, \delta)$-name for all reals y such that $\phi(y, x)$ holds. This is possible, since $\phi \in N^\sharp_{\alpha+2}(x)$. Thus, there is a strategy σ such that one of the following holds:

1. Whenever $y \in \mathbb{R}$ is a play in which Player I plays by σ, there is a countable iterate N of $N^\sharp_{\alpha+2}(x)$ and an N-generic $g \subset \mathrm{Coll}(\omega, j(\delta))$ such that $y \in N[g]$ and $N[g] \models \phi(y, x)$, where j is the embedding.

2. Whenever $y \in \mathbb{R}$ is a play in which Player II plays by σ, there is a countable iterate N of $N^\sharp_{\alpha+2}(x)$ and an N-generic $g \subset \mathrm{Coll}(\omega, j(\delta))$ such that $y \in N[g]$ and $N[g] \not\models \phi(y, x)$, where j is the embedding.

Suppose without loss of generality that the first case holds. We claim σ is a winning strategy for Player I in A. Let y be a play by σ and let N and g be as above. Thus,

$$N[g] \models \phi(y, x).$$

By Lemma 4.2.22 and Lemma 4.3.4, $N[g]$ is $\mathbf{\Sigma}^1_{\alpha+1}$-correct. Therefore $\phi(y,x)$ really holds, as desired. This takes care of the double successor case.

The remaining case is similar. Suppose α is a limit ordinal. Let A be a set in $\Sigma^1_\alpha(x)$, for some $x \in \mathbb{R}$, so A is definable from x by a formula ϕ in $\Sigma^1_\alpha(x)$, say

$$\phi(y,x) = \bigvee_{\beta<\alpha} \phi_\beta(y,x).$$

By hypothesis $N^\sharp_{\alpha+1}(x)$ exists. Let

$$\kappa = \sup\{\mathsf{Ord} \cap N^\sharp_\beta(N^\sharp_{\alpha+1}(x)|\delta) : \beta < \alpha\}.$$

Apply Neeman's theorem again, this time to $N^\sharp_{\alpha+1}(x)$, using its Woodin cardinal δ and setting $\dot{A}$ equal to the canonical $\mathrm{Coll}(\omega,\delta)$-name for all reals y such that

$$(N^\sharp_{\alpha+1}(x)|\kappa)[g] \models \phi^*(y,x),$$

where $\phi \mapsto \phi^*$ is the mapping obtained in Claim 4.3.5 (which is defined from x and g via the proofs of Claim 4.3.5 and Lemma 4.2.22).

If δ is the Woodin cardinal of $N^\sharp_{\alpha+1}(x)$ and $g \subset \mathrm{Coll}(\omega,\delta)$ is $N^\sharp_{\alpha+1}(x)$-generic, then $N^\sharp_{\alpha+1}(x)[g]$ is essentially a premouse of the form $N^\sharp_\alpha(x_g)$ for some x_g, by Lemma 4.2.22, and, by the same argument,

$$(N^\sharp_{\alpha+1}(x)|\kappa)[g]$$

is essentially a model of the form $L^F_1(x_g)$, where F is the operator

$$a \mapsto \bigcup_{\beta<\alpha} N^\sharp_\beta(a).$$

By Claim 4.3.5, this model computes truth in $\mathbf{\Sigma}^1_\alpha$, as witnessed by the mapping $\phi \mapsto \phi^*$, whereby an argument as before completes the proof. □

Corollary 4.3.6. *Suppose that $N^\sharp_{\alpha+1}(x)$ exists for almost all $x \in \mathbb{R}$ and that $\alpha < \omega_1^x$. Then*

$$N^\sharp_{\alpha+1}(x) \models \mathbf{\Delta}^1_\alpha\text{-determinacy.}$$

Proof. If $A_0, A_1 \in \mathbf{\Sigma}^1_\alpha$, then the formula

$$\forall y\,(y \in A_0 \leftrightarrow y \notin A_1)$$

is in $\mathbf{\Pi}^1_{\alpha+1}$, so it follows from Lemma 4.3.4 that

$$\mathcal{P}(\mathbb{R})^{N^\sharp_{\alpha+1}(x)} \cap \mathbf{\Delta}^1_\alpha = (\mathcal{P}(\mathbb{R}) \cap \mathbf{\Delta}^1_\alpha)^{N^\sharp_{\alpha+1}(x)}.$$

By Theorem 4.3.1, sets in $\mathbf{\Sigma}^1_\alpha$ are determined. It is easy to check that determinacy for sets in $\mathbf{\Delta}^1_\alpha$ is expressible (schematically) by formulae in $\mathbf{\Sigma}^1_{\alpha+1}$, so the result follows from Lemma 4.3.4. □

4.4 Π^1_α-Iterability

We will need to define a notion of iterability which is simply definable, yet suffices for the comparison of small premice. This is the natural extension of Steel's [Ste95] notion of Π^1_n-iterability. The results in this section can be summarized as follows: under reasonable circumstances, the conclusion of the comparison theorem holds for two premice which are not of class S_α if one of them is ω_1-iterable and the other is $\Pi^1_{1+\alpha}$-iterable.

Definition 4.4.1. Suppose δ is a cutpoint of M. The notion of Π^1_α-iterability is defined as follows.

1. Suppose α is a successor ordinal. M is Π^1_α-iterable above δ if Player II has a winning strategy in the αth simple iteration game over M and δ (defined below).

2. Suppose α is a limit ordinal. Then M is Π^1_α-iterable above δ if whenever β, γ, N are such that

   (a) $\beta < \alpha$,

   (b) $\gamma \in M$ and $\delta \leq \gamma$,

   (c) $N \trianglelefteq M$ and N is not of class S_β above γ,

   (d) N has γ as cutpoint and is k-sound above γ for all k such that $\gamma < \rho_k(N)$,

   Then N is $\Pi^1_{1+\beta}$-iterable above γ.

Let T be an iteration tree and α be an ordinal. A cofinal branch b of T is *α-good* if whenever N is the direct limit along b or N is the αth linear iterate of an initial segment of the direct limit along b by a single extender and its images, then α belongs to the wellfounded part of N. Note that the set of all pairs (x, y) such that

if x codes an ordinal α and a tree T on a premouse M, then y is α-good

is Σ^1_1.

Let $n \in \mathbb{N}$. The nth *simple* iteration game over M and δ is defined[5] for M that have δ as cutpoint and that are (k-)sound above δ, for $2 \leq n$, and is defined according as n is even or odd. In this game, players I and II cooperate to define models M_k and ordinals δ_k, where $0 \leq k \leq n$. There are certain rules that the players need to follow during the game.

If n is odd, then the nth game (of length $n-1$) is played as follows:

1. We begin with $M_0 = M$ and $\delta_0 = \delta$.

2. In round k, for $0 \leq k \leq n-2$, Player I plays a countable, normal, putative iteration tree T on M_k above δ_k. Player II

   (a) accepts T, if it has a last, wellfounded model; or

   (b) plays a cofinal, wellfounded branch b_k through T such that

   $$b \in \Delta^1_n(M_k, T).$$

   In the first case, set M_{k+1} to be the last model of T_k, and

   $$\delta_{k+1} = \sup\{\mathrm{LTH}(E_{\alpha+1}) : \alpha + 1 < \mathrm{LTH}(T_k)\};$$

   in the second case, set M_{k+1} to be the direct limit along b and let

   $$\delta_{k+1} = \sup\{\mathrm{LTH}(E_{\alpha+1}) : \alpha \in b\}.$$

3. If all rounds have been played and no player has lost, then Player II wins.

If n is even and nonzero, then the nth game (of length $n-1$) is played as follows:

1. We begin with $M_0 = M$ and $\delta_0 = \delta$.

2. If $0 \leq k < n-2$ is even, then Player I plays a countable, normal, putative iteration tree T_k on M_k above δ_k and some $x_k \in \mathbb{R}$. Player II then

   (a) accepts T_k, if it has a last, wellfounded model; or

   (b) plays a cofinal, wellfounded branch b_k through T_k

[5] This is the game $\mathcal{I}(M, \delta, n-1)$ in [Ste95].

In the first case, set M_{k+1} to be the last model of T_k, and

$$\delta_{k+1} = \sup\{\mathrm{LTH}(E_{\alpha+1}) : \alpha + 1 < \mathrm{LTH}(T_k)\};$$

in the second case, set M_{k+1} to be the direct limit along b and let

$$\delta_{k+1} = \sup\{\mathrm{LTH}(E_{\alpha+1}) : \alpha \in b\}.$$

3. If $k = n - 2$, then the game is played as above, except that we do not require b to be wellfounded. If so, the game ends and Player II wins if, and only if, one of the following holds:
   (a) x_k does not code an ordinal; or
   (b) the last model played is x_k-good.
4. If $0 < k < n-2$ is odd, then the rules are as in the case that k is even, except that we require

$$T \in \Delta^1_{n-k}(x_{k-1}, T_{k-1}, b_{k-1}).$$

Let λ be a countable limit ordinal and $n \in \mathbb{N}$. The $\lambda + 2n + 1$th game is defined similarly to the $2n + 1$th game, except that it has an additional round ($2n + 1$ in total):

1. We begin with $M_0 = M$ and $\delta_0 = \delta$.
2. In round k, for $0 \leq k < n$, Player I plays a countable, normal, putative iteration tree T on M_k above δ_k. Player II
   (a) accepts T, if it has a last, wellfounded model; or
   (b) plays a cofinal, wellfounded branch b_k through T such that

$$b \in \Delta^1_{\lambda+2n+1}(M_k, T).$$

   In the first case, set M_{k+1} to be the last model of T_k, and

$$\delta_{k+1} = \sup\{\mathrm{LTH}(E_{\alpha+1}) : \alpha + 1 < \mathrm{LTH}(T_k)\};$$

   in the second case, set M_{k+1} to be the direct limit along b and let

$$\delta_{k+1} = \sup\{\mathrm{LTH}(E_{\alpha+1}) : \alpha \in b\}.$$

3. If all rounds have been played and no player has lost, then Player II wins if, and only if, M_{2n+1} is Π^1_λ-iterable.

Let $0 < n$. The $\lambda + 2n$th game is defined similarly to the $2n$th game, except that it has an additional round ($2n$ in total):

1. We begin with $M_0 = M$ and $\delta_0 = \delta$.

2. If $0 \leq k \leq 2n-2$ is even, then Player I plays a countable, normal, putative iteration tree T_k on M_k above δ_k and some $x_k \in \mathbb{R}$. Player II then

   (a) accepts T_k, if it has a last, wellfounded model; or

   (b) plays a cofinal, wellfounded branch b_k through T_k

   In the first case, set M_{k+1} to be the last model of T_k, and

   $$\delta_{k+1} = \sup\{\textsc{lth}(E_{\alpha+1}) : \alpha + 1 < \textsc{lth}(T_k)\};$$

   in the second case, set M_{k+1} to be the direct limit along b and let

   $$\delta_{k+1} = \sup\{\textsc{lth}(E_{\alpha+1}) : \alpha \in b\}.$$

3. If $0 < k < 2n$ is odd, then the rules are as in the case that k is even, except that we require

   $$T \in \Delta^1_{\lambda+2n-k}(x_{k-1}, T_{k-1}, b_{k-1}).$$

4. After round $2n-1$ has been played, the game ends. Player II wins if, and only if, M_{2n} is Π^1_λ-iterable.

Lemma 4.4.2. *Let α be a nonzero even ordinal recursive in x.*

1. *Suppose that sets in $\mathbf{\Delta}^1_\beta$ are determined whenever $\beta < \alpha$ is even. Then the set $\{(M,\delta) : M$ is Π^1_α-iterable above $\delta\}$ is $\Pi^1_\alpha(x)$.*

2. *Suppose that sets in $\mathbf{\Delta}^1_\alpha$ are determined. Then the set $\{(M,\delta) : M$ is $\Pi^1_{\alpha+1}$-iterable above $\delta\}$ is $\Pi^1_{\alpha+1}(x)$.*

Proof. For finite n this is proved in [Ste95]. Suppose λ is a countable limit ordinal and $n \in \mathbb{N}$. Clearly being Π^1_λ-iterable is a $\Pi^1_\lambda(x)$ property. Now, if n is odd, then M being $\Pi^1_{\lambda+n}$-iterable above δ is a formula of the form

$$\forall T_0\, \exists b_0 \in \Delta_{\lambda+n}(M, T_0) \ldots \exists b_{2n} \in \Delta_{\lambda+n}(M_{n-1}, T_{n-1}) M_n \text{ is } \Pi^1_\lambda\text{-iterable} \vee F, \tag{4.1}$$

where F denotes the possible outcomes of the game in which it stops before reaching the end. The formula

$$\exists b_{2n} \in \Delta_{\lambda+n}(M_{n-1}, T_{n-1}) M_n \text{ is } \Pi^1_\lambda\text{-iterable}$$

is $\Pi^1_{\lambda+n}(x)$ by Lemma 4.2.9, which we can apply since n is odd. It follows that the formula

$$\forall T_{2n}\, \exists b_{2n} \in \Delta_{\lambda+n}(M_{n-1}, T_{n-1}) M_n \text{ is } \Pi^1_\lambda\text{-iterable.}$$

is also $\Pi^1_{\lambda+n}(x)$. Inductively, and using Lemma 4.2.9 at each stage, the formula (4.1) is $\Pi^1_{\lambda+n}(x)$. If n is even, the proof is similar; we illustrate the case $n = 2$. Here, M being $\Pi^1_{\lambda+n}$-iterable is a formula of the form

$$\forall T_0\, \forall x_0\, \exists b_0\, \forall T_1 \in \Delta_{\lambda+1}(x_0, T_0, b_0)\, \exists b_1\, M_2 \text{ is } \Pi^1_\lambda\text{-iterable} \vee F, \tag{4.2}$$

where F is as above. Here,

$$\forall b_1\, M_2 \text{ is not } \Pi^1_\lambda\text{-iterable}$$

is $\Pi^1_{\lambda+1}(x)$, so that

$$\exists T_1 \in \Delta_{\lambda+1}(x_0, T_0, b_0)\, \forall b_1\, M_2 \text{ is not } \Pi^1_\lambda\text{-iterable}$$

is $\Pi^1_{\lambda+1}(x)$ by Lemma 4.2.9. It follows that (4.2) is $\Pi^1_{\lambda+2}(x)$. □

We shall now state the Π^1_α-comparison lemma 4.4.3 below. It is the extension of [Ste95, Lemma 2.2] to infinite α.

Lemma 4.4.3 (Π^1_α-comparison)**.** *Let α be an infinite ordinal. Assume that $N^\sharp_\alpha(x)$ exists for almost all $x \in \mathbb{R}$. Let M and N be countable premice such that*

1. *there are no Woodin cardinals in M nor in N above δ,*
2. *M and N have a common cutpoint δ,*
3. *M and N are not of class S_α above δ, and*
4. *M is k-sound above δ for all k such that $\delta < \rho_k(M)$. Similarly for N.*

Then the following hold:

1. *Suppose that $\alpha = \beta + 1$ and M is ω_1-iterable above δ. Let T be a normal iteration tree on M above δ of countable limit length λ and let b the unique cofinal wellfounded branch of T such that $Q(b, T)$ is ω_1-iterable above $\delta(T)$. Then b is the unique wellfounded cofinal branch c through T such that $Q(c, T)$ is $\Pi^1_{1+\beta}$-iterable above $\delta(T)$.*

2. *Suppose that α is a limit ordinal and M is ω_1-iterable above δ. Let T be a normal iteration tree on M above δ of countable limit length λ and let b the unique cofinal wellfounded branch of T such that $Q(b, T)$ is ω_1-iterable above $\delta(T)$. Then there exists $\gamma < \alpha$ such that*

   (a) *the first extender in T is applied to an initial segment of M not of class $S_{\gamma+1}$ above some δ_0 which is either δ or the largest Woodin cardinal of M below $\mathrm{CRIT}(T)$, and*

   (b) *b is the unique cofinal branch wellfounded branch c through T such that $1 \leq \gamma$ implies that $Q(c, T)$ is $\Pi^1_{1+\gamma}$-iterable above $\delta(T)$.*

3. *Suppose that M is ω_1-iterable above δ. Then M is $\Pi^1_{1+\alpha}$-iterable above δ.*

4. *Suppose that M is ω_1-iterable above δ and that N is $\Pi^1_{1+\alpha}$-iterable above δ. Suppose moreover that $M|\delta = N|\delta$. Then, there are a normal iteration tree T on M and a putative iteration tree U on N, both of successor length $\lambda + 1$ such that one of the following holds:*

   (a) *$M^T_{\lambda+1} \trianglelefteq M^U_{\lambda+1}$ (so $\mathsf{Ord}^{M^T_{\lambda+1}}$ belongs to the wellfounded part of $M^U_{\lambda+1}$) and T is non-dropping, or*

   (b) *$M^U_{\lambda+1} \trianglelefteq M^T_{\lambda+1}$ (so $M^U_{\lambda+1}$ is wellfounded) and U is non-dropping.*

   If, in addition, M and N project to or across δ, then one of the following holds:

   (a) *$M \trianglelefteq N$, or*

   (b) *$N \trianglelefteq M$.*

Moreover, if M and N do have Woodin cardinals, but M is $\omega_1 + 1$-iterable above δ in $N^\sharp_\alpha(z)$ for almost all z, then the conclusions hold as well.

Proof Sketch. We assume that α is infinite. The proof is by induction. Suppose that the lemma holds for all $\gamma < \alpha$ and let W be a premouse of the form $N^\sharp_\alpha(x_W)$ in which M and N are countable. We first consider the case that α is a successor ordinal $\beta + 1$. We need to prove (1), (3), and (4).

Claim 4.4.4. *If M has no Woodin cardinals, then M is $\omega_1^W + 1$-iterable in W.*

Proof. An argument as in Claim 4.2.20 shows that it is ω_1-iterable. Then, if $g \subset \mathrm{Coll}(\omega, \omega_1^W)$ is W-generic, the same argument shows that M is $\omega_1^{W[g]}$-iterable in $W[g]$ and since $\mathrm{Coll}(\omega, \omega_1^W)$ is a homogeneous partial order, the claim follows. □

Let us work in W for the remainder of the proof (of the successor case). The point is that W is correct about $\mathbf{\Sigma}^1_\alpha$ statements by Lemma 4.3.4, so the hypotheses of the lemma hold in W, and it suffices to show that the conclusions do too. Clauses (1) and (3) are proved as in [Ste95, Lemma 2.2].

Clause (4) is also proved as in [Ste95, Lemma 2.2], but perhaps one wants to be careful, since one needs to ensure that the comparison between M and N is short. The proof is as follows: one attempts to compare M and N by least disagreement, defining a sequence of trees $\{T_\gamma, U_\gamma\}_\gamma$ along the way. At limit stages, $T_{\gamma+1}$ is defined by choosing the branch singled out by the iteration strategy of M. Corollary 4.3.6 gives us $\mathbf{\Delta}^1_\beta$-determinacy in W, which is enough to apply Lemma 4.2.9 for all even ordinals smaller than α, regardless of whether α is even or odd. Thus, by (1), the branch is $\Delta^1_{\beta+1}(T_\gamma, U_\gamma)$. The tree $U_{\gamma+1}$ is defined similarly:

1. if α is odd, then $U_{\gamma+1}$ is given by the unique cofinal, wellfounded branch b through U_γ—if it exists—such that the direct limit along b is Π^1_β-iterable. If b exists, then $U_{\gamma+1}$ is $\Delta^1_{\beta+1}(T_{\gamma+1}, U_\gamma)$.

2. if α is even, then let P be the last model of $T_{\gamma+1}$, if it exists, or the direct limit along the branch chosen for T_γ, otherwise. Then $U_{\gamma+1}$ is given by choosing the unique branch b through U_γ—if it exists—such that Player II can win the αth simple iteration game after Player I has begun the game playing U_γ and $x = (T_{\gamma+1}, P)$ and Player II has responded with b. The next move of the game requires Player I to choose a tree in $\Delta^1_\beta(x, U_\gamma, b)$, so Player II having a winning strategy is a $\Sigma^1_\beta(x, U_\gamma, b, M)$ condition, by Lemma 4.2.9. If b exists (and is unique), then $U_{\gamma+1}$ is $\Delta^1_\beta(T_{\gamma+1}, U_\gamma)$.

The construction continues until it attempts to define an object which does not exist. By construction, the set of all pairs (T_γ, U_γ) is $\Sigma^1_\alpha(M, N)$.

Claim 4.4.5. *There is some $\eta < \omega_1$ such that (T_η, U_η) is not defined.*

Proof. Suppose otherwise. We use the facts that the sequence of such trees is a $\Sigma^1_\alpha(M,N)$ set and that W is $\mathbf{\Sigma}^1_\alpha$-correct. The point is that if $g \subset \mathrm{Coll}(\omega, \omega_1^W)$ is W-generic, then $W[g]$ is also $\mathbf{\Sigma}^1_\alpha$-correct (this follows from the proof of Lemma 4.3.4). Thus, the sequence

$$\{(T_\gamma, U_\gamma) : \gamma < \omega_1^W\}$$

is an initial segment of the sequence

$$\{(T_\gamma, U_\gamma) : \gamma < \hat{\gamma}\}^{W[g]}$$

(where $\hat{\gamma}$ is the length of the sequence in $W[g]$). In $W[g]$, there are cofinal wellfounded branches through $\bigcup_{\gamma<\omega_1} T_\gamma$ and $\bigcup_{\gamma<\omega_1} U_\gamma$, and so they belong to W, by the homogeneity of the collapse algebra. But this leads to a contradiction, e.g., by the argument of the claim in the proof of Steel [Ste10, Theorem 3.11]. □

If one assumes that M and N project to or across δ, the second conclusion in clause (4) follows from the argument of [Ste95, Lemma 2.2] (one must argue for infinite successor even ordinals as Steel does for finite odd numbers, and vice-versa). Without this assumption, all one obtains from the argument is the first part of clause (4). This finishes the proof of the lemma when α is a successor ordinal.

Suppose now that α is a limit ordinal. We need to prove (2), (3), and (4). To prove (2), note that since M is not of class S_α above δ, then one of the following holds:

1. M has no total extenders with critical point above δ, or
2. there is $\gamma < \alpha$ such that M is not of class S_γ above δ.

If there is $\gamma < \alpha$ such that M is not of class S_γ above δ, then M is $\Pi^1_{1+\gamma}$-iterable above δ, by Π^1_α-iterability, so (2) follows by induction hypothesis. Suppose that M has no total extenders with critical point above δ. It follows that every iteration tree on M above δ has a first model of the form M_0 such that $M_0 \lhd M$. Let T be such an iteration tree. By the remark just given, M_0 is not of class S_γ above δ, for some $\gamma < \alpha$. Let δ_0 be the largest Woodin cardinal of M_0 below $\mathrm{CRIT}(T)$, or $\delta_0 = \delta$ if there are no such cardinals.

The induction hypothesis applied to M_0 above δ_0 yields the result. M_0 might have Woodin cardinals above δ_0; however, the definition of a normal iteration tree dictates that extenders must be applied to the largest initial

segment possible, so M_0 must project to or across the critical point of T, which is enough for the proof of the successor case of this lemma to go through.

Clause (3) is immediate from (2). For (4), let η be least such that

$$M|\eta \neq N|\eta.$$

(We may assume such an η exists, for otherwise there is nothing to prove.) As before, one of the following holds:

1. there is $\gamma < \alpha$ such that M is not of class S_γ above δ; or
2. M is in every class S_γ above δ for $\gamma < \alpha$, but has no total extenders with critical point above δ.

Similarly for N. We consider all four combinations of 1 and 2 for M and N. If both M and N are not of class S_γ above δ for some $\gamma < \alpha$, then we are done by the induction hypothesis.

A second possibility is that none of M and N have any total extenders, but both are of class S_γ above δ for all $\gamma < \alpha$. If so, then a comparison argument shows that

$$N^\sharp_\gamma(M|\delta) \lhd M$$

for all $\gamma < \alpha$. [For each γ, M has a least initial segment of class S_γ above δ, and it must project to or across δ; since it is a proper initial segment, it is sound above δ and hence must be equal to $N^\sharp_\gamma(M|\delta)$.]

Let γ be least such that $N|\eta$ (and thus $M|\eta$) is not of class S_γ above δ and let N' be the smallest initial segment of N of class S_γ above δ. By the induction hypothesis applied to $\gamma + 1$, N' can be compared with $N^\sharp_\gamma(N|\delta)$, so they must be equal. Hence,

$$N' \lhd M,$$

contrary to the choice of η.

A third possibility is that M is not of class S_γ above δ for some least $\gamma < \alpha$, and N is of class S_γ above δ for all $\gamma < \alpha$. We can easily reduce to the case in which γ is a successor ordinal, so let us assume that. By the argument of the previous case,

$$N^\sharp_\gamma(M|\gamma) \lhd N.$$

But then M and $N^\sharp_\gamma(M|\gamma)$ can be compared in the usual way, so M must embed into an initial segment of an iterate of $N^\sharp_\gamma(M|\gamma)$. The resulting tree on $N^\sharp_\gamma(M|\gamma)$ can be viewed as a tree on N, which yields the desired outcome.

The remaining case is analogous, except that it uses the induction hypothesis. Moreover, a similar argument with the added assumption that both M and N project to or across δ, yields the conclusion that

1. $M \trianglelefteq N$, or
2. $N \trianglelefteq M$,

again using the induction hypothesis. This completes the proof of Lemma 4.4.3. □

We state a consequence of Lemma 4.4.3 that we will need. By the results of the following sections, the hypothesis can be weakened to σ-projective determinacy only.

Theorem 4.4.6. *Suppose that σ-projective determinacy holds and $N^\sharp_\alpha(x)$ exists for almost all $x \in \mathbb{R}$. Then, for almost all $x \in \mathbb{R}$,*

$$N^\sharp_\alpha(x) \models \text{“there is a } \sigma\text{-projective wellordering of } \mathbb{R}\text{.”}$$

Proof. Suppose first that α is a successor ordinal. For $y, z \in \mathbb{R}$, write $y \preceq z$ if there is a (countable) Π^1_α-iterable, sound x-premouse M with $\rho(M) = \omega$ which satisfies the hypotheses of Lemma 4.4.3 (with $\delta = \omega$) such that $y, z \in M$ and y is constructed before z in M. By Lemma 4.4.3, if y and z belong to $N^\sharp_\alpha(x)$, then $y \preceq z$ if, and only if, for *every* x-premouse $M \in N^\sharp_\alpha(x)$ as above, y is constructed before z in M. Thus, $\preceq \cap N^\sharp_\alpha(x)$ is in $\Delta^1_{\alpha+1}(x)$, so it can be computed in $N^\sharp_\alpha(x)$ by Lemma 4.3.4. It follows that $\preceq \cap N^\sharp_\alpha(x)$ is a wellordering of $\mathbb{R} \cap N^\sharp_\alpha(x)$ in

$$(\mathbf{\Delta}^1_{\alpha+1})^{N^\sharp_\alpha(x)}.$$

The limit case is immediate, since by definition α is recursive in x, so the sequence

$$\{N^\sharp_\beta(x) : \beta < \alpha\}$$

belongs to $N^\sharp_\alpha(x)$ and

$$N^\sharp_\alpha(x)|\omega_1^{N^\sharp_\alpha(x)} \models \mathsf{ZFC} \text{ - Powerset} + V = L\Big(\bigcup_{\beta<\alpha} N^\sharp_\beta(x)\Big).$$

This proves the theorem. □

4.5 Definable Determinacy

In this section, we begin working towards the proof of Theorem 12.2.6. It will make the proof significantly easier that we assume σ-projective determinacy throughout (as opposed to, say, $\Sigma^1_{\alpha+1}$-determinacy). For the rest of this section, we assume:

Inductive hypothesis σ-projective determinacy holds and $N^\sharp_\alpha(x)$ exists and is ω_1-iterable for almost all $x \in \mathbb{R}$.

The inductive hypothesis thus should be added to every lemma stated in this section. In order to mine consistency strength, we need to find models M of class S_α such that

$$M \models \mathsf{OD}\text{-determinacy}.$$

These models will be those of the form $Lp^\alpha(x)$. We have the following correctness lemma for $Lp^\alpha(x)$.

Lemma 4.5.1. *Suppose $\alpha < \omega_1^x$. Then, $Lp^\alpha(x)$ is $\mathbf{\Sigma}^1_\alpha$-correct.*

Proof. This is immediate from Lemma 4.3.4 and Lemma 4.2.24, for successor α, and from Lemma 4.3.4 and Claim 4.3.5, for limit α. □

We will need the following property of models of the form $Lp^\alpha(x)$.

Lemma 4.5.2. *Suppose $\alpha < \omega_1^x$ and $\alpha < \omega_1^y$. Suppose moreover that $x \in L[y]$ and $y \in L[x]$. Then*

1. *$|Lp^\alpha(y)| = |Lp^\alpha(x)|$,*
2. *Let P be the model of length ω_1^V computed in $Lp^\alpha(y)$ using the definition of $Lp^\alpha(x)$. Then $P = Lp^\alpha(x)$.*

Proof. We begin with (1). If α is a limit ordinal, then this follows from Lemma 4.2.27. Suppose now that α is a successor, say, $\alpha = \beta + 1$.

Since $x \in L[y]$, $x \in y^\sharp$ (where $y^\sharp$ is regarded as a y-premouse). It is easy to verify that

$$|N^\sharp_\alpha(y)| = |N^\sharp_\alpha(y^\sharp)|.$$

Since $x \in L[y]$ and $y \in L[x]$, an ordinal is an x-indiscernible if, and only if, it is a y-indiscernible; thus, $|x^\sharp| = |y^\sharp|$. As above,

$$|N^\sharp_\alpha(x)| = |N^\sharp_\alpha(x^\sharp)|.$$

It follows that

$$|N^\sharp_\alpha(y)| = |N^\sharp_\alpha(x)|, \tag{4.3}$$

from which (1) follows, using Lemma 4.2.24.

For (2), the limit case again follows from Lemma 4.2.27. Suppose that $\alpha = \beta + 1$ and that (2) fails. Let

$$P_\iota = Lp^\alpha_\iota(x)$$

for $\iota < \omega_1^V$ and let

$$Q_\iota = (Lp^\alpha_\iota(x))^{Lp^\alpha(y)},$$

for $\iota < \mathsf{Ord}^{Lp^\alpha(y)} = \omega_1^V$ (the definition of $(Lp^\alpha_\iota(x))^{Lp^\alpha(y)}$ involves countable iterability in $Lp^\alpha(y)$). Let ι be least such that

$$P_{\iota+1} \neq Q_{\iota+1}.$$

There are two cases. If $P_{\iota+1} \not\subset Q_{\iota+1}$, then there is some x-premouse P such that

1. P extends Q_ι and has Ord^{Q_ι} as cutpoint,
2. P is ω_1-iterable above Ord^{Q_ι},
3. P is not of class S_α above Ord^{Q_ι},
4. P is sound above Ord^{Q_ι},
5. P projects to or accross Ord^{Q_ι},
6. either $P \notin Lp^\alpha(y)$ or P is not countably iterable above Ord^{Q_ι} in $Lp^\alpha(y)$.

By (1), $P \in Lp^\alpha(y)$, so P is not countably iterable in $Lp^\alpha(y)$. However, since P projects to or across Ord^{Q_ι}, its iteration strategy above Q_ι is given by Q-structures which are not of class S_β, and those belong to $Lp^\alpha(y)$—a contradiction.

Hence, $Q_{\iota+1} \not\subset P_{\iota+1}$, so there is some x-premouse P such that

1. P extends Q_ι and has Ord^{Q_ι} as cutpoint,
2. P is countably iterable above Ord^{Q_ι} in $Lp^\alpha(y)$,
3. P is not of class S_α above Ord^{Q_ι},
4. P is sound above Ord^{Q_ι},

5. P projects to or accross Ord^{Q_ι},

6. P is not ω_1-iterable above Ord^{Q_ι} in V.

Let M be the ω_1th iterated ultrapower of $N^\sharp_\alpha(y)$ by its topmost measure and its images, so that

$$M|\omega_1 = Lp^\alpha(y)$$

by Lemma 4.2.24. Since

$$M \cap \mathbb{R} = Lp^\alpha(y) \cap \mathbb{R},$$

P is ω_1^M-iterable in M. Let T and U be trees resulting from a comparison of P and $Lp^\alpha(x)$ by least disagreement inside M. If either of those trees is nontrivial, they must drop (in the case of P, it must drop in model or degree, since P projects to or across Ord^{Q_ι} and has no total extenders below Ord^{Q_ι}; in the case of $Lp^\alpha(x)$, it must drop in model, since $Lp^\alpha(x)$ has no total extenders). The usual argument from the proof of the comparison theorem then shows that one of T or U must be trivial.

On the one hand, T cannot be nontrivial, for then there is an iterate P^* of P of which $Lp^\alpha(x)$ is an initial segment. Since $Lp^\alpha(x)$ has no total extenders, normality of U and coherence of the extender sequence of P implies that all extenders used in T must be (equivalent to) measures of Mitchell order 0. Thus, T is definable over $Lp^\alpha(x)^{Lp^\alpha(y)}$ from ordinal parameters, so

$$Lp^\alpha(x) \subset Lp^\alpha(x)^{Lp^\alpha(y)}.$$

It then follows from (1) that

$$Lp^\alpha(x) = Lp^\alpha(x)^{Lp^\alpha(y)},$$

contrary to the assumption. On the other hand U also cannot be nontrivial, for otherwise there is an iterate Q^* of $Lp^\alpha(x)$ of which P is an initial segment; but this implies that P is ω_1-iterable, contrary to the assumption. □

We can now prove that the structure $Lp^\alpha(x)$ is a model of OD-determinacy. This is done by means of a Kechris-Solovay–type game, as is usual. We begin with the limit case.

Lemma 4.5.3. *Let α be a limit ordinal. For all $x \in \mathbb{R}$, there is some x_0 such that whenever $z \in \mathbb{R}$ is recursive in x_0,*

$$Lp^\alpha(z) \models \mathsf{OD}_x\text{-}\textit{determinacy.}$$

Proof. Let $x \in \mathbb{R}$ be arbitrary. We first show that every real x_0 is recursive in some z such that

$$Lp^{\alpha}(z) \models \mathsf{OD}_x\text{-determinacy.}$$

Suppose otherwise. Then, there is some $z \in \mathbb{R}$ such that whenever z is recursive in y, then

$$Lp^{\alpha}(y) \not\models \mathsf{OD}_x\text{-determinacy.}$$

Consider the following game, G:

I	$u,$	a
II	$v,$	b

Here, players I and II alternate (ω-many) turns playing natural numbers and construct reals $u, v, a, b \in \mathbb{R}$. Player I wins if, and only if, there exist a countable ordinal ζ and a sequence[6] $\{P_{\xi,\gamma} : \xi \leq \zeta, \gamma < \alpha\}$ such that for each ξ,

1. $P_{0,0} = u \oplus v$;
2. $P_{\xi,\gamma+1}$ is a Π^1_α-iterable $u \oplus v$-premouse extending $P_{\xi,\gamma}$ and it is of class S_γ above $P_{\xi,\gamma+1}$ but not of class $S_{\gamma+1}$ above $P_{\xi,\gamma+1}$;
3. $P_{\xi,\gamma+1}$ projects to or across $\mathsf{Ord} \cap P_{\xi,0}$ and has no Woodin cardinals;
4. If γ is a limit ordinal, then $P_{\xi,\gamma} = \bigcup_{\eta<\gamma} P_{\eta,\eta}$;
5. If ξ is a limit ordinal, then $P_{\xi,0} = \bigcup_{\eta<\xi} P_{\eta,0}$;
6. There is some $\gamma < \alpha$ and an initial segment P of $P_{\zeta,\gamma}$ such that

$$P \models \mathsf{ZFC} + \neg\mathsf{OD}_x\text{-determinacy, and}$$

$$P \models \forall w \in \mathbb{R}\,(V = |Lp^{\alpha}(w \oplus (u \oplus v))|). \tag{4.4}$$

7. $a \oplus b \in P_\zeta$ and $z \in P_\zeta$.
8. letting P be as above, we have $a \oplus b \in A$, where A is the least non-determined ordinal-definable set of reals in P.

[6] See Definition 4.2.26.

The game is certainly σ-projective and thus determined. We will reach a contradiction by showing that no player can have a winning strategy. For this, suppose that Player I has a winning strategy σ. By the definition of the game, z is recursive in σ, so

$$Lp^\alpha(\sigma) \not\models \mathsf{OD}_x\text{-determinacy}.$$

By Lemma 4.2.27,

$$|Lp^\alpha(\sigma)| = |Lp^\alpha(u \oplus \sigma)| \text{ for every } u \in \mathbb{R} \cap Lp^\alpha(\sigma)$$

and, in fact,

$$Lp^\alpha(\sigma) \models \forall u \in \mathbb{R}\,(V = |Lp^\alpha(u \oplus \sigma)|).$$

Let us say that a countable premouse N satisfies $(*)$ if the following hold for some $z \in \mathbb{R}$:

1. N is a sound z-premouse,
2. N is not of class S_α and
$$N \models \mathsf{ZFC} + \neg\mathsf{OD}_x\text{-determinacy};$$
3. N satisfies (4.4) with z in place of $u \oplus v$, i.e., $N \models \forall u \in \mathbb{R}\,(V = |Lp^\alpha(u \oplus z)|)$.

By the comment above, $Lp^\alpha(\sigma)$ satisfies $(*)$.

We now define a real r and a r-premouse N. Let N_0 be the smallest initial segment of $Lp^\alpha(\sigma)$ that satisfies $(*)$. If for every $z_0 \in N_0$, no proper initial segment of $Lp^\alpha(z_0 \oplus \sigma)$ satisfies $(*)$, then we set $N = N_0$. Otherwise, pick such a z_0 and define N_1 to be the smallest initial segment of $Lp^\alpha(z_0 \oplus \sigma)$ satisfying $(*)$. It follows that

$$\mathsf{Ord}^{N_1} < \mathsf{Ord}^{N_0}.$$

If one continues on to define N_2, N_3, etc., there is $i \in \mathbb{N}$ such that $N_i = N_{i+1} =: N$ and N is a r-premouse for some $r \in \mathbb{R}$.

Let A_N be the least non-determined set of reals in OD_x^N. There is a natural strategy for Player I in A_N, namely,

$$\tau(b) = a \text{ if, and only if, } \exists u \in \mathbb{R} \cap N\, \sigma(r, b) = (u, a)$$

Clearly $\tau \in N$. Let $b \in N$ and $a \oplus b$ be a play by τ; we claim $a \oplus b \in A_N$. Let $u \in \mathbb{R} \cap N$ be such that $\sigma(r, b) = (u, a)$. Since σ is winning for Player I in G, there are a countable ordinal ζ and a sequence $\{P_{\xi,\gamma} : \xi \leq \zeta, \gamma < \alpha\}$ such that for each ξ,

1. $P_{0,0} = u \oplus r$;
2. $P_{\xi,\gamma+1}$ is a Π^1_α-iterable $u \oplus r$-premouse extending $P_{\xi,\gamma}$ and it is of class S_γ above $P_{\xi,\gamma+1}$ but not of class $S_{\gamma+1}$ above $P_{\xi,\gamma+1}$;
3. $P_{\xi,\gamma+1}$ projects to or across $\mathsf{Ord} \cap P_{\xi,0}$ and has no Woodin cardinals;
4. If γ is a limit ordinal, then $P_{\xi,\gamma} = \bigcup_{\eta<\gamma} P_{\eta,\eta}$;
5. If ξ is a limit ordinal, then $P_{\xi,0} = \bigcup_{\eta<\xi} P_{\eta,0}$;
6. There is some $\gamma < \alpha$ and an initial segment P of $P_{\zeta,\gamma}$ such that $P \models (*)$.
7. $a \oplus b \in P_\zeta$ and $z \in P_\zeta$.
8. $a \oplus b \in A$, where A is the least non-determined ordinal-definable set of reals in P.

We show inductively that each $P_{\xi,\gamma}$ is an initial segment of $Lp^\alpha(u \oplus r)$. Suppose we have shown that $P_{\xi,\gamma} \lhd Lp^\alpha(u \oplus r)$. Since $Lp^\alpha(u \oplus r)$ is closed under the operation

$$a \mapsto N^\sharp_{\gamma+1}(a),$$

and $P_{\xi,\gamma} \lhd Lp^\alpha(u \oplus r)$ by induction hypothesis, it follows from Lemma 4.4.3 (applied to $\gamma+2$ instead of $\gamma+1$, and with $\delta = \mathsf{Ord} \cap P_{\xi,0}$) that

$$P_{\xi,\gamma+1} \lhd N^\sharp_{\gamma+1}(P_{\xi,\gamma}) \lhd Lp^\alpha(u \oplus r),$$

as desired. Hence, if γ is such that for some initial segment P of $P_{\zeta,\gamma}$, we have

$$P \models (*)$$

and P is the minimal such initial segment, then $P \lhd Lp^\alpha(u \oplus r)$, so $|P| = |N|$, by the minimality condition on N. Since $a \oplus b$ belongs to the least non-determined set of reals in P, it belongs to A_N, as claimed.

The case that it is Player II who has a winning strategy σ in G is similar. As before, it induces a strategy τ for the least non-determined game ordinal-definable from x on an r-premouse N defined as above, for some $r \in \mathbb{R}$. This strategy is given by

$$\tau(a) = b \text{ if, and only if, } \exists v \in \mathbb{R} \cap N\, \sigma(r, a) = (v, b).$$

If so, one reaches a contradiction as in the case that Player I has a winning strategy. The only difference is that we must show that Player II does not

win vacuously, i.e., that she does not win because there exist no sequences $\{P_{\xi,\gamma} : \xi \leq \zeta, \gamma < \alpha\}$ as in the rules of the game. Given a play (r, v, a, b), such a sequence can be easily extracted from $Lp^\alpha(r \oplus v)$ using Definition 4.2.26 (the sequence given there does not quite work, as it includes models with Woodin cardinals, but one can obtain the desired sequence by adding a level of constructibility at each stage).

Now, to see that OD_x-determinacy holds of almost every real, notice that otherwise every real x is recursive in some z for which there is a sequence $\{P_{\xi,\gamma} : \xi \leq \zeta, \gamma < \alpha\}$, for some ordinal ζ, such that

1. $P_{0,0} = z$;
2. $P_{\xi,\gamma+1}$ is a Π^1_α-iterable z-premouse extending $P_{\xi,\gamma}$ and it is of class S_γ above $P_{\xi,\gamma+1}$ but not of class $S_{\gamma+1}$ above $P_{\xi,\gamma+1}$;
3. $P_{\xi,\gamma+1}$ projects to or across $\mathsf{Ord} \cap P_{\xi,0}$ and has no Woodin cardinals;
4. If γ is a limit ordinal, then $P_{\xi,\gamma} = \bigcup_{\eta<\gamma} P_{\eta,\eta}$;
5. If ξ is a limit ordinal, then $P_{\xi,0} = \bigcup_{\eta<\xi} P_{\eta,0}$;
6. There is some $\gamma < \alpha$ and an initial segment P of $P_{\zeta,\gamma}$ such that $P \models (*)$.

Arguing as above, one can show that every element of such a sequence is an initial segment of $Lp^\alpha(z)$. However, the conjunction of all these statements is certainly σ-projective, so by σ-projective determinacy, it must fail for almost all $x \in \mathbb{R}$, and we have just shown that this is not the case. This completes the proof of the lemma. □

The corresponding version of Lemma 4.5.3 for successor ordinals can be proved directly for the models $N^\sharp_{\alpha+1}(x)$. This is sufficient (and necessary), by Lemma 4.2.24. The proof is similar:

Lemma 4.5.4. *Let α be infinite. For all $x \in \mathbb{R}$, there is some x_0 such that whenever x_0 is recursive in z,*

$$N^\sharp_{\alpha+1}(z) \models \mathsf{OD}_x\text{-}\textit{determinacy.}$$

Proof. We will show that the conclusion holds for a Turing-cofinal set of x_0. One can argue as in the proof of Lemma 4.5.3, using σ-projective determinacy, to show that it then holds of almost all $x_0 \in \mathbb{R}$. Before proving the lemma, we begin with an observation:

Let α be recursive in some $x \in \mathbb{R}$ and let δ be the least Woodin cardinal of $N^\sharp_{\alpha+1}(x)$. Then δ is a limit of measurable cardinals in $N^\sharp_{\alpha+1}(x)$ and, since $N^\sharp_{\alpha+1}(x)|\delta$ has no inner model with a Woodin cardinal that is closed under

$$a \mapsto N^\sharp_\alpha(a),$$

$K(x)^{N^\sharp_{\alpha+1}(x)}|\Omega_0$ exists for all $\Omega_0 < \delta$ which are measurable in $N^\sharp_{\alpha+1}(x)$. As in Schindler [Sch06], one can define $K(x)^{N^\sharp_{\alpha+1}(x)}|\delta$ in $N^\sharp_{\alpha+1}(x)$ as the union of the "partial $K(x)$s." By an argument of Steel (cf. [Sch06, Lemma 1.1] and Schimmerling-Steel [SS99, Theorem 2.3]) we have

$$N^\sharp_{\alpha+1}(x)|\delta = K(x)^{N^\sharp_{\alpha+1}(x)}|\delta.$$

Now, we argue that if x is recursive in $y \in N^\sharp_{\alpha+1}(x)$, then

$$|N^\sharp_{\alpha+1}(x)|\delta| = |K(y)^{N^\sharp_{\alpha+1}(x)}|\delta|.$$

To see this, suppose otherwise. Since x is recursive in y,

$$x \in K(y)^{N^\sharp_{\alpha+1}(x)}|\delta.$$

Let ξ index the least extender witnessing a disagreement between $N^\sharp_{\alpha+1}(x)$ and

$$P := K(x)^{K(y)^{N^\sharp_{\alpha+1}(x)}}.$$

The index ξ must be countable, for otherwise $N^\sharp_{\alpha+1}(x)$ and P have the same reals. This would imply that $N^\sharp_{\alpha+1}(x)$ and $K(y)^{N^\sharp_{\alpha+1}(x)}$ have the same reals, whereby the extender sequences of $N^\sharp_{\alpha+1}(x)$ and $K(y)^{N^\sharp_{\alpha+1}(x)}$ agree above $\omega_1^{N^\sharp_{\alpha+1}(x)}$, contrary to the assumption that the models were different. Thus, ξ is countable. But then let $\xi \leq \zeta_0$ and $\xi \leq \zeta_1$ be countable and such that

$$\rho_\omega^{J^{N^\sharp_{\alpha+1}(x)}_{\zeta_0}} = \rho_\omega^{J^P_{\zeta_1}} = \omega.$$

If so, then it follows from Lemma 4.4.3 that

1. $J^{N^\sharp_{\alpha+1}(x)}_{\zeta_0} \trianglelefteq J^P_{\zeta_1}$, or
2. $J^P_{\zeta_1} \trianglelefteq J^{N^\sharp_{\alpha+1}(x)}_{\zeta_0}$,

either of which is again a contradiction. Thus,

$$|N^{\sharp}_{\alpha+1}(x)|\delta| = |K(y)^{N^{\sharp}_{\alpha+1}(x)}|\delta|,$$

as claimed.

This time, let us say that a countable premouse M satisfies $(*)$ if there is some $z \in \mathbb{R}$ such that:

1. M is a sound z-premouse with no Woodin cardinals;
2. M is not of class $S_{\alpha+1}$ and
$$M \models \mathsf{ZFC} + \neg\mathsf{OD}_x\text{-determinacy};$$
3. If z is recursive in $y \in M$, then $|M| = |K(y)^M|$ and, in fact, if
$$z \leq_T x_0 \leq_T \ldots \leq_T x_k,$$
$x_k \in M$, and
$$P = K(x_k)^{K(x_{k-1})^{\cdot^{\cdot^{\cdot^{K(x_0)^M}}}}},$$
then $|P| = |M|$.

If y is a real, we write δ_y for the smallest Woodin cardinal of $N^{\sharp}_{\alpha+1}(y)$. By the observation above, if y is a real and $N^{\sharp}_{\alpha+1}(y) \models \neg\mathsf{OD}_x$-determinacy, then $N^{\sharp}_{\alpha+1}(y)|\delta_y$ satisfies $(*)$.

Now, assume towards a contradiction that there is some $x_0 \in \mathbb{R}$ such that $N^{\sharp}_{\alpha+1}(y)|\delta_y$ satisfies $(*)$ whenever x_0 is recursive in y. Consider the following game G:

I	$u,$	a
II	$v,$	b

Here, players I and II alternate turns playing natural numbers and construct reals $u, v, a, b \in \mathbb{R}$. Player I wins if, and only if, there exists an $u \oplus v$-premouse M such that

1. M satisfies $(*)$;
2. No proper initial segment of M satisfies $(*)$;
3. M is $\Pi^1_{\alpha+1}$-iterable;
4. $a \oplus b \in M$, $x_0 \in M$;

5. $a \oplus b \in A$, where A is the least non-determined OD_x set of reals in M.

The game is certainly σ-projective, so it is determined. Suppose first that Player I has a winning strategy σ. Since x_0 is recursive in σ, $N_{\alpha+1}(\sigma)|\delta_\sigma$ satisfies $(*)$. Let N_0 be the least initial segment of $N_{\alpha+1}(\sigma)$ that satisfies $(*)$. Suppose first that whenever $w \in \mathbb{R} \cap N_0$, $\sigma \leq_T w$, and $\xi < \mathsf{Ord}^{N_0}$,

$$K(w)^{N_0}|\xi \not\models (*);$$

in this case, let $N = N_0$. Otherwise choose such w_0 and ξ_0 and let

$$N_1 = K(w_0)^{N_0}|\xi_0.$$

By repeating this process sufficiently many times, we arrive at some $z \in \mathbb{R}$ and a z-premouse N such that whenever $w \in \mathbb{R} \cap N$ and $\xi < \mathsf{Ord}^N$,

$$K(w)^N|\xi \not\models (*),$$

and moreover $\sigma \in N$.

N is not of class $S_{\alpha+1}$, has no Woodin cardinals, and is ω_1-iterable. Let A be the least non-determined OD_x set of reals in N and define a strategy τ for Player I in A by

$\tau(b) = a$ if, and only if, there exists $u \in \mathbb{R}$ such that $\sigma(z, b) = (u, a)$.

Since τ is recursive in σ, $\tau \in N$. Let $a \oplus b \in \mathbb{R} \cap N$ be a play by τ. We claim $a \oplus b \in A$. Choose $u \in \mathbb{R} \cap N$ such that $\sigma(z, b) = (u, a)$. Since σ is a winning strategy for Player I in G, there exists an $(u \oplus z)$-premouse M such that

1. M satisfies $(*)$;
2. No proper initial segment of M satisfies $(*)$;
3. M is $\Pi^1_{\alpha+1}$-iterable;
4. $a \oplus b \in M$, $x_0 \in M$;
5. $a \oplus b \in A_M$, where A_M is the least non-determined OD_x set of reals in M.

We first compare M with $N^\sharp_{\alpha+1}(u \oplus z)$. Applying Lemma 4.4.3 (with $\delta = \omega$) to M and $N^\sharp_{\alpha+1}(u \oplus z)|\delta_{u\oplus z}$, we obtain a normal iteration tree T on the premouse $N^\sharp_{\alpha+1}(u \oplus z)|\delta_{u\oplus z}$ and a putative iteration tree U on M, both of successor length $\lambda + 1$ such that one of the following holds:

1. $M^T_{\lambda+1} \trianglelefteq M^U_{\lambda+1}$ (so $\mathsf{Ord}^{M^T_{\lambda+1}}$ belongs to the wellfounded part of $M^U_{\lambda+1}$) and T is non-dropping, or

2. $M^U_{\lambda+1} \trianglelefteq M^T_{\lambda+1}$ (so $M^U_{\lambda+1}$ is wellfounded), and U is non-dropping.

The first case cannot occur, for then $M^U_{\lambda+1}$ and has as proper initial segment a non-dropping iterate of $N^\sharp_{\alpha+1}(u \oplus z)|\delta_{u\oplus z}$, which (since x_0 is recursive in z) has an initial segment satisfying $(*)$, contrary to the minimality of M, so the second case occurs.

By the minimality conditions, $M^U_{\lambda+1}$ is the smallest initial segment Q of $M^T_{\lambda+1}$ such that

$$Q \models (*)$$

and, since σ is winning strategy for Player I in G, $a \oplus b$ must belong to the least non-determined OD_x set of reals in $M^U_{\lambda+1}$. By elementarity, $a \oplus b$—which is not moved by the iteration embedding—belongs to the least non-determined OD_x set of reals of the smallest initial segment P of $N^\sharp_{\alpha+1}(u \oplus z)$ such that

$$P \models (*).$$

Let P be as above. It is an initial segment of $N^\sharp_{\alpha+1}(u \oplus z)$, so it is iterable. We compare it with $K(u \oplus z)^N$. Since $N \models (*)$,

$$|K(u \oplus z)^N| = |N|,$$

and, by definition of N, no proper initial segment of $K(u \oplus z)^N$ satisfies $(*)$. It follows that $K(u \oplus z)^N$ and P must have a common iterate. Since reals cannot be moved by the iteration embedding, we have by elementarity that $a \oplus b$ belongs to the least non-determined OD_x set of reals in $K(u \oplus z)^N$, which is the least non-determined OD_x set of reals in N. Therefore, τ is a winning strategy for Player I in A—a contradiction.

Similarly, one reaches a contradiction if Player II has a winning strategy in G. As in Lemma 4.5.3, one needs to show that Player II does not win vacuously, but this can be done. □

The main consequence of the last two lemmata that we will need is as follows.

Lemma 4.5.5. *For all $x \in \mathbb{R}$ there is some x_0 such that whenever x_0 is recursive in $z \in \mathbb{R}$,*

$$\mathsf{HOD}^{Lp^\alpha(z)}_x \models \text{“}\omega_2^{Lp^\alpha(z)} \text{ is an inaccessible cardinal.”}$$

Proof. Let x_0 be so that $Lp^\alpha(x_0) \models \mathsf{OD}_x$-determinacy and suppose that x_0 is recursive in z. Recall that Θ_0 is the supremum of all ordinals α onto which there is an ordinal-definable surjection from $\mathbb{R}$. We first need to show that

$$Lp^\alpha(z) \models \Theta_0 = \omega_2.$$

The Continuum Hypothesis holds in $Lp^\alpha(z)$, so $\Theta_0^{Lp^\alpha(z)} \leq \omega_2^{Lp^\alpha(z)}$. Notice that in $Lp^\alpha(z)$ there is a wellorder of $\mathbb{R} \cap Lp^\alpha(z)$ definable from z and ordinal parameters. For successor α, this follows from Theorem 4.4.6; for limit α, this follows from the construction of $Lp^\alpha(z)$. Thus, if $\gamma < \omega_2^{Lp^\alpha(z)}$, there is a surjection ρ_z from $\mathbb{R}$ onto γ in $\mathsf{OD}_z^{Lp^\alpha(z)}$. But then one can average over the reals to obtain such a function with no real parameters, i.e., letting $\phi(z, y, \eta, \vec{\beta})$ be the formula defining ρ_z (so $\rho_z(y) = \eta$ if, and only if, $\phi(z, y, \eta, \vec{\beta})$ holds), one sets

$$\rho(y) = \begin{cases} \eta, \text{ if, } \phi((y)_0, (y)_1, \eta, \vec{\beta}) \text{ holds of a unique } \eta \in \mathsf{Ord}, \\ 0, \text{ otherwise.} \end{cases}$$

This shows that

$$Lp^\alpha(z) \models \Theta_0 = \omega_2.$$

This, together with Lemmata 4.5.3 and 4.5.4, combines with the argument of Moschovakis's Coding Lemma to finish the proof. Cf. Claim 2 in the proof of [MSW16, Theorem 2.24] for details. (Cf. also [KW10, Remark 3.5].) □

4.6 The Strength of σ-Projective Determinacy

In this section, we prove Theorem 12.2.6. One of the implications we proved already; it remains to show that σ-projective determinacy implies that the classes S_α are all non-empty. Because of the induction hypothesis we employ, we actually need to show that $N_\alpha^\sharp(x)$ exists and is ω_1-iterable for all α and almost all $x \in \mathbb{R}$; this will take little additional work. The limit cases of the induction are immediate; it remains to take care of the (infinite) successor case. Fortunately, most of the work has been done already. What remains is almost identical to the proof of Müller-Schindler-Woodin [MSW16, Theorem 3.20]. There are some superficial differences, so we sketch the argument, but refer the reader to [MSW16, Theorem 3.20] for the full proof. We continue assuming:

Inductive hypothesis σ-projective determinacy holds and $N^\sharp_\alpha(x)$ exists and is ω_1-iterable for almost all $x \in \mathbb{R}$. In particular, the conclusion of Lemma 4.5.5 holds.

Let $x, x_0 \in \mathbb{R}$ be a pair of reals for which the conclusion of Lemma 4.5.5 holds, i.e., whenever x_0 is recursive in z,

$$\mathsf{HOD}_x^{Lp^\alpha(z)} \models \text{“}\omega_2^{Lp^\alpha(z)} \text{ is an inaccessible cardinal.”}$$

Assume, additionally, that α is recursive in x. By the inductive hypothesis, if $\beta \leq \alpha$, then $N^\sharp_\beta(x)$ exists and, in fact, $N^\sharp_\beta(P)$ exists whenever P is a countable premouse containing x. Let F be the mouse operator

$$a \mapsto N^\sharp_\alpha(a),$$

defined on almost all of HC.

If $x \leq_T z$ and $a = z$ is a real or a is a z-premouse, one would like to define the countably certified construction $K^{c,F}(a)$ and the relativized core model $K^F(a)$. The problem here is that the inductive hypothesis ensures the existence of $N^\sharp_\alpha(a)$ only for *countable* a. The remedy for this is to proceed with the argument inside an inner model.

Lemma 4.6.1. *Suppose that x_0 is recursive in z. Then, there is an inner model M_z of* ZFC *containing z such that*

1. $\mathsf{Ord} \cap M_z = \omega_1$,
2. *M_z is $\mathbf{\Sigma}^1_{\alpha+1}$-correct,*
3. *M_z is closed under F and $F(a)$ is countably iterable in M_z for all $a \in M_z$.*

Proof Sketch. The proof is like that of Claim 2 in [MSW16, Theorem 3.20]. Roughly, M_z is the smallest model of ZFC of height ω_1 closed under F and a σ-projective (in fact, $\mathbf{\Pi}^1_{\alpha+1}$ or $\mathbf{\Pi}^1_{\alpha+2}$) uniformizing function for the universal $\mathbf{\Pi}^1_{\alpha+1}$ set. The crucial difference is that we are assuming full σ-projective determinacy, which implies the uniformization property for $\Pi^1_\beta(z')$ for all odd β (granted z' is reasonable). This is what allows us to prove the necessary correctness, regardless of whether α is even or odd. The resulting model is seen to be definable by a set in $\mathbf{\Pi}^1_{\alpha+\varepsilon}$. By σ-projective determinacy, there is no uncountable σ-projective sequence of pairwise distinct reals, and this is what allows us to prove that M_z is a model of ZFC. □

We would like to show that

$$M_z \models \text{“}N^{\sharp}_{\alpha+1}(x) \text{ exists.”} \tag{4.5}$$

If so, then an argument as in [MSW16] would show that $N^{\sharp}_{\alpha+1}(x)$ is the same as computed in almost all M_y, for $y \in \mathbb{R}$ (this uses the correctness of M_y and Lemma 4.4.3) and is ω_1-iterable in V, which would complete the proof of Theorem 12.2.6.

Hence, we suppose that the negation of (4.5) holds. Working in M_z, we can apply Schindler-Steel [SS14, Theorem 1.3.20] (making use of Jensen-Steel [JS13, Theorem 1.1] to rid ourselves of the large-cardinal assumption) to obtain the existence (and iterability) of the core model $K^F(x)$.

The remainder of the argument consists in reaching a contradiction from the assumption that

$$K_x := (K^F(x))^{M_z}$$

exists. In M_z, the core model K_x satisfies *weak covering*. Thus, one can find singular limit cardinals γ in M_z of uncountable cofinality such that

$$(\gamma^+)^{K_x} = (\gamma^+)^{M_z}.$$

Fix any such γ.

Lemma 4.6.2. *There is a real $y \geq_T z$ such that:*

1. $(\gamma^+)^{M_z} = \omega_2^{Lp^\alpha(y)}$,
2. $K_x|(\gamma^+)^{M_z} \in Lp^\alpha(y)$, *and*
3. *(the universe of) $Lp^\alpha(y)$ is a generic extension of $Lp^\alpha(K_x|(\gamma^+)^{M_z})$.*

Proof. Cf. Subclaim 2 in the proof of [MSW16, Theorem 3.20]. The fact that γ has uncountable cofinality in M_z is used in the proof. In particular, it implies that

$$\omega_1^{M_z} \leq \mathrm{cof}\,(\gamma)^{Lp^\alpha(K_x|(\gamma^+)^{M_z})},$$

since $Lp^\alpha(K_x|(\gamma^+)^{M_z})$ is a definable class from the point of view of M_z (this follows from the fact that M_z is closed under F and from Definition 4.2.26—for limit α—and Lemma 4.2.24—for successor α). □

Using the notion of “certificate” from Mitchell-Schindler [MS04], one can make sense of

$$(K^{c,F}(x))^{Lp^\alpha(y)}.$$

To prove the analog of Subclaim 4 in the proof of [MSW16, Theorem 3.20], we need to argue slightly differently, since the proof in [MSW16, Theorem 3.20] uses a lemma whose counterpart we have not proved.

Lemma 4.6.3. *$(K^{c,F}(x))^{Lp^\alpha(y)}$ is definable over $\mathsf{HOD}_x^{Lp^\alpha(y)}$*

Proof. Working in $Lp^\alpha(y)$, and given a real u, one can define the model

$$Lp^\alpha(u)^{Lp^\alpha(y)}$$

of height ω_1^V (cf. Lemma 4.5.2), as well as its extender sequence, E_u (which can be amenably and uniformly coded by a set of ordinals). Because we use the construction of [MS04] (there are no large cardinals in $Lp^\alpha(y)$), $(K^{c,F}(x))^{Lp^\alpha(y)}$ can be defined from x and (y, E_y), the latter of which is a class from the point of view of $Lp^\alpha(y)$. By Lemma 4.5.2(2),

$$Lp^\alpha(y) \models V = Lp^\alpha(y).$$

Let $u \in \mathbb{R} \cap Lp^\alpha(y)$ be arbitrary such that

$$Lp^\alpha(y) \models V = Lp^\alpha(u)$$

and let K^u be the result of defining

$$(K^{c,F}(x))^{Lp^\alpha(y)}$$

using (u, E_u) instead of (y, E_y). We claim that $K^u = (K^{c,F}(x))^{Lp^\alpha(y)} (= K^y)$. As in [MSW16], this will imply that the construction is not sensitive to the choice of u, so that the parameter y in the definition of $(K^{c,F}(x))^{Lp^\alpha(y)}$ can be dispensed with; this will suffice.

The pair $(u, E_u \restriction \omega_1^{Lp^\alpha(y)})$ is definable over the structure

$$(L_{\omega_1^{Lp^\alpha(y)}}[y, E_y], y, E_y),$$

since $\omega_1^{Lp^\alpha(y)}$-iterability for countable premice in $Lp^\alpha(y)$ is witnessed by countable Q-structures. In fact, (u, E_u) is definable from u in that structure. Conversely, since

$$|Lp^\alpha(y)| = |Lp^\alpha(u)^{Lp^\alpha(y)}|,$$

$y \in Lp^\alpha(u)|\omega_1^{Lp^\alpha(y)}$, so the pair $(y, E_y \restriction \omega_1^{Lp^\alpha(y)})$ is definable over the structure

$$(L_{\omega_1^{Lp^\alpha(y)}}[u, E_u], u, E_u),$$

from y. It follows that

$$L_{\omega_1^{Lp^\alpha(y)}}[u, E_u] = L_{\omega_1^{Lp^\alpha(y)}}[y, E_y],$$

so the extender sequences of both structures agree above $\omega_1^{Lp^\alpha(y)}$. This implies that

$$L_\kappa[u, E_u] = L_\kappa[y, E_y],$$

whenever $\omega_1^{Lp^\alpha(y)} \leq \kappa$.

Using the terminology of [MS04], we show that an extender F is certified with respect to E_y if, and only if, it is so with respect to E_u (here and in the forthcoming argument, we use terminology from [MS04, Section 1]). In [MS04], the definition of "certificate" depends on a class of ordinals A such that $V = L[A]$, because it depends on the notion of "Σ_{1+}-elementarity," which depends on A. We write $X \prec_{1+}^A Y$ for Σ_{1+}-substructurehood, to emphasize this A.

The main point is that, by the remarks above, we have

$$X \prec_{1+}^{(y,E_y)} L_\kappa[y, E_y] \text{ if, and only if, } X \prec_{1+}^{(u,E_u)} L_\kappa[y, E_y],$$

provided $\omega_1^{Lp^\alpha(y)} \leq \kappa$, and both u and y belong to X.

We now carry on with the proof. Suppose F is certified with respect to E_u, $\text{CRIT}(F) = \kappa$, $\text{LTH}(F) = \nu$, and $U \prec_{1+}^{(y,E_y)} L_{\nu^+}[y, E_y]$, with U countable in $Lp^\alpha(y)$. We need to find some

$$\tau : U \to_{1+}^{(y,E_y)} L_\kappa[y, E_y]$$

witnessing that F is countably complete with respect to U.

Let $\hat{U}$ be a countable (in $Lp^\alpha(y)$) Σ_{1+}-elementary (with respect to (y, E_y)) substructure of $L_{\nu^+}[y, E_y]$ containing all elements of U, as well as u. Since ν^+ is an uncountable cardinal, we have

$$U \prec_{1+}^{(y,E_y)} \hat{U} \prec_{1+}^{(y,E_y)} L_{\nu^+}[u, E_u],$$

as well as

$$\hat{U} \prec_{1+}^{(u,E_u)} L_{\nu^+}[u, E_u],$$

so, because F is certified with respect to E_u, there is some

$$\tau : \hat{U} \to_{1+}^{(u,E_u)} L_\kappa[y, E_y]$$

witnessing that F is countably complete with respect to $\hat{U}$. As before,

$$\tau : \hat{U} \to_{1+}^{(y,E_y)} L_\kappa[y, E_y],$$

so that whenever $\vec{x} \in U$ and ϕ is Σ_{1+},

$$\begin{aligned}(U, y, E_y) \models \phi(\vec{x}) \text{ if, and only if, } (\hat{U}, y, E_y) &\models \phi(\vec{x}) \\ \text{if, and only if, } (L_\kappa[y, E_y], y, E_y) &\models \phi(\tau[\vec{x}]),\end{aligned}$$

and so

$$\tau : U \to_{1+}^{(y, E_y)} L_\kappa[y, E_y],$$

as desired.

Finally, it is clear that τ witnesses that F is countably complete with respect to U; thus, F is certified with respect to E_y. The converse argument goes through symmetrically. □

By Lemma 4.6.3 and the choice of x and y,

$$(K^{c,F}(x))^{Lp^\alpha(y)} \models \text{“}\omega_2^{Lp^\alpha(y)} \text{ is an inaccessible cardinal.”}$$

By [MSW16, Lemma 3.31], $(K^{c,F}(x))^{Lp^\alpha(y)}$ is universal with respect to premice of height $\leq \omega_2^{Lp^\alpha(y)}$ with which it can be compared.

One needs to prove the following lemma, which corresponds to Subclaim 3 in the proof of [MSW16, Theorem 3.20]. The proof is the same, but we sketch it.

Lemma 4.6.4. *$K_x|(\gamma^+)^{M_z}$ is fully iterable in $Lp^\alpha(y)$.*

Proof Sketch. The core model $K_x|(\gamma^+)^{M_z}$ is not of class $S_{\alpha+1}$ and is iterable in M_z by the Q-structure iteration strategy. Moreover, by the argument of Lemma 4.4.3 in M_z, this strategy selects unique branches whose associated Q-structures are Π^1_α-iterable (one needs the proof, rather than the lemma itself, because M_z is only closed under $a \mapsto N^\sharp_\alpha(a)$; on the other hand, its Q-structures are $\omega_1 + 1$-iterable in M_z). Because all such Q-structures are not of class S_α and are premice over initial segments of iterates of $K_x|(\gamma^+)^{M_z}$, they belong to $Lp^\alpha(K_x|(\gamma^+)^{M_z})$, provided the iteration tree does too. It follows that $K_x|(\gamma^+)^{M_z}$ is ω_1-iterable in $Lp^\alpha(K_x|(\gamma^+)^{M_z})$. To prove the lemma, one has to show that it is also ω_1-iterable in generic extensions of $Lp^\alpha(K_x|(\gamma^+)^{M_z})$. This uses $\mathbf{\Sigma}^1_\alpha$-correctness of $Lp^\alpha(K_x|(\gamma^+)^{M_z})$ (Lemma 4.5.1).

For example, to prove that $K_x|(\gamma^+)^{M_z}$ is iterable in $Lp^\alpha(y)$ as in Subclaim 3 in the proof of [MSW16, Theorem 3.20], one defines a tree T_0 in $Lp^\alpha(K_x|(\gamma^+)^{M_z})$ that projects to a universal $\mathbf{\Pi}^1_\alpha$ set in $Lp^\alpha(K_x|(\gamma^+)^{M_z})$ and all its generic extensions by partial orders of size at most $(\gamma^+)^{M_z}$. One then defines a tree T_1 searching for witnesses to the failure of countable

iterability of $K_x|(\gamma^+)^{M_z}$ by the Q-structure iteration strategy. Specifically, T_1 searches for

1. the transitive collapse of a countable elementary substructure $\bar{K}$ of $K_x|(\gamma^+)^{M_z}$ and the collapse embedding π,

2. a putative iteration tree T on $\bar{K}$ by the Q-structure iteration strategy,

3. Q-structures not of class S_α witnessing the clause above and a branch of T_0 witnessing the Π^1_α-iterability of the Q-structures,

4. A countable transitive model of a sufficient fragment of ZFC containing all objects above and witnessing that T either has a last illfounded model, or else no cofinal branch b such that $Q(T) \trianglelefteq M^T_b$.

One shows as in Subclaim 3 in the proof of [MSW16, Theorem 3.20] that the tree T_1 is wellfounded in $Lp^\alpha(K_x|(\gamma^+)^{M_z})$, so it too is in $Lp^\alpha(y)$, by the absoluteness of wellfoundedness. This uses the fact that $Lp^\alpha(y)$ is a generic extension of $Lp^\alpha(K_x|(\gamma^+)^{M_z})$, so T_0 projects to the complete $\mathbf{\Pi}^1_\alpha$-set in $Lp^\alpha(y)$, whereby T_1 is the same as defined in $Lp^\alpha(y)$ and in $Lp^\alpha(K_x|(\gamma^+)^{M_z})$; thus $K_x|(\gamma^+)^{M_z}$ is countably iterable in $Lp^\alpha(y)$. □

After this, one shows as in [MSW16] (cf. Subclaims 5 and 6 in the proof of Theorem 3.20) that $(K^{c,F}(x))^{Lp^\alpha(y)}$ is sufficiently iterable inside $Lp^\alpha(y)$ to be compared with $K_x|(\gamma^+)^{M_z}$. A universality argument shows that $K_x|(\gamma^+)^{M_z}$ embeds into an initial segment M of an iterate of $(K^{c,F}(x))^{Lp^\alpha(y)}$ and that either the comparison must use compatible extenders at some stage, or

$$(\gamma^+)^{M_z} = \omega_2^{Lp^\alpha(y)}$$

remains a successor cardinal in M. In either case, one obtains a contradiction.

5

Long Games and σ-Projective Sets

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5.1 Introduction

Let ω^ω denote the space of infinite sequences of natural numbers with the product topology, i.e., the topology generated by basic (cl)open sets of the form

$$O(s) = \{x \in \omega^\omega : \ x \text{ extends } s\},$$

where $s \in \omega^{<\omega}$. As usual, we will refer to the elements of ω^ω as *reals*. Given a subset A of ω^ω, the *payoff set*, we consider the Gale-Stewart game G of length ω as follows:

This chapter was written jointly with S. Müller and P. Schlicht.

I	x_0		x_2		$\dots$	
II		x_1		x_3		$\dots$

for $x_0, x_1, \dots \in \omega$.

Two players, I and II, alternate turns playing $x_0, x_1, \dots \in \omega$ to produce an element $x = (x_0, x_1, \dots)$ of ω^ω. Player I wins if and only if $x \in A$; otherwise, Player II wins. One can likewise define longer games by considering subsets of ω^α, where α is a countable ordinal. If so, we will again regard ω^α as a product of discrete spaces. A game is *determined* if one of players I and II has a winning strategy. A set $A \subset \omega^\omega$ is said to be determined if the corresponding game is.

These games have been studied extensively; under suitable set-theoretic assumptions, one can prove various classes of them to be determined. One often studies the determinacy of pointclasses given in terms of definability (a general reference is Moschovakis [Mos09]). A pointclass central to this chapter is the following:

Definition 5.1.1. The pointclass of *σ-projective* sets is the smallest pointclass closed under complements, countable unions,[1] and projections.

In this chapter, we consider the following classes of games, and their interplay:

1. games of fixed countable length α whose payoff is σ-projective;
2. games of variable length $<\alpha+\omega^2$ whose payoff is a pointclass containing the clopen sets and contained in the σ-projective sets;
3. games of countable length α with payoff in other σ-algebras.

Later, Neeman extensively studied long games and their connection to large cardinals in [Nee04]. These results are based on his earlier work in [Nee95] and [Nee02] on games of length ω, where he started connecting moves in long games with moves in iteration games. He showed in [Nee04] for example that the determinacy of games with fixed countable length and analytic payoff set follows from the existence of Woodin cardinals. Moreover, he also analyzed games of *continuously coded* length (see also [Nee05]) and games of length up to a *locally uncountable* ordinal. In [Nee07] he even showed determinacy for open games of length ω_1, indeed for a larger class of games of length ω_1, from large cardinals. That the determinacy of arbitrary games of length ω_1 is inconsistent is due to Mycielski and has been known for a long time (see [Myc64]).

[1] Of course, one only considers unions of sets in the same space, as is usual.

As for the converse, the determinacy of infinite games implies the existence of inner models with large cardinals (cf. e.g., [Har78, KW10, Tra13, Uhl16, MSW16] and others).

Summary of results We begin in Section 5.2 by introducing a class of games of variable length below ω^2. We call these games Γ-*simple*, where Γ is a pointclass. We show that σ-projective determinacy implies the determinacy of Γ-simple games of length ω^2 where Γ is the pointclass of all σ-projective sets. In Section 5.3 we introduce a class of games we call *decoding games*. These are used to show that σ-projective determinacy follows from simple clopen determinacy of length ω^2. The proof also shows directly that simple σ-projective determinacy of length ω^2 follows from simple clopen determinacy of length ω^2

In Section 5.4, we prove simple clopen determinacy of length ω^2 (and thus σ-projective determinacy of length ω) from optimal large cardinal assumptions. The proof is level-by-level. An alternative, purely inner-model-theoretic proof of σ-projective determinacy can be found [Agug]; our proof, however, requires little inner model theory beyond the definition of the large-cardinal assumption. Roughly, it consists in repeatedly applying a theorem of Neeman [Nee04] to reduce a simple clopen game of length ω^2 to an iteration game on an extender model with many partial extenders. The difference here is that players are allowed to drop gratuitously in the iteration game finitely many times to take advantage of the partial extenders in the model.

Finally, in Section 5.5, we exhibit some additional applications of the proof in Section 5.4. Specifically we prove from (likely optimal) large cardinal assumptions that σ-projective games of length $\omega \cdot \theta$ are determined, where θ is a countable ordinal. We also prove, from a hypothesis slightly beyond projective determinacy, that games in the smallest σ-algebra containing the projective sets are determined.

5.2 Simple games of length ω^2

We begin by noting a result on the determinacy of games of length[2] $< \omega^2$:

Theorem 5.2.1 (folklore)**.** *The following are equivalent:*

1. *All projective games of length ω are determined.*

[2]Here and below, we identify the spaces ω^{ω^2}, $(\omega^\omega)^\omega$ and $\omega^{\omega\times\omega}$, as well as $\omega^{\omega\cdot n}$ and $(\omega^\omega)^n$.

2. *All projective games of length $\omega \cdot n$ are determined, for all $n < \omega$.*

3. *All clopen games of length $\omega \cdot n$ are determined, for all $n < \omega$.*

Instead of providing a proof of Theorem 5.2.1, we refer the reader to the proof of Theorem 12.2.6 below, an easy adaptation of which suffices (cf. Remark 5.2.5).

Our first result is the analog of Theorem 5.2.1 for σ-projective games. Although the equivalence between the first two items remains true if one replaces "projective" with "σ-projective," the one between the last two does not. Instead, one needs to consider a larger class of games that are still decided in less than ω^2 rounds, in the sense that for any $x \in \omega^{\omega^2}$ there is $n \in \omega$ such that for all $y \in \omega^{\omega^2}$, if $y \restriction \omega \cdot n = x \restriction \omega \cdot n$, then

$$x \text{ is a winning run for Player I if and only if } y \text{ is.}$$

Note that every game of length $\omega \cdot n$ can be seen as a game of this form, for any $n \in \omega$. For the definition below, we adapt the convention that if $n \in \omega$, then a subset A of $\omega^{\omega \cdot n}$ can be identified with

$$\{x \in \omega^{\omega^2} : x \restriction \omega \cdot n \in A\}.$$

Definition 5.2.2. Let Γ be a collection of subsets of ω^{ω^2} (each $A \in \Gamma$ identified with a subset of $\omega^{\omega \cdot n}$ for some $n \in \omega$ as above). A game of length ω^2 is Γ-*simple* if it is obtained as follows:

1. For every $n \in \omega$, games in Γ of fixed length $\omega \cdot n$ are Γ-simple.

2. Let $n \in \omega$ and for each $i \in \omega$ let G_i be a Γ-simple game. Then the game G obtained as follows is Γ-simple: Players I and II take turns playing natural numbers for $\omega \cdot n$ moves, i.e., n rounds in games of length ω. Afterwards, Player I plays some $i \in \omega$. Players I and II continue playing according to the rules of G_i (keeping the first $\omega \cdot n$ natural numbers they have already played, but not i).

3. Let $n \in \omega$ and for each $i \in \omega$ let G_i be a Γ-simple game. Then the game G obtained as follows is Γ-simple: Players I and II take turns playing natural numbers for $\omega \cdot n$ moves, i.e., n rounds in games of length ω. Afterwards, Player II plays some $i \in \omega$. Players I and II continue playing according to the rules of G_i (keeping the first $\omega \cdot n$ natural numbers they have already played, but not i).

We are mainly interested in games of length ω^2 which are *simple clopen*, i.e., which are Γ-simple for $\Gamma = \mathbf{\Delta}^0_1$ the collection of clopen sets in the topology defined above. Let us start by noting that every simple clopen game has in fact a payoff set which is clopen in $\omega^{\omega\times\omega}$ (this fact will not be needed, but it justifies our terminology).

Lemma 5.2.3. *Let G be a simple clopen game of length ω^2 with payoff set B. Then B is clopen in $\omega^{\omega\times\omega}$.*

Proof. We prove this by induction on the definition of simple clopen games. In the case that G is a clopen game of fixed length $\omega \cdot n$ it is clear that B is clopen. So suppose that we are given simple clopen games G_i, $i < \omega$ with payoff sets B_i. Moreover, suppose for notational simplicity that $n = 1$, i.e., the players play one round of length ω before Player I plays $i \in \omega$ to decide which rules to follow. Inductively, we can assume that every B_i is clopen in $\omega^{\omega\times\omega}$. Let G be the game obtained by applying (2) in Definition 5.2.2. For $x \in \omega^{\omega\times\omega}$, write x^* for

$$x \restriction \omega^\frown x \restriction [\omega + 1, \omega^2).$$

Then x is a winning run for Player I in G iff

$$x \in \bigcup_{i\in\omega} \left\{ y \in \omega^{\omega\times\omega} : y(\omega) = i \wedge y^* \in B_i \right\}.$$

Each B_i is open, so the payoff set B of G is open. Additionally, $x \in B$ if, and only if,

$$x \in \bigcap_{i\in\omega} \left\{ y \in \omega^{\omega\times\omega} : y(\omega) \neq i \vee y^* \in B_i \right\}.$$

Each B_i is closed, so B is closed. Therefore, B is clopen.

The argument for applying (3) in Definition 5.2.2 is analogous. □

The main fact about simple clopen games is that their determinacy determinacy is already equivalent to determinacy of Γ-simple games where Γ is the pointclass of all σ-projective sets:

Theorem 5.2.4. *The following are equivalent:*

1. *All σ-projective games of length ω are determined.*
2. *All simple σ-projective games of length ω^2 are determined.*
3. *All simple clopen games of length ω^2 are determined.*

The proof of σ-projective determinacy of length ω from simple clopen determinacy will take place in the next section. For now, we content ourselves with showing that σ-projective determinacy of length ω implies simple clopen determinacy of length ω^2, although the proof we give easily adapts to show simple σ-projective determinacy of length ω^2 from the same hypothesis. It will be convenient for the future to introduce the definition of the *game rank* of a simple clopen game.

To each simple clopen game G of length ω^2 we associate a countable ordinal $\mathrm{gr}(G)$, the game rank of G, by induction on the definition of simple clopen games. If G is a game of fixed length $\omega \cdot n$, then $\mathrm{gr}(G) = n$. If G is obtained from games $G_0, G_1, \ldots$, and from an ordinal $\omega \cdot n$ as in Definition 5.2.2, we let

$$\mathrm{gr}(G) = \sup\{\mathrm{gr}(G_i) + \omega : i \in \omega\} + n.$$

Remark 5.2.5. The proof of Theorem 12.2.6 is local. We leave the computation of the precise complexity bounds to the curious reader, but we mention that e.g., the proof shows the equivalence among

1. the determinacy of games of length ω which are $\mathbf{\Pi}^1_n$ for some $n \in \omega$;
2. the determinacy of simple clopen games of length ω^2 of rank $<\omega$
3. the determinacy of simple games of rank $<\omega$ which are $\mathbf{\Pi}^1_n$ for some $n \in \omega$.

Thus, Theorem 12.2.6 generalizes Theorem 5.2.1.

The reason why we have chosen to define the game rank this way is that it will make some arguments by induction easier later on. Let us consider some examples: if $\mathrm{gr}(G) = \omega$, then G is essentially a game in which an infinite collection of games, each of bounded length, are given, and one of the players begins by deciding which one of them they will play. If $\mathrm{gr}(G) = \omega + 1$, then the game is similar, except that the player does not decide which game they will play until the first ω moves have been played. More generally, a game has limit rank if and only if it begins with one player choosing one among a countably infinite collection of games that can be played.

Remark 5.2.6. Let G be a simple clopen game of rank α and p be a partial play of G. Denote by G_p the game that results from G after p has been played. Then G_p is a simple clopen game of rank $\leq \alpha$.

Clearly, every simple clopen game has a countable rank.

Proposition 5.2.7. *Suppose that all σ-projective games of length ω are determined. Then all simple clopen games of length ω^2 are determined.*

Proof. Let us say that two games G and H are *equivalent* if the following hold:

1. Player I has a winning strategy in G if and only if she has one in H; and
2. Player II has a winning strategy in G if and only if she has one in H.

We prove the proposition by induction on the game rank of a simple clopen game G. In the case that the game rank is a successor ordinal, we additionally show that the game is equivalent to a σ-projective game of length ω (this is clear in the limit case). Suppose that α is a limit ordinal and this has been shown for games of rank $<\alpha$. Let $\alpha + n$ be the rank of G, where $n \in \omega$. If $n = 0$, then the result follows easily: by the definition of game rank, the rules of G dictate that one player must begin by choosing one amongst an infinite sequence of games G^i. If that player has a winning strategy in any one of them, then choosing that game will guarantee a win in G; otherwise, the induction hypothesis yields a winning strategy for the other player in each game G^i and thus in G.

If $0 < n$, say, $n = k + 1$, then one argues as follows. Given a partial play p of G, we denote by G_p the game G after p has been played. Consider the following game, H:

1. Players I and II alternate ω many turns to produce a real number x.
2. Afterwards, the game ends. Player I wins if and only if

   $\exists x_1 \in \omega^\omega \, \forall y_1 \in \omega^\omega \, \exists x_2 \in \omega^\omega \, \ldots \, \forall y_k \in \omega^\omega$ Player I has a winning strategy in G_p,

   where $p = \langle x, x_1 * y_1, \ldots, x_k * y_k \rangle$.[3]

Claim 5.2.8. *H is equivalent to G.*

Granted the claim, it is easy to prove the proposition, for, letting p be as above, the rules of G_p dictate that one of the players must choose one amongst an infinite sequence of games G^i. Assume without loss of generality that this is Player I. By induction hypothesis, the set of p such that there exists i so that Player I has a winning strategy in G^i is σ-projective. Hence, the payoff set of H is σ-projective, as was to be shown.

It remains to prove the claim. Thus, let $0 \leq m \leq k$ and consider the following game H_m:

[3]Here, $x * y$ denotes the result of facing off the strategies coded by the reals x and y.

1. Players I and II alternate $\omega \cdot (m+1)$ many turns to produce real numbers $z_0, \dots, z_m$.

2. Afterwards, the game ends. Player I wins if and only if

   $\exists x_{m+1}\, \forall y_{m+1}\, \exists x_{m+2}\, \dots\, \forall y_k$ Player I has a winning strategy in the game G_p,

   where $p = \langle z_0, \dots, z_m, x_{m+1} * y_{m+1}, \dots, x_k * y_k \rangle$.

Thus, $H = H_0$. By induction on $q = k - m$ (i.e., by downward induction on m), we show that H_m is equivalent to G. A simple modification of the argument shows that $(H_m)_p$ is equivalent to G_p, for each $p \in (\omega^\omega)^l$ and each $l \leq m+1$; this is possible because G_p is a simple clopen game of rank $\leq \alpha + n$. We will use the equivalence between $(H_m)_p$ and G_p as part of the induction hypothesis.

We have shown (by the induction hypothesis for the proposition) that G is equivalent to H_k. The same argument applied to G_p shows that G_p is equivalent to $(H_k)_p$ for every $p \in (\omega^\omega)^{k+1}$ and thus that G_p is determined. Moreover, Player I wins a run $p \in (\omega^\omega)^{m+1}$ of H_m if and only if she has a winning strategy in $(H_{m+1})_p$ and Player II wins a run $p \in (\omega^\omega)^{m+1}$ of H_m if and only if Player I does not have a winning strategy in $(H_{m+1})_p$; however, $(H_{m+1})_p$ is determined for every $p \in (\omega^\omega)^{m+1}$, as it is a σ-projective game of length ω (this follows from an argument as above). Moreover, the induction hypothesis (for the claim) shows that a player has a winning strategy in $(H_{m+1})_p$ if and only if she has one in G_p. This shows that a player has a winning strategy in H_m if and only if she has one in G, as was to be shown. □

5.3 Decoding games

Our first goal in this section is to prove the following proposition.

Proposition 5.3.1. *Suppose simple clopen games of length ω^2 are determined. Then σ-projective games of length ω are determined.*

In order to prove Proposition 5.3.1, we introduce a representation of σ-projective sets.

Definition 5.3.2. Fix an enumeration

$$\{A_{n+1} : n \in \omega\}$$

of all basic open and basic closed sets in each $(\omega^\omega)^k$, for $1 \leq k < \omega$. Suppose $A \subset (\omega^\omega)^k$ is σ-projective. A *σ-projective code* $[A]$ of A is defined inductively as follows:

1. If A is basic open, then $[A] = \langle n \rangle$, where $A = A_n$.
2. If A is basic closed, then $[A] = \langle n \rangle$, where $A = A_n$.
3. If $A = \bigcup_i A_i$, then $[A] = \langle [A_0], [A_1], \ldots \rangle$.
4. If $A = \bigcap_i A_i$, then $[A] = \langle 0, [A_0], [A_1], \ldots \rangle$.
5. If $A = (\omega^\omega)^k \setminus B$ for some $k \in \omega$, then $[A] = \langle 1, [B] \rangle$.
6. If $A = p[B]$, then $[A] = \langle 2, [B] \rangle$.
7. If $A = u[B]$, then $[A] = \langle 3, [B] \rangle$.

Here, $u[B]$ denotes the dual of the projection,

$$u[B] = \{x \in \omega^\omega : \forall y \, (x, y) \in B\}.$$

A σ-projective code of a given set is not unique. In fact,

Lemma 5.3.3. *Let A be σ-projective.*

1. *A has a σ-projective code in which no complements and no basic open sets appear.*
2. *A has a σ-projective code in which no complements and no basic closed sets appear.*

Proof. Given a σ-projective code for A, one obtains, by a simple application of de Morgan's laws, a σ-projective code in which complements are only applied to basic open sets or to basic closed sets. Since every basic open set is clopen, basic open sets can be replaced by a countable intersection of basic closed sets in the projective code; similarly, basic closed sets can be replaced by a countable union of basic open sets. □

Definition 5.3.4. Suppose $A \subset (\omega^\omega)^n$ for some $1 \leq n < \omega$ is σ-projective and fix a code $[A]$ for A. The *decoding game for A* (with respect to $[A]$) is a game of length ω^2 according to the following rules: In the first n rounds, which we will call the *preparation*, Players I and II start by alternating turns playing $\omega \cdot n$ natural numbers to obtain reals $x_1, x_2, \ldots, x_n \in \omega^\omega$. Afterwards, they proceed according to the σ-projective code $[A]$ of A via the following recursive definition:

1. If A is basic open, say $A = A_k$, then the game is over, i.e., further moves are not relevant. Player I wins if and only if $(x_1, x_2, \ldots, x_n) \in A_k$.

2. If A is basic closed, say $A = A_k$, then the game is over, i.e., further moves are not relevant. Player I wins if and only if $(x_1, x_2, \ldots, x_n) \in A_k$.

3. If $A = \bigcup_i A_i$, so that $[A] = \langle [A_0], [A_1], \ldots \rangle$, then Player I plays some $k \in \omega$. The game continues from the current play (without the last move, k) with the rules of the decoding game with respect to $[A_k]$.

4. If $A = \bigcap_i A_i$, so that $[A] = \langle 0, [A_0], [A_1], \ldots \rangle$, then Player II plays some $k \in \omega$. The game continues from the current play (without the last move, k) with the rules of the decoding game with respect to $[A_k]$.

5. If $A = (\omega^\omega)^k \setminus B$ for some $k \in \omega$, so that $[A] = \langle 1, [B] \rangle$, then the game continues from the current play with the rules of the decoding game with respect to $[B]$, except that the roles of Players I and II are reversed.

6. If $A = p[B]$, so that $[A] = \langle 2, [B] \rangle$, then Player I plays some $y \in \omega^\omega$ in ω moves of the game, where the moves of Player II are not relevant. The game continues from the current play, together with y, using the rules of the decoding game with respect to $[B]$.

7. If $A = u[B]$, so that $[A] = \langle 3, [B] \rangle$, then Player II plays some $y \in \omega^\omega$ in ω moves if the game, where the moves of Player I are not relevant. The game continues from the current play, together with y, using the rules of the decoding game with respect to $[B]$.

Clause (5) of the preceding definition will not be used below, but we have defined it in the natural way nonetheless.

Lemma 5.3.5. *Let $A \subset (\omega^\omega)^n$ for some $1 \leq n < \omega$ be σ-projective and fix a code $[A]$ for A. A player has a winning strategy in the game with payoff set A if and only if the player has a winning strategy in the decoding game for A with respect to $[A]$.*

Proof. Assume first that Player I has a winning strategy σ in the game G_A with winning set A. Then the following describes a winning strategy for Player I in the decoding game for A with respect to $[A]$. In the first n rounds of the game, Player I follows the strategy σ. Since σ is a winning strategy

in G_A, the players produce a sequence of reals $(x_1, \dots, x_n) \in A$. In the following rounds, Player I follows the rules of the decoding game according to $[A]$, playing witnesses to the fact that $(x_1, \dots, x_n) \in A$. This means that, e.g., if $[A] = \langle [A_0], [A_1], \dots \rangle$, i.e., $A = \bigcup_i A_i$, then Player I plays $k \in \omega$ such that $(x_1, \dots, x_n) \in A_k$. The strategy for the other cases is defined similarly. This yields a winning strategy for Player I in the decoding game for A with respect to $[A]$.

Now assume that Player I has a payoff strategy σ in the decoding game for A with respect to $[A]$. Then the restriction of σ to the first n rounds of the game is a winning strategy for Player I in the game G_A. Similarly for Player II. □

Lemma 5.3.6. *Let $A \subset (\omega^\omega)^k$ be σ-projective, for some $1 \leq k < \omega$. Then, for every σ-projective code $[A]$ for A in which no complements appear, the decoding game given by $[A]$ is simple clopen.*

Proof. Choose a code $[A]$ for A in which no complements appear. We will show by induction on the definition of simple clopen games, that the game obtained as in (6) or (7) in Definition 5.3.4 is simple clopen again. This will finish the proof as games obtained by clauses (1)-(4) in Definition 5.3.4 are clearly simple clopen, by the definition of simple clopen games.

Let us introduce some notation for this proof. If $x \in \omega^\omega$, we write x^I for the sequence of digits of x in even positions, and x^{II} for the sequence of digits of x in odd positions. Thus, if x results from a run of a Gale-Stewart game, then x^I is the sequence of moves of Player I and x^{II} that of Player II.

So let G be a simple clopen game with payoff set $B \subseteq (\omega^\omega)^\omega$ and consider the game $G^{P,k}$ which is defined as follows. Players I and II start by alternating turns playing $\omega \cdot k$ natural numbers to define reals $x_1, \dots, x_k \in \omega^\omega$. Then players I and II alternate moves to play some real x_{k+1}. Finally, Players I and II alternate again to produce reals $z_1, z_2, \dots$ so that $(x_1, \dots, x_k, x^I_{k+1}, z_1, z_2, \dots) \in (\omega^\omega)^\omega$ and we say that $(x_1, \dots, x_k, x_{k+1}, z_1, z_2, \dots)$ is a winning run for Player I in $G^{P,k}$ iff $(x_1, \dots, x_k, x^{\mathrm{I}}_{k+1}, z_1, z_2, \dots) \in B$. That means if B^* denotes the payoff set of $G^{P,k}$, we have $(x_1, \dots, x_k, x_{k+1}, z_1, z_2, \dots) \in B^*$ iff $(x_1, \dots, x_k, x^{\mathrm{I}}_{k+1}, z_1, z_2, \dots) \in B$. We aim to show that $G^{P,k}$ is simple clopen again.

Suppose first that G is a clopen game of some fixed length $\omega \cdot n$, i.e., we consider G as a game of length ω^2 but only the first $\omega \cdot n$ moves are relevant for the payoff set. Fix some $k \in \omega$. In particular, B is a clopen set in $\omega^{\omega \times \omega}$. We can naturally write $B = B_0 \times B_1 \times B_2$ for clopen sets $B_0 \subseteq (\omega^\omega)^k$,

$B_1 \subseteq \omega^\omega$ and $B_2 \subseteq (\omega^\omega)^\omega$. By definition, the payoff set for the game $G^{P,k}$ is $B_0 \times (B_1)_{\mathrm{I}} \times B_2$, where

$$(B_1)_{\mathrm{I}} = \{x \in \omega^\omega : x^{\mathrm{I}} \in B_1\},$$

which is clopen. Moreover, in $G^{P,k}$ again only the first $\omega \cdot n$ moves are relevant, so $G^{P,k}$ is a clopen game of fixed length $\omega \cdot n$ and in particular simple clopen.

Now suppose that G is a simple clopen game obtained by condition (2) in Definition 5.2.2, i.e., we are given simple clopen games G_i, $i < \omega$, and after Player I and II take turns producing $x_1, \dots, x_k, x_{k+1}$, Player I plays some natural number i and the players continue according to the rules of G_i (keeping the moves which produced $x_1, \dots, x_k, x_{k+1}$). That means for every $i < \omega$, some sequence $(x_1, \dots, x_k, x_{k+1}, i, z_1, z_2, \dots)$ is a winning run for Player I in G iff $(x_1, \dots, x_k, x_{k+1}, z_1, z_2, \dots)$ is a winning run for Player I in G_i. We can assume inductively that the games $G_i^{P,k}$ for $i < \omega$ (obtained as above) are simple clopen and we aim to show that $G_i^{P,k}$ is simple clopen. Consider the simple clopen game G^* which is obtained by applying (2) in Definition 5.2.2 to the games $G_i^{P,k}$ and the natural number $k+1$. Suppose Players I and II produce $(x_1, \dots, x_k, x_{k+1}, i, z_1, z_2, \dots)$ in a run of G^*. Then

$$\begin{aligned}
&(x_1, \dots, x_k, x_{k+1}, i, z_1, z_2, \dots) \text{ is a winning run for Player I in } G^* \\
\text{iff } &(x_1, \dots, x_k, x_{k+1}, z_1, z_2, \dots) \text{ is a winning run for Player I in } G_i^{P,k} \\
\text{iff } &(x_1, \dots, x_k, x^{\mathrm{I}}_{k+1}, z_1, z_2, \dots) \text{ is a winning run for Player I in } G_i \\
\text{iff } &(x_1, \dots, x_k, x^{\mathrm{I}}_{k+1}, i, z_1, z_2, \dots) \text{ is a winning run for Player I in } G \\
\text{iff } &(x_1, \dots, x_k, x_{k+1}, i, z_1, z_2, \dots) \text{ is a winning run for Player I in } G^{P,k},
\end{aligned}$$

where the first equivalence holds by definition of G^*, the second equivalence holds by inductive hypothesis, the third equivalence by choice of G, and the fourth equivalence by definition of $G^{P,k}$. Hence $G^{P,k}$ and G^* are equal and $G^{P,k}$ is a simple clopen game, as desired.

The argument for simple clopen games obtained by condition (3) in Definition 5.2.2 is analogous. □

With Lemmata 5.3.5 and 5.3.6, Proposition 5.3.1 is proved. To finish the proof of Theorem 12.2.6 one needs to show:

Proposition 5.3.7. *Suppose either that simple clopen games of length ω^2 are determined or that σ-projective games of length ω are determined. Then simple σ-projective games of length ω^2 are determined.*

Proposition 5.3.7 can be proved directly either by the method of the proof of Proposition 5.3.1, or by that of Proposition 5.2.7. In the second case, one need only carry out a straightforward adaptation, as was remarked before. In the first case, a simple σ-projective game G is reduced to the simple clopen game in which two players play the game G, producing a sequence $x \in (\omega^\omega)^n$ for some $n \in \omega$ such that there is a σ-projective set $A \subset (\omega^\omega)^n$ with the property that Player I wins G iff $x \in A$, no matter how the players continue playing the rest of G. After this, the players play the decoding game for A to determine who the winner is in a clopen way.

We close this section with a useful characterization of the σ-projective sets, although it will not be used.

Proposition 5.3.8 (Folklore)**.** *A set $A \subset \mathbb{R}$ is σ-projective if and only if it belongs to $L_{\omega_1}(\mathbb{R})$.*

Proof. Clearly, $L_{\omega_1}(\mathbb{R})$ is closed under countable sequences and $\mathcal{P}(\mathbb{R}) \cap L_{\omega_1}(\mathbb{R})$ is closed under projections.

Conversely, by induction on $\alpha < \omega_1$, one sees that $L_\alpha(\mathbb{R})$ and the satisfaction relation for $L_\alpha(\mathbb{R})$ are coded by σ-projective sets of reals:

1. $L_0(\mathbb{R}) = \mathbb{R} = V_{\omega+1}$ is coded by itself. The satisfaction relation S_0 for $V_{\omega+1}$ is given by
$$S_0(\phi, \vec{a}) \leftrightarrow V_{\omega+1} \models \phi(\vec{a}),$$
for ϕ a formula in the language of set theory and $\vec{a}$ a finite sequence of reals. Since every formula in the language of set theory is in the class Σ_n for some n, S_0 belongs to the pointclass $\bigcup_{n<\omega} \Sigma^1_n$ (i.e., the pointclass of countable unions of projective sets) and is thus σ-projective.

2. Suppose that a σ-projective code C_α for $L_\alpha(\mathbb{R})$ has been defined and that a σ-projective satisfaction predicate S_α for $L_\alpha(\mathbb{R})$ relative to C_α has been defined. Suppose that $\vec{a} \in (C_\alpha)^n$ and
$$\phi = \exists y_1 \, \forall y_2, \ldots Q y_m \, \phi_0(x_1, \ldots, x_n, y_1, \ldots, y_m)$$
(where Q is a quantifier and ϕ_0 is Δ_0) is a Σ_m-formula with n free variables. Then, letting
$$C_\alpha^{\phi,\vec{a}} = \{x \in \mathbb{R} : \exists y_1 \in C_\alpha \, \forall y_2 \in C_\alpha \, \ldots \, Q y_m \in C_\alpha \, S_\alpha(\phi_0, \vec{a}, y_1, \ldots, y_m)\},$$
a σ-projective code $C_{\alpha+1}$ for $L_{\alpha+1}(\mathbb{R})$ can be defined by the disjointed union
$$C_{\alpha+1} = C_\alpha \dot\cup \{\hat{C}_\alpha^{\phi,\vec{a}} : \phi \text{ is a formula and } \vec{a} \in (C_\alpha)^n\},$$

where $\hat{C}_\alpha^{\phi,\vec{a}}$ is a real number coding[4] the set $C_\alpha^{\phi,\vec{a}}$. A satisfaction relation $S_{\alpha+1}$ can be defined from this: for atomic formulae, we set

$$S_{\alpha+1}(\cdot \in \cdot, x, y) \leftrightarrow (x, y \in C_\alpha \wedge S_\alpha(\cdot \in \cdot, x, y)) \vee (x \in C_\alpha \wedge \exists \phi\, \exists \vec{a} \subset (C_\alpha)^{\mathrm{lth}(a)}\, y = \hat{C}_\alpha^{\phi,\vec{a}} \wedge S_\alpha(\phi, \vec{a}, x)).$$

For other formulae, this is done as above, so the satisfaction relation $S_{\alpha+1}$ is seen to belong to the pointclass $\bigcup_{n<\omega} \Sigma^1_n(C_{\alpha+1}, S_\alpha)$ and is thus by using the inductive hypothesis σ-projective.

3. Suppose that a σ-projective code C_α for $L_\alpha(\mathbb{R})$ has been defined and that a σ-projective satisfaction predicate S_α for $L_\alpha(\mathbb{R})$ has been defined for every $\alpha < \lambda$, where λ is a countable limit ordinal. Then C_λ can be defined as the disjoint (countable) union of $C_{\alpha+1} \setminus C_\alpha$, for $\alpha < \lambda$. Thus C_λ is σ-projective. The satisfaction relation S_λ can be defined as above.

This completes the proof. □

5.4 Determinacy from large cardinals

In this section, we prove level-by-level that the existence of certain iterable inner models with Woodin cardinals implies simple clopen determinacy of length ω^2, which in turn implies σ-projective determinacy of length ω. A different proof of this latter result can be found in Chapter 4, but the proof we give here has the advantage that it requires almost no inner-model-theoretic background. Another advantage is that it easily generalizes to yield further results (cf. the next section).

The idea of this proof is to enhance a simple clopen game of length ω^2 by presenting each player with a fine-structural model that can be manipulated to obtain information about the game. This method of proof is due to Neeman [Nee04]; the difference in our context is that the models considered will contain many partial measures and, in addition to taking iterated ultrapowers, we will allow the players to remove end-segments of their models during the game so as to make the measures total and obtain new information. Although this determinacy proof could have been framed directly in terms of (the decoding game for) σ-projective sets, it seems somewhat more natural to consider simple clopen games of length ω^2 instead and proceed by induction on the game rank.

[4] E.g., it could be the tuple $(\phi, \vec{a}, \alpha)$.

Before we start with the proof, let us recall the relevant inner model theoretic notions we will need. In what follows, we will work with premice and formulae in the language of relativized premice $\mathcal{L}_{\mathrm{pm}} = \{\dot{\in}, \dot{E}, \dot{F}, \dot{x}\}$, where $\dot{E}$ is a predicate for a sequence of extenders, $\dot{F}$ is a predicate for an extender, and $\dot{x}$ is a predicate for a real number over which we construct the premouse.

For some real x, a *potential x-premouse* is a model $M = (J^{\vec{E}}_\eta, \in, \vec{E} \restriction \eta, E_\eta, x)$, where $\vec{E}$ is a fine extender sequence[5] and η an ordinal or $\eta = \mathsf{Ord}$ (see Section 2 in [Ste10] for details). We say that such a potential x-premouse M is *active* iff $E_\eta \neq \emptyset$. If $\nu \leq \eta$, we write $M|\nu = (J^{\vec{E}}_\nu, \in, \vec{E} \restriction \nu, E_\nu, x)$ for the corresponding initial segment of M. A potential x-premouse M is called an *x-premouse* if every proper initial segment of M is ω-sound. If it does not lead to confusion, we will sometimes drop the x and just call M a premouse. Informally, an *x-mouse* is an iterable x-premouse. We will avoid this term as the notion of iterability is ambiguous, but ω_1-iterability suffices for all our arguments.[6] Here we say that an x-premouse M is *ω_1-iterable* if it is iterable for countable stacks of normal trees of length $< \omega_1$ (see Section 4.1 of [Ste10] for a formal definition).

We consider premice belonging to various *smallness classes*:

Definition 5.4.1. Let M be an ω_1-iterable premouse and $\delta \in M \cap \mathsf{Ord}$.

1. We say M is of class S_0 above δ if M is a proper class or active.

2. We say M is of class $S_{\alpha+1}$ above δ if there is some ordinal $\delta_0 > \delta$ and some $N \trianglelefteq M$ with $\delta_0 < N \cap \mathsf{Ord}$ such that N is of class S_α above δ_0 and δ_0 is Woodin in N.

3. Let λ be a countable limit ordinal. We say M is of class S_λ above δ if $\lambda < \omega_1^M$ and M is of class S_α above δ for all $\alpha < \lambda$.

Moreover, we say M is of class S_α if it is of class S_α above 0.

Example 5.4.2. Let $n \geq 1$ and recall that a premouse M is called n-small if for every κ which is a critical point of an extender on the sequence of extenders of M, $M|\kappa \not\models$ "there are n Woodin cardinals". Moreover, we say M is 0-small if M is an initial segment of L. Let $M_n^\sharp$ denote the unique countable, sound, ω_1-iterable premouse which is not n-small, but all of whose proper initial segments are n-small, if it exists and is unique. If $M_n^\sharp$ exists

[5] See Definition 2.4 in [Ste10], which goes back to Section 1 in [MS94] and [SSZ02].

[6] We will only need *weak iterability*, in the sense of [Nee04].

and is unique for all $n \in \omega$, then – in particular – every $x \in \omega^\omega$ has a sharp. Let $N^\sharp_\omega$ denote the smallest active premouse extending

$$\bigcup_{n\in\omega} M^\sharp_n.$$

Then $N^\sharp_\omega$ is of class S_ω. If there is a premouse of class $S_{\omega+1}$, then it contains $N^\sharp_\omega$ and in fact $N^\sharp_\omega$ is countable in it.

Note that if M is a premouse of class S_α, then M is aware of this, since α is countable in M. If M is of class S_α and not a proper class, then M is active and we can obtain a proper class model by iterating the active extender of M and its images out of the universe. By convention, we shall say that this proper class model is also of class S_α.

We recall the definition of the game rank of a simple clopen game. If G is a game of fixed length $\omega \cdot n$, then $\mathrm{gr}(G) = n$. If G is obtained from games $G_0, G_1, \ldots$, and from an ordinal $\omega \cdot n$ as in Definition 5.2.2, we let

$$\mathrm{gr}(G) = \sup\{\mathrm{gr}(G_i) + \omega : i \in \omega\} + n.$$

Remark 5.4.3. For every simple game G and initial play p of G, let G_p denote the rest of the game G after p has been played. Let $p_0, p_1, \ldots$ be a sequence of initial plays of G starting with the empty sequence $p_0 = \emptyset$ such that p_{i+1} end-extends p_i, and either $\mathrm{gr}(G_{p_i})$ is a successor ordinal with $\mathrm{gr}(G_{p_i}) = \mathrm{gr}(G_{p_{i+1}}) + 1$ or $\mathrm{gr}(G_{p_i})$ is a limit ordinal. Then this sequence has to be of finite length $k+1$ and we can choose k large enough such that $\mathrm{gr}(G_{p_k}) = 1$.

Theorem 5.4.4. *Suppose that $\gamma < \omega_1$ and for every $y \in \omega^\omega$ there is an $x \geq_T y$ such that there is a proper class x-premouse of class S_γ which is a model of* ZFC*. Then every simple clopen game G such that $\mathrm{gr}(G) \leq \gamma$ is determined.*

In the proof, we are going to use premice of class S_γ to apply the following theorem of Neeman multiple times.

Theorem 5.4.5 (Neeman, Theorem 2A.2 in [Nee04])**.** *There are binary formulae ϕ_{I} and ϕ_{II} in the language of set theory such that the following hold for any transitive, weakly iterable*[7] *premouse M which is a model of* ZFC*:*

Let $\delta < \omega_1$ be a Woodin cardinal in M and let $\dot{A} \in M$ and $\dot{B} \in M$ be $Coll(\omega, \delta)$-names for a subset of ω^ω. Then,

[7] As defined in Appendix A, Iterability in [Nee04]. Note that ω_1-iterability implies weak iterability.

1. *If $M \models \phi_{\mathrm{I}}[\delta, \dot{A}]$, there is a strategy σ for Player I in a game of length ω such that whenever x is a play by σ, there is a non-dropping iterate N of M with embedding j and an N-generic $g \subset Coll(\omega, j(\delta))$ such that $x \in N[g]$ and $x \in j(\dot{A})[g]$.*

2. *If $M \models \phi_{\mathrm{II}}[\delta, \dot{B}]$, there is a strategy τ for Player II in a game of length ω such that whenever x is a play by τ, there is a non-dropping iterate N of M with embedding j and an N-generic $g \subset Coll(\omega, j(\delta))$ such that $x \in N[g]$ and $x \in j(\dot{B})[g]$.*

3. *Otherwise, there is an M-generic $g \subset Coll(\omega, \delta)$ and an $x \in M[g]$ such that $x \notin \dot{A}[g]$ and $x \notin \dot{B}[g]$.*

We now proceed to:

Proof of Theorem 5.4.4. We may as well assume that γ is infinite, since otherwise G has fixed length $<\omega^2$, so it is determined by the results of [Nee95] and [Nee02] (see also the introduction of [Nee04]).

Let G be a simple clopen game with $\mathrm{gr}(G) \leq \gamma$, say $\mathrm{gr}(G) = \alpha$. We assume that α is a successor ordinal, since the limit case is similar. Let $r \in \omega^\omega$ code all parameters used in the definition of G. We may as well assume that r belongs to the Turing cone given by the hypothesis of the theorem.

To each non-terminal play p of G corresponds a game

$$G_p := \text{ the game } G \text{ after } p \text{ has been played.}$$

Clearly, G_p is a simple game and $\mathrm{gr}(G_p) \leq \alpha$. Given $y \geq_T r$, a y-premouse M, we select $\delta \in \mathsf{Ord}^M$ and define formulae ϕ^p_{I} and ϕ^p_{II}, and two sets $\dot{W}^p_{\mathrm{I}}$ and $\dot{W}^p_{\mathrm{II}}$, each of which is either a $\mathrm{Coll}(\omega, \delta)$-name for a set of real numbers or a set of natural numbers. The definition is by induction on $\mathrm{gr}(G_p)$ (not on p!). Formally, these names and sets of course depend on the model M in which they are defined, so we sometimes write $\dot{W}^p_{\mathrm{I}}(M)$ and $\dot{W}^p_{\mathrm{II}}(M)$ to make this explicit. But for simplicity and readability we will omit this whenever it does not lead to confusion.

At the same time, we show that there are names, names for names, etc., for these sets such that their interpretation with respect to generics $g_0, \ldots, g_n$ for some $n \in \omega$ for collapsing Woodin cardinals of M is a set of the same form with respect to the model $M[g_0] \ldots [g_n]$ instead of M. More precisely, if M is a model as above and M has (at least) $n+1$ Woodin cardinals, then let $\mathbb{P}_n = \mathbb{P}(\delta_0, \ldots, \delta_n)$ be the forcing iteration of length $n+1$ collapsing the ordinals $\delta_0, \delta_1, \ldots, \delta_n$ to ω one after the other, where δ_0 is

the least Woodin cardinal in M and δ_{i+1} is the least Woodin cardinal in M above δ_i for all $0 \leq i < n$, i.e., $\mathbb{P}_0 = \mathrm{Coll}(\omega, \delta_0)$ and for all $0 \leq i < n$,

$$\mathbb{P}_{i+1} = \mathbb{P}_i * \mathrm{Coll}(\omega, \delta_{i+1})\check{}\,,$$

where $\mathrm{Coll}(\omega, \delta_{i+1})\check{} \in M$ is the canonical $\mathbb{P}_i$-name for $\mathrm{Coll}(\omega, \delta_{i+1})$. This is of course equivalent to the product and to the $\mathrm{Coll}(\omega, \delta_n)$, but we are interested in the step-by-step collapse.

We consider nice $\mathbb{P}_n$-names $\dot{p}$ for finite sequences of reals with the property that the game rank of $G_{\dot{p}}$ is decided by the empty condition.[8] By induction on the game rank of $G_{\dot{p}}$, we will specify for every $n < \omega$ such that M has at least $n+1$ Woodin cardinals and every such $\mathbb{P}_n$-name $\dot{p}$, a $\mathbb{P}_n$-name $\dot{B}_n^{\dot{p},I}$ in M such that whenever G is $\mathbb{P}_n$-generic over M,

$$\dot{B}_n^{\dot{p},I}[G] = \dot{W}_{\mathrm{I}}^p(M[G]),$$

where $p = \dot{p}[G]$. We will call this name $\dot{B}_n^{\dot{p},I}$ the *good* $\mathbb{P}_n$*-name for* $\dot{W}_{\mathrm{I}}^p$. We will also define analogous names $\dot{B}^{\dot{p},II}$ for $\dot{W}_{\mathrm{II}}^p$. We now proceed to the recursive definition of ϕ_{I}^p, ϕ_{II}^p, $\dot{W}_{\mathrm{I}}^p$, $\dot{W}_{\mathrm{II}}^P$, $\dot{B}_n^{p,I}$, and $\dot{B}_n^{p,II}$.[9]

Case 1. $\mathrm{gr}(G_p) = 2$ (this is the base case).

Let $y \geq_T r$ and let M be any y-premouse which is a model of ZFC and of class S_1. Assume without loss of generality that M is minimal, in the sense that no proper initial segment of M is a model of ZFC of class S_1. Let δ be the least Woodin cardinal in M, so that M is of class S_0 above δ, and let $p \in M$. We define the $\mathrm{Coll}(\omega, \delta)$-names for sets of reals $\dot{W}_{\mathrm{I}}^p$ and $\dot{W}_{\mathrm{II}}^p$ by[10]

$$\dot{W}_{\mathrm{I}}^p = \dot{W}_{\mathrm{I}}^p(M) = \{(\dot{y}, q) \mid q \Vdash^M_{\mathrm{Coll}(\omega,\delta)} \text{ Player I has a winning strategy in } G_{\check{p}^\frown \dot{y}}\}$$

and

$$\dot{W}_{\mathrm{II}}^p = \dot{W}_{\mathrm{II}}^p(M) = \{(\dot{y}, q) \mid q \Vdash^M_{\mathrm{Coll}(\omega,\delta)} \text{ Player II has a winning strategy in } G_{\check{p}^\frown \dot{y}}\}.$$

Let ϕ_{I}^p and ϕ_{II}^p be the formulae given by Neeman's theorem (Theorem 5.4.5) such that the following hold:

[8] Note that the game rank of G_p depends only on G and finitely many digits of p.

[9] For every p, ϕ_{I}^p will be one of three formulae (there is one possibility for the successor case and two for the limit case). We will however use the notation ϕ_{I}^p to make the presentation uniform. Similarly for ϕ_{II}^p.

[10] Here and below, all names for real are assumed to be nice.

1. If $M \models \phi^p_{\mathrm{I}}[\dot{W}^p_{\mathrm{I}}]$, there is a strategy σ for Player I in a game of length ω such that whenever x is a play by σ, there is an iterate N of M, an elementary embedding $j\colon M \to N$ and an N-generic $h \subset \mathrm{Coll}(\omega, j(\delta))$ such that $x \in N[h]$ and $x \in j(\dot{W}^p_{\mathrm{I}})[h]$.

2. If $M \models \phi^p_{\mathrm{II}}[\dot{W}^p_{\mathrm{II}}]$, there is a strategy τ for Player II in a game of length ω such that whenever x is a play by τ, there is an iterate N of M, an elementary embedding $j\colon M \to N$ and an N-generic $h \subset \mathrm{Coll}(\omega, j(\delta))$ such that $x \in N[h]$ and $x \in j(\dot{W}^p_{\mathrm{II}})[h]$.

3. Otherwise, there is an M-generic $g \subset \mathrm{Coll}(\omega, \delta)$ and an $x \in M[g]$ such that $x \notin \dot{W}^p_{\mathrm{I}}[g]$ and $x \notin \dot{W}^p_{\mathrm{II}}[g]$.

Now, we specify the good $\mathbb{P}_n$-names $\dot{B}^{\dot{p},I}_n$ as claimed above, for models with enough Woodin cardinals below δ so that $\mathbb{P}_n$ is defined. For $n = 0$, suppose that M is a premouse with Woodin cardinals $\delta_0 < \delta$ and is of class S_0 above δ and minimal above δ.[11] Let $\dot{p}$ be a nice $\mathrm{Coll}(\omega, \delta_0)$-name for a finite sequence of reals and set

$$\dot{B}^{\dot{p},I}_0 = \{((\dot{y}, \dot{q}), q_0) \mid q_0 \Vdash_{\mathrm{Coll}(\omega,\delta_0)} \dot{q} \Vdash_{\mathrm{Coll}(\omega,\delta)^{\check{}}} \text{Player I has a winning strategy in } G_{\check{p}^\frown \dot{y}}\}.$$

The good $\mathbb{P}_n$-names $\dot{B}^{\dot{p},I}_n$ for $n > 0$ and similar names $\dot{B}^{\dot{p},II}_n$ for $\dot{W}^p_{\mathrm{II}}$ are defined analogously.

Case 2. $\mathrm{gr}(G_p) = \gamma + 1$ for some $\gamma \geq 2$.

Let $y \geq_T r$ and let M be any y-premouse which is a model of ZFC of class $S_{\gamma+1}$. Assume without loss of generality that M is minimal, in the sense that no proper initial segment of M is a model of ZFC of class $S_{\gamma+1}$. Let δ be the least Woodin cardinal in M, so that M is of class S_γ above δ, and let $p \in M$. Then let

$$\dot{W}^p_{\mathrm{I}} = \dot{W}^p_{\mathrm{I}}(M) = \{(\dot{y}, q) \mid q \Vdash^M_{\mathrm{Coll}(\omega)} \phi^{\check{p}^\frown \dot{y}}_{\mathrm{I}}[\dot{B}]\},$$

where $\dot{B} \in M$ is the good $\mathrm{Coll}(\omega, \delta)$-name with respect to $\check{p}^\frown \dot{y}$, so that whenever G is $\mathrm{Coll}(\omega, \delta)$-generic over M, $\dot{B}[G] = \dot{W}^{p^\frown y}_{\mathrm{I}}(M[G])$, where $p^\frown y = (\check{p}^\frown \dot{y})[G]$. We also let

$$\dot{W}^p_{\mathrm{II}} = \dot{W}^p_{\mathrm{II}}(M) = \{(\dot{y}, q) \mid q \Vdash^M_{\mathrm{Coll}(\omega,\delta)} \phi^{\check{p}^\frown \dot{y}}_{\mathrm{II}}[\dot{B}]\},$$

[11] A case of interest will be that of such M which are initial segments of a model N of some class S_β which is minimal above δ_0. In such an N, δ need not be a cardinal.

for $\dot{B}$ analogous as above. Note that $\dot{B}$ is a good name for $\dot{W}_{\mathrm{I}}^{p^\frown y}$ or $\dot{W}_{\mathrm{II}}^{p^\frown y}$ respectively and hence already defined since for every real y, $\mathrm{gr}(G_{p^\frown y}) = \gamma$. Let ϕ_{I}^p and ϕ_{II}^p be the formulae given by Neeman's theorem (Theorem 5.4.5) such that the following hold:

1. If $M \models \phi_{\mathrm{I}}^p[\dot{W}_{\mathrm{I}}^p]$, there is a strategy σ for Player I in a game of length ω such that whenever x is a play by σ, there is an iterate N of M, an elementary embedding $j\colon M \to N$ and an N-generic $h \subset \mathrm{Coll}(\omega, j(\delta))$ such that $x \in N[h]$ and $x \in j(\dot{W}_{\mathrm{I}}^p)[h]$.

2. If $M \models \phi_{\mathrm{II}}^p[\dot{W}_{\mathrm{II}}^p]$, there is a strategy τ for Player II in a game of length ω such that whenever x is a play by τ, there is an iterate N of M, an elementary embedding $j\colon M \to N$ and an N-generic $h \subset \mathrm{Coll}(\omega, j(\delta))$ such that $x \in N[h]$ and $x \in j(\dot{W}_{\mathrm{II}}^p)[h]$.

3. Otherwise, there is an M-generic $g \subset \mathrm{Coll}(\omega, \delta)$ and an $x \in M[g]$ such that $x \notin \dot{W}_{\mathrm{I}}^p[g]$ and $x \notin \dot{W}_{\mathrm{II}}^p[g]$.

Now, we specify the good $\mathbb{P}_n$-names $\dot{B}_n^{\dot{p},I}$. For $n = 0$, suppose M is a premouse with Woodin cardinals $\delta_0 < \delta$ and that M is of class S_γ above δ and minimal above δ. Let $\dot{p} \in M$ be a nice $\mathrm{Coll}(\omega, \delta_0)$-name for a finite sequence of reals. Set

$$\dot{B}_0^{\dot{p},I} = \{((\dot{\dot{y}}, \dot{q}), q_0) \mid q_0 \Vdash_{\mathrm{Coll}(\omega,\delta_0)} \dot{q} \Vdash_{\mathrm{Coll}(\omega,\delta)^\cdot} \phi_{\mathrm{I}}^{\check{\dot{p}}^\frown \dot{\dot{y}}}[\dot{B}]\},$$

where $\dot{B}$ is a good $\mathbb{P}(\delta_0, \delta)$-name for $\dot{W}_{\mathrm{I}}^p$. That means $\dot{B}$ is such that whenever G is $\mathbb{P}(\delta_0, \delta)$-generic over M, $\dot{B}[G] = \dot{W}_{\mathrm{I}}^{p^\frown y}(M[G])$, where $p^\frown y = (\check{\dot{p}}^\frown \dot{y})[G]$. The $\mathbb{P}_n$-names $\dot{B}_n^{\dot{p},I}$ for $n > 0$ and similar names $\dot{B}_n^{\dot{p},II}$ for $\dot{W}_{\mathrm{II}}^p$ are defined analogously.

Case 3.

$\mathrm{gr}(G_p) = \lambda$ is a limit ordinal and the rules of G dictate that, after p, it is Player I's turn.

Let $y \geq_T r$ and let M be any y-premouse which is a model of ZFC and of class S_λ. Let $p \in M$. Then let $\dot{W}_{\mathrm{I}}^p = \dot{W}_{\mathrm{I}}^p(M)$ be the set of all $k \in \omega$ such that there is an $\eta < M \cap \mathsf{Ord}$ such that $M|\eta$ is an active y-premouse of class $S_{\mathrm{gr}(G_{p^\frown k})}$ which is minimal, in the sense that no proper initial segment thereof is of class $S_{\mathrm{gr}(G_{p^\frown k})}$, and, if we let M^* be the result of iterating the active extender of $M|\eta$ out of the universe, then

$$M^* \models \phi_{\mathrm{I}}^{p^\frown k}[\delta^*, \dot{W}_{\mathrm{I}}^{p^\frown k}(M^*)].$$

This makes sense because for all $k \in \omega$, $\mathrm{gr}(G_{p^\frown k}) < \mathrm{gr}(G_p)$, so the set $\dot{W}_{\mathrm{I}}^{p^\frown k}(M^*)$ has been defined for all M^* as above. Moreover, $\dot{W}_{\mathrm{I}}^p$ belongs to M, since the formulae $\phi_I^{p^\frown k}(\dot{W}_{\mathrm{I}}^{p^\frown k}[M^*])$, together with their parameters $\dot{W}_{\mathrm{I}}^{p^\frown k}(M^*)$, are definable[12] uniformly in p, $\mathrm{gr}(G_p)$, and η, as can be shown inductively by following this construction and the proof of Theorem 5.4.5 (cf. the proof of [Nee04, Theorem 1E.1]).

We let $\phi_{\mathrm{I}}^p[\dot{W}_{\mathrm{I}}^p(M)]$ be the formula asserting that $\dot{W}_{\mathrm{I}}^p$ is non-empty. Similarly, let $\dot{W}_{\mathrm{II}}^p = \dot{W}_{\mathrm{II}}^p(M)$ be the set of all $k \in \omega$ such that whenever $\eta < M \cap \mathsf{Ord}$ is such that $M|\eta$ is an active y-premouse of class $S_{\mathrm{gr}(G_{p^\frown k})}$ which is minimal, then

$$M^* \models \phi_{\mathrm{II}}^{p^\frown k}[\dot{W}_{\mathrm{II}}^{p^\frown k}(M^*)],$$

for M^* defined as above. We let $\phi_{\mathrm{II}}^p[\dot{W}_{\mathrm{II}}^p(M)]$ be the formula asserting that $\dot{W}_{\mathrm{II}}^p$ is equal to ω. As before, the set $\dot{W}_{\mathrm{II}}^p$ belongs to M.

The good $\mathbb{P}_n$-names $\dot{B}_n^{\dot{p},I}$ and $\dot{B}_n^{\dot{p},II}$ are defined as before, for premice M which are of class S_λ above finitely many Woodin cardinals.

Case 4. $\mathrm{gr}(G_p) = \lambda$ is a limit ordinal and the rules of G dictate that, after p, it is Player II's turn.

Let $y \geq_T r$ and let M be any y-premouse which is a model of ZFC and of class S_λ. Let $p \in M$. Then let $\dot{W}_{\mathrm{I}}^p(M)$ be the set of all $k \in \omega$ such that there is an $\eta < M \cap \mathsf{Ord}$ such that $M|\eta$ is an active y-premouse of class $S_{\mathrm{gr}(G_{p^\frown k})}$ which is minimal, in the sense that no proper initial segment of $M|\eta$ is of class $S_{\mathrm{gr}(G_{p^\frown k})}$, and, if we again let M^* be the result of iterating the active extender of $M|\eta$ out of the universe, then

$$M^* \models \phi_{\mathrm{I}}^{p^\frown k}[\dot{W}_{\mathrm{I}}^{p^\frown k}(M^*)].$$

Again, $\dot{W}_{\mathrm{I}}^p$ is in M. We let $\phi_{\mathrm{I}}^p[\dot{W}_{\mathrm{I}}^p(M)]$ be the formula asserting that $\dot{W}_{\mathrm{I}}^p$ is equal to ω. Similarly, let $\dot{W}_{\mathrm{II}}^p = \dot{W}_{\mathrm{II}}^p(M)$ be the set of all $k \in \omega$ such that whenever $\eta < M \cap \mathsf{Ord}$ is such that $M|\eta$ is an active y-premouse of class $S_{\mathrm{gr}(G_{p^\frown k})}$ which is minimal, then

$$M^* \models \phi_{\mathrm{II}}^{p^\frown k}[\dot{W}_{\mathrm{II}}^{p^\frown k}(M^*)],$$

for M^* defined as above. We let $\phi_{\mathrm{II}}^p[\dot{W}_{\mathrm{II}}^p(M)]$ be the formula asserting that $\dot{W}_{\mathrm{II}}^p$ is non-empty.

[12] Recall that the structure M includes a predicate for its extender sequence.

The good $\mathbb{P}_n$-names $\dot{B}_n^{\dot{p},I}$ and $\dot{B}_n^{\dot{p},II}$ are defined as before, for premice M which are of class S_λ above finitely many Woodin cardinals.

This completes the definition of the names $\dot{W}_{\mathrm{I}}^p$, $\dot{W}_{\mathrm{II}}^p$, $\dot{B}_n^{\dot{p},I}$, $\dot{B}_n^{\dot{p},II}$, and the formulae ϕ_{I}^p and ϕ_{II}^p. Continuing with the proof of Theorem 5.4.4, we prove a technical claim which shows that these names behave well under elementary embeddings.

Claim 5.4.6. *Suppose* $\mathrm{gr}(G)$ *is a successor ordinal, let* M *be a proper class premouse which is a model of* ZFC *of class* $S_{\mathrm{gr}(G)}$ *and minimal, in the sense that no proper initial segment of* M *is a model of* ZFC *of class* $S_{\mathrm{gr}(G)}$.[13] *Let* $j\colon M \to N$ *be an elementary embedding. Let* $p \in M$ *be a finite sequence of reals in* M. *Then*

$$j(\dot{W}_{\mathrm{I}}^p(M)) = \dot{W}_{\mathrm{I}}^p(N)$$

and, analogously,

$$j(\dot{W}_{\mathrm{II}}^p(M)) = \dot{W}_{\mathrm{II}}^p(N).$$

Proof. Let δ denote the least Woodin cardinal of M. We will in fact show that whenever $\dot{p} \in M$ is a $\mathbb{P}_n$-name for a finite sequence of reals such that $\mathrm{gr}(G_{\dot{p}})$ is decided by the empty condition, we have

$$j(\dot{B}_n^{\dot{p},I}) = (\dot{B}_n^{j(\dot{p}),I})^N,$$

and

$$j(\dot{B}_n^{\dot{p},II}) = (\dot{B}_n^{j(\dot{p}),II})^N,$$

i.e., if $\dot{B} \in M$ is a good $\mathbb{P}_n$-name such that whenever G is $\mathbb{P}_n$-generic over M,

$$\dot{B}[G] = \dot{W}_{\mathrm{I}}^{\dot{p}[G]}(M[G]),$$

then $j(\dot{B}) \in N$ is a good $j(\mathbb{P}_n)$-name such that whenever H is $j(\mathbb{P}_n)$-generic over N,

$$j(\dot{B})[H] = \dot{W}_{\mathrm{I}}^{j(\dot{p})[H]}(N[H])$$

and similarly for $\dot{B}_n^{\dot{p},II}$.

This will yield the claim if applied to $n = 0$, as, by definition,

$$\dot{W}_{\mathrm{I}}^p(M) = \{(\dot{y}, q) : q \Vdash^M_{\mathrm{Coll}(\omega,\delta)} \phi_{\mathrm{I}}^{\check{p}^\frown \dot{y}}[\dot{B}_0^{\check{p}^\frown \dot{y},I}]\}.$$

[13] Since $\mathrm{gr}(G)$ is a successor ordinal and M is minimal, M has a Woodin cardinal.

Hence,

$$\begin{aligned} j(\dot{W}_{\mathrm{I}}^{p}(M)) &= \{(\dot{y},q) : q \Vdash^{N}_{\mathrm{Coll}(\omega,j(\delta))} \phi_{\mathrm{I}}^{\check{p}^\frown \dot{y}}[j(\dot{B}_0^{\check{p}^\frown \dot{y},I})]\} \\ &= \{(\dot{y},q) : q \Vdash^{N}_{\mathrm{Coll}(\omega,j(\delta))} \phi_{\mathrm{I}}^{\check{p}^\frown \dot{y}}[(\dot{B}_0^{j(\check{p}^\frown \dot{y}),I})^N]\} \\ &= \dot{W}_{\mathrm{I}}^{p}(N), \end{aligned}$$

and similarly for $\dot{B}_n^{\dot{p},II}$.

By the definition of simple games, $\mathrm{gr}(G_p)$ depends only on G, the length of p, and finitely many values of p (those in which the players determine the subgames played). Thus, if $\dot{p}$ is a $\mathbb{P}_n$-name in M for a finite sequence of reals such that $\Vdash^{M}_{\mathbb{P}_n} \mathrm{gr}(G_{\dot{p}}) = \gamma + 1$ for some countable ordinal γ and if $\dot{y}$ is a $\mathbb{P}(\delta_0, \ldots, \delta_n, \delta)$-name for a real, then

$$\Vdash^{M}_{\mathbb{P}(\delta_0,\ldots,\delta_n,\delta)} \mathrm{gr}(G_{\check{p}^\frown \dot{y}}) = \gamma.$$

The claim is proved simultaneously for all premice M and all $\dot{p} \in M$, by induction on the game rank of $\mathrm{gr}(G_p)$. We prove it for the names $\dot{B}_n^{\dot{p},I}$ for Player I; the other part is similar. We proceed by cases:

First suppose that $\gamma = 2$ and let $n < \omega$ and $\dot{p} \in M$ be a $\mathbb{P}_n$-name for a finite sequence of reals such that $\Vdash^{M}_{\mathbb{P}_n} \mathrm{gr}(G_{\dot{p}}) = 2$. Let $\dot{B} = \dot{B}_n^{\dot{p},I} \in M$, so that whenever G is $\mathbb{P}_n$-generic over M,

$$\dot{B}[G] = \dot{W}_{\mathrm{I}}^{\dot{p}[G]}(M[G]).$$

By definition,

$$\Vdash^{M} \ldots \Vdash \dot{B} = \{(\dot{y},q) \mid q \Vdash_{\mathrm{Coll}(\omega,\delta)} \text{ Player I has a winning strategy in } G_{\check{p}^\frown \dot{y}}\}.$$

By elementarity,

$$\Vdash^{N} \ldots \Vdash j(\dot{B}) = \{(\dot{y},q) \mid q \Vdash_{\mathrm{Coll}(\omega,j(\delta))} \text{ Player I has a winning strategy in } G_{j(\check{p})^\frown \dot{y}}\}.$$

But this implies that $j(\dot{B})$ is a good $j(\mathbb{P}_n)$-name such that whenever H is $j(\mathbb{P}_n)$-generic over N,

$$j(\dot{B})[H] = \dot{W}_{\mathrm{I}}^{j(\dot{p})[H]}(N[H]).$$

This finishes the case $\gamma = 2$.

If $\gamma = \beta + 1$, where $\beta > 1$, let again $n < \omega$, $\dot{p} \in M$ be a $\mathbb{P}_n$-name for a finite sequence of reals such that $\Vdash^{M}_{\mathbb{P}_n} \mathrm{gr}(G_{\dot{p}}) = \beta + 1$ and let $\dot{B} = \dot{B}_n^{\dot{p},I} \in M$, so that whenever G is $\mathbb{P}_n$-generic over M,

$$\dot{B}[G] = \dot{W}_{\mathrm{I}}^{\dot{p}[G]}(M[G]).$$

This means that

$$\Vdash^M \ldots \Vdash \dot{B} = \{(\dot{y}, q) \mid q \Vdash_{\mathrm{Coll}(\omega,\delta)} \phi_{\mathrm{I}}^{\check{\dot{p}}^\frown \dot{y}}[\dot{C}]\},$$

where $\dot{C} \in M$ is a good $\mathbb{P}(\delta_0, \ldots, \delta_n, \delta)$-name such that whenever g is $\mathbb{P}(\delta_0, \ldots, \delta_n, \delta)$-generic over M,

$$\dot{C}[g] = \dot{W}_{\mathrm{I}}^{(\check{\dot{p}}^\frown \dot{y})[g]}(M[g]).$$

By inductive hypothesis, and using that $\Vdash^M_{\mathbb{P}(\delta_0,\ldots,\delta_n,\delta)} \mathrm{gr}(G_{\check{\dot{p}}^\frown \dot{y}}) = \beta < \mathrm{gr}(G_{\check{\dot{p}}})$, we have that $j(\dot{C}) \in N$ is a good $\mathbb{P}(j(\delta_0), \ldots, j(\delta_n), j(\delta))$-name such that whenever h is $\mathbb{P}(j(\delta_0), \ldots, j(\delta_n), j(\delta))$-generic over N,

$$j(\dot{C})[h] = \dot{W}_{\mathrm{I}}^{j(\check{\dot{p}}^\frown \dot{y})[h]}(N[h]).$$

Since, by elementarity,

$$\Vdash^N \ldots \Vdash j(\dot{B}) = \{(\dot{y}, q) \mid q \Vdash_{\mathrm{Coll}(\omega, j(\delta))} \phi_{\mathrm{I}}^{j(\check{\dot{p}}^\frown \dot{y})}[j(\dot{C})]\},$$

it follows that $j(\dot{B})$ is a good $j(\mathbb{P}_n)$-name such that for every $j(\mathbb{P}_n)$-generic H over N,

$$j(\dot{B})[H] = \dot{W}_{\mathrm{I}}^{j(\dot{p})[H]}(N[H]),$$

as desired.

If γ is a limit ordinal, let again $n < \omega$, $\dot{p} \in M$ be a $\mathbb{P}_n$-name for a finite sequence of reals such that $\Vdash^M_{\mathbb{P}_n} \mathrm{gr}(G_{\dot{p}}) = \gamma$. Let $\dot{B} = \dot{B}_n^{\dot{p},I} \in M$, so that whenever G is $\mathbb{P}_n$-generic over M,

$$\dot{B}[G] = \dot{W}_{\mathrm{I}}^{\dot{p}[G]}(M[G]).$$

This means that

$$\begin{aligned}\Vdash^M \ldots \Vdash \dot{B} = \{k \in \omega \mid &\text{there is an initial segment } M^* \text{ minimal of class} \\ &S_{\mathrm{gr}(G_{\dot{p}^\frown k})} \text{ which satisfies } \phi_{\mathrm{I}}^{\dot{p}^\frown k}[\dot{C}]\},\end{aligned}$$

where $\dot{C} \in M^*$ is a good $\mathbb{P}_n$-name such that whenever g is $\mathbb{P}_n$-generic over M^*,

$$\dot{C}[g] = \dot{W}_{\mathrm{I}}^{(\dot{p}^\frown k)[g]}(M^*[g]).$$

By inductive hypothesis, $j(\dot{C}) \in N|(j(M^*) \cap \mathsf{Ord})$ is a good $j(\mathbb{P}_n)$-name such that whenever h is $j(\mathbb{P}_n)$-generic over $N|(j(M^*) \cap \mathsf{Ord})$,

$$j(\dot{C})[h] = \dot{W}_{\mathrm{I}}^{j(\dot{p}^\frown k)[h]}(N|(j(M^*) \cap \mathsf{Ord})[h]).$$

Since, by elementarity,

$$\Vdash^N \ldots \Vdash j(\dot{B}) = \{k \in \omega \mid \text{there is an initial segment } N^* \text{ minimal of class } S_{\mathrm{gr}(G_{\dot{p}^\frown k})} \text{ which satisfies } \phi_{\mathrm{I}}^{j(\dot{p}^\frown k)}[j(\dot{C})]\},$$

it follows that $j(\dot{B})$ is a good $j(\mathbb{P}_n)$-name such that for every $j(\mathbb{P}_n)$-generic H over N,

$$j(\dot{B})[H] = \dot{W}_{\mathrm{I}}^{j(\dot{p})[H]}(N[H]),$$

as desired.

The argument for $\dot{W}_{\mathrm{II}}^p(M)$ is analogous. This completes the proof of the claim. □

Now we turn to the proof of the following claim, from which the theorem follows. Recall that $\alpha = \mathrm{gr}(G)$.

Claim 5.4.7. *Let M be an r-premouse which of* ZFC *of class S_α but has no proper initial segment of class S_α. Let δ denote the least Woodin cardinal in M. Then*

1. *If $M \models \phi_{\mathrm{I}}^{\emptyset}[\dot{W}_{\mathrm{I}}^{\emptyset}]$, then Player I has a winning strategy in G.*
2. *If $M \models \phi_{\mathrm{II}}^{\emptyset}[\dot{W}_{\mathrm{II}}^{\emptyset}]$, then Player II has a winning strategy in G.*
3. *$M \models \phi_{\mathrm{I}}^{\emptyset}[\dot{W}_{\mathrm{I}}^{\emptyset}] \vee \phi_{\mathrm{II}}^{\emptyset}[\dot{W}_{\mathrm{II}}^{\emptyset}]$.*

Proof. If M is active, then we may identify it with the proper class sized iterated ultrapower by its top extender and its images, so we may assume that M is a model of ZFC. Note that M always has a Woodin cardinal, as α is a successor ordinal. We first prove (3). Suppose that $M \not\models \phi_{\mathrm{I}}^{\emptyset}[\dot{W}_{\mathrm{I}}^{\emptyset}]$ and $M \not\models \phi_{\mathrm{II}}^{\emptyset}[\dot{W}_{\mathrm{II}}^{\emptyset}]$. We inductively construct a run of the game G by its initial segments p_m together with reals y_m, y_m-premice M^{p_m} which are proper class models of ZFC and ordinals δ_{p_m}. Let p_0 be the empty play, $y_0 = r$, and $M_0 = M$. Inductively, suppose that p_m, y_m, and M^{p_m} have been defined and that

$$M^{p_m} \not\models \phi_{\mathrm{I}}^{p_m}[\dot{W}_{\mathrm{I}}^{p_m}] \text{ and } M^{p_m} \not\models \phi_{\mathrm{II}}^{p_m}[\dot{W}_{\mathrm{II}}^{p_m}],$$

and let δ_{p_m} be the least Woodin cardinal of M^{p_m}, if it exists.

If $\mathrm{gr}(G_{p_m}) = \gamma + 1$ is a successor ordinal for some $\gamma \geq 2$, then, by Neeman's theorem (Theorem 5.4.5), there is an M^{p_m}-generic $g_m \subset \mathrm{Coll}(\omega, \delta_{p_m})$ and a $y \in M^{p_m}[g_m]$ such that $y \notin \dot{W}_{\mathrm{I}}^{p_m}[g_m]$ and $y \notin \dot{W}_{\mathrm{II}}^{p_m}[g_m]$. This means that

$$M^{p_m}[g_m] \not\models \phi_{\mathrm{I}}^{p_m^\frown y}[\dot{W}_{\mathrm{I}}^{p_m^\frown y}] \text{ and } M^{p_m}[g_m] \not\models \phi_{\mathrm{II}}^{p_m^\frown y}[\dot{W}_{\mathrm{II}}^{p_m^\frown y}].$$

Let $p_{m+1} = p_m^\frown y$, $M^{p_{m+1}} = M^{p_m}[g_m]$. Here $M^{p_m}[g_m]$ can be rearranged as a y_{m+1}-premouse for some real y_{m+1} coding y_m and g_m as g_m collapses a cutpoint of the y_m-premouse M^{p_m} (see [FNS10] for the fine-structural details). We will always consider $M^{p_{m+1}}$ as such a y_{m+1}-premouse.

Suppose that $\mathrm{gr}(G_{p_m})$ is a limit ordinal and the rules of G dictate that after p_m, it is Player I's turn. By inductive hypothesis we have $M^{p_m} \not\models \phi_{\mathrm{I}}^{p_m}[\dot{W}_{\mathrm{I}}^{p_m}]$, so there is no active initial segment $M^{p_m}|\nu$, $\nu \in \mathsf{Ord}$, of M^{p_m} of class $S_{\mathrm{gr}(G_{p_m^\frown k})}$ which is minimal and satisfies $\phi_{\mathrm{I}}^{p_m^\frown k}[\dot{W}_{\mathrm{I}}^{p_m^\frown k}]$, for any $k \in \omega$. Moreover, $M^{p_m} \not\models \phi_{\mathrm{II}}^{p_m}[\dot{W}_{\mathrm{II}}^{p_m}]$. Hence, there is some $k \in \omega$ and an active initial segment of M^{p_m} of class $S_{\mathrm{gr}(G_{p_m^\frown k})}$ which is minimal and does not satisfy $\phi_{\mathrm{II}}^{p_m^\frown k}[\dot{W}_{\mathrm{II}}^{p_m^\frown k}]$, where δ^* is defined analogous as above. Fix such a k and such an active initial segment $M^{p_m}|\eta$ of M^{p_m} and let $M^{p_{m+1}}$ be the proper class model resulting from iterating the active extender of $M^{p_m}|\eta$ out of the universe. Set $p_{m+1} = p_m^\frown k$ and $y_{m+1} = y_m$. Then,

$$M^{p_{m+1}} \not\models \phi_{\mathrm{I}}^{p_{m+1}}[\dot{W}_{\mathrm{I}}^{p_{m+1}}] \text{ and } M^{p_{m+1}} \not\models \phi_{\mathrm{II}}^{p_{m+1}}[\dot{W}_{\mathrm{II}}^{p_{m+1}}].$$

The case that $\mathrm{gr}(G_{p_m})$ is a limit ordinal and the rules of G dictate that it is Player II's turn after p_m is similar.

Finally, suppose that $\mathrm{gr}(G_{p_m}) = 2$. By Neeman's theorem (Theorem 5.4.5), there is some M^{p_m}-generic $g_m \subset \mathrm{Coll}(\omega, \delta_{p_m})$ and some $y \in M_m[g_m]$ such that $y \notin \dot{W}_{\mathrm{I}}^{p_m}[g_m]$ and $y \notin \dot{W}_{\mathrm{II}}^{p_m}[g_m]$. This means that

1. $M^{p_m}[g_m] \models$ "Player I does not have a winning strategy in $G_{p_m^\frown y}$", and
2. $M^{p_m}[g_m] \models$ "Player II does not have a winning strategy in $G_{p_m^\frown y}$".

However, $\mathrm{gr}(G_{p_m^\frown y}) = 1$, so $G_{p_m^\frown y}$ is a clopen game of length ω. This is a contradiction, as $M^{p_m}[g_m]$ is a model of ZFC, so it certainly satisfies clopen determinacy. This proves (3).

We now prove (1); the proof of (2) is similar.

Let M be as in the statement of the claim and suppose $M \models \phi_{\mathrm{I}}^{\emptyset}[\dot{W}_{\mathrm{I}}^{\emptyset}]$. We will describe a winning strategy σ for Player I in G (in V) as a concatenation of strategies σ_m for different rounds of G. Playing against arbitrary moves of Player II, we will for $m \geq 1$ inductively construct plays p_m for initial segments of G according to σ together with

- reals y_m such that $y_{m+1} \geq_T y_m$ and $y_0 = r$,
- y_m-premice M^{p_m} which are proper class models of ZFC of class $S_{\mathrm{gr}(G_{p_m})}$, none of whose initial segments are of class $S_{\mathrm{gr}(G_{p_m})}$, and such that $p_m \in M^{p_m}$,

- premice N^{p_m} together with iteration embeddings $j_m \colon M^{p_m} \to N^{p_m}$,
- if M^{p_m} has a Woodin cardinal and δ_{p_m} is the least Woodin cardinal in M^{p_m}, $\mathrm{Coll}(\omega, j(\delta_{p_m}))$-generics g_{p_m} over N^{p_m}.

In case M^{p_m} does not have a Woodin cardinal, we will let g_{p_m} be undefined and let $M^{p_{m+1}} = N^{p_m}$. In the other case, we let $M^{p_{m+1}} = N^{p_m}[g_{p_m}]$. Finally, we will stop the construction after finitely many steps when $\mathrm{gr}(G_{p_m}) = 1$.

Let p_0 be the empty play and $M^{p_0} = M$. We will inductively argue that

$$M^{p_m} \models \phi_{\mathrm{I}}^{p_m}[\dot{W}_{\mathrm{I}}^{p_m}(M^{p_m})],$$

where $\dot{W}_{\mathrm{I}}^{p_m}(M^{p_m})$ is the $\mathrm{Coll}(\omega, \delta_{p_m})$-name or set of natural numbers in M^{p_m} defined above.

Assume inductively that p_n and M^{p_n} with $M^{p_n} \models \phi_{\mathrm{I}}^{p_n}[\dot{W}_{\mathrm{I}}^{p_n}(M^{p_n})]$ are already constructed for all $n \leq m$ and that $\mathrm{gr}(G_{p_m}) \geq 2$. To construct p_{m+1} and $M^{p_{m+1}}$ we distinguish the following cases.

Assume first that $\mathrm{gr}(G_{p_m}) = \gamma + 1$ for some $\gamma \geq 2$. Since

$$M^{p_m} \models \phi_{\mathrm{I}}^{p_m}[\dot{W}_{\mathrm{I}}^{p_m}(M^{p_m})],$$

it follows from Neeman's theorem (Theorem 5.4.5) that there is a strategy σ_m for Player I in a game of length ω such that whenever x is a play by σ_m, there is an iterate N^{p_m} of M^{p_m}, an elementary embedding $j \colon M^{p_m} \to N^{p_m}$ and an N^{p_m}-generic $h \subset \mathrm{Coll}(\omega, j(\delta_{p_m}))$ such that $x \in N^{p_m}[h]$ and $x \in j(\dot{W}_{\mathrm{I}}^{p_m}(M^{p_m})[h]$. Hence, by Claim 5.4.6, $x \in \dot{W}_{\mathrm{I}}^{p_m}(N^{p_m})[h]$. By definition,

$$N^{p_m}[h] \models \phi_{\mathrm{I}}^{p_m \widehat{\ } x}[\dot{W}_{\mathrm{I}}^{p_m \widehat{\ } x}(N^{p_m}[h])].$$

Let $p_{m+1} = p_m \widehat{\ } x$, $M^{p_{m+1}} = N^{p_m}[h]$, and $g_{p_m} = h$. Note that $p_{m+1} \in M^{p_{m+1}}$ and as before $N^{p_m}[h]$ can be rearranged as a y_{m+1}-premouse for some real y_{m+1} coding y_m and h (see [FNS10] for the fine structural details). Again, we will always consider $M^{p_{m+1}}$ as such a y_{m+1}-premouse.

Now suppose that $\mathrm{gr}(G_{p_m}) = \lambda$ is a limit ordinal and the rules of G dictate that, after p_m, it is Player I's turn. By assumption, $M^{p_m} \models \phi_{\mathrm{I}}^{p_m}[\dot{W}_{\mathrm{I}}^{p_m}(M^{p_m})]$. So

$$M^{p_m} \models \dot{W}_{\mathrm{I}}^{p_m}(M^{p_m}) \neq \emptyset.$$

Hence, there is a natural number k and an ordinal η such that $M^{p_m}|\eta$ is an active initial segment of M^{p_m} of class $S_{\mathrm{gr}(G_{p_m \widehat{\ } k})}$ and is minimal, and, if we let N^{p_m} be the result of iterating the active extender of $M^{p_m}|\eta$ out of the

universe, then $N^{p_m} \models \phi_{\mathrm{I}}^{p_m^\frown k}[\dot{W}_{\mathrm{I}}^{p_m^\frown k}(N^{p_m})]$. Let $p_{m+1} = p_m^\frown k$, $y_{m+1} = y_m$, and $M^{p_{m+1}} = N^{p_m}$. Moreover, let σ_m be the strategy which tells Player I to play k.

For the other limit case suppose that $\mathrm{gr}(G_{p_m}) = \lambda$ is a limit ordinal and the rules of G dictate that, after p_m, it is Player II's turn. In this case we can let $\sigma_m = \emptyset$ as Player I is not playing, i.e., we only have to react to what Player II is playing in the next round. Suppose that Player II plays some natural number k and let $p_{m+1} = p_m^\frown k$. Since $M^{p_m} \models \phi_{\mathrm{I}}^{p_m}[\dot{W}_{\mathrm{I}}^{p_m}(M^{p_m})]$, we have

$$M^{p_m} \models \dot{W}_{\mathrm{I}}^{p_m}(M^{p_m}) = \omega.$$

In particular, $k \in \dot{W}_{\mathrm{I}}^{p_m}(M^{p_m})$. Thus there is an ordinal η such that $M^{p_m}|\eta$ is an active initial segment of M^{p_m}, minimal of class $S_{\mathrm{gr}(G_{p_m^\frown k})}$, and if we let N^{p_m} denote the result of iterating the active extender of $M^{p_m}|\eta$ out of the universe, then $N^{p_m} \models \phi_{\mathrm{I}}^{p_m^\frown k}[\dot{W}_{\mathrm{I}}^{p_m^\frown k}(N^{p_m})]$. Let $M^{p_{m+1}} = N^{p_m}$, and $y_{m+1} = y_m$.

Finally, assume that $\mathrm{gr}(G_{p_m}) = 2$. Since $M^{p_m} \models \phi_{\mathrm{I}}^{p_m}[\dot{W}_{\mathrm{I}}^{p_m}(M^{p_m})]$, by Neeman's theorem (Theorem 5.4.5), there is a strategy σ_m for Player I in a game of length ω such that whenever x is a play by σ_m, there is an iterate N^{p_m} of M^{p_m}, an elementary embedding $j\colon M^{p_m} \to N^{p_m}$ and an N^{p_m}-generic $h \subset \mathrm{Coll}(\omega, j(\delta_{p_m}))$ such that $x \in N[h]$ and $x \in j(\dot{W}_{\mathrm{I}}^{p_m}(M^{p_m}))[h]$. Then, by Claim 5.4.6, $x \in \dot{W}_{\mathrm{I}}^{p_m}(N^{p_m}))[h]$. Therefore,

$$N^{p_m}[h] \models \text{ Player I has a winning strategy in } G_{p_m^\frown x}.$$

We let $p_{m+1} = p_m^\frown x$ and stop the construction. Since $\mathrm{gr}(G_{p_m^\frown x}) = 1$, $G_{p_m^\frown x}$ is a clopen game of length ω. As $N^{p_m}[h]$ is a proper class model of ZFC, we can use absoluteness of winning strategies for clopen games of length ω to obtain that Player I has a winning strategy in $G_{p_m^\frown x}$ in V. Let σ_{m+1} be a strategy for Player I witnessing this.

This process describes a winning strategy σ for Player I in G by concatenating the strategies σ_i for $1 \leq i \leq m+1$, as desired. □

This finishes the proof of Theorem 5.4.4. □

5.5 Further applications

In this section, we present some additional applications of the proof of Theorem 5.4.4. Since the proofs are similar, we simply sketch the differences.

5.5.1 Longer games

We begin by noting that the results presented so far generalize to longer games. In particular:

Theorem 5.5.1. *Let θ be a countable ordinal. Suppose that for each $\alpha < \omega_1$ and each $y \in \mathbb{R}$ there is some $x \geq_T y$ and an x-premouse M of class S_α above some λ below which there are θ Woodin cardinals in M. Then σ-projective games of length $\omega \cdot \theta$ are determined.*

The theorem is a consequence of a more general result akin to Theorem 5.4.4, namely, the determinacy of simple σ-projective games of length $\omega \cdot (\theta + \omega)$, in the following sense:

Definition 5.5.2. Let Γ be a collection of subsets of $\omega^{\omega \cdot \theta + \omega^2}$ (with each $A \in \Gamma$ identified with a subset of $\omega^{\omega \cdot \theta + \omega \cdot n}$ for some $n \in \omega$ as in Definition 5.2.2). A game of length $\omega \cdot \theta + \omega^2$ is Γ-*simple* if it is obtained as follows:

1. For every $n \in \omega$, games in Γ of fixed length $\omega \cdot \theta + \omega \cdot n$ are Γ-simple.

2. Let $n \in \omega$ and for each $i \in \omega$ let G_i be a Γ-simple game. Then the game G obtained as follows is Γ-simple: Players I and II take turns playing natural numbers for $\omega \cdot \theta + \omega \cdot n$ moves, i.e., $\theta + n$ rounds in games of length ω. Afterwards, Player I plays some $i \in \omega$. Players I and II continue playing according to the rules of G_i (keeping the first $\omega \cdot \theta + \omega \cdot n$ natural numbers they have already played).

3. Let $n \in \omega$ and for each $i \in \omega$ let G_i be a Γ-simple game. Then the game G obtained as follows is Γ-simple: Players I and II take turns playing natural numbers for $\omega \cdot \theta + \omega \cdot n$ moves, i.e., $\theta + n$ rounds in games of length ω. Afterwards, Player II plays some $i \in \omega$. Players I and II continue playing according to the rules of G_i (keeping the first $\omega \cdot \theta + \omega \cdot n$ natural numbers they have already played).

The notion of *game rank* in this context is defined as before: if G is a game of fixed length $\omega \cdot \theta + \omega \cdot n$, then $\mathrm{gr}(G) = n$. If G is obtained from games $G_0, G_1, \ldots$, and from an ordinal $\omega \cdot \theta + \omega \cdot n$ as in Definition 5.5.2, we let

$$\mathrm{gr}(G) = \sup\{\mathrm{gr}(G_i) + \omega : i \in \omega\} + n.$$

Theorem 5.5.3. *Let θ be a countable ordinal. Suppose that for each $\alpha < \omega_1$ and each $y \in \mathbb{R}$ there is some $x \geq_T y$ and an x-premouse M of class S_α above some λ below which there are θ Woodin cardinals in M. Then, simple σ-projective games of length $\omega \cdot (\theta + \omega)$ are determined.*

Proof Sketch. The theorem is proved like Theorem 5.4.4: first, by arguing as in Proposition 5.3.7, one sees that is suffices to prove determinacy for simple clopen games of length $\omega \cdot (\theta + \omega)$. Let G be a game of successor rank α definable from a parameter x. Given $y \geq_T x$ and an active y-premouse M of class S_α, one defines sets $\dot{W}^p_{\mathrm{I}}(M)$ and $\dot{W}^p_{\mathrm{II}}(M)$ and formulae ϕ^p_{I} and ϕ^p_{II} by induction on $\mathrm{gr}(G_p)$ as in the proof of Theorem 5.4.4, provided $\mathrm{gr}(G_p) < \mathrm{gr}(G)$.

The difference is as follows: in the proof of Theorem 5.4.4, $\mathrm{gr}(G) = \mathrm{gr}(G_p)$ occurs when $p = \emptyset$; here, it happens when p is a θ-sequence of reals. Instead of defining $\dot{W}^p_{\mathrm{I}}$ in this case, we define only $\dot{W}^\emptyset_{\mathrm{I}}$ and $\dot{W}^\emptyset_{\mathrm{II}}$. These are names for sets of $\theta+1$-sequences of reals and are defined as in Case 2 in the proof of Theorem 5.4.4; namely,

$$\dot{W}^\emptyset_{\mathrm{I}} = \{(\dot{p}, q) | q \Vdash^M_{\mathrm{Coll}(\omega,\delta)} \phi^{\dot{p}}_{\mathrm{I}}[\dot{W}^{\dot{p}}_{\mathrm{I}}]\},$$

where $\dot{p}$ is a name for a $\theta+1$-sequence of reals, δ is the least Woodin cardinal of M above λ, and δ^* is the second Woodin cardinal of M above λ, if it exists, and ω, otherwise. (Note that δ exists since, by assumption, α is a successor ordinal). $\dot{W}^\emptyset_{\mathrm{II}}$ is defined similarly. Once these names have been defined, one applies Theorem 2A.2 of [Nee04] (the general version of Theorem 5.4.5) so that (using the fact that M has θ Woodin cardinals below λ) one obtains formulae ϕ_{I} and ϕ_{II} with parameters $\dot{W}^\emptyset_{\mathrm{I}}$ and $\dot{W}^\emptyset_{\mathrm{II}}$, such that one of the following holds:

1. If $M \models \phi_{\mathrm{I}}[\dot{W}^\emptyset_{\mathrm{I}}]$, there is a strategy σ for Player I in a game of length $\omega \cdot \theta + \omega$ such that whenever $\vec{x}$ is a play by σ, there is a non-dropping iterate N of M with an embedding j and an N-generic $g \subset \mathrm{Coll}(\omega, j(\delta))$ such that $\vec{x} \in N[g]$ and $\vec{x} \in j(\dot{W}^\emptyset_{\mathrm{I}})[g]$.

2. If $M \models \phi_{\mathrm{II}}[\dot{W}^\emptyset_{\mathrm{II}}]$, there is a strategy τ for Player II in a game of length $\theta \cdot \omega + \omega$ such that whenever $\vec{x}$ is a play by τ, there is a non-dropping iterate N of M with an embedding j and an N-generic $g \subset \mathrm{Coll}(\omega, j(\delta))$ such that $\vec{x} \in N[g]$ and $\vec{x} \in j(\dot{W}^\emptyset_{\mathrm{II}})[g]$.

3. Otherwise, there is an M-generic $g \subset \mathrm{Coll}(\omega, \delta)$ and an $\vec{x} \in M[g]$ such that $\vec{x} \notin \dot{W}^\emptyset_{\mathrm{I}}[g]$ and $\vec{x} \notin \dot{W}^\emptyset_{\mathrm{II}}[g]$.

An argument as in the proof of Theorem 5.4.4 yields the following claim:

Claim 5.5.4. *Let M be an x-premouse and $\lambda \in M$ be an ordinal such that M has θ Woodin cardinals below λ. Suppose that M is of class S_α above λ and no proper initial segment of M is of class S_α above λ, where α is*

a successor ordinal. Let δ denote the least Woodin cardinal of M above λ. Then

1. *If $M \models \phi_{\mathrm{I}}^{\emptyset}[\dot{W}_{\mathrm{I}}^{\emptyset}]$, then Player I has a winning strategy in G.*
2. *If $M \models \phi_{\mathrm{II}}^{\emptyset}[\dot{W}_{\mathrm{II}}^{\emptyset}]$, then Player II has a winning strategy in G.*
3. *$M \models \phi_{\mathrm{I}}^{\emptyset}[\dot{W}_{\mathrm{I}}^{\emptyset}] \vee \phi_{\mathrm{II}}^{\emptyset}[\dot{W}_{\mathrm{II}}^{\emptyset}]$.*

The theorem is now immediate from the claim. □

It seems very likely that the hypotheses of Theorem 5.5.1 are optimal. However, the proof in [Agug] does not seem to adapt easily to show this.

5.5.2 σ-algebras

The proof of Theorem 5.4.4 adapts to prove the determinacy of various σ-algebras. As an example, we consider the smallest σ-algebra containing all projective sets. We show that a sufficient condition for its determinacy is the existence of x-premice of class $S_{\omega+1}$ for every $x \in \mathbb{R}$.

Theorem 5.5.5. *Suppose that for each $x \in \omega^\omega$ there is an x-premouse of class $S_{\omega+1}$. Then, every set in the smallest σ-algebra on ω^ω containing the projective sets is determined.*

Proof Sketch. Let A belong to the σ-algebra in the statement. Let

$$\{\mathbf{\Sigma}^0_\alpha(\mathbf{\Pi}^1_\omega) : \alpha < \omega_1\}$$

denote the Borel hierarchy built starting from sets which are countable intersections of projective sets, i.e., $\mathbf{\Sigma}^0_0(\mathbf{\Pi}^1_\omega) = \mathbf{\Pi}^1_\omega$ consists of all countable intersections of projective sets and $\mathbf{\Sigma}^0_\alpha(\mathbf{\Pi}^1_\omega)$ consists of all countable unions of sets each of which is the complement of a set in $\mathbf{\Sigma}^0_\beta(\mathbf{\Pi}^1_\omega)$ for some $\beta < \alpha$. Standard arguments show that these pointclasses constitute a hierarchy of sets in the smallest σ-algebra containing the projective sets. It follows that there is $\alpha < \omega_1$ and $x \in \omega^\omega$ such that $A \in \Sigma^0_\alpha(\Pi^1_\omega)(x)$, i.e., such that A belongs to $\mathbf{\Sigma}^0_\alpha(\mathbf{\Pi}^1_\omega)$ and that A has a σ-projective code (in the sense of Section 5.3) which is recursive in x.

Let $[A]$ be a σ-projective code for A which is recursive in x and in which no complements appear. To show that A is determined, it suffices to show that the decoding game for $[A]$ is determined. Let us denote this game by G. It is a simple clopen game; it is not quite of rank $\omega + 1$, so determinacy

does not follow immediately from Theorem 5.4.4, but it follows from the proof:

Given a partial play p of the game and a model N, we define sets $\dot{W}_{\mathrm{I}}^p(N)$ and $\dot{W}_{\mathrm{II}}^p(N)$, formulae $\phi_{\mathrm{I}}[\dot{W}_{\mathrm{I}}^p(N)]$ and $\phi_{\mathrm{II}}[\dot{W}_{\mathrm{II}}^p(N)]$, and names $\dot{B}_n^{\dot{p},I}$ and $\dot{B}_n^{\dot{p},II}$ as in the proof of Theorem 5.4.4. The cases where $\mathrm{gr}(G_p) \leq \omega$ are exactly as in the proof of Theorem 5.4.4. The remaining cases are very slightly different—for this, let us define the notion of *subrank*. Recall that G has the following rules:

1. Players I and II begin by alternating ω many rounds to play a real number $y \in \omega^\omega$.

2. Afterwards, they alternate a finite amount of turns as follows: letting $[A_0] = [A]$ and supposing $[A_n]$ has been defined, $[A_n]$ is a σ-projective code for a union or an intersection of sets, say $\{B_i : i \in \omega\}$, with each B_i of smaller (Borel) rank. By the rules of the decoding game, one of the players needs to play a natural number i, thus selecting a σ-projective code $[B_i]$ for B_i; we let $[A_{n+1}] = [B_i]$.

3. Eventually, a stage n^* is reached in which $[A_{n^*}]$ is a code for a projective set; at this point the players continue with the rules of the decoding game for $[A_{n^*}]$.

Given a play p of G in which a player needs to play a natural number i and $[A_n]$ has been defined as above, we say that the *game subrank* of G_p is the least γ such that $A_n \in \Sigma^0_\gamma(\Pi^1_\omega)(x)$ or $A_n \in \Pi^0_\gamma(\Sigma^1_\omega)(x)$.

We now continue the definition of the sets $\dot{W}_{\mathrm{I}}^p(N)$ and $\dot{W}_{\mathrm{II}}^p(N)$, formulae $\phi_{\mathrm{I}}[\dot{W}_{\mathrm{I}}^p(N)]$ and $\phi_{\mathrm{II}}[\dot{W}_{\mathrm{II}}^p(N)]$, and names $\dot{B}_n^{\dot{p},I}$ and $\dot{B}_n^{\dot{p},II}$. Assume p is such that $\omega < \mathrm{gr}(G_p)$, so that the sets have not been defined already; we proceed by induction on the subrank of G_p:

Case 4. The subrank of G_p is defined and nonzero and the rules of G dictate that, after p, it is Player I's turn.

Let $y \geq_T x$ and let M be any y-premouse which is a model of ZFC and of class S_ω. Let $p \in M$. Then let $\dot{W}_{\mathrm{I}}^p = \dot{W}_{\mathrm{I}}^p(M)$ be the set of all $k \in \omega$ such that

$$M \models \phi_{\mathrm{I}}^{p^\frown k}[\dot{W}_{\mathrm{I}}^{p^\frown k}(M)].$$

This makes sense because for all $k \in \omega$, the subrank of $G_{p^\frown k}$ is smaller than the subrank of G_p, so the set $\dot{W}_{\mathrm{I}}^{p^\frown k}(M)$ has been defined. Moreover, $\dot{W}_{\mathrm{I}}^p$

belongs to M, since $[A]$ is recursive in x and the formulae $\phi_I^{p^\frown k}(\dot{W}_{\mathrm{I}}^{p^\frown k}[M])$, as well as the sets $\dot{W}_{\mathrm{I}}^{p^\frown k}(M)$, are definable uniformly in p, as can be shown inductively by following this construction and the proof of Theorem 5.4.5 (cf. the proof of [Nee04, Theorem 1E.1]).

We let $\phi_{\mathrm{I}}^p[\dot{W}_{\mathrm{I}}^p(M)]$ be the formula asserting that $\dot{W}_{\mathrm{I}}^p$ is non-empty. Similarly, let $\dot{W}_{\mathrm{II}}^p = \dot{W}_{\mathrm{II}}^p(M)$ be the set of all $k \in \omega$ such that

$$M \models \phi_{\mathrm{II}}^{p^\frown k}[\dot{W}_{\mathrm{II}}^{p^\frown k}(M)].$$

We let $\phi_{\mathrm{II}}^p[\dot{W}_{\mathrm{II}}^p(M)]$ be the formula asserting that $\dot{W}_{\mathrm{II}}^p$ is equal to ω. As before, the set $\dot{W}_{\mathrm{II}}^p$ belongs to M.

The good $\mathbb{P}_n$-names $\dot{B}_n^{\dot{p},I}$ and $\dot{B}_n^{\dot{p},II}$ are defined as in the other cases.

Case 5. The subrank of G_p is defined and nonzero and the rules of G dictate that, after p, it is Player II's turn.
This case is similar to the preceding one.

Case 6. p is the empty play.

Let M be any x-premouse which is a model of ZFC and of class $S_{\omega+1}$ and let δ be the smallest Woodin cardinal of M. In this case, $\dot{W}_{\mathrm{I}}^{\emptyset}$ is the canonical $\mathrm{Coll}(\omega, \delta)$-name for all reals p such that $\phi_I^p(\dot{W}_{\mathrm{I}}^p[\dot{B}])$ holds, where $\dot{B} \in M$ is a good $\mathrm{Coll}(\omega, \delta)$-name with respect to $\dot{p}$, so that whenever G is $\mathrm{Coll}(\omega, \delta)$-generic over M, $\dot{B}[G] = \dot{W}_{\mathrm{I}}^p(M[G])$, where $p = \dot{p}[G]$.

The name $\dot{W}_{\mathrm{II}}^{\emptyset}$ is defined analogously. The formulae $\phi_{\mathrm{I}}^{\emptyset}(\dot{W}_{\mathrm{I}}^{\emptyset})$ and $\phi_{\mathrm{II}}^{\emptyset}(\dot{W}_{\mathrm{II}}^{\emptyset})$ are obtained by applying Neeman's theorem to $\dot{W}_{\mathrm{I}}^{\emptyset}$ and $\dot{W}_{\mathrm{II}}^{\emptyset}$.

This completes the definition of the names and formulae. After this, an argument as in the proof of Theorem 5.4.4 yields the following claim:

Claim 5.5.6. *Let M be an x-premouse which is of class $S_{\omega+1}$ but has no proper initial segment of class $S_{\omega+1}$. Let δ denote the least Woodin cardinal in M. Then*

1. *If $M \models \phi_{\mathrm{I}}^{\emptyset}[\dot{W}_{\mathrm{I}}^{\emptyset}]$, then Player I has a winning strategy in G.*
2. *If $M \models \phi_{\mathrm{II}}^{\emptyset}[\dot{W}_{\mathrm{II}}^{\emptyset}]$, then Player II has a winning strategy in G.*
3. *$M \models \phi_{\mathrm{I}}^{\emptyset}[\dot{W}_{\mathrm{I}}^{\emptyset}] \vee \phi_{\mathrm{II}}^{\emptyset}[\dot{W}_{\mathrm{II}}^{\emptyset}]$.*

The theorem is now immediate from the claim. □

6

Long Projective Determinacy

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6.1 Introduction

We study the consistency strength of determinacy for games of length ω^2 with payoff in various pointclasses Γ. Specifically, given a set $A \subset \omega^{\omega^2}$, i.e., a set of sequences of natural numbers of length ω^2, with $A \in \Gamma$, consider the following game:

I	n_0		n_2		$\ldots$	n_ω		$\ldots$
II		n_1		n_3	$\ldots$		$n_{\omega+1}$	$\ldots$

This chapter was written jointly with S. Müller.

Players I and II alternate turns playing natural numbers to produce some $x = (n_0, n_1, \dots, n_\omega, n_{\omega+1}, \dots) \in \omega^{\omega^2}$. Player I wins such a run x of the game if, and only if, $x \in A$; otherwise, Player II wins. We study the strength of the statement that games of this form are determined, i.e., that one of the players has a winning strategy. For all nontrivial classes Γ, this question is independent of Zermelo-Fraenkel set theory with the Axiom of Choice (ZFC); for some of them, however, it is known to follow from natural strengthenings of ZFC, namely, from assumptions on the existence of *large cardinals*.

Recall that the projective subsets of a Polish space are those obtainable from Borel sets in finitely many stages by applying complements and projections from a finite power of the space. We are mainly interested in the case where Γ is a projective pointclass, i.e., $\mathbf{\Pi}^1_n$ for some natural number n, but we will also consider the cases in which Γ is equal to $\complement^{\mathbb{R}}\mathbf{\Pi}^1_1$ or a σ-projective pointclass.[1]

The study of games of length ω^2 is motivated by the folklore result that projective determinacy for games of length ω implies projective determinacy for games of length α, for any $\alpha < \omega^2$. It is also not difficult to see that $\mathbf{\Pi}^1_{n+1}$-determinacy of length ω^2 follows from analytic determinacy for games of length $\omega \cdot (\omega + n)$, for natural numbers n.[2]

Analytic determinacy for games of length $\omega \cdot (\omega + n)$ was proved by Neeman [Nee04, Theorem 2A.3] from a large cardinal hypothesis. Specifically, he assumed the existence of a weakly iterable model of set theory with $\omega + n$ Woodin cardinals, along with a sharp for the model. In fact, he proved this result for analytic games of any fixed countable length $\omega \cdot (\theta + 1)$, from corresponding assumptions. This naturally yields the question whether his results are optimal, i.e., whether the determinacy of these long games implies the existence of models with certain numbers of Woodin cardinals. In light of this, we prove the following theorem:

Theorem 6.1.1. *Suppose that $\mathbf{\Pi}^1_{n+1}$-determinacy for games of length ω^2 holds and let $x \in \omega^\omega$ be arbitrary. Then, there is a proper class model M of ZFC with $\omega + n$ Woodin cardinals such that $x \in M$.*

The Woodin cardinals of the model we construct in the proof of Theorem 6.1.1 are in reality countable. Moreover, the model is a premouse, i.e., fine

[1] The pointclass of all σ-projective sets is the smallest pointclass closed under complements, countable unions, and projections, where countable unions refer to sets which are subsets of the same product space.

[2] In fact, an argument as in [AMS, Proposition 2.7] (see Chapter 5 with a more careful analysis of the complexity of the payoff sets (using projective determinacy) shows that these two determinacy hypotheses are equivalent.

structural, and we can in fact construct it such that it is active, i.e., it has a sharp on top. As a corollary of the theorem and the results from [Nee04], we obtain an equiconsistency:

Corollary 6.1.2. *The following are equiconsistent:*

1. ZFC + *Projective Determinacy for games of length* ω^2.
2. ZFC + {"*there are* $\omega + n$ *Woodin cardinals*": $n \in \omega$}.

The proof of Theorem 6.1.1 has two main parts: in the first one, the hypothesis is shown to imply that for a closed and unbounded set of countable sets of reals A, there is a fine structural model of the Axiom of Determinacy with n Woodin cardinals whose set of real numbers is precisely A. In the second part we use a Prikry-like partial order to force over these models and obtain via a translation procedure an infinite sequence of Woodin cardinals below the already existing ones.

Background The situation for games of length ω is well understood: There is a tight connection beween determinacy for these games and the existence of inner models with large cardinals. Martin [Mar75] showed that Borel games are determined in ZFC. Contrary to that, determinacy for $\mathbf{\Sigma}^1_1$ (i.e., analytic) games cannot be proved in ZFC alone. By theorems of Martin [Mar70] and Harrington [Har78] $\mathbf{\Sigma}^1_1$ games are determined if, and only if, $x^\sharp$ exists for every $x \in \omega^\omega$. Martin and Steel [MS89] proved, for each $n \in \omega$, that the existence of n Woodin cardinals with a measurable cardinal above implies the determinacy of all $\mathbf{\Sigma}^1_{n+1}$ sets. Woodin improved this for odd n by showing that the existence of $M^\sharp_n(x)$, the canonical active ω_1-iterable inner model with n Woodin cardinals constructed over x, for all reals x suffices to show $\mathbf{\Sigma}^1_{n+1}$ determinacy. Afterwards, Neeman improved this in [Nee95] even further and showed that for all n, the existence of $M^\sharp_n(x)$ for all reals x implies determinacy of all $\eth^n(<\omega^2 - \mathbf{\Pi}^1_1)$ sets. Concerning the other direction, Woodin (see [MSW16]) showed that if $\mathbf{\Sigma}^1_{n+1}$ games are determined, then $M^\sharp_n(x)$ exists for all reals x, thus establishing a level-by-level characterization of projective determinacy in terms of the existence of inner models with large cardinals. Similar characterizations are known for σ-projective games of length ω (see [Agug] and [AMS]). Determinacy for games of length ω with payoff in $\eth^{\mathbb{R}}\mathbf{\Pi}^1_1$ is equivalent to $\mathsf{AD}^{L(\mathbb{R})}$ (see [MS08]).

Woodin showed that the existence of ω^2 Woodin cardinals under choice is equiconsistent with $\mathsf{AD}^+ + \mathsf{DC}$ and the existence of a normal fine measure

on $\mathcal{P}_{\omega_1}(\mathbb{R})$ (see Remark 9.98 in [Woo10]). This implies that determinacy of analytic games of length ω^3 implies the consistency of ω^2 Woodin cardinals under choice. In fact, a similar result holds for ω^α Woodin cardinals and determinacy of analytic games of length $\omega^{1+\alpha}$ for all $1 < \alpha < \omega$ and all limit ordinals $\omega \leq \alpha < \omega_1$. For details see Theorems 2.2.1 and 2.3.11 in [Tra13], as well as [Tra14] and [Tra15].

Further results The proof of Theorem 6.1.1 uses game arguments based on techniques from Martin and Steel [MS08] and [MSW16], tracing back to arguments of H. Friedman [Fri71] and Kechris and Solovay [KS85] as well as inner model theoretic methods based on the unpublished notes [Stea]. The method of the proof of Theorem 6.1.1 can also be used to show the following two generalizations:

Theorem 6.1.3. *Suppose that $\Im^{\mathbb{R}}\mathbf{\Pi}^1_1$-determinacy for games of length ω^2 holds and let $x \in \omega^\omega$ be arbitrary. Then, there is a proper class model M of* ZFC *with $\omega + \omega$ Woodin cardinals such that $x \in M$.*

As above, it is not hard to show that determinacy of analytic games of length $\omega \cdot (\omega + \omega)$ implies determinacy of $\Im^{\mathbb{R}}\mathbf{\Pi}^1_1$ games of length ω^2. Our methods also generalize to games with σ-projective payoff and premice of class S_α (see Chapters 4 and 5).

Theorem 6.1.4. *Suppose that σ-projective determinacy for games of length ω^2 holds and let $x \in \omega^\omega$ be arbitrary. Then, for every $\alpha < \omega_1$, there is a proper class model M of* ZFC *with ω Woodin cardinals with supremum λ which is of class S_α above λ and such that $x \in M$.*

Outline In Section 6.2, we establish conventions and recall some known facts about extender models which will be used later on. Focusing first on the case $n = 1$, Section 6.3 contains the main argument and shows that determinacy for games of length ω^2 with $\mathbf{\Pi}^1_2$ payoff implies the existence of fine structural models of the Axiom of Determinacy with one Woodin cardinal. In Section 6.4, we argue that we can in fact assume that this fine structural model satisfies DC and AD^+. In Section 6.5, we show how to obtain a model of ZFC with $\omega + 1$ Woodin cardinals from this model. Finally, in Section 6.6 we explain how to carry out the modifications needed to prove Theorems 6.1.3 and 6.1.4.

We would like to thank John Steel for valuable discussions related to this paper, in particular to Lemma 6.3.8, and for making the unpublished notes [Stea] available to us.

6.2 Preliminaries

For basic set theoretic definitions and results we refer to [Kan08] and [Mos09]. Moreover, we work with canonical, fine structural models with large cardinals, called *premice*. We refer the reader to e.g., [Ste10] for an introduction, and to [MS94], [SSZ02], and [Ste08b] for additional background. As in [Ste08b] we will consider relativized premice constructed over arbitrary sets X. Let $\mathcal{L}_{\mathrm{pm}} = \{\dot{\in}, \dot{E}, \dot{F}, \dot{X}\}$ denote the language of relativized premice, where $\dot{E}$ is the predicate for the extender sequence, $\dot{F}$ is the predicate for the top extender, and $\dot{X}$ is the predicate for the set over which we construct the premouse.

We say an *X-premouse* $M = (J^{\vec{E}}_\alpha, \in, \vec{E}, E_\alpha, X)$ for $\vec{E} = (\dot{E})^M$, $E_\alpha = (\dot{F})^M$, and $X = (\dot{X})^M$ is *active* if $E_\alpha \neq \emptyset$. Otherwise, we say M is *passive*. We let $M|\gamma = (J^{\vec{E}}_\gamma, \in, \vec{E} \restriction \gamma, E_\gamma, X)$ for $\gamma \leq M \cap \mathsf{Ord}$. Moreover, we write $M||\gamma$ for the passive initial segment of M of height γ, i.e. $M||\gamma = (J^{\vec{E}}_\gamma, \in, \vec{E} \restriction \gamma, \emptyset, X)$, for $\gamma \leq M \cap \mathsf{Ord}$. In particular, $M||\mathsf{Ord}^M$ denotes the premouse N which agrees with M except that we let $(\dot{F})^N = \emptyset$. We say an ordinal η is a *(strong) cutpoint* of M if there is no extender E on the M-sequence with $\mathrm{CRIT}(E) \leq \eta \leq \mathrm{lh}(E)$.

An arbitrary X-premouse might not satisfy the Axiom of Choice, but it can be construed as an ordinary premouse which satisfies the Axiom of Choice if a well-order of X is added generically. In particular, every X-premouse M is *well-ordered mod X*, i.e., for every set $Y \in M$ there is an ordinal α and a surjection h in M such that $h\colon X \times \alpha \twoheadrightarrow Y$. In this chapter we shall mainly be interested in X-premice which satisfy the Axiom of Determinacy (and hence not the full Axiom of Choice), where $X \in \mathcal{P}_{\omega_1}(\mathbb{R})$, i.e., X is a countable set of reals.

Throughout this chapter, we work under the assumption that all premice are tame, i.e., that there is no extender on the sequence of a premouse overlapping a Woodin cardinal. This results in no loss of generality, as otherwise the conclusions of the theorems in Section 6.1 hold.

In the rest of this section we summarize some facts about premice which we are going to need later. Most of these can be found in [MSW16] and [Uhl16] for x-premice for a real x and they straightforwardly generalize to X-premice for arbitrary countable sets of reals X. Fix such a set $X \in \mathcal{P}_{\omega_1}(\mathbb{R})$

for the rest of this section.

Definition 6.2.1. Let M be an X-premouse and let $\alpha < \omega_1$. Then we say that M is *α-small* if, and only if, for every ordinal $\kappa \leq M \cap \mathsf{Ord}$ such that there is an extender with critical point κ on the M-sequence, in $M|\kappa$ there is no set of ordinals W of order-type α such that every ordinal in W is a Woodin cardinal in $M|\kappa$.

If it exists and is unique, we denote the ω_1-iterable, countable, sound X-premouse which is not α-small, but all of whose proper initial segments are α-small, by $M_\alpha^\sharp(X)$. In this case $M_\alpha(X)$ denotes the result of iterating the top most measure of $M_\alpha^\sharp(X)$ and its images out of the universe.

We will now argue that under certain conditions there is a comparison lemma for n-small X-premice. This will be used in the proof of Theorem 6.3.1. For premice over a real x this can be found for example in [MSW16, Lemma 2.11]; the argument there generalizes to premice over countable sets of reals. For the reader's convenience we will briefly sketch the main ideas and recall the statements. The following notion will be important in what follows to ensure that $\mathcal{Q}$-structures exist.

Definition 6.2.2. Let M be an X-premouse and let δ be a cardinal in M or $\delta = M \cap \mathsf{Ord}$. We say that δ *is not definably Woodin over* M if, and only if, there exists an ordinal $\gamma \leq M \cap \mathsf{Ord}$ such that $\gamma \geq \delta$ and either

1. over $M|\gamma$ there exists an $r\Sigma_n$-definable set $A \subset \delta$ for some $n < \omega$ such that for no $\kappa < \delta$ do the extenders on the M-sequence witness that κ is strong up to δ with respect to A, or
2. $\rho_n(M|\gamma) < \delta$ for some $n < \omega$.

For several iterability arguments to follow we need our premice to satisfy the following property, which, as a fine structural argument shows, is preserved during iterations. This in turn ensures that $\mathcal{Q}$-structures exist in iterations of a premouse M satisfying it.

Definition 6.2.3. Let M be an X-premouse. We say M has *no definable Woodin cardinals* if, and only if, for all $\delta \leq M \cap \mathsf{Ord}$ we have that δ is not definably Woodin over M.

The proof of [MSW16, Lemma 2.11] generalizes to X-premice and shows the following lemma. Here the hypothesis that $M_n^\sharp(x)$ exists for all $x \in \mathbb{R}$ is used to compare the countable premice M and N inside the model $M_n^\sharp(x)$ for a real x coding M and N. The fact that the iteration strategies for M and N are guided by $\mathcal{Q}$-structures ensures that their restriction to trees in

$H_{\delta_x}^{M_n^\sharp(x)}$, where δ_x is the least Woodin cardinal in $M_n^\sharp(x)$, is in $M_n^\sharp(x)$ and hence ω_1^V-iterability suffices for comparison in this situation.

Lemma 6.2.4. *Suppose that $M_n^\sharp(x)$ exists for all $x \in \mathbb{R}$. Let M and N be countable ω_1-iterable X-premice such that every proper initial segment of M and N is n-small and they both do not have definable Woodin cardinals. Then there are iterates M^* and N^* of M and N respectively such that one of the following holds:*

1. *M^* is an initial segment of N^* and there is no drop on the main branch in the iteration from M to M^*,*

2. *N^* is an initial segment of M^* and there is no drop on the main branch in the iteration from N to N^*.*

In the statement of Lemma 6.2.4 the assumption that M and N do not have definable Woodin cardinals can be replaced by the assumption that M and N do not have Woodin cardinals; see the remark after the proof of Lemma 2.11 in [MSW16].

There is a variant of this lemma for the case that one of the premice is only Π^1_n-iterable as introduced in Definitions 1.4 and 1.6 in [Ste95]. In this case, the last model of the iteration tree on that premouse need not be fully well-founded, but the argument from [Ste95, Lemma 2.2] (see also Corollary 2.15 in [MSW16]) yields that we still have a comparison lemma.

Lemma 6.2.5. *Suppose that $M_n^\sharp(x)$ exists for all $x \in \mathbb{R}$. Let M and N be countable n-small and solid X-premice which both do not have Woodin cardinals. Moreover, assume that M is ω_1-iterable and N is Π^1_{n+1}-iterable. Then there is an iteration tree $\mathcal{T}$ on M and a putative [3] iteration tree $\mathcal{U}$ on N of length $\lambda+1$ for some limit ordinal λ such that one of the following holds:*

1. *$\mathcal{M}_\lambda^\mathcal{T}$ is an initial segment of $\mathcal{M}_\lambda^\mathcal{U}$ and there is no drop on the main branch through $\mathcal{T}$. In this case $\mathcal{M}_\lambda^\mathcal{U}$ need not be fully well-founded, but it is well-founded up to $\mathcal{M}_\lambda^\mathcal{T} \cap \mathsf{Ord}$.*

2. *$\mathcal{M}_\lambda^\mathcal{U}$ is an initial segment of $\mathcal{M}_\lambda^\mathcal{T}$ and there is no drop on the main branch through $\mathcal{U}$. In this case $\mathcal{M}_\lambda^\mathcal{U}$ is fully well-founded and $\mathcal{U}$ is an iteration tree.*

[3] We say that $\mathcal{U}$ is a *putative iteration tree* if it satisfies all properties of an iteration tree, but we allow the last model, if it exists, to be ill-founded.

Finally, note that by a standard argument $\mathbf{\Pi}^1_2$-determinacy, or equivalently that $M_1^\sharp(x)$ exists and is ω_1-iterable for all $x \in \mathbb{R}$, implies that $M_1^\sharp(X)$ exists and is ω_1-iterable for any countable set of reals X.

6.3 Models of the Axiom of Determinacy with a Woodin Cardinal

Recall that $\mathcal{P}_{\omega_1}(\mathbb{R})$ denotes the set of all countable sets of reals. We consider models of the form $M_n(A)$, where $n \in \omega$ and $A \in \mathcal{P}_{\omega_1}(\mathbb{R})$. It is not hard to see that—provided they exist—many of these structures are models of the Axiom of Choice. Our first theorem shows that, if games of length ω^2 with $\mathbf{\Pi}^1_{n+1}$ payoff are determined, then many of these structures are models of the Axiom of Determinacy and we can in addition have that $M_n(A) \cap \mathbb{R} = A$.

Theorem 6.3.1. *Let $n \in \omega$ and suppose that determinacy for $\mathbf{\Pi}^1_{n+1}$ games of length ω^2 holds. Then, there is a club $\mathcal{C} \subset \mathcal{P}_{\omega_1}(\mathbb{R})$ such that for all $A \in \mathcal{C}$, $M_n^\sharp(A)$ exists, is ω_1-iterable, $M_n^\sharp(A) \cap \mathbb{R} = A$, and*

$$M_n(A) \models \mathsf{ZF} + \mathsf{AD}.$$

To simplify the notation we will from this point on only consider the case $n = 1$. The general case $n \in \omega$ can be shown by straightforward modifications of the proof we give for $n = 1$ below. Moreover, we will focus on the case $x = 0$ in the statement of Theorem 6.1.1 as the proof relativizes easily. For the rest of this section assume that V is a model of ZFC and Projective Determinacy, i.e., that $M_n^\sharp(x)$ exists for all reals x. Note that the hypothesis of Theorem 6.1.1, determinacy for $\mathbf{\Pi}^1_{n+1}$ games of length ω^2, trivially implies Projective Determinacy since after the first ω moves the following rounds can be used to "play witnesses for projections". Whenever we are assuming a stronger determinacy hypothesis, we will point it out explicitly.

Before moving on to the proof of Theorem 6.3.1, we recall some basic model-theoretic facts we will need later. In addition to the language of premice, $\mathcal{L}_{\mathrm{pm}} = \{\dot\in, \dot E, \dot F, \dot X\}$, we will now also consider the language $\mathcal{L}_{\mathrm{pm}}(\{\dot x_i : i \in \omega\})$ resulting from enhancing $\mathcal{L}_{\mathrm{pm}}$ with constants $\dot x_i$, for $i \in \omega$. For an $\mathcal{L}_{\mathrm{pm}}(\{\dot x_i : i \in \omega\})$-model $\mathcal{M} = (M, \in, \dot E^{\mathcal{M}}, \dot F^{\mathcal{M}}, \dot X^{\mathcal{M}}, \{\dot x_i^{\mathcal{M}} : i < \omega\})$ we write $\mathcal{M} \restriction \mathcal{L}_{\mathrm{pm}}$ for the restriction $(M, \in, \dot E^{\mathcal{M}}, \dot F^{\mathcal{M}}, \dot X^{\mathcal{M}})$ of the model $\mathcal{M}$ to the smaller language $\mathcal{L}_{\mathrm{pm}}$.

Let $\mathcal{M} = (M, \in, \dot E^{\mathcal{M}}, \dot F^{\mathcal{M}}, \dot X^{\mathcal{M}}, \{x_i : i \in \omega\})$ for $x_i = \dot x_i^{\mathcal{M}}$, $i \in \omega$, be a model in the enhanced language, so in particular $\{x_i : i \in \omega\} \subseteq M$. The

definable closure of $\{x_i : i \in \omega\}$ in $\mathcal{M} \restriction \mathcal{L}_{\mathrm{pm}}$ is defined to be the submodel

$$(\bar{M}, \in, \dot{E}^{\mathcal{M}} \cap \bar{M}, \dot{F}^{\mathcal{M}} \cap \bar{M}, \dot{X}^{\mathcal{M}} \cap \bar{M})$$

of $\mathcal{M} \restriction \mathcal{L}_{\mathrm{pm}}$ where $\bar{M}$ consists of all $a \in M$ such that for some $k < \omega$ and some $\mathcal{L}_{\mathrm{pm}}$-formula $\phi(v, v_0, \ldots, v_k)$,

$\mathcal{M} \restriction \mathcal{L}_{\mathrm{pm}} \models$ "$\phi[a, x_0, \ldots, x_k]$ and there is a unique x such that $\phi[x, x_0, \ldots, x_k]$."

For sufficiently nice theories T, the definable closure of a model of T does not depend on the model itself but only on the theory.

Lemma 6.3.2. *Suppose $T \supset \mathsf{ZF}$ is a complete, consistent theory in the language $\mathcal{L}_{\mathrm{pm}}(\{\dot{x}_i : i \in \omega\})$ with the property that whenever*

$$\mathcal{M} = (M, \in, \dot{E}^{\mathcal{M}}, \dot{F}^{\mathcal{M}}, \dot{X}^{\mathcal{M}}, \{\dot{x}_i^{\mathcal{M}} : i \in \omega\})$$

is a model of T and $\mathcal{N}^{\mathcal{M}}$ is the definable closure of $\{\dot{x}_i^{\mathcal{M}} : i \in \omega\}$ in $\mathcal{M} \restriction \mathcal{L}_{\mathrm{pm}}$, then

1. *for each $i \in \omega$, $\mathcal{M} \models \dot{x}_i^{\mathcal{M}} \in \mathbb{R}$,*
2. *$\mathcal{N}^{\mathcal{M}} \prec \mathcal{M} \restriction \mathcal{L}_{\mathrm{pm}}$.*

Then $\mathcal{N}^{\mathcal{M}}$ does not depend on $\mathcal{M}$, i.e., if $\mathcal{P}$ is another model of T with properties (1) *and* (2)*, then $\mathcal{N}^{\mathcal{M}}$ and $\mathcal{N}^{\mathcal{P}}$ are isomorphic.*

Proof. Let T be as in the statement and let

$$\mathcal{M} = (M, \in, \dot{E}^{\mathcal{M}}, \dot{F}^{\mathcal{M}}, \dot{X}^{\mathcal{M}}, \{\dot{x}_i^{\mathcal{M}} : i \in \omega\})$$

and

$$\mathcal{P} = (P, \in, \dot{E}^{\mathcal{P}}, \dot{F}^{\mathcal{P}}, \dot{X}^{\mathcal{P}}, \{\dot{x}_i^{\mathcal{P}} : i \in \omega\})$$

be two models of T. Since T is complete, $\mathcal{M}$ and $\mathcal{P}$ are elementarily equivalent with respect to the language $\mathcal{L}_{\mathrm{pm}}(\{\dot{x}_i : i \in \omega\})$. If $\mathcal{N}^{\mathcal{M}}$ and $\mathcal{N}^{\mathcal{P}}$ are the respective definable closures of $\mathcal{M} \restriction \mathcal{L}_{\mathrm{pm}}$ and $\mathcal{P} \restriction \mathcal{L}_{\mathrm{pm}}$, the natural function ρ given by

$$\begin{aligned} &\text{the unique } a \in M \text{ such that } \mathcal{M} \restriction \mathcal{L}_{\mathrm{pm}} \models \phi[a, \dot{x}_1^{\mathcal{M}}, \ldots, \dot{x}_k^{\mathcal{M}}] \\ \mapsto\ &\text{the unique } b \in P \text{ such that } \mathcal{P} \restriction \mathcal{L}_{\mathrm{pm}} \models \phi[b, \dot{x}_1^{\mathcal{P}}, \ldots, \dot{x}_k^{\mathcal{P}}] \end{aligned}$$

for some $k < \omega$ and some $\mathcal{L}_{\mathrm{pm}}$-formula $\phi(x, v_1, \ldots, v_k)$ is an isomorphism from $\mathcal{N}^{\mathcal{M}}$ to $\mathcal{N}^{\mathcal{P}}$. This follows from the following observations:

1. Since $\dot{x}_i$ are constants interpreted by reals and T is complete, they have the same interpretation in $\mathcal{M}$ and $\mathcal{P}$, i.e., for all $i \in \omega$, $\dot{x}_i^{\mathcal{M}} = \dot{x}_i^{\mathcal{P}}$.

2. If $x \in \mathcal{N}^{\mathcal{M}}$, then there is an $\mathcal{L}_{\mathrm{pm}}$-formula $\psi(v, v_1, \dots, v_k)$ such that $\mathcal{M} \restriction \mathcal{L}_{\mathrm{pm}} \models \psi[x, \dot{x}_1^{\mathcal{M}}, \dots, \dot{x}_k^{\mathcal{M}}]$ and x is unique with this property in $\mathcal{M} \restriction \mathcal{L}_{\mathrm{pm}}$. Thus, if $x \in \dot{E}^{\mathcal{M}}$,

   $\mathcal{M} \models$ "x is the unique element satisfying $\psi[x, \dot{x}_1^{\mathcal{M}}, \dots, \dot{x}_k^{\mathcal{M}}] \wedge x \in \dot{E}^{\mathcal{M}}$,"

   and so by considering the $\mathcal{L}_{\mathrm{pm}}(\{\dot{x}_i : i \in \omega\})$-sentence "there exists a unique x with $\psi(x, \dot{x}_1, \dots, \dot{x}_k) \wedge x \in \dot{E}$",

   $\mathcal{P} \models$ "$\rho(x)$ is the unique element satisfying $\psi[\rho(x), \dot{x}_1^{\mathcal{P}}, \dots, \dot{x}_k^{\mathcal{P}}] \wedge \rho(x) \in \dot{E}^{\mathcal{P}}$",

   hence $\rho(x) \in \dot{E}^{\mathcal{P}}$. If $x \notin \dot{E}^{\mathcal{M}}$, we have $x \notin \dot{E}^{\mathcal{P}}$ by the same argument.

The argument for the other predicates is analogous; hence, all predicates are interpreted the same way. Therefore, $\mathcal{N}^M$ and $\mathcal{N}^P$ are indeed isomorphic. □

We will now set a general context in which we prove the following two lemmas, as we want to apply them in both the proof of Theorem 6.3.1 and the proof of Lemma 6.3.8 below, for different formulae φ.

Definition 6.3.3. Let $X \in \mathcal{P}_{\omega_1}(\mathbb{R})$ and let $\mathcal{N}$ be a countable X-premouse. Let φ be an $\mathcal{L}_{\mathrm{pm}}$-formula without free variables.

1. We say $\mathcal{N}$ is a *φ-witness* if, and only if, $\mathcal{N}$ is 1-small, $\mathcal{N} \vDash \mathsf{ZF}$, there are no Woodin cardinals in $\mathcal{N}$, and $\mathcal{N} \vDash \varphi$.

2. We say $\mathcal{N}$ is a *minimal φ-witness* if, and only if, $\mathcal{N}$ is a φ-witness and no proper initial segment of $\mathcal{N}$ is a φ-witness, i.e., whenever $\mathcal{P}$ is a proper initial segment of $\mathcal{N}$ satisfying ZF + "there are no Woodin cardinals", then $\mathcal{P} \nvDash \varphi$.

Lemma 6.3.4. *Let $X \in \mathcal{P}_{\omega_1}(\mathbb{R})$ and suppose that $\mathcal{N}$ is a countable Π^1_2-iterable X-premouse which is a minimal φ-witness for some $\mathcal{L}_{\mathrm{pm}}$-formula φ. Moreover, assume that there is another countable X-premouse $\mathcal{M}$ which is a φ-witness and ω_1-iterable. Then $\mathcal{N}$ is in fact ω_1-iterable.*

Proof. Let x be a real coding $\mathcal{M}$ and $\mathcal{N}$ and consider the coiteration of $\mathcal{M}$ and $\mathcal{N}$ inside $M_1^\sharp(x)$ in the sense of Lemma 6.2.5 using ω_1-iterability for $\mathcal{M}$ and Π^1_2-iterability for $\mathcal{N}$. Let $\mathcal{T}$ and $\mathcal{U}$ be the resulting iteration trees on $\mathcal{M}$ and $\mathcal{N}$ with final models $\mathcal{M}^*$ and $\mathcal{N}^*$ respectively. We will show that

$\mathcal{N}$ cannot win this comparison, i.e., $\mathcal{N}^* \trianglelefteq \mathcal{M}^*$ and there is no drop on the main branch through $\mathcal{U}$. This implies that $\mathcal{N}$ is elementarily embeddable into the ω_1-iterable premouse $\mathcal{M}^*$ and thus ω_1-iterable.

Assume first that $\mathcal{M}^* = \mathcal{N}^*$, there is no drop on the main branch through $\mathcal{T}$, and there is at least one drop on the main branch through $\mathcal{U}$. Then $\mathcal{M}^*$ is (by elementarity) a model of ZF, contrary to the fact that $\rho_\omega(\mathcal{N}^*) < \mathcal{N}^* \cap \mathsf{Ord}$.

Finally assume, again towards a contradiction, that $\mathcal{M}^* \triangleleft \mathcal{N}^*$ and there is no drop on the main branch through $\mathcal{T}$. The model $\mathcal{N}^*$ need not be fully well-founded, but this does not affect the rest of the argument as we shall work in the well-founded part of $\mathcal{N}^*$.

Notice that $\mathcal{M}^*$ is a proper initial segment of $\mathcal{N}^*$ which (by elementarity) satisfies ZF, "there are no Woodin cardinals," and φ. Therefore, it cannot be that there is no drop in model on the main branch through $\mathcal{U}$, by elementarity and the minimality of $\mathcal{N}$. Assume for simplicity that there is exactly one drop in model on the main branch through $\mathcal{U}$, say, at level $\beta + 1 < \lambda$ (the general case is similar: if there is more than one drop, we repeat the argument). Using the notation from [Ste10, Section 3.1], the fact that there is a drop in model at stage $\beta + 1$ implies that $\mathcal{M}^*_{\beta+1}$ is a proper initial segment of $\mathcal{M}^{\mathcal{U}}_\xi$, where ξ is the U-predecessor of $\beta+1$ and $\mathcal{M}^*_{\beta+1}$ is the model to which the next extender on the main branch through $\mathcal{U}$ is applied. So by elementarity between $\mathcal{M}^*_{\beta+1}$ and $\mathcal{N}^*$, there is an ordinal α^* witnessing the failure of the minimality property for $\mathcal{M}^*_{\beta+1}$, i.e., the following hold:

1. $\alpha^* < \mathcal{M}^*_{\beta+1} \cap \mathsf{Ord} < \mathcal{M}^{\mathcal{U}}_\xi \cap \mathsf{Ord}$,
2. $\mathcal{M}^*_{\beta+1}|\alpha^*$ is a model of ZF with no Woodin cardinals, and
3. $\mathcal{M}^*_{\beta+1}|\alpha^* \vDash \varphi$.

But $\mathcal{M}^*_{\beta+1}$ is an initial segment of $\mathcal{M}^{\mathcal{U}}_\xi$, so the same holds for $\mathcal{M}^{\mathcal{U}}_\xi|\alpha^*$. Now by elementarity again—this time between $\mathcal{N}$ and $\mathcal{M}^{\mathcal{U}}_\xi$—this failure of the minimality property also holds for $\mathcal{N}$, contradicting the fact that $\mathcal{N}$ is a minimal φ-witness. □

Lemma 6.3.5. *Let $X \in \mathcal{P}_{\omega_1}(\mathbb{R})$ and let $\mathcal{M}$ and $\mathcal{N}$ be ω_1-iterable countable X-premice which are minimal φ-witnesses for some $\mathcal{L}_{\mathrm{pm}}$-formula φ. Then $\mathcal{M}$ and $\mathcal{N}$ have a common iterate and on both sides of the iteration there is no drop in model on the main branch through the iteration tree.*

Proof. Let $\mathcal{T}$ and $\mathcal{U}$ be the iteration trees of length $\lambda + 1$ for some ordinal λ on $\mathcal{M}$ and $\mathcal{N}$ respectively obtained from a successful comparison in the

sense of Lemma 6.2.4. Write $\mathcal{M}^* = \mathcal{M}_\lambda^{\mathcal{T}}$ and $\mathcal{N}^* = \mathcal{M}_\lambda^{\mathcal{U}}$ for the last models of the iteration trees. We cannot have $\mathcal{M}^* \trianglelefteq \mathcal{N}^*$, by the argument of Lemma 6.3.4. Similarly, the alternative $\mathcal{N}^* \trianglelefteq \mathcal{M}^*$ leads to a contradiction, so we must have $\mathcal{N}^* = \mathcal{M}^*$.

Only one side of the comparison can drop; assume that there is a drop in model on the main branch through $\mathcal{U}$. The case that the main branch through $\mathcal{T}$ drops is analogous. As in the proof of Lemma 6.3.4, we assume for simplicity that there is exactly one drop in model along the main branch through $\mathcal{U}$, say at stage $\beta + 1 < \lambda$; the general case is dealt with similarly by repeating the argument. By elementarity, $\mathcal{M}^* = \mathcal{N}^*$ and $\mathcal{M}^*_{\beta+1}$ are φ-witnesses. Moreover, as $\mathcal{N}$ is a minimal φ-witness, by elementarity the same holds for $\mathcal{M}_\xi^{\mathcal{U}}$, where ξ is as in the proof of Lemma 6.3.4 the U-predecessor of $\beta + 1$. But $\mathcal{M}^*_{\beta+1} \trianglelefteq \mathcal{M}_\xi^{\mathcal{U}}$, contradicting the minimality property for $\mathcal{M}_\xi^{\mathcal{U}}$. Therefore, both sides of the comparison do not drop in model. □

We are now going to define a collection of games of length ω^2 which are generalizations of the game in [MS08, Lemma 3]. The argument there goes back to ideas in [Fri71] allowing one of the two players in the game to play the theory of a model with certain properties in addition to the usual moves. In the proofs of Theorem 6.3.1 and Lemma 6.3.8 below we will consider two different instances of games from this collection where Player I plays a complete and consistent theory in the language of premice with additional constant symbols.

Before we give the definition of the games, recall that if $X \in \mathcal{P}_{\omega_1}(\mathbb{R})$ and $\mathcal{M}$ is an X-premouse, then analogously to the existence of a definable well-order in L, there is a uniformly definable X-parametrized family of well-orders the union of whose ranges is $\mathcal{M}$ (cf. [Ste08b, Proposition 2.4]). More specifically, we can fix a formula $\theta(\cdot, \cdot, \cdot)$ in the language of premice $\mathcal{L}_{\mathrm{pm}}$ such that for any such X and any X-premouse $\mathcal{M}$, the following hold:

1. for any $x \in \mathcal{M}$, there is some $\alpha \in \mathsf{Ord}^{\mathcal{M}}$ and some $r \in X$ such that
$$\mathcal{M} \models \theta(\alpha, r, x);$$
2. for all $r \in X$ and $\alpha \in \mathsf{Ord}^{\mathcal{M}}$ there is at most one $x \in \mathcal{M}$ such that
$$\mathcal{M} \models \theta(\alpha, r, x).$$

Moreover, fix recursive bijections m and n assigning an odd number > 1 to each $\mathcal{L}_{\mathrm{pm}}(\{\dot{x}_i : i \in \omega\})$-formula φ such that m and n have disjoint recursive ranges and for every φ, $m(\varphi)$ and $n(\varphi)$ are larger than $\max\{i : \dot{x}_i \text{ occurs in } \varphi\}$.

Definition 6.3.6. Let φ and $\psi(x_0, a, b)$ be $\mathcal{L}_{\mathrm{pm}}$-formulae and let $\mathcal{G}_{\varphi,\psi}$ denote the following game of length ω^2 on ω: Fix some enumeration $(\phi_i \colon i \in \omega)$ of all $\mathcal{L}_{\mathrm{pm}}(\{\dot{x}_i : i \in \omega\})$-formulae such that $\dot{x}_i$ does not appear in ϕ_j if $j \leq i$. Then a typical run of $\mathcal{G}_{\varphi,\psi}$ looks as follows:

I	x_0	a	v_0, x_1		v_1, x_3		$\ldots$
II		b		x_2		x_4	$\ldots$

1. Player I starts by playing some parameter $x_0 \in \omega^\omega$;
2. Players I and II take turns playing natural numbers to construct reals $a, b \in \omega^\omega$;
3. Players I and II take turns, respectively playing sequences of natural numbers (v_i, x_{2i+1}) and x_{2i+2} in ω^ω, for $i \in \omega$. We ask that $v_i \in \{0, 1\}$.

Here v_i will be interpreted as the truth value of the formula ϕ_i from the enumeration fixed above. This can be thought of as Player I either accepting or rejecting the formula ϕ_i. If so, the play determines a complete theory T in the language $\mathcal{L}_{\mathrm{pm}}(\{\dot{x}_i : i \in \omega\})$.

Player I wins the game $\mathcal{G}_{\varphi,\psi}$ if, and only if,

1. $x_1 = a \oplus b$.
2. For each $i \in \omega$, T contains the sentence $\dot{x}_i \in \omega^\omega$ and, moreover, for each $j, m \in \omega$, T contains the sentence $\dot{x}_i(m) = j$ if, and only if, $x_i(m) = j$.
3. For every formula $\phi(x)$ with one free variable in the expanded language $\mathcal{L}_{\mathrm{pm}}(\{\dot{x}_i : i \in \omega\})$, and $m(\phi)$ and $n(\phi)$ as fixed above, T contains the statements

$$\exists x\, \phi(x) \to \exists x\, \exists \alpha\, (\phi(x) \wedge \theta(\alpha, \dot{x}_{m(\phi)}, x)),$$

$$\exists x\, (\phi(x) \wedge x \in \dot{X}) \to \phi(\dot{x}_{n(\phi)}).$$

4. T is a complete, consistent theory such that for every countable model $\mathcal{M}$ of T and every model $\mathcal{N}^*$ which is the definable closure of $\{x_i : i < \omega\}$ in $\mathcal{M} \restriction \mathcal{L}_{\mathrm{pm}}$, $\mathcal{N}^*$ is well-founded and if $\mathcal{N}$ denotes the transitive collapse of $\mathcal{N}^*$,
   (a) $\mathcal{N}$ is an X-premouse, where $X = \{x_i \colon i \in \omega\}$,
   (b) $\mathcal{N}$ is a minimal φ-witness,

(c) $\mathcal{N}$ is Π^1_2-iterable in the sense of [Ste95, Definition 1.6], and
(d) $\mathcal{N} \vDash \psi(x_0, a, b)$.

If Player I plays according to all these rules, he wins the game. In this case there is a unique premouse $\mathcal{N}_p$ as in (4) associated to the play $p = (x_0, a \oplus b, v_0, x_1, x_2, \dots)$ of the game. Otherwise, Player II wins.

Remark 6.3.7. Rule (3) in the game $\mathcal{G}_{\varphi,\psi}$ ensures that if $\mathcal{M}$ is a model of the theory T, then the definable closure of $\{x_i : i \in \omega\}$ in $\mathcal{M} \restriction \mathcal{L}_{\mathrm{pm}}$ is an elementary substructure of $\mathcal{M} \restriction \mathcal{L}_{\mathrm{pm}}$ (by the Tarski-Vaught criterion) by the following argument: Suppose $\exists x \phi(x)$ holds in $\mathcal{M}$. Then rule (3) ensures that $\exists x\, \exists \alpha\, (\phi(x) \wedge \theta(\alpha, \dot{x}_{m(\phi)}, x))$. Now, the formula $\phi(x) \wedge \exists \alpha(\theta(\alpha, \dot{x}_{m(\phi)}, x) \wedge \forall \beta \in \alpha\, \neg \exists y(\phi(y) \wedge \theta(\beta, \dot{x}_{m(\phi)}, y)))$ uniquely defines a witness x for $\phi(x)$ (the minimal witness according to the well-order given by $\theta(\cdot, \dot{x}_{m(\phi)}, \cdot)$). Hence, rule (4) can be followed by Player I by playing an appropriate theory T, as then the model $\mathcal{N}$ is uniquely determined by it, by Lemma 6.3.2.

To prove Theorem 6.3.1, we first need to show the following lemma. We thank John Steel for pointing out to us that it can be proved via a modification of our argument for Theorem 6.3.1.

Lemma 6.3.8. *Suppose that $\mathbf{\Pi}^1_2$ games of length ω^2 are determined. Then there is a club $\mathcal{C}^* \subset \mathcal{P}_{\omega_1}(\mathbb{R})$ such that for all $A \in \mathcal{C}^*$,*

$$\mathbb{R} \cap M_1(A) = A.$$

Proof. Assume towards a contradiction that the statement of the lemma fails. Thus, there is a stationary set of sets $A \in \mathcal{P}_{\omega_1}(\mathbb{R})$ such that

$$A \subsetneq \mathbb{R} \cap M_1(A).$$

Let

$$\varphi \equiv \text{“there is a real } y \text{ which is not in } \dot{X}\text{”}$$

and

$$\psi(x_0, a, b) \equiv \text{“there is a real definable from } x_0 \text{ which is not in } \dot{X} \text{ and}$$
$$\text{if } z_0 \text{ is the least real definable from } x_0 \text{ which is not in } \dot{X},$$
$$\text{then its } b_0\text{th digit is } a_1\text{”},$$

where $a = (a_0, a_1, \dots)$ and $b = (b_0, b_1, \dots)$ with $a_i, b_i \in \omega$ for all $i \in \omega$. This will only be applied in X-premice $\mathcal{M}$ for some $X \in \mathcal{P}_{\omega_1}(\mathbb{R})$ with $x_0 \in X$ and “least” refers to the least real in the well-order of elements of $\mathcal{M}$ definable from x_0 which is given by $\theta(\cdot, x_0, \cdot)$.

Consider the game $\mathcal{G}_{\varphi,\psi}$, i.e., after Player I plays the parameter x_0, the only relevant moves are the following: Player II plays a natural number b_0 asking Player I for the b_0th digit of the least real definable from x_0 which is not going to be in X and Player II answers by playing a_1. Afterwards they continue playing the rest of $X = \{x_0, x_1, \dots\}$ and the theory of a φ-witness.

The winning condition in this game $\mathcal{G}_{\varphi,\psi}$ is Π^1_2, so it is determined and we can distinguish the following two cases to obtain a contradiction by arguing that no player can have a winning strategy.

Case 1: Player I has a winning strategy σ in $\mathcal{G}_{\varphi,\psi}$.

Let W be the transitive collapse of a countable elementary substructure Y of some large V_κ such that $\sigma \in Y$ and let π denote the inverse of the collapse embedding, i.e.,

$$\pi\colon W \cong Y \prec V_\kappa.$$

Since $\mathbb{R}^W$ is countable, it follows that $M_1^\sharp(\mathbb{R}^W)$ exists and is ω_1-iterable. Since the set of $\mathbb{R}^W$ for such elementary substructures W is a club in $\mathcal{P}_{\omega_1}(\mathbb{R})$, we may assume that

$$\mathbb{R}^W \subsetneq \mathbb{R} \cap M_1(\mathbb{R}^W).$$

The game $\mathcal{G}_{\varphi,\psi}$ can be defined in W. Let $\bar{\sigma} \in W$ be such that $\pi(\bar{\sigma}) = \sigma$, i.e., $\bar{\sigma} = \sigma \cap W$. By elementarity,

$$W \vDash \text{“}\bar{\sigma} \text{ is a winning strategy for Player I in } \mathcal{G}_{\varphi,\psi}\text{.”}$$

Let h be a well-ordering of $\mathbb{R}^W$ in V of order-type ω. Consider a play of the game $\mathcal{G}_{\varphi,\psi}$ in V in which Player II plays some $b \in \omega^\omega$ and $x_2, x_4, \dots$ according to h and Player I plays according to the winning strategy σ. Every proper initial segment of the play is in the domain of $\bar{\sigma}$. It follows that the real part $(x_0, x_1, x_2, \dots)$ of the play, say p, enumerates $\mathbb{R}^W$. Furthermore, p is consistent with σ, whereby p is won by Player I. This means that p determines a Π^1_2-iterable $\mathbb{R}^W$-premouse $\mathcal{N}_p$ which is a minimal φ-witness.

Let $\delta_{\mathbb{R}^W}$ denote the Woodin cardinal in $M_1(\mathbb{R}^W)$. Since we chose W so that $\mathbb{R}^W \subsetneq \mathbb{R} \cap M_1(\mathbb{R}^W)|\delta_{\mathbb{R}^W}$ and since satisfying φ for the $\mathbb{R}^W$-premice $\mathcal{N}_p$ and $M_1(\mathbb{R}^W)|\delta_{\mathbb{R}^W}$ means having a real which is not in $\mathbb{R}^W$, we have that $M_1(\mathbb{R}^W)|\delta_{\mathbb{R}^W}$ is an ω_1-iterable φ-witness. Thus, Lemma 6.3.4 implies that $\mathcal{N}_p$ is ω_1-iterable as well.

Let $x_0 \in \omega^\omega$ be the first move given by σ (so x_0 is also the first move given by $\bar{\sigma}$). Moreover, let $a_0 \in \omega$ be the first move of Player I after x_0 given by σ (and $\bar{\sigma}$). Let τ be the real defined by

$$\tau(n) = \bar{\sigma}(x_0, a_0, n),$$

for all possible moves $n \in \omega$ of Player II for b_0. We claim that τ is the least real in $\mathcal{N}_p$ not in $\mathbb{R}^W$ which is definable from x_0. This will be a contradiction, since $\tau \in W$ as $\bar{\sigma} \in W$ and $x_0 \in W$.

Let τ' be the least real in $\mathcal{N}_p$ not in $\mathbb{R}^W$ which is definable from x_0. Assume that $\tau' \neq \tau$ and choose some $n_0 \in \omega$ such that $\tau(n_0) \neq \tau'(n_0)$. Let q be the play of the game $\mathcal{G}_{\varphi,\psi}$ in which Player I plays according to $\bar{\sigma}$ and Player II plays some $b \in \omega^\omega$ with first digit n_0 and then h as above. As Player I plays according to $\bar{\sigma}$ and hence according to σ, this is a winning play for Player I. Let $\mathcal{N}_q$ be the corresponding model. In particular, $\mathcal{N}_q \vDash \psi(x_0, a, b)$, i.e. the least real in $\mathcal{N}_q$ not in $\mathbb{R}^W$ which is definable from x_0 has $a_1 = \bar{\sigma}(x_0, a_0, n_0)$ as n_0th digit.

By the rules of the game, $\mathcal{N}_q$ is a Π^1_2-iterable $\mathbb{R}^W$-premouse which is a minimal φ-witness. Hence, Lemma 6.3.4 yields (as in the case of $\mathcal{N}_p$) that $\mathcal{N}_q$ is in fact ω_1-iterable. So Lemma 6.3.5 implies that $\mathcal{N}_p$ and $\mathcal{N}_q$ coiterate to a common model and there is no drop in model on the main branch through the trees on both sides of the coiteration. Let $\mathcal{T}$ on $\mathcal{N}_p$ and $\mathcal{U}$ on $\mathcal{N}_q$ be iteration trees witnessing this. As $\mathcal{N}_q$ and $\mathcal{N}_p$ both satisfy $\psi(x_0, a, b)$, they contain a real different from all reals in $\mathbb{R}^W$ which is definable from x_0. In fact, the least such real τ' is the same in $\mathcal{N}_p$ and $\mathcal{N}_q$. But by choice of q, the n_0th digit of this real is $\tau'(n_0) = \bar{\sigma}(x_0, a_0, n_0) = \tau(n_0)$, which is the desired contradiction.

Case 2: Player II has a winning strategy σ in $\mathcal{G}_{\varphi,\psi}$.

Let W be the transitive collapse of a countable elementary substructure Y of some large V_κ such that $\sigma \in Y$ and let π denote the inverse of the collapse embedding. Moreover, let $\bar{\sigma} \in W$ be such that $\pi(\bar{\sigma}) = \sigma$, i.e., $\bar{\sigma} = \sigma \cap W$. Since $\mathbb{R}^W$ is countable, it follows that $M_1^\sharp(\mathbb{R}^W)$ exists and is ω_1-iterable in V. As before, by our hypothesis we may assume that

$$\mathbb{R}^W \subsetneq \mathbb{R} \cap M_1(\mathbb{R}^W).$$

Let $\mathcal{Q} = M_1(\mathbb{R}^W)|\alpha$, where α is least such that $\mathcal{Q} \vDash \mathsf{ZF}+$ "there are no Woodin cardinals" and $\mathcal{Q}$ contains a real which is not in $\mathbb{R}^W$. Let $\mathcal{N}^{*,\mathcal{Q}}$ be the definable closure of $\mathbb{R}^W$ in $\mathcal{Q}$ and $\mathcal{N}^{\mathcal{Q}}$ the transitive collapse of $\mathcal{N}^{*,\mathcal{Q}}$. Then $\mathcal{N}^{\mathcal{Q}} \prec \mathcal{Q}$. Thus, there is some real z in $\mathcal{N}^{\mathcal{Q}}$ which is not in $\mathbb{R}^W$ such that z is definable in $\mathcal{N}^{\mathcal{Q}}$ from some real $x_0 \in \mathbb{R}^W$. We shall ask Player I to begin every play of the game $\mathcal{G}_{\varphi,\psi}$ by playing this real x_0. Assume without loss of generality that z is the least real in $\mathcal{N}^{\mathcal{Q}} \setminus \mathbb{R}^W$ definable from x_0 according to the well-order defined by $\theta(\cdot, x_0, \cdot)$.

Consider the play p in $\mathcal{G}_{\varphi,\psi}$ in which Player II plays according to $\bar{\sigma}$ (and hence according to the winning strategy σ) and Player I plays:

1. $x_0 \in \mathbb{R}^W$, in the first round,
2. $a_1 = z(b_0)$, in response to Player II playing $b_0 \in \omega$ according to σ,
3. other, arbitrary, natural numbers $a_0, a_2, a_3, \ldots$,
4. some enumeration h of $\mathbb{R}^W$ in order-type ω with $h \in V$ as in Case 1, together with the theory of $\mathcal{Q}$ in the language $\mathcal{L}_{\mathrm{pm}}(\{\dot{x}_i : i \in \omega\})$, where the constants $\dot{x}_i$ are interpreted by the reals $x_i \in \mathbb{R}^W$ according to p, satisfying rules (1), (2), and (3) of the game $\mathcal{G}_{\varphi,\psi}$.

Arguing as before, one shows that the reals in played in p enumerate $\mathbb{R}^W$. It follows that the model $\mathcal{N}^{\mathcal{Q}}$ witnesses that p is a winning play for Player I, which contradicts the fact that σ is a winning strategy for Player II. This proves the lemma. □

Finally, we are ready to prove Theorem 6.3.1.

Proof of Theorem 6.3.1. Assume towards a contradiction that we have

$$M_1(B)|\delta_B \not\models \mathsf{AD}$$

for a stationary set of sets $B \in \mathcal{P}_{\omega_1}(\mathbb{R})$, where δ_B denotes the Woodin cardinal in $M_1(B)$. Let

$$\varphi \equiv \text{“}\dot{X} = \mathbb{R} + \neg\mathsf{AD}\text{”}$$

and

$$\begin{aligned}\psi(x_0, a, b) \equiv \text{“there is a non-determined set of reals definable from } x_0 \text{ and}\\ \text{if } Z \text{ is the least such set in the well-order relative to } \dot{X},\\ \text{then } a \oplus b \in Z\text{”}.\end{aligned}$$

This will only be applied in X-premice $\mathcal{M}$ for some $X \in \mathcal{P}_{\omega_1}(\mathbb{R})$ with $x_0 \in X$ and as in the proof of Lemma 6.3.8 “least” refers to the least set in the well-order of elements of $\mathcal{M}$ definable from x_0 which is given by $\theta(\cdot, x_0, \cdot)$.

In this case, the game $\mathcal{G}_{\varphi,\psi}$ is a variant of the Kechris-Solovay game in [KS85] (see also the game in [MSW16, Lemma 2.3]) adapted as a model game. The winning condition is Π^1_2, so the game $\mathcal{G}_{\varphi,\psi}$ is determined. We will obtain a contradiction by arguing that no player can have a winning strategy.

Case 1: Player I has a winning strategy σ in $\mathcal{G}_{\varphi,\psi}$.

Let W be the transitive collapse of a countable elementary substructure Y of some large V_κ with $\sigma \in Y$ and let π denote the inverse of the collapse embedding, i.e.

$$\pi \colon W \cong Y \prec V_\kappa.$$

Since $\mathbb{R}^W = \mathbb{R} \cap W$ is countable, it follows that $M_1^\sharp(\mathbb{R}^W)$ exists and is ω_1-iterable (in V). By Lemma 6.3.8, the set of $\mathbb{R}^W$ for W as above with the additional property that $M_1(\mathbb{R}^W) \cap \mathbb{R} = \mathbb{R}^W$ is club in $\mathcal{P}_{\omega_1}(\mathbb{R})$. By assumption, we may thus choose W so that $M_1(\mathbb{R}^W) \cap \mathbb{R} = \mathbb{R}^W$ and in addition

$$M_1(\mathbb{R}^W)|\delta_{\mathbb{R}^W} \not\models \mathsf{AD}.$$

Note that the game $\mathcal{G}_{\varphi,\psi}$ can be defined in W and let $\bar{\sigma} \in W$ be such that $\pi(\bar{\sigma}) = \sigma$, i.e., $\bar{\sigma} = \sigma \cap W$. By elementarity,

$$W \models \text{“}\bar{\sigma} \text{ is a winning strategy for Player I in } \mathcal{G}_{\varphi,\psi}\text{.”}$$

Let h be a well-ordering of $\mathbb{R}^W$ in V of order-type ω. Clearly, every proper initial segment of h is in W. Consider a play of the game $\mathcal{G}_{\varphi,\psi}$ in V in which Player II plays some arbitrary $b^* \in \omega^\omega$ and $x_2, x_4, \dots$ according to h and Player I plays according to the winning strategy σ. Every proper initial segment of the play is in the domain of $\bar{\sigma}$. It follows that the real part $(x_0, x_1, x_2, \dots)$ of the play, say p, enumerates $\mathbb{R}^W$. Futhermore, p is consistent with σ, whereby p is won by Player I. This means that p determines a Π^1_2-iterable $\mathbb{R}^W$-premouse $\mathcal{N}_p$ which is a minimal φ-witness. In particular, $\mathcal{N}_p \cap \mathbb{R} = \mathbb{R}^W$. Since both $\mathcal{N}_p$ and $M_1(\mathbb{R}^W)|\delta_{\mathbb{R}^W}$ are $\mathbb{R}^W$-premice and $M_1(\mathbb{R}^W)|\delta_{\mathbb{R}^W}$ is a φ-witness, $\mathcal{N}_p$ is ω_1-iterable by Lemma 6.3.4.

Let $x_0 \in \omega^\omega$ be the first move given by σ (so x_0 is also the first move given by $\bar{\sigma}$). Let $Z = Z(x_0, \mathcal{N}_p)$ denote the least non-determined set of reals in $\mathcal{N}_p$ which is definable from x_0. This exists since $\mathcal{N}_p \vDash \psi(x_0, a^*, b^*)$, where a^* is the sequence of natural numbers Player I plays after x_0 in response to b^* according to σ. There is a natural strategy τ for Player I in $G(Z)$—the Gale-Stewart game with winning condition Z played in $\mathcal{N}_p$—which is induced by $\bar{\sigma}$. Let τ be the unique strategy such that for $a, b \in \omega^\omega \cap \mathcal{N}_p$,

$$a = \tau(b) \text{ if, and only if, } (x_0, a) = \bar{\sigma}(b).$$

Note that $\tau \in W$ as $\bar{\sigma}, x_0 \in W$ and, since the reals of $\mathcal{N}_p$ are those of W, we also have $\tau \in \mathcal{N}_p$.

We claim that τ is a winning strategy for Player I (in the game $G(Z)$ in $\mathcal{N}_p$), which will contradict the fact that the set Z is non-determined in

$\mathcal{N}_p$. Let $a \oplus b \in \mathbb{R}^W$ be a play by τ. Let q be the play of the game $\mathcal{G}_{\varphi,\psi}$ in which Player I plays according to $\bar{\sigma}$ and Player II plays b and then h as above. As Player I plays according to $\bar{\sigma}$ and hence according to σ, this is a winning play for Player I. Let $\mathcal{N}_q$ be the corresponding model. In particular, $\mathcal{N}_q \vDash \psi(x_0, a, b)$, i.e. $a \oplus b \in Z(x_0, \mathcal{N}_q)$, where $Z(x_0, \mathcal{N}_q)$ denotes the least non-determined set of reals in $\mathcal{N}_q$ which is definable from x_0.

By the rules of the game, $\mathcal{N}_q$ is a Π^1_2-iterable $\mathbb{R}^W$-premouse which is a minimal φ-witness. Hence Lemma 6.3.4 yields that $\mathcal{N}_q$ is in fact ω_1-iterable. So we can apply Lemma 6.3.5 to $\mathcal{N}_q$ and $\mathcal{N}_p$.

Let $\mathcal{T}$ on $\mathcal{N}_q$ and $\mathcal{U}$ on $\mathcal{N}_p$ be iteration trees witnessing the conclusion of Lemma 6.3.5. As $\mathcal{N}_q$ and $\mathcal{N}_p$ both satisfy $\psi(x_0, \cdot, \cdot)$, there is a non-determined set of reals definable from x_0 in both $\mathcal{N}_q$ and $\mathcal{N}_p$. As x_0 is not moved by the iteration embeddings and $Z = Z(x_0, \mathcal{N}_p)$ is the least such set in the well-order relative to x_0 defined by $\theta(\cdot, x_0, \cdot)$ in $\mathcal{N}_p$, by elementarity Z is also the least such set in the well-order relative to x_0 defined by $\theta(\cdot, x_0, \cdot)$ in $\mathcal{N}_q$, i.e. $Z = Z(x_0, \mathcal{N}_p) = Z(x_0, \mathcal{N}_q)$. Thus, $a \oplus b \in Z$. Hence, τ is a winning strategy for Player I, contrary to the fact that Z is non-determined in $\mathcal{N}_p$.

Case 2: Player II has a winning strategy σ in $\mathcal{G}_{\varphi,\psi}$.

As before, let W be the transitive collapse of a countable elementary substructure Y of some large V_κ with $\sigma \in Y$ and let π denote the inverse of the collapse embedding, i.e.

$$\pi \colon W \cong Y \prec V_\kappa.$$

Then $M_1^\sharp(\mathbb{R}^W)$ exists and is ω_1-iterable in V. As before, we may choose W so that

$$M_1(\mathbb{R}^W) \cap \mathbb{R} = \mathbb{R}^W \text{ and } M_1(\mathbb{R}^W)|\delta_{\mathbb{R}^W} \not\models \mathsf{AD}.$$

Let $\bar{\sigma} = \sigma \cap W$, so that $\bar{\sigma} \in W$ and

$$W \models \text{“}\bar{\sigma} \text{ is a winning strategy for Player II in } \mathcal{G}_{\varphi,\psi}\text{.”}$$

Let $\mathcal{Q} = M_1(\mathbb{R}^W)|\alpha$, where α is least such that there are no Woodin cardinals in $\mathcal{Q}$ and $\mathcal{Q} \models \mathsf{ZF} + \neg\mathsf{AD}$. Let $\mathcal{N}^{*,\mathcal{Q}}$ be the definable closure of $\mathbb{R}^W$ in $\mathcal{Q}$ and $\mathcal{N}^{\mathcal{Q}}$ the transitive collapse of $\mathcal{N}^{*,\mathcal{Q}}$. Then $\mathcal{N}^{\mathcal{Q}} \prec \mathcal{Q}$ and $\mathcal{N}^{\mathcal{Q}}$ is ω_1-iterable because it is elementary embedded in the ω_1-iterable premouse $\mathcal{Q}$. Moreover, there is some non-determined set in $\mathcal{N}^{\mathcal{Q}}$ witnessing $\mathcal{N}^{\mathcal{Q}} \vDash \neg\mathsf{AD}$, and this set is definable in $\mathcal{Q}$ and hence in $\mathcal{N}^{\mathcal{Q}}$ from some real $x_0 \in \mathbb{R}^W$. We shall ask Player I to begin every play of the game $\mathcal{G}_{\varphi,\psi}$ by playing this

real x_0. Let $Z(x_0, \mathcal{N}^{\mathcal{Q}})$ denote the least non-determined set definable from x_0 in $\mathcal{N}^{\mathcal{Q}}$ according to the well-order relative to $\mathbb{R}^W$ defined by $\theta(\cdot, x_0, \cdot)$.

Since $\bar{\sigma}$ is a winning strategy for Player II in the game $\mathcal{G}_{\varphi,\psi}$ in W, in particular $\bar{\sigma}$ wins against all plays in which Player I begins by playing x_0. There is a natural strategy $\tau \in \mathcal{N}^{\mathcal{Q}}$ for Player II for the Gale-Stewart game inside $\mathcal{N}^{\mathcal{Q}}$ with payoff set $Z(x_0, \mathcal{N}^{\mathcal{Q}})$, namely, the unique strategy such that

$$b = \tau(a) \text{ if, and only if, } b = \bar{\sigma}(x_0, a).$$

Note that $\tau \in \mathcal{N}^{\mathcal{Q}}$ as $\bar{\sigma}, x_0 \in W$ and τ can be coded by a real in $\mathbb{R}^W$.

We claim that τ is a winning strategy for Player II in $\mathcal{N}^{\mathcal{Q}}$. Let $a \oplus b$ be a play by τ in the Gale-Stewart game $G(Z(x_0, \mathcal{N}^{\mathcal{Q}}))$ in $\mathcal{N}^{\mathcal{Q}}$. Then $a \oplus b \in \mathcal{N}^{\mathcal{Q}} \cap \mathbb{R} = \mathbb{R}^W$. Consider the corresponding play p in $\mathcal{G}_{\varphi,\psi}$ in V extending τ in which Player II keeps playing according to $\bar{\sigma}$ (and hence according to the winning strategy σ) and Player I continues by playing some enumeration h of $\mathbb{R}^W$ in order-type ω with $h \in V$ in addition to the theory T of $\mathcal{Q}$ in the language $\mathcal{L}_{\mathrm{pm}}(\{\dot{x}_i : i \in \omega\})$, where the constants $\dot{x}_i$ are interpreted by the reals $x_i \in \mathbb{R}^W$ according to p. We ask Player I to organize these moves in such a way that they satisfy rules (1), (2), and (3) of the game $\mathcal{G}_{\varphi,\psi}$.

Let $\mathcal{M}$ be some model of T, let $\mathcal{N}_p^*$ be the definable closure of $\mathbb{R}^W$ in $\mathcal{M} \restriction \mathcal{L}_{\mathrm{pm}}$ as in rule (4) in the definition of $\mathcal{G}_{\varphi,\psi}$, and let $\mathcal{N}_p$ be the transitive collapse of $\mathcal{N}_p^*$. By Lemma 6.3.2, $\mathcal{N}_p^*$ and $\mathcal{N}^{*,\mathcal{Q}}$ are isomorphic and thus $\mathcal{N}_p = \mathcal{N}^{\mathcal{Q}}$. Therefore, the theory T of the model $\mathcal{Q}$ played together with the reals in $\mathbb{R}^W$ satisfies all rules of the game $\mathcal{G}_{\varphi,\psi}$ except perhaps for the very last one (4d), i.e., Player I cannot lose because of having played the wrong theory but only because $a \oplus b \notin Z(x_0, \mathcal{N}_p)$, where $Z(x_0, \mathcal{N}_p)$ denotes the least non-determined set in $\mathcal{N}_p$ definable from x_0 in the well-order relative to $\mathbb{R}^W$ defined by $\theta(\cdot, x_0, \cdot)$. In fact, this must be the case, as Player II played according to σ, which is a winning strategy.

However, $\mathcal{N}_p = \mathcal{N}^{\mathcal{Q}}$ and in particular $Z(x_0, \mathcal{N}_p) = Z(x_0, \mathcal{N}^{\mathcal{Q}})$. Hence, $a \oplus b \notin Z(x_0, \mathcal{N}^{\mathcal{Q}})$. Therefore, τ is a winning strategy for Player II in the Gale-Stewart game $G(Z(x_0, \mathcal{N}^{\mathcal{Q}}))$ in $\mathcal{N}^{\mathcal{Q}}$, contrary to the fact that $Z(x_0, \mathcal{N}^{\mathcal{Q}})$ was not determined. This finishes the proof of Theorem 6.3.1. □

6.4 Dependent Choices and AD^+

By Theorem 6.3.1 we obtain a countable set of reals A such that $M_1(A)$ is an A-premouse constructed over its reals and $M_1(A) \vDash \mathsf{ZF} + \mathsf{AD}$ from the assumption that all $\mathbf{\Pi}^1_2$ games of length ω^2 are determined. We aim to show that there is a model with $\omega + 1$ Woodin cardinals from this hypothesis.

First, we have that in fact $M_1(A) \vDash \mathsf{DC}$ by the following theorem, which is a special case of [MÏ18, Theorem 1.1].

Theorem 6.4.1. *Let $A \in \mathcal{P}_{\omega_1}(\mathbb{R})$ be a countable set of reals such that $M_1(A) \cap \mathbb{R} = A$ and $M_1(A) \vDash \mathsf{ZF} + \mathsf{AD}$. Then $M_1(A) \vDash \mathsf{DC}$.*

In what follows we argue that we can in fact assume that $M_1(A)$ satisfies AD^+, a strengthening of AD isolated by Woodin, and $\Theta = \theta_0$, i.e. that the Solovay sequence is trivial. We recall the definitions and basic facts for the reader's convenience.

Assuming AD, recall that

$$\Theta = \sup\{\beta : \text{there is a surjection } f\colon \mathbb{R} \to \beta\},$$

and

$$\theta_0 = \sup\{\beta : \text{there is an } \mathsf{OD} \text{ surjection } f\colon \mathbb{R} \to \beta\}.$$

Definition 6.4.2. A set $A \subset \mathbb{R}$ is *∞-Borel* if there is a set of ordinals S and a formula φ in the language of set theory such that

$$A = \{x \in \mathbb{R} : L[S, x] \models \varphi(S, x)\}.$$

Definition 6.4.3. *Ordinal determinacy* is the assertion that whenever $A \subset \omega^\omega$, $\lambda < \Theta$, and $f : {}^\omega\lambda \to \omega^\omega$ is a continuous function, where ${}^\omega\lambda$ is viewed as a product of discrete spaces, the set $f^{-1}[A]$ is determined.

Definition 6.4.4 (Woodin)**.** AD^+ is the conjunction of the following three statements:

1. $\mathsf{DC}_{\mathbb{R}}$,
2. Every $A \subset \mathbb{R}$ is ∞-Borel,
3. Ordinal determinacy.

By a theorem of Woodin (see Section 8.1 in [KW10]) the theory "$\mathsf{ZF} + \mathsf{DC}_{\mathbb{R}} + \mathsf{AD} + \neg\mathsf{AD}^+$" has high consistency strength; more precisely, it proves $\mathrm{Con}(\mathsf{AD}_{\mathbb{R}})$ and hence by a theorem of Steel (see Corollary 1.21 in [Stea]) the existence of a premouse with a cardinal λ which is a limit of Woodin cardinals and of $<\lambda$-strong cardinals. Since this is much stronger than the consistency of $\omega + 1$ Woodin cardinals, we shall assume that for any A as above in fact $M_1(A) \vDash \mathsf{ZF} + \mathsf{DC} + \mathsf{AD}^+$.

Similarly, by another theorem of Woodin (see Theorem 15.1 in [Ste08c]) the theory "$\mathsf{ZF} + \mathsf{AD}^+ + \theta_0 < \Theta$" has high consistency strength; its consistency implies the existence of a model with a cardinal λ which is a limit of

Woodin cardinals and a cardinal κ which is $<\lambda$-strong. Again, this is much stronger than the consistency of $\omega+1$ Woodin cardinals, so we shall assume that $M_1(A) \vDash \mathsf{ZF} + \mathsf{DC} + \mathsf{AD}^+ + \Theta = \theta_0$.

Finally, we also have that $M_1(A)$ satisfies *Mouse Capturing* (MC), i.e., that for any two countable transitive sets x and y such that $x \subseteq y$ and $x \in \mathsf{OD}_{y\cup\{y\}}$, x is contained in an ω_1-iterable y-premouse (see for example [Ste08c], [Sar13] or [ST] for more on Mouse Capturing and the consistency strength of its negation).

6.5 $\omega + 1$ Woodin cardinals

In this section we will use the results from the previous sections to construct a premouse with $\omega + 1$ Woodin cardinals. More precisely, we prove the following theorem:

Theorem 6.5.1. *Suppose there is some $A \in \mathcal{P}_{\omega_1}(\mathbb{R})$ such that*

1. *$M_1^\sharp(A)$ exists,*
2. *$\mathbb{R} \cap M_1(\mathbb{R}) = A$, and*
3. *$M_1(A) \vDash \mathsf{AD}$.*

Then there is an active premouse with $\omega + 1$ Woodin cardinals.

Using the argument of Section 6.4, for the rest of this chapter, we fix a countable set of reals A such that $M_1(A)$ is an ω_1-iterable A-premouse with $M_1(A) \cap \mathbb{R} = A$ and

$$M_1(A) \vDash \mathsf{ZF} + \mathsf{DC} + \mathsf{AD}^+ + \Theta = \theta_0 + \mathsf{MC}.$$

The rest of this section is devoted to the proof of Theorem 6.5.1. Most of the proof in this section closely follows ideas from Section 3 in the unpublished notes [Stea]. See also Sections 6.5 and 6.6 in [SW16] for a similar argument applied to $L(\mathbb{R})$ or [SS15]. We start by introducing some notation, generalizing ideas from [SW16, Section 3] and [ST16] to our context. Suppose that A is as in the statement of Theorem 6.5.1 and work inside $V = M_1(A)$.

The proof of Theorem 6.5.1 can be split in several parts. After we recall some useful properties of models of determinacy, we define suitable premice. From these we will, by pseudo-comparison and pseudo-genericity iteration, obtain models which we can use in a Prikry-like forcing to construct a model

with ω Woodin cardinals. Then we argue that this model can be rearranged into a premouse on top of which we can perform a $\mathcal{P}$-construction to add one more Woodin cardinal.

6.5.1 Some properties of models of determinacy

The following standard lemma will be useful later on:

Lemma 6.5.2. *There is a Σ^2_1 scale $\vec{\phi}$ on a Σ^2_1 set which is universal for $\mathbf{\Sigma}^2_1$ such that, letting T be the tree obtained from $\vec{\phi}$, we have for any countable transitive set a,*

$$\begin{aligned}\mathcal{P}(a)\cap L[T,a] &= \mathcal{P}(a)\cap \mathsf{OD}_{a\cup\{a\}}\\ &= \{b\in H(\omega_1) : b \text{ belongs to an } \omega_1\text{-iterable } a\text{-premouse}\}.\end{aligned}$$

Proof. Since $\Theta=\theta_0$, every set of reals is ordinal-definable from a real. Let $U\subset\mathbb{R}^2$ be any Σ^2_1 set that is universal for $\mathbf{\Sigma}^2_1$. Woodin has shown that $\mathbf{\Sigma}^2_1$ has the scale property under AD^+ (see [Ste08a, Theorem 7.1] for a proof of this in derived models). Let T be a tree on $\omega\times\omega\times\delta^2_1$ obtained from a Σ^2_1-scale on U (thus T projects to U).

Now, suppose $b\in\mathcal{P}(a)\cap\mathsf{OD}_{a\cup\{a\}}$. By Mouse Capturing, b belongs to an ω_1-iterable premouse over a. The set

$$A=\{M : M \text{ is an } \omega_1\text{-iterable premouse over } a \text{ and } b\in M\}$$

belongs to $\Sigma^2_1(a)$, so there is an a' recursive in a such that $A=U_{a'}$. It follows that $T_{a'}$, the part of T projecting to $U_{a'}$, is illfounded both in V and in $L[T,a]$. Since from any member of A one can easily recover b, it follows that $b\in L[T,a]$.

Conversely, every real in $L[T,a]$ is definable from a and ordinal parameters. This is because the reals of $L[T,a]$ do not depend on the choice of the universal set U nor on the scale on U (this follows e.g., from [Mos09, Exercise 8G.29], see also [HK81]).

Hence, we have shown that

$$\begin{aligned}\mathcal{P}(a)\cap L[T,a] &= \mathcal{P}(a)\cap \mathsf{OD}_{a\cup\{a\}}\\ &= \{b\in H(\omega_1) : b \text{ belongs to an } \omega_1\text{-iterable } a\text{-premouse}\},\end{aligned}$$

as desired. □

Fix a Σ^2_1 set U which is universal for $\mathbf{\Sigma}^2_1$ in $V=M_1(A)$ and a tree T as above for the rest of this section.

6.5.2 Suitable premice

We begin by isolating a class of models suitable for our purposes.

Definition 6.5.3. Suppose b is a countable transitive set. We write

$$Lp(b) = \bigcup \{M : M \text{ is a sound } \omega_1\text{-iterable } b\text{-premouse such that } \rho_\omega(M) = b\}.$$

Moreover, we inductively define $Lp^1(b) = Lp(b)$,

$$Lp^{n+1}(b) = Lp(Lp^n(b)),$$

and

$$Lp^\omega(b) = \bigcup_{n<\omega} Lp^n(b).$$

Remark 6.5.4. Recall that we are working inside $M_1(A)$, which is a model of AD, so the club filter is an ultrafilter on ω_1. This can be used to show that ω_1-iterability already implies ω_1+1-iterability (see e.g. [Ste10, Lemma 7.11] for details), so that any two ω_1-iterable b-premice M and N as in the definition of $Lp(b)$ can be successfully compared and line up, i.e $M \trianglelefteq N$ or $N \trianglelefteq M$. Therefore, $Lp(b)$ is a well-defined premouse.

In the definitions below, let a be an arbitrary countable transitive set.

Definition 6.5.5. We say that an a-premouse M is *suitable* if, and only if, there is an ordinal δ such that

1. M is a model of ZFC - "Replacement" and $M \cap \mathsf{Ord} = \sup_{n<\omega}(\delta^{+n})^M$,
2. δ is the unique Woodin cardinal in M, and
3. M is *full*, i.e. for every cutpoint[4] η in M, $Lp(M|\eta) \trianglelefteq M$.

If M is suitable, we denote its Woodin cardinal by δ_M.

Lemma 6.5.6. *Let M be a countable a-premouse and x_M a real coding M. Then for any real $z \geq_T x_M$, the statement "M is suitable" is absolute between V and $L[T,z]$.*

Proof. Suppose not, say M is suitable in $L[T,z]$ but there is some $\eta < M \cap \mathsf{Ord}$ and a sound ω_1-iterable $M|\eta$-premouse N in V with $\rho_\omega(N) = M|\eta$ such that $N \not\trianglelefteq M$. This statement is $\mathbf{\Sigma}^2_1$ and hence such a counterexample would also exist in $L[T,z]$. By the same argument, if we suppose that M

[4] In the case where η is not a cutpoint, we refer to the *-transformation in [Ste08c, Section 7].

is suitable in V, every such ω_1-iterable $M|\eta$-premouse in $L[T, z]$ is also ω_1-iterable in V. Hence the statement "M is suitable" is absolute between V and $L[T, z]$. □

Definition 6.5.7. Let $\mathcal{T}$ be a normal iteration tree on a suitable a-premouse M of length $< \omega_1^V$. Then we say that $\mathcal{T}$ is *correctly guided* if, and only if, for every limit ordinal $\lambda < \mathrm{lh}(\mathcal{T})$, if b is the branch choosen through $\mathcal{T} \restriction \lambda$ in $\mathcal{T}$, then $\mathcal{Q}(b, \mathcal{T} \restriction \lambda)$ exists and $\mathcal{Q}(b, \mathcal{T} \restriction \lambda) \trianglelefteq Lp(\mathcal{M}(\mathcal{T} \restriction \lambda))$.

Definition 6.5.8. Let $\mathcal{T}$ be a normal iteration tree on a suitable a-premouse M of length $< \omega_1^V$. Then we say that $\mathcal{T}$ is *short* if, and only if, $\mathcal{T}$ is correctly guided and if $\mathcal{T}$ has limit length, then $\mathcal{Q}(\mathcal{T})$ exists, and $\mathcal{Q}(\mathcal{T}) \trianglelefteq Lp(\mathcal{M}(\mathcal{T}))$. If $\mathcal{T}$ is correctly guided but not short, then it is said to be *maximal*.

As in [ST16], we define the notion of being suitability-strict in order to make the proofs of Lemmas 6.5.12 and 6.5.13 below work.

Definition 6.5.9. Let M be a suitable a-premouse and let $\mathcal{T}$ be a normal iteration tree on M of length $< \omega_1^V$. Then we say that $\mathcal{T}$ is *suitability-strict* if, and only if, for all $\alpha < \mathrm{lh}(\mathcal{T})$,

1. if $[0, \alpha]_T$ does not drop then $\mathcal{M}_\alpha^\mathcal{T}$ is suitable, and
2. if $[0, \alpha]_T$ drops then no $\mathcal{R} \trianglelefteq \mathcal{M}_\alpha^\mathcal{T}$ is suitable.

Definition 6.5.10. Let M be a suitable a-premouse. Then we say that M is *short tree iterable* if, and only if, whenever $\mathcal{T}$ is a short tree on M of length $< \omega_1^V$,

1. $\mathcal{T}$ is suitability-strict,
2. if $\mathcal{T}$ has a last model, then every putative iteration tree $\mathcal{U}$ extending $\mathcal{T}$ such that $\mathrm{lh}(\mathcal{U}) = \mathrm{lh}(\mathcal{T}) + 1$ has a well-founded last model, and
3. if $\mathcal{T}$ has limit length, then there exists a cofinal well-founded branch b through $\mathcal{T}$ such that $\mathcal{Q}(b, \mathcal{T}) = \mathcal{Q}(\mathcal{T})$.

Definition 6.5.11. Suppose M is a suitable a-premouse. We say R is a *pseudo-normal iterate* of M if, and only if, R is suitable and there is a normal iteration tree $\mathcal{T}$ on M such that either $\mathcal{T}$ has successor length and R is the last model of $\mathcal{T}$ or $\mathcal{T}$ is maximal and $R = Lp^\omega(\mathcal{M}(\mathcal{T}))$.

As usual, this notion can easily be generalized to stacks of normal trees, but we omit the technical details. The interested reader can find them in a different setting for example in [SW16], [Sar14], or [MS17]. The following

lemmas are the analogues of Theorems 3.14 and 3.16 in [SW16]. The proofs are similar to the ones in [SW16] and [ST16] and use absoluteness as in the proof of Lemma 6.5.6; we omit further details.

Lemma 6.5.12 (Pseudo-comparison)**.** *Suppose M and N are countable short tree iterable, suitable a-premice. Then, they have a common pseudo-normal iterate R such that $R \in L[T,z]$, where z is a real coding M and N. Moreover, $\delta_R \leq (\max\{\delta_M, \delta_N\}^+)^{L[T,z]} = \omega_1^{L[T,z]}$.*

Lemma 6.5.13 (Pseudo-genericity iteration)**.** *Let M be a countable short tree iterable, suitable a-premouse and y an arbitrary real. Then, there is a non-dropping pseudo-normal iterate R of M such that $R \in L[T,z]$, where $z \geq_T y$ is a real coding M, y is generic over R for Woodin's extender algebra at δ_R, and $\delta_R \leq (\delta_M^+)^{L[T,z]} = \omega_1^{L[T,z]}$.*

6.5.3 The models $\mathcal{R}_a^x$

As before, let a be an arbitrary countable transitive set. Let x be a real such that $a \in L[x]$. We consider the simultaneous pseudo-comparison of all short tree iterable, suitable a-premice coded by some real $z \leq_T x$. We carry out this pseudo-comparison while at the same time performing a pseudo-genericity iteration making every $z \leq_T x$ generic over the common part of the final model. Note that there are only countably many reals $z \leq_T x$ for every fixed real x and let $\mathcal{R}^- = \mathcal{R}_a^{x,-}$ denote the resulting model. I.e., either $\mathcal{R}^-$ is the common last model of the iteration trees obtained by the process described above, or all of these trees are maximal and $\mathcal{R}^-$ is the common part model of one (and hence all) of these trees. Moreover, let $\mathcal{R} = \mathcal{R}_a^x = Lp(\mathcal{R}_a^{x,-})$ and $\delta = \mathcal{R}_a^{x,-} \cap \mathsf{Ord}$.

Lemma 6.5.14. *We have the following properties.*

1. *As an $\mathcal{R}^-$-premouse, no level of $\mathcal{R}$ projects across δ,*
2. *$\delta_{\mathcal{R}} = \delta$ is a Woodin cardinal in $\mathcal{R}$,*
3. *$\mathcal{P}(\delta) \cap \mathcal{R} = \mathcal{P}(\delta) \cap \mathsf{OD}_{\mathcal{R}^- \cup \{\mathcal{R}^-\}}$,*
4. *$\mathcal{P}(a) \cap \mathcal{R} = \mathcal{P}(a) \cap \mathsf{OD}_{a \cup \{a\}}$, and*
5. *$\omega_1^{L[T,x]} = \delta$.*

Proof. We carry out the pseudo-comparisons and pseudo-genericity iterations to obtain $\mathcal{R}^-$ within $L[T,x]$ in the sense of Lemmas 6.5.12 and 6.5.13.

Claim 6.5.15. *The pseudo-comparisons reach a limit stage in which all of the iteration trees are maximal.*

Proof. If the pseudo-comparisons reach a limit stage in which one iteration tree $\mathcal{T}$ is maximal, this already implies that all iteration trees are maximal since they agree on their common part model and thus a short iteration tree $\mathcal{U}$ would provide a $\mathcal{Q}$-structure $\mathcal{Q}(\mathcal{U}) \trianglelefteq Lp(\mathcal{M}(\mathcal{U})) = Lp(\mathcal{M}(\mathcal{T}))$ for $\mathcal{T}$, contradicting the maximality of $\mathcal{T}$.

Therefore, we can suppose toward a contradiction that all iteration trees occuring in the pseudo-comparisons are short. Then the pseudo-comparsions are in fact comparisons and they end successfully using the short tree iteration strategies. They give rise to a last model $\mathcal{R}^*$ such that every $z \leq_T x$ is generic over $\mathcal{R}^*$ for Woodin's extender algebra. Moreover, the main branches through all iteration trees in the comparisons are non-dropping and we have elementary iteration embeddings

$$j_{\mathcal{N}} \colon \mathcal{N} \to \mathcal{R}^*$$

for each short tree iterable, suitable a-premouse $\mathcal{N}$ coded by some real $z \leq_T x$. In particular, $\mathcal{R}^*$ is suitable, witnessed by a Woodin cardinal $\delta^* = \delta_{\mathcal{R}^*}$.

The proof of the comparison lemma (cf. e.g., the claim in the proof of [Ste10, Theorem 3.11]) shows that, if a coiteration terminates successfully, the comparison process lasts at most countably many steps in $L[T, x]$, and so $\mathcal{R}^*$ is countable in $L[T, x]$. By construction, x is generic over $\mathcal{R}^*$ for Woodin's extender algebra at δ^*, so we shall write $\mathcal{R}^*[x]$ for the corresponding generic extension.

Subclaim 6.5.16. $\mathbb{R} \cap L[T, x] \subseteq \mathbb{R} \cap \mathcal{R}^*[x]$.

Proof. Recall that Woodin's extender algebra at δ^* has the δ^*-c.c. and hence $((\delta^*)^+)^{\mathcal{R}^*} = ((\delta^*)^+)^{\mathcal{R}^*[x]}$. Let $\gamma = ((\delta^*)^+)^{\mathcal{R}^*}$. Consider the countable set $\mathcal{R}^*[x]|\gamma$ and the model $L[T, \mathcal{R}^*[x]|\gamma]$. Since $x \in \mathcal{R}^*[x]|\gamma$, we have that $L[T, x] \subseteq L[T, \mathcal{R}^*[x]|\gamma]$. Now, Lemma 6.5.2 implies that every real y in $L[T, \mathcal{R}^*[x]|\gamma]$, belongs to an ω_1-iterable $\mathcal{R}^*[x]|\gamma$-premouse N^y. By taking an initial segment if necessary, we can assume that N^y is sound and $\rho_\omega(N^y) \leq \gamma$. Since $L[T, x] \subseteq L[T, \mathcal{R}^*[x]|\gamma]$, it suffices to show that every real y in an ω_1-iterable $\mathcal{R}^*[x]|\gamma$-premouse N^y belongs to $\mathcal{R}^*[x]$.

Let $\bar{N}^y = \mathcal{P}^{N^y}(\mathcal{R}^*|\gamma)$ be the $\mathcal{R}^*|\gamma$-premouse obtained as the result of a $\mathcal{P}$-construction above $\mathcal{R}^*|\gamma$ inside N^y in the sense of [SS09] or [Ste08b, Section 3]. As the size of the extender algebra at δ^* is small, $\bar{N}^y$ is again a premouse and by definability of the forcing $\rho_\omega(\bar{N}^y) \leq \gamma$ (see for example

[Ste08b, Section 3] for a similar argument). Therefore, the suitability of $\mathcal{R}^*$ yields $\bar{N}^y \trianglelefteq \mathcal{R}^*$ and hence $N^y = \bar{N}^y[x] \trianglelefteq \mathcal{R}^*[x]$, where $\bar{N}^y[x]$ and $\mathcal{R}^*[x]$ are construed as $\mathcal{R}^*[x]|\gamma$-premice. Thus $y \in \mathcal{R}^*[x]$, as desired. □

Since $\mathcal{R}^*$ is countable in $L[T,x]$, in particular δ^*, the Woodin cardinal in $\mathcal{R}^*$, is countable in $L[T,x]$. Using the subclaim, this implies that δ^* is countable in $\mathcal{R}^*[x]$. But the extender algebra at δ^* in $\mathcal{R}^*$ has the δ^*-c.c., so δ^* remains a cardinal in $\mathcal{R}^*[x]$—a contradiction. □

Since all iteration trees are maximal, we have $\mathcal{R}^- = \mathcal{M}(\mathcal{T})$, where $\mathcal{T}$ is an iteration tree of limit length on a suitable a-premouse coded by some real $z \leq_T x$. Then $\mathcal{R} = Lp(\mathcal{R}^-)$ satisfies (1) and (2).

For (3), note that $\mathcal{P}(\delta) \cap \mathsf{OD}_{\mathcal{R}^- \cup \{\mathcal{R}^-\}}$ is the set of all subsets of δ which belong to an ω_1-iterable $\mathcal{R}^-$-premouse. Since $\mathcal{R} = Lp(\mathcal{R}^-)$ these correspond to initial segments of $\mathcal{R}$.

This also implies that every subset of a in $\mathsf{OD}_{a \cup \{a\}}$ belongs to $\mathcal{R}$ since $\mathsf{OD}_{a \cup \{a\}} \subseteq \mathsf{OD}_{\mathcal{R}^- \cup \{\mathcal{R}^-\}}$. For the other inclusion in (4), suppose that b is a subset of a and $b \in \mathcal{R}$. As no new subsets of a are added during the iteration, $b \in N$ for some short tree iterable, suitable a-premouse N coded by some real $z \leq_T x$. But then $N|\eta$ is ω_1-iterable for every ordinal η such that $\rho_\omega(N|\eta) = a$ as, in these cases, $\mathcal{Q}$-structures exist. Hence, b belongs to an ω_1-iterable a-premouse and hence to $\mathsf{OD}_{a \cup \{a\}}$.

Finally, (5) follows by the same argument as in the proof of the subclaim since the assumption that $\delta < \omega_1^{L[T,x]}$ together with (1) and (2) suffices to derive the contradiction. □

The construction of $\mathcal{R}_a^x$ depends only on the Turing degree of x, i.e., $x \equiv_T y$ implies $\mathcal{R}_a^x = \mathcal{R}_a^y$. Therefore, we will also write $\mathcal{R}_a^d$ for $\mathcal{R}_a^x$, if $d = [x]_T$.

6.5.4 Prikry-like forcing a premouse

We define a Prikry-like partial order $\mathbb{P}$ to add a premouse with infinitely many Woodin cardinals. Let $\mathcal{D}$ denote the set of all Turing degrees and let μ denote the Martin measure on $\mathcal{D}$. Further, let $\mathcal{D}^m$ be the set of all increasing sequences of Turing degrees of length m and μ_m be the measure on $\mathcal{D}^m$ induced by the product of μ. More precisely, let $\mu_0 = \mu$ and assume inductively that μ_k is already defined on $\mathcal{D}^k$ for some $k < m$. Then we let for any $X \in \mathcal{D}^{k+1}$, $\mu_{k+1}(X) = 1$ if, and only if, for μ_0-a.e. d_0 and μ_k-a.e. $(d_1, \ldots, d_k)$, $(d_0, \ldots, d_k) \in X$.

To define the Prikry-like partial order $\mathbb{P}$, we first define a sequence of premice along an increasing sequence of Turing degrees. For $\vec{d} = (d_0, \dots, d_m) \in \mathcal{D}^{m+1}$, we let

$$\mathcal{Q}_0^{\vec{d}}(a) = \mathcal{R}_a^{d_0}$$

if d_0 is large enough so that $\mathcal{R}_a^{d_0}$ is defined, and recursively

$$\mathcal{Q}_{i+1}^{\vec{d}}(a) = \mathcal{R}^{d_{i+1}}_{\mathcal{Q}_i^{\vec{d}}(a)}$$

for $i < m$, if d_{i+1} is large enough so that $\mathcal{R}^{d_{i+1}}_{\mathcal{Q}_i^{\vec{d}}(a)}$ is defined.

Recall that U is a Σ^2_1 set which is universal for $\mathbf{\Sigma}^2_1$ in $V = M_1(A)$. Now, the conditions in $\mathbb{P}$ are of the form $(s, \vec{X})$, where

1. $s = (\mathcal{S}_0, \dots, \mathcal{S}_n)$ is a sequence of premice such that for some $\vec{d}_s \in \mathcal{D}^{n+1}$, $\mathcal{S}_i = \mathcal{Q}_i^{\vec{d}_s}(\emptyset)$ for all $i \leq n$; and
2. $\vec{X} = (X_k \colon k < \omega) \in L(U, A)$[5] is a sequence of sets such that for all $k < \omega$,
   (a) X_k is a collection of $(k+1)$-sequences of premice, and
   (b) $(\mathcal{Q}_0^{\vec{d}}(\mathcal{S}_n), \dots, \mathcal{Q}_k^{\vec{d}}(\mathcal{S}_n)) \in X_k$ for μ_{k+1}-a.e. $\vec{d} \in \mathcal{D}^{k+1}$.

We call s the stem of the condition $(s, \vec{X})$. For two conditions $(s, \vec{X})$ and $(r, \vec{Y})$ in $\mathbb{P}$ with $s = (\mathcal{S}_0, \dots, \mathcal{S}_n)$ and $r = (\mathcal{R}_0, \dots, \mathcal{R}_m)$, we let $(s, \vec{X}) \leq (r, \vec{Y})$ if, and only if, one of the following holds:

1. $s = r$ and $X_i \subseteq Y_i$ for all $i < \omega$; or
2. for some $k < \omega$ and a sequence $(\mathcal{Q}_0^{\vec{d}}(\mathcal{R}_m), \dots, \mathcal{Q}_k^{\vec{d}}(\mathcal{R}_m)) \in Y_k$ given by some $\vec{d} \in \mathcal{D}^{k+1}$,
   (a) $s = r^\frown(\mathcal{Q}_0^{\vec{d}}(\mathcal{R}_m), \dots, \mathcal{Q}_k^{\vec{d}}(\mathcal{R}_m))$, and
   (b) for all $i < \omega$ and sequences $\vec{e} \in \mathcal{D}^{i+1}$ such that $(\mathcal{Q}_0^{\vec{e}}(\mathcal{S}_n), \dots, \mathcal{Q}_i^{\vec{e}}(\mathcal{S}_n))$ is defined and belongs to X_i, we have

$$(\mathcal{Q}_0^{\vec{d}^\frown\vec{e}}(\mathcal{R}_m), \dots, \mathcal{Q}_k^{\vec{d}^\frown\vec{e}}(\mathcal{R}_m), \mathcal{Q}_{k+1}^{\vec{d}^\frown\vec{e}}(\mathcal{R}_m), \dots, \mathcal{Q}_{k+i+1}^{\vec{d}^\frown\vec{e}}(\mathcal{R}_m)) \in Y_{k+i+1}.$$

if, and only if, $\vec{X} = (X_k \colon k < \omega)$, where each X_k is a set of stems each of which is compatible with $s = (\mathcal{S}_0, \dots, \mathcal{S}_n)$ and the following hold:

[5]The reason for requiring $\vec{X} \in L(U, A)$ will become apparent in the proof of Lemma 6.5.24.

1. $\vec{X}$ is closed under initial segments in the following sense: whenever $(x_0, \dots, x_k) \in X_k$ for some $k < \omega$, then $(x_0, \dots, x_i) \in X_i$ for all $i < k$; and

2. for all $k < \omega$ and all $x \in X_k$,

$$\left\{ d \in \mathcal{D} : x^\frown(\mathcal{R}^d_{\mathcal{S}_n}) \in X_{k+1} \right\} \in \mu.$$

The next lemma shows that $\mathbb{P}$ has the Prikry property. As the proof is analogous to e.g., the proof of Corollary 6.39 in [SW16], we omit it.

Lemma 6.5.17. *Let $(s, \vec{X}) \in \mathbb{P}$ be a condition and Λ a countable set of sentences in the forcing language. Then there is some $(s, \vec{Y}) \leq (s, \vec{X})$ such that $(s, \vec{Y})$ decides ϕ, for all $\phi \in \Lambda$.*

Now fix a G which is $\mathbb{P}$-generic over $M_1(A)$ and let $\vec{\mathcal{Q}} = (\mathcal{Q}_n : n < \omega)$ be the union of the stems of conditions in G. Write δ_n for the largest cardinal in $\mathcal{Q}_n$. By definition, all $\mathcal{Q}_n$ are such that $\mathcal{Q}_n = Lp(\mathcal{Q}_n|\delta_n)$ and $Lp^\omega(\mathcal{Q}_n|\delta_n)$ is a suitable premouse, so δ_n is a Woodin cardinal in $\mathcal{Q}_n$. Let

$$\mathcal{Q}_\infty = \bigcup_{n<\omega} \mathcal{Q}_n.$$

Lemma 6.5.18. *The following hold:*

1. *for all $n < \omega$, $\mathcal{P}(\delta_n) \cap L[\vec{\mathcal{Q}}] \subseteq \mathcal{Q}_n$,*

2. *for all $n < \omega$, δ_n is a Woodin cardinal in $L[\vec{\mathcal{Q}}]$,*

3. *$\mathcal{Q}_n = \mathcal{Q}_\infty|(\delta_n^+)^{\mathcal{Q}_\infty}$; hence, $L[\mathcal{Q}_\infty] = L[\vec{\mathcal{Q}}]$.*

Proof. We show (1), from which (2) and (3) follow. Let us first show that $\mathcal{P}(\delta_n) \cap \mathcal{Q}_{n+1} \subseteq \mathcal{Q}_n$; a consequence of this is that $\mathcal{P}(\delta_n) \cap \mathcal{Q}_m \subseteq \mathcal{Q}_n$ whenever $n < m$. To see this, suppose there is a subset a of δ_n which is in $\mathcal{Q}_{n+1}$. By Lemma 6.5.14(4), $\mathcal{P}(\delta_n) \cap \mathcal{Q}_{n+1} = \mathcal{P}(\delta_n) \cap \mathsf{OD}_{\mathcal{Q}_n \cup \{\mathcal{Q}_n\}}$. Lemma 6.5.14(3) implies that $\mathcal{P}(\delta_n) \cap \mathcal{Q}_n = \mathcal{P}(\delta_n) \cap \mathsf{OD}_{\mathcal{Q}_n^- \cup \{\mathcal{Q}_n^-\}}$, but this is equal to $\mathcal{P}(\delta_n) \cap \mathsf{OD}_{\mathcal{Q}_n \cup \{\mathcal{Q}_n\}}$ by definability of $\mathcal{Q}_n = Lp(\mathcal{Q}_n^-)$. Hence, we have $a \in \mathcal{Q}_n$, as desired.

To prove (1), let $a \in \mathcal{P}(\delta_n) \cap L[\vec{\mathcal{Q}}]$. Let $\dot{a}$ be a term defining a from $\vec{\mathcal{Q}}$ and an ordinal parameter in $M_1(A)[G]$. The Prikry property (Lemma 6.5.17) yields a $k < \omega$ and a condition $(s, \vec{X})$ with s of the form $(s_0, \dots, s_k)$, with $n < k$, which decides all statements of the form "$\xi \in \dot{a}$". By genericity we can choose $(s, \vec{X}) \in G$, so that, in particular, $s_i = \mathcal{Q}_i$ for all $i \leq k$.

Claim 6.5.19. *We have $\xi \in a$ if, and only if,*

$$\exists t \exists \vec{Y}(t \text{ is of the form } (t_0,\dots,t_k) \wedge t_i = \mathcal{Q}_i \text{ for all } i \leq k \wedge (t,\vec{Y}) \Vdash \xi \in \dot{a}).$$

Proof. If $\xi \in a$, then the condition $(s,\vec{X})$ is a witness for the displayed equation. Conversely, suppose there is some condition $(t,\vec{Y})$ as in the displayed equation, but $\xi \notin a$. Then we must have $(s,\vec{X}) \Vdash \xi \notin \dot{a}$. Note that $t = s$. Define $\vec{Z}$ by $Z_i = X_i \cap Y_i$ for all $i < \omega$. Then $(s,\vec{Z}) \leq (s,\vec{X})$ and $(s,\vec{Z}) = (t,\vec{Z}) \leq (t,\vec{Y})$. Now, let H be $\mathbb{P}$-generic over $M_1(A)$ such that H contains $(s,\vec{Z})$. Then, in $M_1(A)[H]$, both $\xi \in a$ and $\xi \notin a$ hold, a contradiction. □

The claim yields that $a \in \mathsf{OD}_{\mathcal{Q}_k \cup \{\mathcal{Q}_k\}}$ and thus $a \in \mathsf{OD}_{\mathcal{Q}_k^- \cup \{\mathcal{Q}_k^-\}} \cap \mathcal{P}(\delta_k) = \mathcal{Q}_k \cap \mathcal{P}(\delta_k)$, by Lemma 6.5.14(3). So in particular $a \in \mathcal{Q}_k \cap \mathcal{P}(\delta_n)$. By the remark at the beginning of the proof, this implies $a \in \mathcal{Q}_n$, as desired. □

Write $\lambda = \sup_{n<\omega} \delta_n$ and fix any premouse $\mathcal{P} \models \mathsf{ZFC}$ extending $\mathcal{Q}_\infty$ in which λ remains a cardinal. We form a derived model of $\mathcal{P}$. More precisely, working in $M_1(A)[G]$, let $\mathbb{S}$ be the partial order consisting of sequences $(h_0,\dots,h_k)$ such that for all $0 \leq n \leq k$, $h_n \in M_1(A)$ is $\mathrm{Coll}(\omega,\delta_n)$-generic over $\mathcal{Q}_n$. The order on $\mathbb{S}$ is sequence extension. Fix $\hat{h}$ $\mathbb{S}$-generic over $M_1(A)[G]$, let $(h_n : n < \omega)$ be the induced sequence, and let h be given by $h(n,m) = h_n(m)$. We may abuse notation and identify $\hat{h}$ with h and with the corresponding $\mathrm{Coll}(\omega,{<}\lambda)$-generic filter over $M_1(A)[G]$, since $(\delta_n : n < \omega)$ is definable from $\mathcal{Q}_\infty$ by the previous lemma.

Using this h, we can build the derived model of $\mathcal{P}$: Write

$$\mathbb{R}^*_h = \bigcup_{n\in\omega} \mathbb{R} \cap \mathcal{P}[h \restriction \delta_n],$$

and

$$Hom^*_h = \{p[S] \cap \mathbb{R}^*_h | \exists n < \omega(\mathcal{P}[h \restriction \delta_n] \vDash S \text{ is a } {<}\lambda\text{-absolutely complemented tree})\}.$$

Note that $\mathbb{R}^*_h$, and Hom^*_h only depend on $\mathcal{Q}_\infty$ and h, not on the full premouse $\mathcal{P}$, so this notation makes sense. For this reason, we sometimes do not distinguish between $\mathcal{P}$ and $\mathcal{Q}_\infty$ in what follows.

6.5.5 Preparation for adding extenders on top

Our next goal is to *add the extenders of* $M_1(A)$ on top of $\mathcal{Q}_\infty$ while preserving the Woodin cardinals of $\mathcal{Q}_\infty$ to obtain a model with $\omega+1$ Woodin cardinals. This will be done via a $\mathcal{P}$-construction. For this, some preparation is needed— we need to show e.g.,

$$L_\xi[\mathcal{Q}_\infty][h] = (M_1(A)|\xi)[G][h]$$

for some ordinal ξ below the Woodin cardinal of $M_1(A)$.

We first need the following lemmas which again essentially can be found also in [Stea]. Recall that we have $V = M_1(A)$ and write $\mathbb{R}^V = M_1(A) \cap \mathbb{R} = A$.

Lemma 6.5.20. $\mathbb{R}^*_h = \mathbb{R}^V$.

Proof. $\mathbb{R}^*_h \subseteq \mathbb{R}^V$ is easy to see as each pair $(\mathcal{Q}_n, h_n)$ is in $M_1(A)$, so let $x \in \mathbb{R}^V$ for the other inclusion. Let $(s, \vec{X}) \in \mathbb{P}$ be an arbitrary condition in the Prikry-like forcing defined above, say $s = (\mathcal{S}_0, \ldots, \mathcal{S}_k)$. For each $n < \omega$ consider

$$Y_n = \{t \in X_n \mid \exists \vec{d} \in \mathcal{D}^{n+1}(x \leq_T d_0 \wedge t = (\mathcal{Q}^{\vec{d}}_0(\mathcal{S}_k), \ldots, \mathcal{Q}^{\vec{d}}_n(\mathcal{S}_k)))\}$$

and note that $(s, \vec{Y}) \leq (s, \vec{X})$. Moreover, since pseudo-genericity iterations are included in the construction of the premice $\mathcal{R}^d_a$, it follows by density that for all $i > k$, x is generic over $\mathcal{Q}_i$ for Woodin's extender algebra at δ_i. Hence $x \in \mathbb{R}^*_h$. □

Lemma 6.5.21. $Hom^*_h = \mathbf{\Delta}^2_1$.

Proof. We first show $\mathbf{\Delta}^2_1 \subseteq Hom^*_h$. Let $B \in \mathbf{\Delta}^2_1$, so that B is, by a theorem of Woodin since AD^+ holds, Suslin and co-Suslin. Choose $z \in \mathbb{R}$ such that $B \in \Delta^2_1(z)$ and choose $i \in \omega$ large enough so that $z \in \mathcal{Q}_i[h_i]$ (this exists by the previous lemma). From T and z, one can construct trees S_0 and S_1 on $\omega \times \kappa$, for some ordinal $\kappa < \boldsymbol{\delta}^2_1$, such that S_0 and S_1 project to B and $\mathbb{R} \setminus B$, respectively. We have

$$S_0, S_1 \in L[T, \mathcal{Q}_j, h_i]$$

for every $j \geq i$. Moreover, by the absoluteness of well-foundedness, if $x \in L[T, \mathcal{Q}_j, h_i] \cap \mathbb{R}$, then

$$L[T, \mathcal{Q}_j, h_i] \models x \in p[S_0] \cup p[S_1].$$

Moreover, if $\mathbb{Q}$ is a small partial order in $L[T, \mathcal{Q}_j, h_i]$ (say, of size strictly less than δ_j) and g is $\mathbb{Q}$-generic over $L[T, \mathcal{Q}_j, h_i]$, then

$$L[T, \mathcal{Q}_j, h_i][g] \models \mathbb{R} = p[S_0] \cup p[S_1],$$

for (since δ_j is countable in V) otherwise there is a $\mathbb{Q}$-generic g over $L[T, \mathcal{Q}_j, h_i]$ with $g \in V$ such that

$$L[T, \mathcal{Q}_j, h_i][g] \models \exists x \in \mathbb{R}\, (x \notin p[S_0] \cup p[S_1]).$$

However, such an x does belong to one of $p[S_0]$ or $p[S_1]$ in V and thus it must do too in $L[T, \mathcal{Q}_j, h_i][g]$ (by the absoluteness of well-foundedness).

We have shown that in $L[T, \mathcal{Q}_j, h_i]$ there are $<\delta_j$-absolutely complementing trees S_0 and S_1 that project to $B \cap L[T, \mathcal{Q}_j, h_i]$ and $(\mathbb{R} \setminus B) \cap L[T, \mathcal{Q}_j, h_i]$, respectively.

The trees S_0 and S_1 might be big (in principle κ might be of arbitrarily large size below $\boldsymbol{\delta}^2_1$). Let M be an elementary substructure of some large $L_\alpha[T, \mathcal{Q}_j, h_i]$ such that

1. $M \in L[T, \mathcal{Q}_j, h_i]$,
2. $L[T, \mathcal{Q}_j, h_i] \models |M| = \delta_j$,
3. $L_{\delta_j}[T, \mathcal{Q}_j, h_i] \subseteq M$, and
4. $S_0, S_1 \in M$.

Let S^j_0 and S^j_1 be the images of S_0 and S_1 under the collapse embedding for M. Then, S^j_0 and S^j_1 are also $<\delta_j$-absolutely complementing and S^j_0 projects to $B \cap L[T, \mathcal{Q}_j, h_i][g]$ whenever g is $\mathbb{Q}$-generic over $L[T, \mathcal{Q}_j, h_i]$ for a partial order $\mathbb{Q}$ in $L[T, \mathcal{Q}_j, h_i]$ of size strictly less than δ_j.

Claim 6.5.22. *For all $j > i$, $S^j_0, S^j_1 \in \mathcal{Q}_\infty[h_i]$.*

Proof. Note that h_i is $\mathrm{Coll}(\omega, \delta_i)$-generic over $\mathcal{Q}_{j+1}$. Recall that by definition, $\mathcal{Q}_{j+1} = \mathcal{R}^d_{\mathcal{Q}_j}$ for some Turing degree d. Therefore we have, for such a d, by an argument similar to that in the proof of Subclaim 6.5.16 that

$$\mathcal{Q}_{j+1}[h_i] = \mathcal{R}^d_{\mathcal{Q}_j[h_i]}.$$

Now, by Lemma 6.5.2, every subset of $\mathcal{Q}_j$ in $L[T, \mathcal{Q}_j, h_i]$ is definable from $\mathcal{Q}_j$, h_i, and ordinal parameters in V. S^j_0 and S^j_1 are essentially subsets

of δ_j in $L[T, \mathcal{Q}_j, h_i]$, so they are definable in V from $\mathcal{Q}_j$, h_i, and ordinal parameters. By Lemma 6.5.14(4),

$$\mathcal{P}(\mathcal{Q}_j[h_i]) \cap \mathcal{R}^d_{\mathcal{Q}_j[h_i]} = \mathcal{P}(\mathcal{Q}_j[h_i]) \cap \mathsf{OD}_{\mathcal{Q}_j[h_i] \cup \{\mathcal{Q}_j[h_i]\}}.$$

This implies that S^j_0 and S^j_1 belong to $Q_{j+1}[h_i]$, which proves the claim. □

It follows that S^j_0 and S^j_1 are $<\delta_j$-absolutely complementing trees in $\mathcal{Q}_\infty[h_i]$ since by an argument as in the proof of Lemma 6.5.18, for every g, which is generic for a partial order of size strictly less than δ_j, $\mathcal{Q}_\infty[g] \cap \mathcal{P}(\delta_j) \subseteq \mathcal{Q}_j[g]$. Now, we aim to piece the trees S^j_0 together to obtain a tree witnessing that $B \in Hom^*_h$. Let S^*_0 be the disjoint union of the trees S^{i+k}_0 for all $k \geq 1$.

Claim 6.5.23. *For all $j > i$, all partial orders $\mathbb{Q} \in \mathcal{Q}_\infty[h_i]$ of size strictly less than δ_j in $\mathcal{Q}_\infty[h_i]$, and all g which are $\mathbb{Q}$-generic over $\mathcal{Q}_\infty[h_i]$,*

$$p[S^*_0]^{\mathcal{Q}_\infty[h_i][g]} = p[S^j_0]^{\mathcal{Q}_\infty[h_i][g]} = B \cap \mathcal{Q}_\infty[h_i][g].$$

Proof. Since S^j_0 and S^j_1 are $<\delta_j$-absolutely complementing trees in $\mathcal{Q}_\infty[h_i]$, we have for all $k, j > i$,

$$p[S^j_0]^{\mathcal{Q}_\infty[h_i]} = B \cap \mathcal{Q}_\infty[h_i] = p[S^k_0]^{\mathcal{Q}_\infty[h_i]}.$$

Hence, $p[S^*_0]^{\mathcal{Q}_\infty[h_i]} = p[S^j_0]^{\mathcal{Q}_\infty[h_i]} = B \cap \mathcal{Q}_\infty[h_i]$.

Recall that if the projections of two trees are disjoint, they remain disjoint in generic extensions. It follows that for any tree S in $\mathcal{Q}_\infty[h_i]$ which has a $<\delta_j$-absolute complement,

$$p[S]^{\mathcal{Q}_\infty[h_i][g]} = \bigcup \{p[S']^{\mathcal{Q}_\infty[h_i][g]} \mid S' \in \mathcal{Q}_\infty[h_i] \text{ is a tree with } \mathcal{Q}_\infty[h_i] \models p[S] = p[S']\},$$

whenever g is $\mathbb{Q}$-generic over $\mathcal{Q}_\infty[h_i]$ for a partial order $\mathbb{Q}$ in $\mathcal{Q}_\infty[h_i]$ of size strictly less than δ_j. So in particular

$$p[S']^{\mathcal{Q}_\infty[h_i][g]} \subseteq p[S^j_0]^{\mathcal{Q}_\infty[h_i][g]}$$

for any tree $S' \in \mathcal{Q}_\infty[h_i]$ which, in $\mathcal{Q}_\infty[h_i]$, projects to the same set as S^j_0.

As argued above, we can apply this to $S' = S^*_0$ and, since the other inclusion holds by definition, obtain

$$p[S^*_0]^{\mathcal{Q}_\infty[h_i][g]} = p[S^j_0]^{\mathcal{Q}_\infty[h_i][g]}$$

for any g as above. This proves the claim, as $p[S^j_0]^{\mathcal{Q}_\infty[h_i][g]} = B \cap \mathcal{Q}_\infty[h_i][g]$. □

We can define S_1^* analogously; it witnesses that S_0^* is $<\lambda$-absolutely complemented in $Q_\infty[h_i]$. The claim implies $B = p[S_0^*] \cap \mathbb{R}_h^*$ and hence $B \in Hom_h^*$. This proves $\mathbf{\Delta}_1^2 \subseteq Hom_h^*$.

We now assume towards a contradiction that $\mathbf{\Delta}_1^2 \neq Hom_h^*$, i.e., that the $\mathbf{\Delta}_1^2$ sets form a proper Wadge initial segment of Hom_h^*. Since Hom_h^* is closed under continuous reducibility and the Σ_1^2 set U which is universal for $\mathbf{\Sigma}_1^2$ is minimal in the Wadge hierarchy above the pointclass $\mathbf{\Delta}_1^2$, it follows that $U \in Hom_h^*$.

By definition of Hom_h^*, there is some $i \in \omega$ and a $<\lambda$-absolutely complemented tree S in $\mathcal{Q}_\infty[h_i]$ such that, since $\mathbb{R}^V = \mathbb{R}_h^*$ by Lemma 6.5.20, $p[S] \cap \mathbb{R}^V = \mathbb{R}^V \setminus U$.

By Lemma 6.5.2, the relation

$$x \notin \mathsf{OD}_{\{y\}}$$

is Π_1^2, so it appears in a section of the complement of U. Using S, we can get a real x such that $x \notin \mathsf{OD}_{\{(\mathcal{Q}_i|\delta_i, h_i)\}}$ but $x \in \mathcal{Q}_\infty[h_i]$. Then in fact $x \in \mathcal{Q}_i[h_i]$ by the argument in the proof of Lemma 6.5.18. However, $\mathcal{Q}_i = Lp(\mathcal{Q}_i|\delta_i)$, so every real in $\mathcal{Q}_i[h_i]$ is definable from $\mathcal{Q}_i|\delta_i$, h_i, and ordinal parameters, which is a contradiction. This completes the proof of the lemma. □

Recall that $\mathbb{R}^V = A$. Define ξ_0 to be the least $\xi > \Theta^{L(U,A)}$ such that $L_\xi(U, A) \vDash \mathsf{ZF}$. Since $V|\xi_0$ is a countably iterable A-premouse, it is easy to see that

$$V|\xi_0 = L_{\xi_0}(U, A).$$

Finally, we have the following agreement between $L[\mathcal{Q}_\infty][h]$ and $L(U, A)[G][h]$ (as classes):

Lemma 6.5.24. $L[\mathcal{Q}_\infty][h] = L(U, A)[G][h]$.

Proof. In $L[\mathcal{Q}_\infty][h]$, one can easily compute the derived model of $L[\mathcal{Q}_\infty]$ associated to h. Thus, using Lemma 6.5.21, it follows that $U \in L[\mathcal{Q}_\infty][h]$. By Lemma 6.5.20, $A = \mathbb{R}^V \in L[\mathcal{Q}_\infty][h]$ and, using $L(U, A)$ and $\mathcal{Q}_\infty$, one can easily define $\mathbb{P}$ and G. Conversely, from G one can easily recover $\mathcal{Q}_\infty$. □

This, together with the observation above, has the following corollary:

Corollary 6.5.25. $L_{\xi_0}[\mathcal{Q}_\infty][h] = V|\xi_0[G][h]$.

6.5.6 Adding extenders on top

Finally, we use a $\mathcal{P}$-construction (see for example [Ste08b, Section 3] or [SS09]) to add extenders witnessing another Woodin cardinal on top of $\mathcal{Q}_\infty$. In Lemma 6.5.26, we first extend Corollary 6.5.25 to ordinals $\xi > \xi_0$ in order to obtain an appropriate background universe W for the $\mathcal{P}$-construction. Using the fact that $V[G][h]$ is a forcing extension of the A-premouse $V = M_1(A)$ by a small forcing, the proof of this lemma is straightforward and similar to the argument in [Ste08b, Section 3], so we omit it.

Lemma 6.5.26. *There is a proper class $(\mathcal{Q}_\infty, h)$-premouse W such that for any $\xi \geq \xi_0$,*

1. *$W|\xi$ has the same universe as $V|\xi[G][h]$,*
2. *for any $k < \omega$, $\rho_k(W|\xi) = \omega$ if, and only if, $\rho_k(V|\xi) = A$, and*
3. *for any $k < \omega$, if $\rho_k(W|\xi) > \omega$, then $\rho_k(W|\xi) = \rho_k(V|\xi)$ and $p_{k+1}(W|\xi) = p_{k+1}(V|\xi)$.*

Let $\mathcal{P} = \mathcal{P}^W(L_{\xi_0}[\mathcal{Q}_\infty])$ be the result of a $\mathcal{P}$-construction above $L_{\xi_0}[\mathcal{Q}_\infty]$ performed inside W and let $\mathcal{P}_\xi$ for $\xi \geq \xi_0$ denote the levels of the $\mathcal{P}$-construction. The following lemma shows that $\mathcal{P}$ is as desired.

Lemma 6.5.27. *The following hold:*

1. *For $\xi \geq \xi_0$, if $X \subseteq \mathcal{P}_\xi$ is definable over $\mathcal{P}_\xi$ with parameters from $\mathcal{P}_\xi$, then $X \cap \mathcal{Q}_\infty|\delta_n \in \mathcal{Q}_\infty$ for all $n < \omega$.*
2. *For all $\xi \geq \xi_0$, $\rho_\omega(\mathcal{P}_\xi) \geq \lambda$.*
3. *$\mathcal{P}$ is a premouse and $\mathcal{P}[h] = W$.*
4. *$\mathcal{P}$ has $\omega + 1$ Woodin cardinals.*

Proof. For the proof of (1) note that $\mathcal{P}_\xi$ is definable over $V|\xi[G]$ from $\mathcal{Q}_\infty$ since the translation between $V|\xi[G][h]$ and $W|\xi$ is definable and h is generic over $V|\xi[G]$ for a homogeneous forcing. Hence, if $X \subseteq \mathcal{P}_\xi$ is definable over $\mathcal{P}_\xi$ with parameters from $\mathcal{P}_\xi$, then $X \in V[G]$. We can assume without loss of generality that X is a set of ordinals and let $\dot{X}_n$ for every $n < \omega$ be a term defining $X \cap \delta_n$ from $\mathcal{Q}_\infty$ and the ordinal parameter ξ. Now we can argue as in the proof of Lemma 6.5.18 to obtain $X \cap \delta_n \in \mathcal{Q}_\infty$, as desired.

Suppose (2) fails and let $\xi \geq \xi_0$ be least such that $\rho_\omega(\mathcal{P}_\xi) = \rho < \lambda$. Say this is witnessed by some set of ordinals $X \subseteq \rho$, which is definable over $\mathcal{P}_\xi$ with parameters from $\mathcal{P}_\xi$, but $X \notin \mathcal{P}_\xi$. Since $\lambda = \bigcup_{n<\omega} \delta_n$, there is some

$n < \omega$ such that $\rho < \delta_n$. This means that $X = X \cap \delta_n \in \mathcal{Q}_\infty$ by (1). But $\mathcal{Q}_\infty \subseteq \mathcal{P}_\xi$, a contradiction.

Finally, (3) and (4) now follow from this and standard properties of the $\mathcal{P}$-construction, see for example [SS09]. □

This finishes the proof of Theorem 6.1.1 and, by putting the active extender of $M_1^\sharp(A)$ on an initial of $\mathcal{P}$, e.g., as in [FNS10, Section 2], also the proof of Theorem 6.5.1.

6.6 Generalizations

We finish this chapter by sketching the modifications of the above argument which are needed to prove Theorems 6.1.3 and 6.1.4.

Proof of Theorem 6.1.3. Instead of Theorem 6.3.1, we now aim to obtain a countable set of reals A such that $M_\omega(A) \cap \mathbb{R} = A$ and $M_\omega(A) \vDash \mathsf{ZF} + \mathsf{AD}$. For this purpose, we replace 1-smallness in the definition of φ-witness (see Definition 6.3.3) by ω-smallness. Moreover, we replace Π^1_2-iterability in the definition of the game $\mathcal{G}_{\varphi,\psi}$ (see Definition 6.3.6) by the notion of *weak iterability* (also called $\partial^{\mathbb{R}}\Pi^1_1$-iterability) as in [Ste10, Definition 7.7]. Note that determinacy of games of length ω^2 with $\partial^{\mathbb{R}}\Pi^1_1$ payoff suffices to show that this game is determined. Furthermore, Theorem 7.10 in [Ste10] suffices to carry out the comparison arguments used to prove Theorem 6.3.1. The proofs in Sections 6.4 and 6.5 straightforwardly generalize to this context. □

Proof of Theorem 6.1.4. Instead of Theorem 6.3.1, we now aim to obtain a countable set of reals A such that there is an A-premouse M of class S_α such that $M \cap \mathbb{R} = A$ and $M \vDash \mathsf{ZF} + \mathsf{AD}$. For this purpose, we replace 1-smallness in the definition of φ-witness (see Definition 6.3.3) by not being of class S_α. Moreover, we replace Π^1_2-iterability in the definition of the game $\mathcal{G}_{\varphi,\psi}$ (see Definition 6.3.6) by the notion of Π^1_α-iterability as in [Agug, Definition 4.1] (see Chapter 4). Note that determinacy of games of length ω^2 with σ-projective payoff suffices to show that this game is determined. Furthermore, Lemmas 2.19 and 4.3 in [Agug] (see Chapter 4) suffice to carry out the comparison arguments used to prove Theorem 6.3.1. The proofs in Sections 6.4 and 6.5 straightforwardly generalize to this context. □

7

Long Clopen Determinacy

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7.1 Introduction

Let us call a subset A of $\mathbb{R}$ *clopen* if $A \setminus \mathbb{Q}$ is clopen in $\mathbb{R} \setminus \mathbb{Q}$. We study games of transfinite length whose payoff set is clopen when viewed as a subset of $\mathbb{R} \setminus \mathbb{Q}$ (henceforth identified with $\mathbb{N}^\mathbb{N}$—the space of infinite sequences of natural numbers, viewed as a product of discrete spaces). More specifically, let α be a countable wellorder and fix a bijection

$$\rho : \mathbb{N} \to \alpha.$$

For simplicity, we will assume that ρ is recursive, although this assumption can be dropped by relativizing to a real parameter. Fix $A \subset \mathbb{R}$ and consider a game of length α in which two players, I and II, alternate turns playing natural numbers

$$\begin{array}{c|ccccccccc} I & n(0) & & n(2) & & \dots & n(\omega) & & & \dots \\ II & & n(1) & & n(3) & \dots & & n(\omega+1) & \dots & \end{array}$$

After α-many rounds have been played, the game ends. Player I wins if, and only if,

$$(n(\rho(0)), n(\rho(1)), n(\rho(2)), \dots) \in A;$$

otherwise, Player II wins. It is a classical theorem of Gale and Stewart [GM53] that if A is clopen and $\alpha = \omega$ (the order-type of $\mathbb{N}$), then the game is determined (in which case we say that A itself is *determined*).

A celebrated result of Martin [Mar75] implies that clopen games of length $\omega+\omega$ are also determined. Moreover, this is the extent of clopen determinacy provable in ZFC, Zermelo-Fraenkel set theory with the Axiom of Choice; i.e., ZFC does not prove that clopen games of length $\omega + \omega + 1$ are determined. This is a theorem of Harrington (see [Har78]). Determinacy for longer clopen games is, however, provable in natural extensions of ZFC by *large cardinal* axioms, and it is natural to ask the extent of clopen determinacy provable in theories stronger than ZFC. For example, clopen determinacy for games of length $\omega^2 + \omega$ is provable in the theory ZFC + "there are infinitely many Woodin cardinals," but clopen determinacy for games of length $\omega^2 + \omega + 1$ is not.

Our work in this chapter takes a different direction, however. Namely, by using completely elementary methods, we prove that every long clopen game is reducible to a more complicated, but shorter, game. Many results in this direction are folklore. For example, it is not hard to show that clopen games of length $\omega + \omega$ are determined if, and only if, Borel games of length ω are determined. The argument generalizes to show that clopen games of length $\alpha+\omega$ are determined if, and only if, Borel games of length α are determined. Similarly, clopen games of even (resp. odd) length $\alpha + 1$ are determined if, and only if, open (resp. closed) games of length α are determined. By combining these two observations, one can shorten clopen games of length $\alpha + \omega + 1$ to analytic games of length α. For similar reasons, the problem of shortening clopen games quickly reduces to the particular case in which the length is additively indecomposable (see Section 7.4 below).

Our first reduction of clopen games is, thus, from length ω^2 to length ω. Below, a set $A \subset \mathbb{R}$ is *σ-projective* if it belongs to the smallest collection of subsets of $\mathbb{R}$ containing all open sets and closed under continuous images, complements, and countable unions.

Theorem 7.1.1. *The following are equivalent:*

1. *Determinacy for clopen games of length ω^2.*

2. *Determinacy for σ-projective games of length ω.*

As a consequence of Theorem 7.1.1 and the main result of Chapter 4, one obtains a characterization of clopen determinacy for games of length ω^2 in terms of large-cardinal assumptions. The implication from (1) to (2) can be found in Chapter 5, i.e., [AMS] (as well as a weak form of the implication from (2) to (1)), so we shall only prove the converse. In light of Theorem 7.1.1, one might conjecture that clopen games of any countable length can be shortened to length ω. However, this is not the case. This is because clopen determinacy for games of length $\omega^2 + \omega + 1$ implies that $\mathbb{R}^\sharp$ exists (this follows from a theorem of Trang [Tra13], together with the observation on analytic games above). Since it is consistent that every set of reals in $L(\mathbb{R})$ is determined, no determinacy assumption for games of length ω can imply clopen determinacy of length $\omega^2 + \omega + 1$ (incidentally, $\omega^2 + \omega + 1$ is the least such ordinal).

However, as we will see, long clopen games can still be shortened. We will speak of some *quantifiers* on $\mathbb{R}$. Projections are usually denoted by the quantifier $\exists$, i.e., one writes $\exists x\, A(x)$ for "$\{x : A(x, y)\}$ is nonempty." Another example is the *game quantifier* given by

$\partial x\, A(x, y)$ if, and only if, Player I has a winning strategy for $\{x : A(x, y)\}$.

Game quantifiers for longer games ∂_α, as well as for games on $\mathbb{R}$, $\partial^{\mathbb{R}}$, are defined similarly.

Definition 7.1.2. Let Q be a quantifier. We denote by $\sigma(Q)$ the smallest σ-algebra on $\mathbb{R}$ containing the open sets and closed under Q.

Similarly, if $\{Q_\iota\}_\iota$ is a collection of quantifiers, we denote by $\sigma(\{Q_\iota\}_\iota)$ the smallest σ-algebra on $\mathbb{R}$ containing the open sets and closed under each Q_ι.

It is easy to see that $\sigma(\exists) = \sigma(\partial)$ is the pointclass of all σ-projective sets. The shortening theorem for games of length ω^3 is as follows:

Theorem 7.1.3. *The following are equivalent:*

1. *Determinacy for clopen games of length ω^3.*
2. *Determinacy for games of length ω^2 with payoff in $\sigma(\partial^{\mathbb{R}})$.*

Theorems 7.1.1 and 7.1.3 are both instances of a general theorem which is stated and proved in Section 7.3.

For expository reasons, we have chosen to include the proof of Theorem 7.1.1 in Section 7.2; it is similar to the general case, but easier. Although we are interested in shortening results for infinite games in their own right,

our main motivation is their applications to proofs of determinacy from large-cardinal hypotheses, such as:

Theorem 7.1.4. *Suppose that there are $\omega \cdot \beta$ Woodin cardinals. Then, clopen games of length $\omega^2 \cdot \beta$ are determined.*

In Section 7.5, we explain how to derive (a strengthening of) Theorem 7.1.4 from the shortening theorems and known techniques. We note that the conclusion of Theorem 7.1.4 would follow from work of Neeman [Nee04] if one additionally assumed the existence of a measurable cardinal above the Woodin cardinals.

7.2 Games of Length ω^2

Let $A \subset \mathbb{R}\times\mathbb{R}$ be clopen and write A_x for the set of all y such that $(x, y) \in A$. Hence,

$$\mathcal{Ø}^{\mathbb{R}}A = \{x : \text{ Player I has a winning strategy in the game on } \mathbb{R} \text{ with payoff } A_x\}.$$

We define

$$\mathcal{Ø}^{\mathbb{R}}\mathbf{\Delta}^0_1 = \{\mathcal{Ø}^{\mathbb{R}}A : \ A \text{ is clopen}\}.$$

Recall that $L_\alpha(\mathbb{R})$ is the αth level of the constructible hierarchy relative to the set of all real numbers. In particular, a subset of $\mathbb{R}$ is σ-projective if, and only if, it belongs to $L_{\omega_1}(\mathbb{R})$ (this is a simple, folklore fact, but a proof can be found in [AMS]).

Lemma 7.2.1. $\mathcal{Ø}^{\mathbb{R}}\mathbf{\Delta}^0_1 \subset L_{\omega_1}(\mathbb{R})$.

The converse of the lemma (for sets of reals) follows from the argument of [AMS].

Proof of the lemma. Suppose that a recursive procedure for coding finite and countably infinite sequences of real numbers by real numbers has been fixed, as well as a recursive relation $\sqsubset$ so that $x \sqsubset y$ if, and only if, x and y code sequences and x is a proper initial segment of y.

Let $A \subset \mathbb{R} \times \mathbb{R}$ be clopen. For each $x \in \mathbb{R}$, there is a game of length ω with moves in $\mathbb{R}$ given by A_x. Let us identify this game with A_x. We shall show that $\mathcal{Ø}^{\mathbb{R}}A \in L_{\omega_1}(\mathbb{R})$. For every $x \in \mathbb{R}$, we define

$$T_x = \Big\{t \in \mathbb{R}^{<\mathbb{N}} : \exists y \in \mathbb{R}^{\mathbb{N}}\, \exists z \in \mathbb{R}^{\mathbb{N}}\, \big(t \sqsubset y \wedge t \sqsubset z \wedge (x, y) \in A \wedge (x, z) \notin A\big)\Big\}.$$

Thus, T_x is the set of "contested" positions in A_x. We define a binary relation on $\mathbb{R}^2$ by

$$(x, y) \prec (w, z) \text{ if, and only if, } y \in \mathbb{R}^{<\mathbb{N}} \wedge x = w \wedge z \in T_w \wedge z \sqsubset y.$$

Since A is clopen, for every $x \in \mathbb{R}$ and every $y \in \mathbb{R}^{\mathbb{N}}$ there is some $n \in \mathbb{N}$ such that for every $z \in \mathbb{R}^{\mathbb{N}}$,

$$y \restriction n = z \restriction n \text{ implies } (y \in A_x \leftrightarrow z \in A_x).$$

It follows that $\prec$ is wellfounded, so it has a rank function, ρ. The rank function is defined by

$$\rho(w, z) = \sup\Big\{\rho(x, y) + 1 : (x, y) \prec (w, z)\Big\}.$$

Now, $\prec$ is analytic, so, by e.g., the Kunen-Martin theorem (see e.g., [Mos09, Theorem 2G.2]), ρ is bounded below ω_1, say, by η. Let us write

$$y \prec_x z \text{ if, and only if, } (x, y) \prec (x, z)$$

and denote by ρ_x the associated rank function. For every $x \in \mathbb{R}$, ρ_x is bounded by η. If z is a partial play of A_x and $\rho_x(z) = 0$, i.e., if z is $\prec_x$-minimal, then it does not belong to T_x, so every extension of z to a full play in the game with payoff A_x is won by the same player. As in the effective proof of (short) clopen determinacy, define:

$$\begin{aligned}
W_0(x) = \Big\{a \in \mathbb{R}^{<\mathbb{N}} : \exists y \in \mathbb{R}\, \forall z \in \mathbb{R}\, \Big(a^\frown y^\frown z \notin T_x \wedge \\
\exists w \in \mathbb{R}^{\mathbb{N}}\, \big(a^\frown y^\frown z \sqsubset w \wedge (x, w) \in A\big)\Big)\Big\}; \\
W_\alpha(x) = \Big\{a \in \mathbb{R}^{<\mathbb{N}} : \exists y \in \mathbb{R}\, \forall z \in \mathbb{R}\, \big(a^\frown y^\frown z \in \bigcup_{\xi<\alpha} W_\xi(x)\big)\Big\}; \\
W_\infty(x) = \bigcup_{\alpha \in \mathsf{Ord}} W_\alpha(x).
\end{aligned}$$

It is not hard to see (in V, where the Axiom of Choice holds) that from a partial play a of even length, Player I has a winning strategy in A_x if $a \in W_\infty(x)$. Conversely, if $a \notin W_\infty(x)$, then Player II has a strategy that prevents Player I from ever reaching a position in $W_\infty(x)$ and ultimately wins. It follows that

$$\mathcal{D}^{\mathbb{R}} A = \Big\{x \in \mathbb{R} : \langle\,\rangle \in W_\infty(x)\Big\}.$$

Let us refer to the least ξ such that $y \in W_\xi(x)$, if any, as the *weight* of y and denote it by $w_x(y)$. The point is that if a has weight ξ, then any extension of a of smaller weight has smaller rank in $\prec_x$. It follows that for every $a \in W_\infty(x)$, $w_x(a) \leq \rho_x(a)$. This is shown by an easy induction on the weight:

Put $\xi = w_x(a)$. Let us assume that $0 < \xi$, so there is $y \in \mathbb{R}$ such that for all $z \in \mathbb{R}$, $a^\frown y^\frown z$ has weight $< \xi$. By minimality of ξ, z can be chosen so that the weight of $a^\frown y^\frown z$ is arbitrarily large below ξ (ξ could be a successor ordinal, in which case this means "equal to the predecessor"). By induction hypothesis,

$$\xi = \sup\Big\{w_x(a^\frown y^\frown z) + 1 : z \in \mathbb{R}\Big\} \leq \sup\Big\{\rho_x(a^\frown y^\frown z) + 1 : z \in \mathbb{R}\Big\}.$$

It cannot be that $a \notin T_x$, for this contradicts the assumption that $w_x(a) \neq 0$; thus, $a \in T_x$ and, similarly, $a^\frown y \in T_x$, in which case we must have

$$\begin{aligned}\xi &\leq \sup\Big\{\rho_x(a^\frown y^\frown z) + 1 : z \in \mathbb{R}\Big\}\\ &= \sup\Big\{\rho_x(a^\frown y^\frown z) + 1 : a^\frown y^\frown z \prec_x a^\frown y\Big\}\\ &= \rho_x(a^\frown y)\end{aligned}$$

It follows that $\rho_x(a)$ is at least $\xi + 1$. This proves the claim. It follows that

$$W_\infty(x) = W_\eta(x)$$

for *every* $x \in \mathbb{R}$.

Finally, the construction of $W_\eta(x)$ can be carried out within $L_{\omega_1}(\mathbb{R})$, uniformly in x, so $\mathcal{D}^{\mathbb{R}}A \in L_{\omega_1}(\mathbb{R})$, as desired. □

To prove Theorem 7.1.1, suppose that σ-projective games of length ω are determined. Let A be a clopen set and consider the game of length ω^2 on $\mathbb{N}$ with payoff A. The argument to follow is a clopen version of one of Blass [Bla75]. Unlike the situation in [Bla75], we do not have too much determinacy to work with; however, we do not need much when dealing with clopen games, and we do not need uniformization.

Consider the following game:

$$\begin{array}{c|ccccc} I & \sigma_0 & & \sigma_1 & & \dots \\ II & & \tau_0 & & \tau_1 & \dots \end{array} \tag{7.1}$$

Here, players I and II take turns playing reals coding strategies for Gale-Stewart games. Player I wins if

$$(\sigma_0 * \tau_0, \sigma_1 * \tau_1, \dots) \in A,$$

where $\sigma * \tau$ denotes the result of facing off the strategies σ and τ; otherwise, Player II wins; i.e., instead of playing the game, the players simply reveal their strategies and use them compute a run of the game. This is a clopen game on reals, so it is determined by the Gale-Stewart Theorem [GM53]. Clearly, if Player I has a winning strategy in this game, then she has one in the long game with payoff A. Suppose instead that Player II has a winning strategy; we claim she has one in the long game.

We will construct a strategy τ for Player II in the long game with the property that every partial play by τ is not a losing play for Player II. Since the game is clopen, there can be no full play in which the winner of the game has not been decided, so τ will be a winning strategy. The strategy is constructed by blocks; first, we define it for plays of finite length. Given $x \in \mathbb{R}$, one may consider the following variant G_x of (7.1):

$$\begin{array}{c|ccccc} I & \sigma_1 & & \sigma_2 & & \dots \\ II & & \tau_1 & & \tau_2 & \dots \end{array}$$

Here, Player I wins if, and only if,

$$(x, \sigma_1 * \tau_1, \dots) \in A;$$

otherwise, Player II wins. This is also a clopen game, so the set

$$W = \{x \in \mathbb{R} : \text{ Player I has a winning strategy in } G_x\}$$

belongs to $\Im^{\mathbb{R}}\mathbf{\Delta}^0_1$, and thus to $L_{\omega_1}(\mathbb{R})$, by the lemma. By hypothesis,

$$L_{\omega_1}(\mathbb{R}) \models \mathsf{AD},$$

and so W is determined. Player I cannot have a winning strategy, for otherwise it could have been used as a first move to obtain a winning strategy in (7.1). Thus, Player II has a winning strategy in W. This will provide the restriction of τ to the first ω-many moves. Given the first ω-many moves, say, a, one repeats the argument above to obtain the restriction of τ to moves of length $\omega \cdot 2$ extending a. Eventually, one defines the response of τ to every $b \in \mathbb{N}^{<\omega^2}$, as desired. This completes the proof of Theorem 7.1.1.

7.3 Games of Length ω^α

In this section we study games of length ω^α. We first assume α is a successor ordinal and show:

Theorem 7.3.1. *Let $2 \leq \alpha < \omega_1$. The following are equivalent:*

1. *Determinacy for clopen games of length $\omega^{\alpha+1}$.*
2. *Determinacy for games of length ω^α with payoff in $\sigma(\partial^{\mathbb{R}}_{\omega^{-1+\alpha}})$.*

We shall only prove that (2) implies (1). As will be pointed out (see the proof of Lemma 7.3.2), the converse again follows from the argument of [AMS].

Let $S = \{(B, i) :$ Player I has a winning strategy in the game of length $\omega^{-1+\alpha}$ on $\mathbb{R}$ with payoff B and $i = 1$, or else Player I does not have a winning strategy in the game of length $\omega^{-1+\alpha}$ on $\mathbb{R}$ with payoff B and $i = 0\}$. Thus, $L(\mathbb{R})[S]$ is the smallest inner model of ZF that "knows" whether each of its games of length $\omega^{-1+\alpha}$ on $\mathbb{R}$ is won by Player I.

Lemma 7.3.2. *Let $2 \leq \alpha < \omega_1$. The following pointclasses all coincide:*

1. $\sigma(\partial^{\mathbb{R}}_{\omega^{-1+\alpha}})$,
2. $\sigma(\partial_{\omega^\alpha})$,
3. $\mathcal{P}(\mathbb{R}) \cap L_{\omega_1}(\mathbb{R})[S]$
4. $\partial^{\mathbb{R}}_{\omega^{-1+\alpha+1}} \mathbf{\Delta}^0_1$.

Proof. First, notice that for any adequate pointclass Γ, the fact that

$$\partial^{\mathbb{R}}_{\omega^{-1+\alpha}} \Gamma = \partial_{\omega^\alpha} \Gamma$$

can be shown easily by considering games like the one in the previous section. This is despite the fact that the two determinacy hypotheses associated to these quantifiers are not in general equivalent. The equality between (1) and (2) now follows by a simple transfinite induction on the construction of the σ-algebras.

Let us prove the equality between (1) and (3). Clearly, $L_{\omega_1}(\mathbb{R})[S]$ contains all reals and contains surjections

$$\pi_\xi : \mathbb{R} \twoheadrightarrow L_\xi(\mathbb{R})[S],$$

for each $\xi < \omega_1$. It follows that it is closed under countable sequences and in particular $\mathcal{P}(\mathbb{R}) \cap L_{\omega_1}(\mathbb{R})[S]$ is closed under countable unions. It is clearly closed under complements and ∂_{ω^α}.

For the converse, we need to verify that every set of reals in $L_{\omega_1}(\mathbb{R})[S]$ belongs to $\sigma(\partial_{\omega^\alpha})$. For this, it suffices to show that for each $\xi < \omega_1$, there

is a prewellordering $\preceq_\xi$ of a subset of $\mathbb{R}$, with $\preceq_\xi$ in $\sigma(\Im_{\omega^\alpha})$, such that if $\equiv_\xi$ denotes the induced equivalence relation and

$$x \prec_\xi y \leftrightarrow x \preceq_\xi y \wedge y \not\preceq_\xi x$$

denotes the strict part of $\preceq_\xi$, then

$$(\mathrm{field}(\preceq_\xi), \prec_\xi, \equiv_\xi)/\equiv_\xi \cong (L_\xi(\mathbb{R})[S], \in, =).$$

This is easily done by induction: for the successor case, suppose $\preceq_\xi$ has been defined and let $i_\xi : \mathrm{field}(\preceq_\xi) \to L_\xi(\mathbb{R})[S]$ be the isomorphism. Suppose moreover that $i_\xi^{-1}[S \cap L_\xi(\mathbb{R})[S]]$ is in $\sigma(\Im_{\omega^\alpha})$. We first define the prewellordering $\preceq^*_{\xi+1}$ as an extension of

$$\Big\{((\xi, x), (\xi, y)) : (x, y) \in \preceq_\xi \Big\}$$

by setting $(\xi, x) \preceq^*_{\xi+1} (\xi + 1, y)$ if, and only if, the following hold:

1. y is a tuple of the form $(\phi, \vec{b})$, where $\vec{b} = b_1, \ldots, b_n$ is a finite collection of reals in $\mathrm{field}(\preceq_\xi)$ and ϕ is a formula of arity n in the language of set theory extended with a predicate symbol $\dot{S}$;

2. The structure $(\mathrm{field}(\preceq_\xi), \prec_\xi)$ satisfies the formula $\phi(b_0, \ldots, b_n)$ if equality is interpreted as $\preceq_\xi$-equivalence and $\dot{S}$ is interpreted as $i_\xi^{-1}[S \cap L_\xi(\mathbb{R})[S]]$.

Then, $\preceq_{\xi+1}$ is defined to be the extensional closure of $\preceq^*_{\xi+1}$, i.e., writing $x \sim y$ if x and y have exactly the same $\prec^*_{\xi+1}$-predecessors, one also sets $x \preceq_{\xi+1} y$ if $x \sim y$ or there are u, v such that $u \sim x$, $v \sim y$, and $u \preceq^*_{\xi+1} v$. It is not hard to verify that this prewellordering is as required. Moreover, if one writes

$$i_{\xi+1} : \mathrm{field}(\preceq_{\xi+1}) \to L_{\xi+1}(\mathbb{R})[S]$$

for the isomorphism, the set $i_{\xi+1}^{-1}[S \cap L_{\xi+1}(\mathbb{R})[S]]$ is in definable from $\preceq_{\xi+1}$ using the quantifier $\Im_{\omega^\alpha}$ and thus belongs to $\sigma(\Im_{\omega^\alpha})$.

It remains to verify the equality between (3) and (4). We shall only prove that

$$\Im^{\mathbb{R}}_{\omega^{-1+\alpha+1}} \mathbf{\Delta}^0_1 \subset \mathcal{P}(\mathbb{R}) \cap L_{\omega_1}(\mathbb{R})[S],$$

The converse can be proved by an argument similar to the one in [AMS, Section 3], using the equivalence of (2) and (3).

The proof is much like that of Lemma 7.2.1. We repeat it with the necessary changes. For notational simplicity, let us assume that α is infinite,

so that $-1+\alpha=\alpha$. Let $A\subset\mathbb{R}\times\mathbb{R}$ be clopen. For each $x\in\mathbb{R}$, there is a game of length $\omega^{\alpha+1}$ with moves in $\mathbb{R}$ given by A_x. We need to show that $\mathcal{D}^{\mathbb{R}}_{\omega^{\alpha+1}}A\in L_{\omega_1}(\mathbb{R})[S]$. Again, assuming some suitable coding of sequences of reals by reals, we define for every $x\in\mathbb{R}$:

$$T_x=\bigcup_{n\in\mathbb{N}}\Big\{t\in\mathbb{R}^{\omega^\alpha\cdot n}:\exists y\in\mathbb{R}^{\omega^{\alpha+1}}\,\exists z\in\mathbb{R}^{\omega^{\alpha+1}}\,\big(t\sqsubset y\wedge t\sqsubset z\wedge(x,y)\in A\wedge(x,z)\notin A\big)\Big\}.$$

Define a binary relation on $\mathbb{R}^2$ by

$$(x,y)\prec(w,z)\text{ if, and only if, }y\in\bigcup_{n\in\mathbb{N}}\mathbb{R}^{\omega^\alpha\cdot n}\wedge x=w\wedge z\in T_w\wedge z\sqsubset y.$$

As before, the fact that A is clopen implies that $\prec$ is wellfounded, so it has a rank function, ρ, which is bounded by some $\eta<\omega_1$, by the analyticity of $\prec$. Define the relation $\prec_x$ and its rank function ρ_x as before and set

$$\begin{aligned}
W_0(x)&=\bigcup_{n\in\mathbb{N}}\Big\{a\in\mathbb{R}^{\omega^\alpha\cdot n}:\mathcal{D}^{\mathbb{R}}_{\omega^\alpha}\vec{y}\,\Big(a^\frown\vec{y}\notin T_x\wedge\\
&\qquad\qquad\exists w\in\mathbb{R}^{\omega^{\alpha+1}}\,\big(a^\frown\vec{y}\sqsubset w\wedge(x,w)\in A\big)\Big)\Big\};\\
W_\alpha(x)&=\bigcup_{n\in\mathbb{N}}\Big\{a\in\mathbb{R}^{\omega^\alpha\cdot n}:\mathcal{D}^{\mathbb{R}}_{\omega^\alpha}\vec{y}\,\big(a^\frown\vec{y}\in\bigcup_{\xi<\alpha}W_\xi(x)\big)\Big\};\\
W_\infty(x)&=\bigcup_{\alpha\in\mathsf{Ord}}W_\alpha(x).
\end{aligned}$$

An argument like the one in Lemma 7.2.1 shows that for each $x\in\mathbb{R}$ and each partial play a, the least ξ such that $y\in W_\xi(x)$, if it exists, is at most $\rho_x(a)$, which implies that

$$W_\infty(x)=W_\eta(x)$$

for every $x\in\mathbb{R}$.

It is easy to see that for every partial play a with length $\omega^\alpha\cdot n$, for some $n\in\mathbb{N}$, Player I has a winning strategy from a in A_x if $a\in W_\infty(x)$. Conversely, if $a\notin W_\infty(x)$, then Player I cannot have a winning strategy from a, for such a strategy would guarantee that every sufficiently long partial play consistent with it belongs to some $W_\alpha(x)$ and thus that $a\in W_\infty(x)$ (we cannot conclude that Player II has a winning strategy, but we do not need to). Now, the set $S_\eta=S\cap L_\eta(\mathbb{R})[S]$ belongs to $L_{\omega_1}(\mathbb{R})[S]$. Using S_η as a parameter, the construction of $W_\eta(x)$ can be carried out within $L_{\omega_1}(\mathbb{R})[S]$, uniformly in x, whereby $\mathcal{D}^{\mathbb{R}}_{\omega^{\alpha+1}}A$ belongs to $L_{\omega_1}(\mathbb{R})[S]$, as desired. □

Lemma 7.3.3. *Let $2 \leq \alpha < \omega_1$. Suppose that all games of length $\omega^{-1+\alpha}$ on $\mathbb{R}$ with payoff in $\sigma(\Im^{\mathbb{R}}_{\omega^{-1+\alpha}})$ are determined. Then, clopen games of length $\omega^{-1+\alpha+1}$ on $\mathbb{R}$ are determined.*

Proof. Let A be a clopen game of length $\omega^{-1+\alpha+1}$ on $\mathbb{R}$ (so A is clopen when viewed as a set of reals). For every partial play p of A, let A_p denote the game A after p has been played. This is also a clopen game of length $\omega^{-1+\alpha+1}$ on $\mathbb{R}$.

We consider the following game, which we denote by G:

$$\begin{array}{c|cccc} I & x_0 & & x_1 & \ldots (< \omega^{-1+\alpha}) \\ II & & y_0 & & y_1 \quad \ldots (< \omega^{-1+\alpha}) \end{array}$$

At the end of the game, a sequence

$$p = (x_0, y_0, x_1, \ldots) \in \mathbb{R}^{\omega^{-1+\alpha}}$$

has been produced. Player I wins if she has a winning strategy in A_p. If Player I has a winning strategy in G, then she clearly has one in A.

Suppose Player I does not have a winning strategy in G. We describe a winning strategy for Player II in A. Note that G is a game of length $\omega^{-1+\alpha}$ on $\mathbb{R}$ with payoff in $\Im^{\mathbb{R}}_{\omega^{-1+\alpha+1}}\mathbf{\Delta}^0_1 = \sigma(\Im^{\mathbb{R}}_{\omega^{-1+\alpha}})$, so G is determined. Thus, Player II has a winning strategy in G, say Σ. Any play consistent with Σ produces an infinite sequence p_0 such that Player I does not have a winning strategy in A_{p_0}.

Let p_0 be any sequence produced this way. Consider the following game, $G(p_0)$:

$$\begin{array}{c|cccc} I & x_0 & & x_1 & \ldots (< \omega^{-1+\alpha}) \\ II & & y_0 & & y_1 \quad \ldots (< \omega^{-1+\alpha}) \end{array} \tag{7.2}$$

At the end of the game, a sequence

$$p = (x_0, y_0, x_1, \ldots) \in \mathbb{R}^{\omega^{-1+\alpha}}$$

has been produced. Player I wins if she has a winning strategy in $A_{p_0^\frown p}$. Player I cannot have a winning strategy in this game, for it would yield a winning strategy in A_{p_0}. As before, this game belongs to $\sigma(\Im^{\mathbb{R}}_{\omega^{-1+\alpha}})$ and is thus determined. It follows that there is a winning strategy $\Sigma(p_0)$ such that any play p_1 consistent with it has the property that Player I does not have a winning strategy in $A_{p_0^\frown p_1}$.

The game A is clopen, so any full play won by Player I is won at a bounded stage. It follows that by repeating the above procedure one obtains a winning strategy for Player II in A. □

If α is an infinite ordinal, then determinacy for games of length ω^α on $\mathbb{N}$ with payoff in $\sigma(\Im^{\mathbb{R}}_{\omega^\alpha})$ implies determinacy for games of length ω^α on $\mathbb{R}$ with payoff in $\sigma(\Im^{\mathbb{R}}_{\omega^\alpha})$, and so the theorem follows from the preceding lemma. Oddly, if α is finite, the proof requires an additional argument:

Lemma 7.3.4. *Let $2 \leq n < \omega$. Suppose that games of length ω^n on $\mathbb{N}$ with payoff in $\sigma(\Im^{\mathbb{R}}_{\omega^{n-1}})$ are determined. Then, clopen games of length ω^{n+1} are determined.*

Proof. Let us assume $n = 2$ for simplicity, so we assume that games of length ω^2 on $\mathbb{N}$ with payoff in $\sigma(\Im^{\mathbb{R}})$ are determined and show that clopen games of length ω^3 are determined. In this proof, unlike before, it will be convenient to maintain the distinction between games and their payoff sets.

Let A be a clopen set and let $G(A)$ be the clopen game of length ω^3 on A. We want to show that $G(A)$ is determined. Let G be the following game on $\mathbb{R}$:

$$\begin{array}{c|ccccc} I & \sigma_0 & & \sigma_1 & & \ldots (< \omega^2) \\ II & & \tau_0 & & \tau_1 & \ldots (< \omega^2) \end{array}$$

Player I wins if, and only if,

$$(\sigma_0 * \tau_0, \sigma_1 * \tau_1, \ldots) \in A.$$

This is a clopen game on $\mathbb{R}$ of length ω^2. Clearly, if I wins G, then I wins $G(A)$. By hypothesis, games of length ω on $\mathbb{R}$ with payoff in $\sigma(\Im^{\mathbb{R}})$ are determined, so Lemma 7.3.3 applies, i.e., clopen games of length ω^2 on $\mathbb{R}$ are determined. Hence, if Player I does not win G, then Player II does. If so, we need to construct a strategy for Player II in $G(A)$. As usual, it will be given by blocks, this time of length ω^2 each. Given $p \in \mathbb{R}^{\mathbb{N}}$ or $p \in \mathbb{N}^{\omega^2}$, we define the game $G(p)$ of length ω^2 with moves in $\mathbb{R}$ by

$$\begin{array}{c|ccccc} I & \sigma_0 & & \sigma_1 & & \ldots (< \omega^2) \\ II & & \tau_0 & & \tau_1 & \ldots (< \omega^2) \end{array}$$

Player I wins if, and only if,

$$(p, \sigma_0 * \tau_0, \sigma_1 * \tau_1, \ldots) \in A.$$

Let $W = \{p : \text{Player I has a winning strategy in } G(p)\}$. Since $G(p)$ is a clopen game on $\mathbb{R}$ of length ω^2 for each p (uniformly), W is a set in $\Im^{\mathbb{R}}_{\omega^2}\mathbf{\Delta}^0_1$. Let H be the game on $\mathbb{R}$ of length ω given by:

$$\begin{array}{c|ccccc} I & \sigma_0 & & \sigma_1 & & \ldots (< \omega) \\ II & & \tau_0 & & \tau_1 & \ldots (< \omega) \end{array}$$

Player I wins if, and only if,

$$(\sigma_0 * \tau_0, \sigma_1 * \tau_1, \ldots) \in W.$$

Note that Player I cannot have a winning strategy in H, since it would easily yield a winning strategy for G. Let H' be the game of length ω^2 with moves in $\mathbb{N}$ and payoff W. Player I cannot have a winning strategy in H', for it would induce a winning strategy in H. Since H' is a game of length ω^2 with moves in $\mathbb{N}$ and payoff in $\mathfrak{D}^{\mathbb{R}}_{\omega^2}\mathbf{\Delta}^0_1$, it is determined, so Player II has a winning strategy in H'. This strategy may act as a non-losing strategy for Player II to use throughout the first ω^2-many moves of $G(A)$. An argument as before completes the proof. □

This completes the proof of Theorem 7.3.1. The analogue for limit ordinals is:

Theorem 7.3.5. *Let λ be a countable limit ordinal. The following are equivalent:*

1. *Determinacy for clopen games of length ω^λ.*
2. *Determinacy for games of length ω^β with payoff in $\sigma(\{\mathfrak{D}^{\mathbb{R}}_{\omega^\iota}\}_{\iota<\lambda})$ for all $\beta < \lambda$.*

Theorem 7.3.5 is proved much like Theorem 7.3.1. The key observation is that, if $\{\lambda_i : i \in \mathbb{N}\}$ is a sequence cofinal in λ, then games of length ω^λ can be decomposed into ω-many blocks the ith of which has length ω^{λ_i}. We note that, in the process of proving Theorem 7.3.5, one shows the following analogue of Lemma 7.3.2, where $S_\lambda = \{(\alpha, B, i) : \alpha < \lambda$ and either Player I has a winning strategy in the game of length ω^α on $\mathbb{R}$ with payoff B and $i = 1$, or else Player I does not have a winning strategy in the game of length ω^α on $\mathbb{R}$ with payoff B and $i = 0\}$.

Lemma 7.3.6. *Let λ be a countable limit ordinal. The following pointclasses all coincide:*

1. $\sigma(\{\mathfrak{D}^{\mathbb{R}}_{\omega^\iota}\}_{\iota<\lambda})$,
2. $\sigma(\{\mathfrak{D}_{\omega^\iota}\}_{\iota<\lambda})$,
3. $\mathcal{P}(\mathbb{R}) \cap L_{\omega_1}(\mathbb{R})[S_\lambda]$
4. $\mathfrak{D}^{\mathbb{R}}_{\omega^\lambda}\mathbf{\Delta}^0_1$.

7.4 Games of Decomposable Length

The proof from the last section (as well as the argument from [AMS]) adapts to shorten games of *any* countable length. For games of length $\theta + 1$ and $\theta + \omega$, this was mentioned in the introduction. Meanwhile, for games of other forms, we obtain the following analogues of Theorems 7.1.1, 7.3.1, and 7.3.1:

Theorem 7.4.1. *Let $\omega \leq \theta < \omega_1$. The following are equivalent:*

1. *Determinacy for clopen games of length $\theta + \omega^2$.*
2. *Determinacy for σ-projective games of length θ.*

Theorem 7.4.2. *Let $2 \leq \alpha < \omega_1$ and $\theta < \omega_1$. The following are equivalent:*

1. *Determinacy for clopen games of length $\theta + \omega^{\alpha+1}$.*
2. *Determinacy for games of length $\max\{\theta, \omega^\alpha\}$ with payoff in $\sigma(\mathfrak{D}^{\mathbb{R}}_{\omega^{-1+\alpha}})$.*

Theorem 7.4.3. *Let λ be a countable limit ordinal and $\theta < \omega_1$. The following are equivalent:*

1. *Determinacy for clopen games of length $\theta + \omega^\lambda$.*
2. *Determinacy for games of length γ with payoff in $\sigma(\{\mathfrak{D}^{\mathbb{R}}_{\omega^\iota}\}_{\iota<\lambda})$ for every $\gamma \in \lambda \cup \{\theta\}$.*

The proofs of these theorems are similar to the ones presented so far. They use (the statements of) Lemmata 7.2.1, 7.3.2 and 7.3.6, as well as the proofs of Theorem 7.1.1 and Lemmata 7.3.3 and 7.3.4. The key observation is that games of length $\theta + \omega^\alpha$ can be divided into ω-many rounds, the first of which has length θ and the rest of which have lengths that add up to ω^α.

7.5 Proofs of Determinacy

Let us finish by mentioning an application of these results to proofs of determinacy. No new ideas are involved beyond the ones presented here, in [AMS], and in [Nee04], so we shall be brief. Let us say that an ordinal α is an ω-limit ordinal if it is of the form $\omega^2 \cdot \beta$ for some β. By work of Neeman [Nee04], analytic games of length $\omega \cdot \alpha$ can be proved to be determined e.g., if one assumes the existence of $-1 + \alpha$ Woodin cardinals and a measurable cardinal above them, or slightly less. If one is only interested in proofs

of clopen—rather than analytic—determinacy for games of a given ω-limit length $\alpha = \omega^2 \cdot \beta$, then one can dispose of the measurable cardinal by first shortening these games using the methods presented and appealing to the technique of [AMS, Section 4]. This involves generalizing the notion of a model of class S_γ ($\gamma < \omega_1$) in a straightforward way:

Definition 7.5.1. Let $x \in \mathbb{R}$ and let M be a countable, ω_1-iterable extender model over x.

1. M is of class S_0^α above δ if $\alpha < \omega_1^x$ and M has an initial segment which is active above δ;

2. M is of class $S_{\gamma+1}^\alpha$ above δ if it has an initial segment N of class S_γ^α above some λ such that there are, in order-type, α-many Woodin cardinals in N within the interval (δ, λ);

3. For a countable limit λ, M is of class S_λ^α above δ if $\lambda < \omega_1^M$ and M has an active initial segment in all classes S_γ^α above δ, for $\gamma < \lambda$;

4. M is of class S_γ^α if it is of class S_γ^α above 0.

When $\alpha = 1$, the classes S_γ^1 generalize the notions of "not n-small" to the σ-projective hierarchy.

The proof of [AMS, Theorem 4.4] (see also [AMS, Theorem 5.3]), together with the shortening results, shows:

Theorem 7.5.2. *Let θ and α be countable ordinals. Suppose that for each $\gamma < \omega_1$ and there is some $x \in \mathbb{R}$ and an extender model M over x which is of class $S_\gamma^{\omega^\alpha}$ above some λ below which there are θ Woodin cardinals in M. Then, every clopen game of length $\omega \cdot (\theta + \omega^{\alpha+1})$ is determined.*

A similar theorem holds for games with length of the form $\omega \cdot \theta + \omega^{\omega \cdot \alpha}$ by considering models of class $S_\gamma^{<\alpha}$, in the natural sense. A comparison argument shows that any countable, ω_1-iterable extender model M (not necessarily of ZFC!) over x with $\alpha \cdot \omega$ Woodin cardinals is of class S_γ^α for *all* $\gamma < \omega_1^x$. This, together with standard methods, implies Theorem 7.1.4 from the introduction.

If one proves clopen determinacy for games of length α in this way, the hypotheses employed are known to be optimal in the cases $\alpha = \omega^2$ [Agug] and $\alpha = \omega^2 \cdot 2$ [AM]. The author conjectures that this is the case for all countable ω-limit α. The consistency strength of determinacy for clopen games of arbitrary countable length α is a very interesting open problem.

8

Long Open Determinacy

Our result on open games of length ω^2 is very similar to the ones in the previous section. It was conjectured by Philipp Schlicht.

Theorem 8.0.1. *The following are equivalent:*

1. *Open games of length ω^2 are determined,*
2. *There is a transitive model of $\mathsf{KP} + \mathsf{AD}$ containing $\mathbb{R}$.*

Proof. Clearly, if M is a transitive model of $\mathsf{KP} + \mathsf{AD}$ containing $\mathbb{R}$, then

$$L(\mathbb{R})^M$$

is also a model of KP containing $\mathbb{R}$. Since

$$\mathcal{P}(\mathbb{R}) \cap L(\mathbb{R})^M \subset M,$$

then it is also a model of AD. Thus, the following are equivalent:

1. There is a transitive model of $\mathsf{KP} + \mathsf{AD}$ containing $\mathbb{R}$.
2. $L_\kappa(\mathbb{R}) \models \mathsf{AD}$, where κ is the least $\mathbb{R}$-admissible ordinal.

It is well known that the inductive sets of reals are precisely those which are Σ_1-definable over $L_\kappa(\mathbb{R})$ with real parameters, where κ is as above. We use the following result:

Theorem 8.0.2 (Moschovakis [Mos72])**.** *The following are equivalent:*

1. *U is an inductive set of reals;*

2. *there is a projective set of reals A such that*

   $x \in U$ *if, and only if,* $\exists x_0 \in \mathbb{R}\, \forall x_1 \in \mathbb{R} \ldots \exists n \in \mathbb{N}\, (x, \langle x_0, \ldots x_n \rangle) \in A$;

3. *there is an analytic set of reals A such that*

   $x \in U$ *if, and only if,* $\exists x_0 \in \mathbb{R}\, \forall x_1 \in \mathbb{R} \ldots \exists n \in \mathbb{N}\, (x, \langle x_0, \ldots x_n \rangle) \in A$;

Let $\mathbf{\Sigma}^0_1$ denote the pointclass of all open sets and

$$\supset^{\mathbb{R}} \mathbf{\Sigma}^0_1 = \{\supset^{\mathbb{R}} U : U \in \mathbf{\Sigma}^0_1\}.$$

Let us rephrase Moschovakis's theorem in a way that is more useful for our purposes.

Lemma 8.0.3. *A set of reals belongs to $\supset^{\mathbb{R}} \mathbf{\Sigma}^0_1$ if, and only if, it is inductive.*

Proof. Suppose U is inductive and let A be an analytic set of reals such that

$$x \in U \text{ if, and only if, } \exists x_0 \in \mathbb{R}\, \forall x_1 \in \mathbb{R} \ldots \exists n \in \mathbb{N}\, (x, \langle x_0, \ldots x_n \rangle) \in A.$$

Fix a closed set B such that

$$A = p[B]$$

and choose a countable collection $\{B_i : i \in \mathbb{N}\}$ of basic clopen sets such that

$$B = \bigcap_{i \in \mathbb{N}} B_i.$$

Consider the following game of length $\leq \omega$ on $\mathbb{R}$:

$$\begin{array}{c|cccc|cc} I & x_0 & & x_2 & \ldots & ! & x^* \\ II & & x_1 & & \ldots & & i \end{array}$$

In the first phase of the game, Players I and II alternate turns playing real numbers. At the end of any given turn n, Player I may decide to transition to the second phase of the game. If so, then Player I must play a real number x^* and Player II must respond with a natural number $i \in \mathbb{N}$. The conditions are:

1. If Player I never decides to move the game forward to the second phase, then Player II wins.

2. Otherwise, if the transition occurs at the end of turn n, then Player I wins if, and only if,

   $$(x, \langle x_0, \ldots, x_n \rangle, x^*) \in B_i.$$

This is clearly an open game. Moreover, Player I has a winning strategy in this game if, and only if, $x \in U$. Thus, letting κ be as above, we have

$$\mathcal{P}(\mathbb{R}) \cap \Sigma_1^{L_\kappa(\mathbb{R})}(\mathbb{R}) \subset \mathcal{D}^{\mathbb{R}}\mathbf{\Sigma}_1^0.$$

Conversely, let $U \in \mathbf{\Sigma}_1^0 \cap \mathcal{P}(\mathbb{R}^2)$. Write

$$U = \bigcup_{i \in \mathbb{N}} U_i$$

as a countable union of basic clopen sets. Thus,

$$x \in \mathcal{D}^{\mathbb{R}}U \text{ if, and only if, } \exists x_0 \in \mathbb{R}\, \forall x_1 \in \mathbb{R} \ldots \exists n \in \mathbb{N}\, (x, \langle x_0, x_1, \ldots\rangle) \in U_n.$$

Since U_n is basic open, it is of the form

$$O(s,t) = \{(x,y) \in \mathbb{R} : (s,t) \sqsubset (x,y)\}.$$

Let k_n be the length of s and t. By our choice of coding of infinite sequences of real numbers by real numbers, the first k_n digits of a sequence $\langle x_0, x_1, \ldots\rangle$ depend only on $x_0, \ldots, x_{k_n}$. We will make use of this. Clearly, the set

$$\begin{aligned} U_n^* = \{(x, \langle k, x_0, \ldots, x_k\rangle) :\ & k \text{ is greater than or equal to the length of the} \\ & \text{unique finite sequences } s \text{ and } t \text{ such that } U_n = O(s,t), \text{ and} \\ & \exists x_{k+1}\, \exists x_{k+2} \ldots (x, \langle x_0, x_1, \ldots\rangle) \in U_n)\}; \end{aligned}$$

is projective. Let

$$A = \{x \in \mathbb{R} : \exists x_0 \in \mathbb{R}\, \forall x_1 \in \mathbb{R} \ldots \exists n, m \in \mathbb{N}\, (x, \langle n, x_0, x_1, \ldots, x_n\rangle) \in U_m^*\}.$$

We claim that A is inductive. To see this, first notice that the collection of all pairs $(x, \langle n, x_0, x_1, \ldots, x_n\rangle)$ belonging to some U_m^* is projective. Moreover, the real $\langle n, x_0, x_1, \ldots, x_{\langle n,m\rangle}\rangle$ coding the tuple $(n, x_0, x_1, \ldots, x_{\langle n,m\rangle})$ agrees on the first n digits with the real $\langle n, x_0, x_1, \ldots, x_n\rangle$ coding the tuple $(n, x_0, x_1, \ldots, x_n)$. Thus, it follows from the definition of U_m^* that

$$(x, \langle n, x_0, x_1, \ldots, x_n\rangle) \in U_m^*$$

implies

$$(x, \langle n, x_0, x_1, \ldots, x_{\langle n,m\rangle}\rangle) \in U_m^*.$$

Hence, A is inductive. Finally, writing k_n for the length of the finite sequences s and t such that $U_n = O(s,t)$,

$$\begin{aligned} x \in A \text{ iff } & \exists x_0 \in \mathbb{R}\, \forall x_1 \in \mathbb{R} \ldots \exists n, m \in \mathbb{N}\, (x, \langle n, x_0, x_1, \ldots, x_n\rangle) \in U_m^* \\ \text{iff } & \exists x_0 \in \mathbb{R}\, \forall x_1 \in \mathbb{R} \ldots \exists n, m \in \mathbb{N}\ k_m \leq n \wedge \\ & \qquad \exists y_{n+1}\, \exists y_{n+2} \ldots (x, \langle x_0, x_1, \ldots, x_n, y_{n+1}, \ldots\rangle) \in U_m \\ \text{iff } & \exists x_0 \in \mathbb{R}\, \forall x_1 \in \mathbb{R} \ldots \exists m \in \mathbb{N}\, (x, \langle x_0, x_1, \ldots\rangle) \in U_m \end{aligned}$$

Here, the first and second equivalences hold by definition. The third equivalence holds because whether $(x, \langle x_0, x_1, \ldots\rangle)$ belongs to U_m depends only on the first k_m digits of $\langle x_0, x_1, \ldots\rangle$ (by definition of k_m), which in turn depend only on $x_0, x_1, \ldots, x_{k_m}$. □

So far, we have shown that

$$\mathcal{P}(\mathbb{R}) \cap \partial^{\mathbb{R}} \mathbf{\Sigma}^0_1 = \mathcal{P}(\mathbb{R}) \cap \Sigma^{L_\kappa(\mathbb{R})}_1,$$

where κ is as above. Thus, if open games of length ω^2 are determined, then games of length ω with payoff in $\partial^{\mathbb{R}} \mathbf{\Sigma}^0_1$ are determined, and thus $L_\kappa(\mathbb{R})$ is a transitive model of $\mathsf{KP} + \mathsf{AD}$ containing $\mathbb{R}$. Conversely, suppose that

$$L_\kappa(\mathbb{R}) \models \mathsf{AD}.$$

Since $L_\kappa(\mathbb{R}) \models \mathsf{KP}$, it satisfies Δ_1-separation, i.e., every subset of $\mathbb{R}$ that is Δ_1-definable over $L_\kappa(\mathbb{R})$ (with parameters) belongs to $L_\kappa(\mathbb{R})$. Thus, hyperprojective sets are determined. By the Kechris-Woodin [KW83] determinacy transfer theorem, inductive sets are determined, so games of length ω on $\mathbb{N}$ with payoff in $\partial^{\mathbb{R}} \mathbf{\Sigma}^0_1$ are determined. We may now argue as in the previous chapter:

Let $A \in \mathbf{\Sigma}^0_1$. Consider the game:

$$\begin{array}{c|ccccc} I & \sigma_0 & & \sigma_1 & & \ldots \\ II & & \tau_0 & & \tau_1 & \ldots \end{array} \tag{8.1}$$

Here, players I and II take turns playing reals coding strategies for Gale-Stewart games. Player I wins if

$$(\sigma_0 * \tau_0, \sigma_1 * \tau_1, \ldots) \in A;$$

otherwise, Player II wins. I.e., instead of playing the game, the players reveal their strategies and use them compute a run of the game. This is an open game on reals, so it is determined by the Gale-Stewart Theorem

[GM53]. Clearly, if Player I has a winning strategy in this game, then she has one in the long game with payoff A. Suppose instead that Player II has a winning strategy in (8.1).

We construct a strategy τ for Player II with the property that every partial play by τ is not a losing play for Player II. Since the game is open, every full play none of whose proper subplays is won by Player I is won by Player II, so τ will be a winning strategy. The strategy is constructed by blocks; first, we define it for plays of finite length. Given $x \in \mathbb{R}$, consider the following variant G_x of (8.1):

$$\begin{array}{c|ccccc} I & \sigma_1 & & \sigma_2 & & \dots \\ II & & \tau_1 & & \tau_2 & \dots \end{array}$$

Here, Player I wins if, and only if,

$$(x, \sigma_1 * \tau_1, \dots) \in A;$$

otherwise, Player II wins. This is also an open game, so the set

$$W = \{x \in \mathbb{R} : \text{ Player I has a winning strategy in } G_x\}$$

belongs to $\mathfrak{D}^{\mathbb{R}}\mathbf{\Sigma}^0_1$ and so it is determined. Player I cannot have a winning strategy, for otherwise it could have been used as a first move to obtain a winning strategy in (8.1). It follows that Player II has a winning strategy in W. This will provide the restriction of τ to the first ω-many moves. Given the first ω-many moves, say, a, one repeats the argument above to obtain the restriction of τ to moves of length $\omega \cdot 2$ extending a. Eventually, one defines the response of τ to every $b \in \mathbb{N}^{<\omega^2}$, as desired. This completes the proof of Theorem 8.0.1. $\square$

9

Long Borel Determinacy

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9.1 Introduction

Given a Borel subset A of the Hilbert space ℓ^2, define a two-player perfect-information, zero-sum game in which Player I and Player II alternate turns playing digits $x_i^0 \in \{0, 1, \ldots, 9\}$ infinitely often, giving rise to a real number

$$x^0 = \sum_{i=0}^{\infty} \frac{x_i^0}{10^{i+1}}.$$

Afterwards, they continue alternating turns to produce real numbers x^1, x^2, and so on. To ensure that the sequence belongs to ℓ^2, we normalize, e.g., by setting

$$y^i = \frac{x^i}{i+1}.$$

Player I wins the game if, and only if, $\vec{y} \in A$; otherwise Player II wins. These are games of transfinite length ω^2. (Recall that ω denotes the order-type of the natural numbers.) It is a natural question whether these games are *determined*, i.e., whether one of the players has a winning strategy.

Given $A \subset [0,1]$, the Gale-Stewart game on A is defined as follows: two players, I and II, alternate infinitely many turns playing digits $x(0), x(1), \ldots$, as before. This play gives rise to a real number $x \in [0,1]$ as before. Player I wins the game if, and only if, $x \in A$; otherwise, Player II wins. We say that a set $A \subset \mathbb{R}$ is *determined* if the game associated to $A \cap [0,1]$ is determined.

Our games of length ω^2 can be recast in terms of Gale-Stewart games: given $A \subset [0,1]$, players I and II alternate infinitely many turns playing digits $x^0_0, x^0_1, x^0_2, \ldots$; afterwards, they alternate infinitely many turns playing digits $x^1_0, x^1_1, x^1_2, \ldots$; and so on. Let

$$\rho : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$$

be a Borel bijection, e.g.,

$$(i,j) \mapsto \frac{1}{2}(i+j)(i+j+1) + j.$$

Put

$$x = \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \frac{x^j_i}{10^{\rho(i,j)+1}}.$$

Player I wins the game if, and only if, $x \in A$; otherwise, Player II wins.

A third and most useful way of studying these games is in terms of the Baire space $\mathbb{N}^{\mathbb{N}}$: given a Borel $A \subset \mathbb{N}^{\mathbb{N}}$, two players alternate ω^2 many turns playing elements of $\mathbb{N}$ and produce an infinite sequence $\{x_i : i \in \mathbb{N}\}$, with $x_i \in \mathbb{N}^{\mathbb{N}}$. As before, by means of a Borel bijection, this sequence is identified with an element $x \in \mathbb{N}^{\mathbb{N}}$. Player I wins the game if, and only if, $x \in A$; otherwise, Player II wins. It is this the way in which we will speak of games of length ω^2 hereafter. Every Polish space is a continuous image of $\mathbb{N}^{\mathbb{N}}$ and Borel sets have enough closure properties that these forms of determinacy are all equivalent.

Zermelo set theory (Z) was introduced in 1908 by E. Zermelo with the goal of reducing Cantor and Dedekind's theories to a small collection of principles. It, together with A. Fraenkel and T. Skolem's refinements, resulted in Zermelo-Fraenkel set theory (ZF). In this chapter, Z consists of the axioms of extensionality, separation, power set, pairing, union, foundation, and infinity; and ZF consists of Z, plus the axiom(s) of replacement. ZC and ZFC are the results of adding the Axiom of Choice to Z and ZF, respectively.

Although ZC is a powerful theory in which a great deal of mathematics can be formalized, there are many examples of theorems that cannot be

proved without the use of the axiom of replacement. An early example of this was that of *Borel determinacy*, i.e., whether all Borel subsets of $\mathbb{R}$ are determined.

Theorem 9.1.1 (Martin [Mar75])**.** *Suppose A is Borel. Then A is determined.*

Even before Martin's Borel determinacy theorem had been proved, it was known that any proof would need essential use of the axiom of replacement:

Theorem 9.1.2 (Friedman [Fri71])**.** *Borel determinacy is not provable in* ZC*.*

Friedman's work showed that any proof of Borel determinacy would require the use of arbitrarily large countable iterations of the power set operator, and Martin's work showed that this suffices. A convenient slogan is that Borel determinacy captures the strength of countably iterated powersets.

It is known that one cannot hope to prove the determinacy of any reasonable class of non-Borel games within ZFC; however, one can extend ZFC by very natural *large cardinal axioms*, or strengthenings of the axiom of infinity, and use them to prove the determinacy of games on the natural numbers. These connections between large cardinal axioms and infinite games have been the focus of extensive study for decades; some of the most famous results include Martin's [Mar70] proof of analytic determinacy, Martin and Steel's [MS89] proof of projective determinacy, and Woodin's [Woo88] proof of the Axiom of Determinacy in $L(\mathbb{R})$, the smallest model of ZF containing all reals and all ordinals. (Recall that the Axiom of Determinacy is the assertion that every Gale-Stewart game (of length ω) is determined. By a result of Mycielski and Steinhaus, it is inconsistent with the Axiom of Choice.) More specifically, Woodin showed that if there are infinitely many Woodin cardinals and a measurable cardinal greater than them, then the Axiom of Determinacy holds in $L(\mathbb{R})$. The measurable cardinal is necessary, in the sense that the existence of infinitely many Woodin cardinals does not imply that the Axiom of Determinacy holds in $L(\mathbb{R})$.

Our main theorem shows that Borel determinacy for games of length ω^2 captures the strength of countably iterated powersets *above a limit of Woodin cardinals.*[1]

[1]In the interest of expedience, one may define the statement "there are infinitely many Woodin cardinals" to mean the following in ZC: there is an infinite sequence $(\delta_0, \delta_1, \ldots)$ of cardinals such that for each $i \in \mathbb{N}$, V_{δ_i} is a model of ZFC and δ_i is a Woodin cardinal in V_{δ_j} whenever $i < j$.

Theorem 9.1.3. *The following are equivalent over* ZFC*:*

1. *Borel determinacy for games of length ω^2;*
2. *For every countable α, there is a countably iterable extender model of* ZC *satisfying "$V_{\lambda+\alpha}$ exists, where λ is a limit of Woodin cardinals."*

Hence, Borel determinacy for games of length ω^2 is a natural example of a theorem whose strength can only be correctly gauged over ZFC in terms of weaker set theories. The models referred to in Theorem 9.1.3 are fine-structural and built over some $x \in \mathbb{R}$. As a consequence thereof, we obtain a result on the provability of Borel determinacy for games of length ω^2 similar to the one for short games:

Corollary 9.1.4. *Borel determinacy for games of length ω^2 is provable in the theory* ZFC *+ "there are infinitely many Woodin cardinals," but not in the theory* ZC *+ "there are infinitely many Woodin cardinals."*

The first part of Corollary 9.1.4 follows from Theorem 9.1.3, together with a theorem of Steel [Ste93] and a reflection argument. Our proof of Theorem 9.1.3 does not quite go through if one replaces ZFC by ZC in the statement, although we conjecture that the equivalence is still provable there. Nonetheless, the proof does go through if one replaces ZFC by ZC, together with the assumption that there are enough ordinals[2] and, in particular, if one assumes that there are infinitely many Woodin cardinals. This yields the second part of Corollary 9.1.4.

The main step towards proving Theorem 9.1.3 is a "shortening" result like the ones in Chapter 7. It shows that Borel games of length ω^2 are determined if, and only if, those in a much more complicated class of games of length ω are. Recall the definition of Gödel's constructible hierarchy over $\mathbb{R}$: $L_0(\mathbb{R})$ is $V_{\omega+1}$ (the collection of all sets of von Neumann rank $\leq\omega$), $L_{\alpha+1}(\mathbb{R})$ is the collection of all sets definable over $L_\alpha(\mathbb{R})$ from parameters in $L_\alpha(\mathbb{R})$, and $L_\lambda(\mathbb{R}) = \bigcup_{\alpha<\lambda} L_\alpha(\mathbb{R})$ if λ is a limit ordinal.

Theorem 9.1.5. *Let ϑ be the least ordinal such that for every countable ordinal α, there is $\xi < \vartheta$ such that $L_\xi(\mathbb{R})$ satisfies "V_α exists." The following are equivalent:*

1. *Borel determinacy for games of length ω^2;*
2. *$L_\vartheta(\mathbb{R})$ is a model of the Axiom of Determinacy.*

[2] It suffices to assume that ϑ_α exists for all $\alpha < \omega_1$ (see below).

It is independent of ZFC whether $L_\vartheta(\mathbb{R})$ is a model of the Axiom of Determinacy or of the Axiom of Choice (or neither), but it is never a model of Zermelo set theory, since $\mathcal{P}(\mathbb{R}) \cap L_\vartheta(\mathbb{R}) \notin L_\vartheta(\mathbb{R})$.

The use of large cardinals is essential in proofs of determinacy by Gödel's second incompleteness theorem, for one can often prove the existence of inner models of set theory satisfying forms of the axiom of infinity from the determinacy of infinite games. Results of this form include Friedman's Theorem 9.1.2 and Theorem 9.1.3, but also Harrington's [Har78] work on analytic determinacy and Woodin's work in projective determinacy and the Axiom of Determinacy (see e.g., [MSW16] and [KW10]). More recent work in this direction includes that of Steel [Stea], Montalbán-Shore [MS11], Welch [Wel11], Trang [Tra13], and others. In order to obtain equivalences such as Theorem 9.1.3, as opposed to equiconsistency results, one needs to involve the techniques of the core model induction. Since we shall be confined within $L(\mathbb{R})$, the arguments we will need can be found in Steel-Woodin [SW16]. Other expositions include Schindler-Steel [SS14] and Wilson [Wil12].

The determinacy of games of countable length with analytic (or projective) payoff has been proved by Neeman [Nee04] from assumptions that are very likely optimal. The optimality for games of sufficiently closed length has been verified by Trang [Tra13] and by Woodin, in unpublished work. Reversals of determinacy hypotheses for projective games of length ω^2 were obtained in [AM] and for clopen games in [Agub]. Games of length ω^2 were first considered in print by Blass [Bla75].

The chapter is structured as follows: Theorem 9.1.5 is proved in Section 9.3. The proof is purely descriptive-set– and recursion-theoretic. It involves the use of model games which incorporate elements from the constructions in Martin and Steel [MS08], Friedman [Fri71], and Martin (see [Mar]). A tool used in the proof (Lemma 9.3.1) is a method of describing initial segments of the $L(\mathbb{R})$-hierarchy by games in which two players determine the truth of a statement in $L(\mathbb{R})$ by challenging each other's claims with auxiliary games. This lemma is proved in Section 9.4. The chapter ends with Section 9.5, in which localizations of arguments from inner model theory are shown to imply Theorem 9.1.3, starting from Theorem 9.1.5.

9.2 Preliminaries

We shall use notions from descriptive set theory and ordinal recursion theory freely. We refer the reader to Moschovakis' book [Mos09] and to Barwise's book [Bar75] for background. Additional background, as well as history, can be found in the chapters from the Handbook of Set Theory by Schindler-Zeman [SZ10], Steel [Ste10], and Koellner-Woodin [KW10].

Our notation is standard. Among the usual abuses of notation in which we shall engage are the identification of $\mathbb{R}$ with the Baire space $\mathbb{N}^{\mathbb{N}}$, as well as the identification of Gale-Stewart games with other infinite zero-sum games without explicitly transforming the payoff set into that of a Gale-Stewart game. We also identify $\mathbb{R}$ with $V_{\omega+1}$. In particular, natural numbers are real numbers.

If σ is a strategy for a short game (i.e., one of length ω) on $\mathbb{N}$, then we may alternately regard σ as a real, as a function on $\mathbb{N}^{\mathbb{N}}$, or as the function on $\mathbb{R}$ which to each $x \in \mathbb{R}$ assigns the unique y which results by playing against x in a way consistent with σ. If σ and τ are two strategies, $\sigma * \tau$ denotes the result of facing them off against each other.

We will often need to consider games of length ω with moves in $\mathbb{R}$. These can be considered as games of length ω^2 on $\mathbb{N}$ in which Player I's moves are ignored throughout the $(2n+1)$th block of ω-many moves, and Player II's moves are ignored throughout the $(2n)$th block of ω-many moves. In particular, if Borel games of length ω^2 are determined, then so too are Borel games of length ω with moves on $\mathbb{R}$; the converse is not true, as the latter statement is provable in ZFC (by Martin's proof of Borel determinacy).

At no point other than Section 9.5 will we need any inner model theory, but we will make use of simple fine-structural facts about $L(\mathbb{R})$. We make use of the Jensen hierarchy over the reals (see [Jen72]). Recall its definition given by:

$$\begin{aligned} J_0(\mathbb{R}) &= V_{\omega+1} \\ J_{\alpha+1}(\mathbb{R}) &= \text{closure of } J_\alpha(\mathbb{R}) \cup \{J_\alpha(\mathbb{R})\} \text{ under rudimentary functions} \\ J_\lambda(\mathbb{R}) &= \bigcup_{\alpha<\lambda} J_\alpha(\mathbb{R}), \text{ for limit } \lambda. \end{aligned}$$

The following definition is perhaps not standard, but will suffice for our purposes:

Definition 9.2.1. Let α and β be ordinals. We say that the *projectum* of $J_\alpha(\mathbb{R})$ is V_β if $\beta < \alpha$ and β is the least ordinal such that

$$V_{\beta+1} \cap \big(J_{\alpha+1}(\mathbb{R}) \setminus J_\alpha(\mathbb{R})\big) \neq \varnothing.$$

Let β be such that V_β is the projectum of $J_\alpha(\mathbb{R})$. We say that $J_\alpha(\mathbb{R})$ *projects to* V_β. Since $\mathbb{R}$ always belongs to $J_\alpha(\mathbb{R})$, β is at least $\omega+1$. If $\beta = \omega+1$, we say that $J_\alpha(\mathbb{R})$ projects to $\mathbb{R}$. If $J_\alpha(\mathbb{R})$ does not project to $\mathbb{R}$, then every subset of $\mathbb{R}$ definable over $\mathcal{P}(\mathbb{R})^{J_\alpha(\mathbb{R})}$ belongs to $J_\alpha(\mathbb{R})$, so

$$(\mathbb{N}, \mathbb{R}, \mathcal{P}(\mathbb{R}) \cap J_\alpha(\mathbb{R}))$$

is a model of third-order arithmetic.

If the projectum of $J_\alpha(\mathbb{R})$ is not defined, then $J_\alpha(\mathbb{R})$ contains all definable subsets of $V_\beta^{J_\alpha(\mathbb{R})}$ for all $\beta < \alpha$, and so

$$J_\alpha(\mathbb{R}) \models \mathsf{ZF}.$$

Proposition 9.2.2. *Suppose that the projectum of $J_\alpha(\mathbb{R})$ is V_β and $\omega+1 < \beta$. Then, there is a surjection*

$$\rho : V_\beta \cap J_\alpha(\mathbb{R}) \twoheadrightarrow J_\alpha(\mathbb{R})$$

in $J_{\alpha+1}(\mathbb{R})$.

Proof. This proof is similar to those in [Jen72, Section 3] and [Ste08c, Section 1]. Suppose that the projectum of $L_\alpha(\mathbb{R})$ is V_β, with $\omega+1 < \beta$, and let ϕ be a formula in the language of set theory such that for some $a \in J_\alpha(\mathbb{R})$, we have

$$\{x \in J_\alpha(\mathbb{R}) : J_\alpha(\mathbb{R}) \models \phi(a, x)\} \in V_{\beta+1} \cap (J_{\alpha+1}(\mathbb{R}) \setminus J_\alpha(\mathbb{R})).$$

Call the set just defined A. Suppose that ϕ is Σ_n. In $L(\mathbb{R})$, every set is definable from a real and ordinal parameters, in fact, by Lemma 1.4 of [Ste08c], there are surjections

$$f_\beta : [\omega \cdot \beta]^{<\omega} \times \mathbb{R} \twoheadrightarrow J_\beta(\mathbb{R})$$

which are uniformly Σ_1 over $J_\beta(\mathbb{R})$. Thus, we may assume that $a \in \mathbb{R}$ by replacing n with a bigger natural number if necessary, and replacing $\phi(a, x)$ with the formula

$$\phi(f_\alpha(\vec{\beta}, a), x) \text{ holds for the least tuple } \vec{\beta} \text{ such that the set}$$
$$\{x : \phi(f_\alpha(\vec{\beta}, a), x)\} \text{ does not exist.}$$

Since $\omega + 1 < \beta$, Lemma 1.7 and Theorem 1.16 of [Ste08c] imply that $J_\alpha(\mathbb{R})$ has a $\mathbf{\Sigma}_n$ Skolem function, i.e., that there is a partial map

$$h : J_\alpha(\mathbb{R}) \times \mathbb{R} \to J_\alpha(\mathbb{R})$$

which is Σ_n in a parameter $p \in J_\alpha(\mathbb{R})$ such that whenever S is a nonempty Σ_n set over $J_\alpha(\mathbb{R})$ with a parameter $q \in J_\alpha(\mathbb{R})$, then

$$h(q, x) \in S \text{ for some } x \in \mathbb{R}.$$

Thus, one can define the Σ_n Skolem hull of $J_\alpha(\mathbb{R})$ with a Σ_n formula using parameters from $J_\alpha(\mathbb{R})$. Let p be the parameter defining h and

$$\mathcal{H} = \mathrm{Hull}_{\Sigma_n}^{J_\alpha(\mathbb{R})}(V_\beta \cap J_\alpha(\mathbb{R}), \{p\}).$$

By Lemma 1.9 of [Ste08c],

1. $\mathcal{H}$ is a Σ_n-elementary substructure of $J_\alpha(\mathbb{R})$, and
2. $\mathcal{H} = f[X^{<\omega}]$ for some partial map f which is Σ_n over $\mathcal{H}$ with parameters in $\mathcal{H}$, where $X = (V_\beta \cap J_\alpha(\mathbb{R})) \cup \{p\}$.

The latter conclusion easily implies that in fact

$$\mathcal{H} = f[X^{<\omega}]$$

for some partial map f as above, where $X = V_\beta \cap J_\alpha(\mathbb{R})$.

Since $\omega + 1 < \beta$, $\mathcal{H}$ contains all reals, so, by condensation, there is $\gamma \leq \alpha$ such that the transitive collapse of $\mathcal{H}$ is $J_\gamma(\mathbb{R})$. Let

$$\pi : J_\gamma(\mathbb{R}) \twoheadrightarrow \mathcal{H}$$

be the collapse embedding, which has critical point $\geq \beta$. By definition, $A \in V_{\beta+1}$, so

$$\text{if } J_\alpha(\mathbb{R}) \models \phi(a, x), \text{ then } x \in V_\beta.$$

Since $V_\beta \cap J_\alpha(\mathbb{R}) \subset J_\gamma(\mathbb{R})$ and $a \in \mathbb{R}$, we have for every $x \in J_\gamma(\mathbb{R})$,

$$\begin{aligned} J_\alpha(\mathbb{R}) \models \phi(a, x) \text{ if, and only if, } & J_\alpha(\mathbb{R}) \models x \in V_{\pi(\beta)} \wedge \phi(a, x), \\ \text{if, and only if, } & J_\gamma(\mathbb{R}) \models x \in V_\beta \wedge \phi(a, x). \end{aligned}$$

This is not entirely straightforward, as $V_\beta \cap J_\alpha(\mathbb{R})$ need not be an element of $J_\gamma(\mathbb{R})$ (or of $J_\alpha(\mathbb{R})$), but for each η, the set

$$\{x \in J_\eta(\mathbb{R}) : x \text{ has rank } \beta\}$$

is uniformly $\Delta_1^{J_\eta(\mathbb{R})}(\beta)$ (provided $\beta < \eta$). Thus, the set A is definable over $J_\gamma(\mathbb{R})$, which implies that $\gamma = \alpha$. Using π and the function f from above, one obtains a partial surjection from $V_\beta \cap J_\alpha(\mathbb{R})$ onto $J_\alpha(\mathbb{R})$ which is definable over $J_\alpha(\mathbb{R})$ by a Σ_n formula with parameters. This easily translates into a Π_{n+1} (with parameters) total surjection f^* from $V_\beta \cap J_\alpha(\mathbb{R})$ onto $J_\alpha(\mathbb{R})$. Clearly f^* belongs to $J_{\alpha+1}(\mathbb{R})$, which completes the proof. □

9.3 Determinacy and $L(\mathbb{R})$

If $A \subset \mathbb{R} \times \mathbb{R}$ is a set and $x \in \mathbb{R}$, we define $A_x = \{y \in \mathbb{R} : (x, y) \in A\}$ and

$$\mathcal{Q}^{\mathbb{R}} A = \{x \in \mathbb{R} : \text{ Player I has a winning strategy in the game with payoff } A_x \text{ and moves in } \mathbb{R}\}.$$

The set $\mathcal{Q}^{\mathbb{R}} A$ is defined similarly for A a subset of other spaces, such as $\mathbb{R}^n$. Letting $\mathcal{Q}^{\mathbb{R}} \mathbf{\Delta}^1_1 = \{\mathcal{Q}^{\mathbb{R}} A : A \text{ is Borel}\}$, we write

$$\gamma^1_1 = \sup\{\alpha : \text{ there is a prewellordering of } \mathbb{R} \text{ in } \mathcal{Q}^{\mathbb{R}} \mathbf{\Delta}^1_1 \text{ of rank } \alpha\}.$$

The ordinal γ^1_1 is usually denoted by $\delta_{\mathcal{Q}^{\mathbb{R}} \mathbf{\Delta}^1_1}$. [3]

Lemma 9.3.1. $\mathcal{P}(\mathbb{R}) \cap L_{\gamma^1_1}(\mathbb{R}) \subset \mathcal{Q}^{\mathbb{R}} \mathbf{\Delta}^1_1$.

The proof of Lemma 9.3.1 will be delegated to the following section. For now, let us assume that it is true.

Remark 9.3.2. Under determinacy, Lemma 9.3.1 can be proved easily by appealing to general Wadge theory. The proof that we shall provide, however, is direct, and goes through in e.g., $\mathsf{ZF} + \mathsf{DC}$.

Definition 9.3.3. We denote by ϑ_α the least ordinal ξ such that $L_\xi(\mathbb{R})$ is a model of

$$\mathsf{KP} + \text{Separation} + \text{“}V_\alpha \text{ exists.”}$$

We also define

$$\vartheta = \sup_{\alpha < \omega_1} \vartheta_\alpha.$$

The main result of this section is the following:

Theorem 9.3.4. *The following are equivalent:*

1. *Borel determinacy for games of length ω^2;*
2. *$L_\vartheta(\mathbb{R}) \models \mathsf{AD}$.*

We need some preliminaries. We will work with structures in the language of set theory with additional constants $\{\dot{x}_i : i \in \mathbb{N}\}$ (which will be interpreted as real numbers). We fix:

[3] Note that $\mathcal{Q}^{\mathbb{R}} \Delta^1_1$ is a much smaller class than $\Delta^{L(\mathbb{R})}_1 = \Delta_{\mathcal{Q}^{\mathbb{R}} \Pi^1_1}$, the class of all sets A such that both A and its complement belong to $\mathcal{Q}^{\mathbb{R}} \Pi^1_1$.

1. a formula $\theta(\cdot,\cdot,\cdot)$ in the language of set theory defining in $L(\mathbb{R})$ an $\mathbb{R}$-parametrized sequence of wellorderings the union of whose domain is $L(\mathbb{R})$ (e.g., the one given by Lemma 1.4 of [Ste08c]), and
2. injective functions $n(\cdot)$ and $m(\cdot)$ assigning natural numbers to formulae in the extended language in such a way that their ranges are recursive and disjoint and whenever $\dot{x}_i$ occurs in a formula ϕ, then $i < m(\phi)$ and $i < n(\phi)$.

Let us begin with an observation:

Lemma 9.3.5. *Suppose that M is a model of KP and M contains an ordinal a isomorphic to α. Then $(V_a)^M$ (which need not belong to M) is wellfounded.*

Proof. Let us identify the wellfounded part of M with its transitive collapse. In particular, we write α for the ordinal a in the statement of the lemma. Although $(V_\alpha)^M$ need not belong to M, it is Δ_1-definable (with parameters) over M, so given $x \in (V_\alpha)^M$, M can determine the rank of x. Thus, if $x \in^M y$ and $y \in^M (V_\alpha)^M$, then

$$\mathrm{rank}^M(x) < \mathrm{rank}^M(y) < \alpha.$$

Since α is wellfounded, $(V_\alpha)^M$ must be too. □

Lemma 9.3.6. *For every $\alpha < \omega_1$, let T_α be the set of all (real numbers coding) complete and consistent theories T in the language of set theory with additional constants $\{\dot{x}_i : i \in \mathbb{N}\}$ such that*

1. *T extends KP + "$\mathbb{R}$ exists" + $V = L(\mathbb{R}) + \{\dot{x}_i \in \mathbb{R} : i \in \mathbb{N}\}$,*
2. *T contains Skolem schemata of the forms*

$$\exists x \in \mathbb{R}\, \phi(x) \to \phi(\dot{x}_{n(\phi)}),$$
$$\exists x\, \phi(x) \to \exists x\, \exists \beta \in \mathsf{Ord}\, (\theta(\beta, \dot{x}_{m(\phi)}, x) \wedge \phi(x));$$

3. *all models of T contain an ordinal isomorphic to α.*

Then T_α is a Borel subset of $\mathbb{R}$.

Proof. Given a complete and consistent theory T in the language of set theory extending KP, we say that M is a *term model* of T if M is the natural $=$-quotient of the collection of all formulae ϕ with one free variable such that

$$\text{"there is a unique } x \text{ satisfying } \phi(x)\text{"} \in T,$$

with the membership predicate and the constants $\dot{x}_i$ having the obvious interpretations. Suppose that T is as in the statement of the lemma and let M be the term model of T. It follows from the Tarski-Vaught criterion and the fact that M satisfies the Skolem schemata that M is in fact a model of T. Moreover, it embeds into every model of T, so every model of T contains an isomorphic copy of $\alpha + 1$ if, and only if, M does.

Fix some wellordering $\preceq$ of $\mathbb{N}$ of length $\alpha + 1$. Thus, the following are equivalent:

1. $T \in T_\alpha$;
2. there is a term model M of T, and a function f such that
   (a) the range of f is the field of $\preceq$,
   (b) the domain of f is the set of $\in^M$-predecessors of an ordinal of M,
   (c) for all $y \in \mathrm{dom}(f)$, $f(y)$ is the $\preceq$-least number greater than $f(x)$ for every $x \in^M y$;
3. for all term models M of T, there is an M-ordinal $m \in M$ such that for all functions f, if
   (a) the range of f is contained in the field of $\preceq$,
   (b) the domain of f is the set of $\in^M$-predecessors of $(m+1)^M$,
   (c) for all $y \in \mathrm{dom}(f)$, $f(y)$ is the $\preceq$-least number greater than $f(x)$ for every $x \in^M y$,

   then $f(m)$ is the $\preceq$-largest number.

Therefore T_α is Borel. □

Lemma 9.3.7. *ϑ is least such that*

$$L_\vartheta(\mathbb{R}) \models \text{“Borel games on } \mathbb{R} \text{ are determined.”}$$

Proof. Clearly ϑ has this property, for $L_\vartheta(\mathbb{R})$ has initial segments with enough iterated powersets of the real numbers for Martin's proof [Mar75] to go through; moreover, the existence of winning strategies for games on $\mathbb{R}$ is upwards absolute between transitive sets that contain $\mathbb{R}$.

Let us verify that ϑ is the least such ordinal. We need to employ model games such as the one of [MS08]. A similar application of the techniques of [MS08] was used in [AM] to produce iterable models of AD with Woodin cardinals. Fortunately, our situation is simpler inasmuch as we do not need

to construct models with large cardinals (yet); the drawback is that we cannot even ask for wellfoundedness in the payoff. One deals with this issue as in Martin's reversal of Borel determinacy (unpublished, but see the exercise in pp. 53–54 of [Mar] for the case of Π^0_4 games on $\mathbb{N}$).

Let $\alpha < \omega_1$ be an infinite successor ordinal. We consider the following game:

$$\begin{array}{c|cccccc} I & x_0, u_0 & & x_2, u_1 & & \dots \\ II & & x_1, v_0 & & x_3, v_1 & \dots \end{array}$$

Here, Players I and II play a countable sequence of reals $\{x_i\}_{i\in\mathbb{N}}$, as well as two countable sequences $u, v \in 2^{\mathbb{N}}$. Fix an enumeration $\{\phi_i : i \in \mathbb{N}\}$ of all formulae in the language of set theory with added constants $\{\dot{x}_i :\in \mathbb{N}\}$. We ask of the players that:

1. $T_I := \{\phi_i : u_i = 1\}$ is a complete, consistent theory in the expanded language, satisfying the following conditions:
   (a) T_I extends the theory KP + Separation + "$\mathbb{R}$ exists" + DC + $V = L(\mathbb{R})$;
   (b) T_I contains the statement $\dot{x}_i \in \mathbb{R}$ for each $i \in \mathbb{N}$, and it contains the statement $\dot{x}_i(n) = m$ if, and only if, $x_i(n) = m$;
   (c) T_I contains the Skolem schemata
   $$\exists x \in \mathbb{R}\, \phi(x) \to \phi(\dot{x}_{n(\phi)}),$$
   $$\exists x\, \phi(x) \to \exists x\, \exists \beta \in \mathsf{Ord}\, (\theta(\beta, \dot{x}_{m(\phi)}, x) \wedge \phi(x));$$
   (d) We ask that all models of T_I contain an ordinal isomorphic to α. If so, then by the proof of Lemma 9.3.6, there must be a formula ϕ_α in the expanded language that defines the αth ordinal in every model of T_I. We demand then that T_I contain the axiom
   $$\text{"if } \phi_\alpha(x) \text{ holds, then } V_x \text{ exists,"}$$
   as well as
   $$\phi_\alpha(x) \wedge \phi(a) \to L_a(\mathbb{R}) \not\models \mathsf{KP} + \text{Separation} + \text{"}V_x \text{ exists,"}$$
   whenever ϕ is a formula and the formula "the unique a satisfying $\phi(a)$ is an ordinal smaller than α" belongs to T_I.
2. $T_{II} := \{\phi_i : v_i = 1\}$ is a complete, consistent theory in the expanded language with the same requirements as above.

If T_I satisfies the properties above and $\mathcal{M}$ is a model of T_I, then the definable closure of $\{x_i^{\mathcal{M}} : i \in \mathbb{N}\}$ in $\mathcal{M}$ is an elementary substructure of $\mathcal{M}$ isomorphic to the term model of T_I. Denote this model by N_I. In particular, N_I satisfies KP + Separation + "V_α exists" and no proper initial segment of N_I does. Similarly for Player II. If the above conditions are met, Player I wins if, and only if, one of the following holds:

1. N_I is isomorphic to an initial segment of N_{II}; *or*
2. there is an ordinal a of N_I such that $L_a(\mathbb{R})^{N_I}$ is isomorphic to an initial segment of N_{II}, but $L_{a+1}(\mathbb{R})^{N_I}$ is not.

In either clause, we do not demand that the initial segment of N_{II} be an element of N_{II}.

Sublemma 9.3.8. *The winning condition is Borel.*

Proof. Let us momentarily reason in

$$\mathsf{KP} + \text{Separation} + \text{“}\mathbb{R}\text{ exists”} + \mathsf{DC} + V = L(\mathbb{R}).$$

Suppose that $J_\xi(\mathbb{R}) \not\models$“$V_\alpha$ exists,” for every ordinal ξ. Thus, for every ξ, the projectum of $J_\xi(\mathbb{R})$ is strictly smaller than V_α. By Proposition 9.2.2, whenever ξ is an ordinal, there is some

$$a \in V_\alpha \cap J_{\xi+1}(\mathbb{R})$$

such that a codes $J_\xi(\mathbb{R})$. Now, let N_I and N_{II} be as in the definition of the game (i.e., suppose that T_I and T_{II} satisfy the conditions necessary in order for N_I and N_{II} to be defined). Now, N_I and N_{II} might have initial segments that satisfy "V_α exists," but they certainly do not have any initial segments satisfying "$V_{\alpha+1}$ exists," so for every initial segment N of either of N_I or N_{II} of (internal) limit length greater than α, we have

$$\text{for all } \xi \in \mathsf{Ord}^N, \text{ there is } a \in (V_{\alpha+1})^N \text{ coding } (J_\xi(\mathbb{R}))^N.$$

(One could easily get around this issue by removing the condition that the theories satisfy KP + Separation, but we have adpoted the rule because this is the theory used in the definition of ϑ_α.)

Back in the real world, we verify that conditions (1) and (2) are Borel. We will show that it is Borel to check whether the elements of $(V_\alpha)^{N_I}$ are (up to isomorphism) those of $(V_\alpha)^N$, for some initial segment N of $(V_\alpha)^{N_{II}}$. This suffices for (1), by the above remark and by replacing α with $\alpha + 1$.

As in the proof of Proposition 9.2.2, one should bear in mind that $(V_\alpha)^N$ need not be an element of N, but it is definable over N in a Borel way by defining an initial segment of the rank function along a Borel presentation of α; viz., a set $x \in N$ has rank α in N if, and only if, $f(x) = \alpha$ for some (all) function(s) f such that

1. the range of f is $\alpha + 1$,

2. the domain of f is a subset of N containing x and is transitive (so, in particular, $\in^N$ is wellfounded in $\mathrm{dom}(f)$),

3. for all $y \in \mathrm{dom}(f)$, $f(y) = \sup\{f(z) : z \in^N y\}$.

Let f_I (in V) be a function assigning to each set in $(V_\alpha)^{N_I}$ its rank in N_I and f_{II} be defined similarly. Consider the following Ehrenfeucht-Fraïssé–like game:

1. Player II begins by selecting an ordinal $a \in N_{II}$. Define $N = (J_a(\mathbb{R}))^{N_{II}}$. For the rest of the game, the players attempt to determine whether $(V_\alpha)^{N_I}$ is equal to $(V_\alpha)^N$ up to isomorphism.

2. During turn 1, Player I selects some $x_0 \in (V_\alpha)^{N_I}$ or some $y_0 \in (V_\alpha)^N$. In the first case, let ξ_0 be least such that $x_0 \in (V_{\xi_0})^{N_I}$; Player II needs to respond with some $y_0 \in (V_{\xi_0})^N$. In the second case, let ξ_0 be least such that $y_0 \in (V_{\xi_0})^N$; Player II needs to respond with some $x_0 \in (V_{\xi_0})^{N_I}$.

3. During turn $n+1$, assuming x_n, y_n, and ξ_n have been defined, Player I plays some $\xi_{n+1} < \xi_n$ and either some $x_{n+1} \in (V_{\xi_{n+1}})^{N_I}$ such that $x_{n+1} \in^{N_I} x_n$ or some $y_{n+1} \in (V_{\xi_{n+1}})^N$ such that $y_{n+1} \in^{N_{II}} y_n$. In the first case, Player II must respond with some $y_{n+1} \in (V_{\xi_{n+1}})^N$ such that $y_{n+1} \in^{N_{II}} y_n$; in the second case, Player II must respond with some $x_{n+1} \in (V_{\xi_{n+1}})^{N_I}$ such that $x_{n+1} \in^{N_I} x_n$.

4. The first player who cannot make a legal move loses the game.

This game can easily be coded by a clopen game of length ω on $\mathbb{N}$ with parameters α, N_I, N_{II}, f_I, and f_{II}; let us denote it by $EF_0(N_I, N_{II}, f_I, f_{II})$ to emphasize this. If there is an ordinal $a \in N_{II}$ such that $(V_\alpha)^{N_I} = (V_\alpha)^{J_a(\mathbb{R})^{N_{II}}}$, then clearly Player II has a winning strategy. Moreover, if there is no such ordinal, then one can easily use the fact that $(V_\alpha)^{N_I}$ and $(V_\alpha)^{N_{II}}$ are wellfounded to construct a winning strategy for Player I.

There is a problem with this game, viz., that one may also want to consider initial segments of N_{II} not of the form $J_a(\mathbb{R})$ for a an ordinal of N_{II}. For this, we define a game $EF(N_I, N_{II}, f_I, f_{II})$ which is like $EF_0(N_I, N_{II}, f_I, f_{II})$, except that Player I begins by selecting an ordinal a_0 of N_I, after which Player II selects $a \in N_{II}$ and they continue as before to determine whether

$$(V_\alpha)^{J_{a_0}(\mathbb{R})^{N_I}} = (V_\alpha)^{J_a(\mathbb{R})^{N_{II}}}.$$

An argument like the preceding one shows that Player II has a winning strategy in $EF(N_I, N_{II}, f_I, f_{II})$ if, and only if, $(V_\alpha)^{N_I} = (V_\alpha)^N$ for some initial segment N of N_{II}.

Now, one sees that the following are equivalent:

1. the elements of $(V_\alpha)^{N_I}$ are those of $(V_\alpha)^N$, for some initial segment N of $(V_\alpha)^{N_{II}}$,

2. there is sequence of surjective functions $\{f_\beta : \beta \leq \alpha\}$ and an initial segment N of N_{II} such that, identifying the wellfounded parts of N_I and N with their transitive collapses,

   (a) f_0 maps the empty set to the empty set,

   (b) for all $\beta \leq \alpha$, $f_{\beta+1} : (V_\beta)^{N_I} \to (V_\beta)^N$ maps each set b to the set $\{f_\beta(a) : a \in^{N_I} b\}$, in the sense of N,

   (c) for all limit $\beta \leq \alpha$, $f_\beta = \bigcup_{\gamma<\beta} f_\gamma$

3. for every pair of functions f_I and f_{II} assigning respectively to each set in $(V_\alpha)^{N_I}$ and $(V_\alpha)^{N_{II}}$ its rank in N_I and N_{II}, Player I does not have a winning strategy in the game $EF(N_I, N_{II}, f_I, f_{II})$.

Since $EF(N_I, N_{II}, f_I, f_{II})$ is a clopen game, Player I not having a winning strategy is a $\mathbf{\Pi}^1_1$ condition. It follows that condition (1) is Borel.

Similarly, to prove that (2) is Borel, it suffices to show that it is Borel to check whether there is $a \in N_I$ such that

1. $(V_\alpha)^{J_a(\mathbb{R})^{N_I}} \subset (V_\alpha)^{N_{II}}$

2. $(V_\alpha)^{J_{a+1}(\mathbb{R})^{N_I}} \not\subset (V_\alpha)^{N_{II}}$

An argument as above shows that the conjunction of these two statements is Borel. This proves the sublemma. □

We have shown that the payoff set of the game is Borel. Moreover, it is easily won by Player I in V, for she can play the theory of $J_{\vartheta_\alpha}(\mathbb{R}) = L_{\vartheta_\alpha}(\mathbb{R})$ and, e.g., only recursive reals. Recall that ϑ_α is least such that

$$J_{\vartheta_\alpha}(\mathbb{R}) \models \mathsf{KP} + \text{Separation} + \text{“}V_\alpha \text{ exists.”}$$

Moreover, as long as Player II plays by copying the theory chosen by Player I (and plays arbitrary reals), then every strategy that ensures that Player I will win demands that she play the theory of $J_{\vartheta_\alpha}(\mathbb{R})$. Thus, the theory of $J_{\vartheta_\alpha}(\mathbb{R})$ can be easily computed from $\mathbb{R}$ and any winning strategy for Player I.

Now let ξ be an ordinal such that the game is determined in $J_\xi(\mathbb{R})$. Then $J_\xi(\mathbb{R})$ contains a winning strategy τ for some player. Since $J_\xi(\mathbb{R})$ contains all plays of the game, τ is also a winning strategy in V, so it must be a winning strategy for Player I. Hence, we must have $\vartheta_\alpha < \xi$, since τ cannot belong to $J_{\vartheta_\alpha}(\mathbb{R})$. Therefore, if $J_\xi(\mathbb{R})$ contains strategies for the game above for each $\alpha < \omega_1$, then it contains every ordinal below ϑ, which is what was to be shown. □

Lemma 9.3.9. *Let $\xi < \vartheta$. Then there is a prewellordering of $\mathbb{R}$ of length greater than ξ definable by a set in $\mathcal{D}^{\mathbb{R}}\mathbf{\Delta}^1_1$.*

Proof. Choose $\alpha < \omega_1$ such that $\xi < \vartheta_\alpha$ and let $(i, x), (j, y)$ be two pairs in $\mathbb{N} \times \mathbb{R}$. The ordinal ϑ_α will play no role in the game other than bounding the complexity of the payoff set. We consider a two-player game $G(\alpha, i, x, j, y)$ given by

I	x_0, u_0					x_2, u_1					$\ldots$
II			x_1, v_0					x_3, v_1			$\ldots$
I					y_1, t_0					y_3, t_1	$\ldots$
II		y_0, s_0					y_2, s_1				$\ldots$

Here, I and II play two runs of the game from Lemma 9.3.7 in parallel, except that their roles are reversed in the second game. As before, we suppose fixed an enumeration $\{\phi_i : i \in \mathbb{N}\}$ of all formulae in the language of set theory with added constants $\{\dot{x}_i : i \in \mathbb{N}\}$. The rules are:

1. If ϕ_j is not a formula with one free variable and with no constant symbols other than $\dot{x}_0$, then Player II loses.

2. If ϕ_i is not a formula with one free variable and with no constant symbols other than $\dot{x}_0$, then Player I loses.

3. Both players have to obey the rules in each subgame.

4. If so, then Player II has to win the second subgame. Moreover, she must set $y_0 = y$ and the theory played must contain the formula "there is a unique ordinal satisfying ϕ_j."

5. If so, then Player I has to win the first subgame. Moreover, she must set $x_0 = x$ and the theory played must contain the formula "there is a unique ordinal satisfying ϕ_i."

If all rules thus far have been obeyed, then we declare the winner according to the following rules (here, we identify the theories played with their term models):

1. Every real played in the first subgame must be played in the second subgame by Player II; otherwise, Player I wins.

2. Every real played in the second subgame must be played in the first subgame by Player I; otherwise, Player II wins.

3. If no winner has been declared thus far, let N_I be the winning model from the first subgame and N_{II} be the winning model from the second subgame (both with α in their wellfounded parts). Then,

   (a) We demand that N_I be isomorphic to an initial segment of N_{II}; otherwise, Player II wins.

   (b) Let a_I be the unique ordinal in N_I satisfying ϕ_i and let a_{II} be the unique ordinal in N_{II} satisfying ϕ_j. Then, Player I wins if for every (some) isomorphism j from N_I to an initial segment N of N_{II}, we have
   $$N_{II} \models j(a_I) < a_{II}.$$
   If not, then Player II wins.

Let ϑ^*_α be the order-type of the set of ordinals definable in $L_{\vartheta_\alpha}(\mathbb{R})$ from a real. Given pairs $(i, x), (j, y)$ as above, if there is not a unique ordinal $\eta < \vartheta_\alpha$ such that
$$L_{\vartheta_\alpha}(\mathbb{R}) \models \phi_j(y, \eta),$$
then Player I has a winning strategy in the game obtained by playing the theory of $L_{\vartheta_\alpha}(\mathbb{R})$ in both subgames. This way, Player I is guaranteed to win the first subgame and Player II is forced to play the theory of $L_{\vartheta_\alpha}(\mathbb{R})$ in the

second subgame, which will cost her the game. Similarly, if η above exists and there is not a unique ordinal $\gamma < \vartheta_\alpha$ such that

$$L_{\vartheta_\alpha}(\mathbb{R}) \models \phi_i(x, \gamma),$$

then Player II has a winning strategy in the game. Otherwise, let γ and η be as above. We claim that

$$\gamma \leq \eta \text{ if, and only if, Player I has a winning strategy in the game.} \quad (9.1)$$

If so, then the game defines a prewellordering of $\mathbb{R}$ of length greater than ϑ^*_α, in which the pairs of highest rank are those whose first coordinate fails the syntactic condition (1)–(2) and the pairs of second-highest rank are those not defining an ordinal in $L_{\vartheta_\alpha}(\mathbb{R})$, i.e., those failing conditions (4)–(5). The facts that condition (1) is checked before condition (2) and that condition (4) is checked before (5) will ensure that the game defines a reflexive relation. Let us now proceed to the proof of equation (9.1).

Suppose $\gamma \leq \eta$. A winning strategy for Player I is defined by setting $x_0 = x$ as appropriate and playing the theory of $L_{\vartheta_\alpha}(\mathbb{R})$ in the first subgame together with all reals played in the second subgame. In the second subgame, Player I also plays the theory of $L_{\vartheta_\alpha}(\mathbb{R})$. This strategy ensures that Player I will win the first subgame, as in Lemma 9.3.7. Moreover, if Player II wishes to have any chance of winning the second subgame, she must also play the theory of $L_{\vartheta_\alpha}(\mathbb{R})$. If so, then this will define two models N_I and N_{II} as in the rules of the game, both wellfounded. Either there will be a real played in the first game and not in the second—in which case Player I wins—or both plays will contain the same reals. If so, the models N_I and N_{II} both embed into $L_{\vartheta_\alpha}(\mathbb{R})$ and are hence wellfounded, so their transitive collapses are of the forms $L_{\alpha_I}(\sigma)$ and $L_{\alpha_{II}}(\sigma)$, for some countable set of reals σ. Since each model satisfies the formulae:

1. KP + Separation,
2. "V_α exists," and
3. "for all $\eta \in \mathsf{Ord}$, $L_\eta(\mathbb{R}) \not\models \mathsf{KP}$ + Separation + 'V_α exists,' "

they must be equal. If Player II has not lost at this point, then we must have $y_0 = y$. The fact that $L_{\alpha_I}(\sigma)$ embeds elementarily into $L_{\vartheta_\alpha}(\mathbb{R})$ implies that the ordinal defined by ϕ_i and x must be at most the ordinal defined by ϕ_j and y in those models, as desired.

Conversely, suppose that Player I has a winning strategy in the game. We consider a run of the game by that strategy in which Player II plays the

theory of $L_{\vartheta_\alpha}(\mathbb{R})$ in the first subgame, as well as in the second, where she also plays all reals from the first subgame. This ensures winning the second subgame and losing the first one. As before, this forces Player I to play the theory of $L_{\vartheta_\alpha}(\mathbb{R})$ in the first game, so the play defines two wellfounded models with the same reals and the same theory, whose transitive collapses must therefore be equal. Since the play must be won by Player I, the ordinal defined by ϕ_i and x must be at most the ordinal defined by ϕ_j and y.

This proves the claim and, together with the remark following equation (9.1) shows that the relation $\preceq$ given by

$$(i,x) \preceq (j,y) \text{ if, and only if, Player I has a winning strategy in } G(\alpha,i,x,j,y)$$

is a prewellordering of length greater than ϑ^*_α. A standard Skolem-hull argument shows that if $\beta < \alpha$, then $\vartheta_\beta < \vartheta^*_\alpha$. Using the fact that the game from Lemma 9.3.7 is Borel, one sees that for every α, $G(\alpha,i,x,j,y)$ is uniformly in $\Delta^1_1(\alpha,\langle i,x\rangle,\langle j,y\rangle)$, which completes the proof. □

Before proving the theorem, we need two more lemmata.

Lemma 9.3.10. *Let $\omega+1<\alpha<\omega_1$. The pointclass $(\mathbf{\Sigma}^2_1)^{L_{\vartheta_\alpha}(\mathbb{R})}$ has the scale property.*

Proof. Let $\gamma = \vartheta_\alpha$. Since $L_\gamma(\mathbb{R})$ is a model of Σ_1-separation, γ is $\mathbb{R}$-nonprojectible, in the sense that there is no total function which is $(\Sigma_1)^{L_\gamma(\mathbb{R})}$ with parameters in $L_\gamma(\mathbb{R})$ and which maps γ injectively into some $\beta<\gamma$. The usual argument for L (see Barwise [Bar75, Theorem 6.8]) shows that γ is recursively $\mathbb{R}$-inaccessible, i.e., $\mathbb{R}$-admissible and a limit of $\mathbb{R}$-admissibles.

To prove the lemma, we first verify that the proof of Martin-Steel [MS08, Lemma 1] goes through in $L_\gamma(\mathbb{R})$ and shows that every $(\supset^{\mathbb{R}}\mathbf{\Pi}^1_1)^{L_\gamma(\mathbb{R})}$ set[4] admits a closed game representation in $L_\gamma(\mathbb{R})$. To see this, first observe that every wellfounded tree T in $L_\gamma(\mathbb{R})$ has a rank, since γ is a limit of $\mathbb{R}$-admissibles. This observation ensures that the proof of the claim in the proof of [MS08, Lemma 1] holds in $L_\gamma(\mathbb{R})$ so that (using the notation from [MS08]) Player I has a winning strategy in G_x if, and only if, she has one in $G^*_{x,\xi}$ for some ξ. Now, each $G^*_{x,\xi}$ is a closed game and $G^*_{x,\xi}\in L_{\xi+1}(\mathbb{R})$. By standard arguments, if Player I has a winning strategy in $G^*_{x,\xi}$, she has one in $L_{\xi^*+1}(\mathbb{R})$ whenever $\xi<\xi^*$ and ξ^* is $\mathbb{R}$-admissible. Clearly, $\Theta^{L_\gamma(\mathbb{R})}$ is also recursively $\mathbb{R}$-inaccessible and, by the Löwenheim-Skolem theorem,

$$L_{\Theta^{L_\gamma(\mathbb{R})}}(\mathbb{R}) \prec_1 L_\gamma(\mathbb{R}),$$

[4] In the choiceless context, $\supset^{\mathbb{R}}$ is to be understood in terms of quasi-strategies.

so it follows that if $x \in \mathbb{R}$ and ξ is least such that Player I has a winning strategy in $G^*_{x,\xi}$, then

$$\xi < \Theta^{L_\gamma(\mathbb{R})}.$$

Denote this least ξ by ξ_x and let λ be the supremum of all ξ_x such that ξ_x is defined; thus, $\lambda \leq \Theta^{L_\gamma(\mathbb{R})}$. This shows that the map $x \mapsto G^*_{x,\lambda}$ belongs to $L_\gamma(\mathbb{R})$. One finishes the argument as in [MS08, Lemma 1].

The statement of [MS08, Lemma 3] also holds true in $L_\gamma(\mathbb{R})$, i.e.,

$$(\supset^{\mathbb{R}} \Pi^1_1)^{L_\gamma(\mathbb{R})} = \Sigma_1(L_\gamma(\mathbb{R}), \{\mathbb{R}\}).$$

Here, the proof needs no modifications other than weakening condition (2) to only require that the model satisfy (say) KP, as opposed to ZF^-. This shows, using the proof of [MS08, Theorem 4], that the pointclass

$$\Sigma_1(L_\gamma(\mathbb{R}), \{\mathbb{R}\})$$

has the scale property. Now the proof of Solovay's basis theorem (e.g., the one in Section 2 of Koellner-Woodin [KW10], replacing their T_0 with some strong-enough theory that holds in $L_{\Theta^{L_\gamma(\mathbb{R})}}(\mathbb{R})$) shows that

$$\Sigma_1(L_\gamma(\mathbb{R}), \{\mathbb{R}\}) = (\Sigma^2_1)^{L_\gamma(\mathbb{R})},$$

from which the lemma follows. □

Lemma 9.3.11. *Suppose that sets in $\supset^{\mathbb{R}} \mathbf{\Delta}^1_1$ are determined. Then, Borel games of length ω^2 are determined.*

Proof. This argument is a local version of the one in Blass [Bla75]. Consider the game on $\mathbb{R}$ of length ω in which two players alternate turns playing strategies for games of length ω:

$$\begin{array}{c|cccccc} I & \sigma_0 & & \sigma_1 & & \dots \\ II & & \tau_0 & & \tau_1 & \dots \end{array}$$

Player I wins if, and only if,

$$(\sigma_0 * \tau_0, \sigma_1 * \tau_1, \dots) \in A,$$

where $\sigma * \tau$ denotes the result of playing the strategies σ and τ against each other.

Call this game G. It is a Borel game (on $\mathbb{R}$), so it is determined. If it is Player I who wins this game, then Player I also wins the game of length

ω^2 with payoff A. Suppose that it is Player II who wins this game; we shall construct a winning strategy for the game of length ω^2 with payoff A for Player II.

Let Σ denote the winning strategy for Player II in G. Σ is essentially a set of reals. By the remark at the beginning of the proof of Lemma 9.3.7, we may assume

$$\Sigma \in L_\vartheta(\mathbb{R}).$$

Recall that $\vartheta = \sup\{\vartheta_\alpha : \alpha < \omega_1\}$. By the minimality of ϑ_α and the $\mathbb{R}$-stability of δ^2_1, we have

$$\vartheta_\alpha < (\delta^2_1)^{\vartheta_{\alpha+1}}$$

for each $\alpha < \omega_1$. This implies by Lemma 9.3.10 that $\mathcal{P}(\mathbb{R}) \cap L_\vartheta(\mathbb{R})$ has the scale property. Since it is closed under quantification by $\mathbb{R}$, Theorem D of [Mos71] implies that it has the uniformization property.

Denote by A^* the set of all sequences p of natural numbers of length $\omega \cdot n$, for some $n \in \mathbb{N}$, such that there is a sequence

$$\vec{\tau} \in \mathbb{R}^n$$

such that p results from applying Σ to $\vec{\tau}$. Put

$$A^*_n = A^* \cap \mathbb{R}^n.$$

For any given $p \in A^*_1$, there might be many strategies τ for Player I witnessing $p \in A^*$; however, A^*_1 is definable from Σ, so it belongs to $L_\vartheta(\mathbb{R})$. Applying the uniformization property, one finds a function

$$f_1 : A^*_1 \to \mathbb{R}$$

such that for each $p \in A^*_1$, $f_1(p)$ is a strategy τ witnessing $p \in A^*$ and, moreover, $f_1 \in L_\vartheta(\mathbb{R})$. Inductively, suppose f_n has been defined. For any given $p \in A^*_{n+1}$, there might be many strategies τ such that $f_n(p \restriction \omega \cdot n)^\frown\tau$ witnesses $p \in A^*$, or none at all; however, A^*_{n+1} is definable from Σ, so it belongs to $L_\vartheta(\mathbb{R})$. Applying the uniformization property, one finds a function

$$f_{n+1} : A^*_{n+1} \to \mathbb{R}^{n+1}$$

with $f_{n+1} \in L_\vartheta(\mathbb{R})$ and such that the following hold for each $p \in A^*_{n+1}$ for which such a τ exists:

1. $f_{n+1}(p)$ is a sequence $\vec{\tau}$ witnessing $p \in A^*$; and
2. $f_{n+1}(p) \restriction m = f_m(p \restriction \omega \cdot m)$ for every $m \leq n+1$.

Let us also define a partial function $f_\omega : \mathbb{R}^\mathbb{N} \to \mathbb{R}^\mathbb{N}$ by

$$f_\omega(p) = \bigcup_{n\in\mathbb{N}} f_n(p \restriction n),$$

if the right-hand side is defined, and put

$$f = \bigcup_{n\leq\omega} f_n.$$

Let us call a sequence p of natural numbers of length $\omega \cdot n$, with $n \in \mathbb{N}$, *promising* if for every $m \leq n$, $p \restriction \omega \cdot m \in A^*$ and this is witnessed by $f(p) \restriction m$. Given a promising sequence p, we shall construct a strategy σ^* for Player II (for a game of length ω on $\mathbb{N}$) such that for every $x \in \mathbb{R}$, $p^\frown(\sigma^*(x))$ is promising. Thus, if p^* is a run of a game of length ω^2 all of whose initial segments of limit length were obtained in this way, then it is a play all of whose initial segments are promising. This means that

$$p^* = f(p^*) * \Sigma(f(p^*)),$$

which implies that $p^* \in A$.

Now, let $p \in \mathbb{R}^n$ be promising and consider the game G^* on $\mathbb{N}$ of length ω:

$$\begin{array}{c|ccccc} I & x(0) & & x(2) & & \dots \\ II & & x(1) & & x(3) & \dots \end{array}$$

Player II wins if, and only if, $p^\frown x$ is promising. We first show that Player I cannot have a winning strategy. Suppose towards a contradiction that τ is a winning strategy for Player I. Thus, if Player I begins a run of G by playing

$$f(p)^\frown\tau,$$

during the first $n+1$ moves and Player II plays by Σ, then she will begin by responding with n moves according to Σ, resulting in a sequence of strategies $\sigma_0, \dots, \sigma_{n-1}$ so that

$$p = (f(p)(0) * \sigma_0, \dots, f(p)(n-1) * \sigma_{n-1})$$

and continue responding with one last move by Σ, say σ. Put $x = \tau * \sigma$. Since the play of G just described is consistent with Σ, we have $p^\frown x \in A^*$. Although it need not be the case that $f(p^\frown x)(n) = \tau$, the construction of f ensures that $f(p^\frown x)$ is a sequence $\vec{\tau}$ witnessing $p^\frown x \in A^*$ and $f(p^\frown x) \restriction n = f(p)$. Thus, the run of G^* in which Player I plays by τ and Player II plays

by σ results in a play x such that $p^\frown x$ is promising, which is a contradiction. Hence, Player I cannot have a winning strategy in G^*. We will show that the payoff set of G^* belongs to $\mathcal{D}^{\mathbb{R}}\mathbf{\Delta}^1_1$; this will imply that it is determined and thus that Player II has a winning strategy, which will finish the proof.

The payoff set of G^* (for Player II) consists of all $x \in \mathbb{R}$ such that $p^\frown x$ is promising. A sequence being promising is defined in terms of Σ and f. Now, each f_n belongs to $L_\vartheta(\mathbb{R})$ and, since

$$\operatorname{cof}(\vartheta) = \omega_1,$$

there is some $\alpha < \omega_1$ such that each f_n belongs to $L_{\vartheta_\alpha}(\mathbb{R})$. Since

$$\left(L_{\Theta^{L_{\vartheta_\alpha}(\mathbb{R})}}(\mathbb{R})\right)^\omega \subset L_{\vartheta_\alpha}(\mathbb{R}),$$

we have $f \in L_{\vartheta_\alpha}(\mathbb{R})$. By choosing α large enough so that $\Sigma \in L_{\vartheta_\alpha}(\mathbb{R})$, we ensure that

$$\{p \in \mathbb{R}^n : p \text{ is promising and } n \in \mathbb{N}\} \in L_{\vartheta_\alpha}(\mathbb{R}).$$

By Lemma 9.3.9, $\vartheta \leq \gamma^1_1$, so that, by Lemma, 9.3.1, the set in the displayed equation belongs to $\mathcal{D}^{\mathbb{R}}\mathbf{\Delta}^1_1$, which completes the proof. □

Woodin has shown that $\mathbb{R}^\sharp$ exists and AD holds in $L(\mathbb{R})$ if, and only if, there is a countably iterable proper-class model of ZFC with infinitely many Woodin cardinals. Neeman [Nee04] has shown that if such a model exists, then all analytic games of length ω^2 are determined (incidentally, the converse follows from a theorem of Trang [Tra13] and a theorem of Martin-Steel [MS08], after appealing to Woodin's theorem). Let us remark that the argument just given yields a direct proof of analytic determinacy for games of length ω^2 which involves no inner model theory.

Theorem 9.3.12. *(Neeman-Woodin) Suppose that $\mathbb{R}^\sharp$ exists and $L(\mathbb{R}) \models \mathsf{AD}$. Then, analytic games of length ω^2 are determined.*

Proof. Given an analytic set A, let G be the game defined as in the proof of Lemma 9.3.11. It is an analytic game on $\mathbb{R}$ and, if won by Player I, then the game of length ω^2 with payoff A is also won by Player I. Since $\mathbb{R}^\sharp$ exists, then analytic games on $\mathbb{R}$ are determined, by the argument in Martin [Mar70], so if Player I does not win A, then Player II does.

Since AD holds in $L(\mathbb{R})$, all sets in $\mathcal{D}^{\mathbb{R}}\mathbf{\Pi}^1_1$ are determined. Moreover, the pointclass $\mathcal{D}^{\mathbb{R}}\mathbf{\Pi}^1_1$ is correctly computed in $L(\mathbb{R})$, so if Player II wins an analytic game on $\mathbb{R}$ (this is a $\mathcal{D}^{\mathbb{R}}\mathbf{\Pi}^1_1$ fact), then she has a winning strategy which

is $(\mathbf{\Delta}^2_1)^{L(\mathbb{R})}$. Hence, the game G^* defined as in the proof of the preceding lemma belongs to $L(\mathbb{R})$ and is thus determined. The rest of the argument is as before. □

Theorem 9.3.4 now follows: if Borel games of length ω^2 are determined, then all sets in $\mathfrak{D}^{\mathbb{R}}\mathbf{\Delta}^1_1$ are determined (in the usual sense). By Lemma 9.3.9, $\vartheta \leq \gamma^1_1$, so

$$L_\vartheta(\mathbb{R}) \models \mathsf{AD} \tag{9.2}$$

by Lemma 9.3.1. Conversely, suppose (9.2) holds. By Lemma 9.3.7 and the fact that the existence of winning strategies for games on $\mathbb{R}$ is upwards absolute from initial segments of $L(\mathbb{R})$ (since they contain all possible plays of the game), we have

$$\mathfrak{D}^{\mathbb{R}}\mathbf{\Delta}^1_1 \subset L_\vartheta(\mathbb{R}). \tag{9.3}$$

Thus, sets in $\mathfrak{D}^{\mathbb{R}}\mathbf{\Delta}^1_1$ are determined (in the usual sense). By the previous lemma, this implies that games of length ω^2 with Borel payoff are determined. Theorem 9.3.4 is thus proved.

Corollary 9.3.13. $\gamma^1_1 = \vartheta$.

Proof. Immediate from Lemma 9.3.1, Lemma 9.3.9, and the observation that Lemma 9.3.7 implies equation (9.3). Note that the three lemmata were proved under no determinacy hypotheses. □

9.4 Proof of Lemma 9.3.1

Let $\alpha < \gamma^1_1$ be an ordinal; we show

$$\mathcal{P}(\mathbb{R}) \cap L_{\alpha+1}(\mathbb{R}) \subset \mathfrak{D}^{\mathbb{R}}\mathbf{\Delta}^1_1.$$

The proof consists in defining a game that determines membership in a set definable over $L_\alpha(\mathbb{R})$; we call it the *definability game*. Readers familiar with Tanaka's [Tan90] proof of Steel's [Ste77] classical result on the equivalence between arithmetical transfinite recursion and clopen determinacy should find similarities.

Fix some prewellordering $\preceq$ of $\mathbb{R}$ in $\mathfrak{D}^{\mathbb{R}}\mathbf{\Delta}^1_1$ of length $\alpha + 1$ or greater. To ease notation, assume that $\preceq$ is defined without real parameters. The definability game will involve playing *representations* of elements of $L_\alpha(\mathbb{R})$. These are defined inductively: pairs $(0, x)$, where $x \in \mathbb{R}$, are representations for x. Let $\beta + 1 \leq \alpha$, $b \in L_\beta(\mathbb{R})$ and

$$B = \{x \in L_\beta(\mathbb{R}) : L_\beta(\mathbb{R}) \models \hat{\phi}(x, b)\}$$

be an element of $L_{\beta+1}(\mathbb{R})$. If x_b codes a representation (ψ, y, y_b) of b and $|y|_\preceq < \beta$, then the triple

$$(\hat{\phi}, x_\beta, x_b), \tag{9.4}$$

is a representation of B, whenever x_β is any real of rank β in $\preceq$.[5]

Lemma 9.4.1. *Let α and $\preceq$ be as above. Then, every set in $L_\alpha(\mathbb{R})$ has a representation.*

Proof. By induction on ordinals $\beta \leq \alpha$. □

Let us also inductively define a *proto-representation* to be either a pair $(0, x)$ as above, or a triple as in (9.4) whose first coordinate is a formula and third coordinate is a proto-representation, i.e., one drops the requirement on the $\preceq$-rank of the second coordinate.

Being a proto-representation is easy to check—the class of all proto-representations is Σ^0_1. Being a genuine representation, on the other hand, is more complicated, as, if $(\hat{\phi}, x_\beta, x_b)$ is as in (9.4), then one needs to check that x_b in fact codes a representation for an element of $L_\beta(\mathbb{R})$.

In principle, one can have highly inefficient representations in which one defines simple objects in a complicated way. We will denote by Rep_α the set of codes of all representations of the form $(0, x)$ and of the form $(\hat{\phi}, x_\beta, x_b)$, where x_β has rank $<\alpha$ in $\preceq$. Belonging to Rep_α does not guarantee that a representation is optimal, but it guarantees that, in a sense, it is not "more complicated than α."

Lemma 9.4.2. *Let $\alpha < \gamma^1_1$, $\preceq$, and Rep_α be as above. If x_α has rank α in $\preceq$, then $\mathrm{Rep}_\alpha \in \mathfrak{I}^{\mathbb{R}}\Delta^1_1(x_\alpha)$.*

Proof. It is easy to determine whether a real number codes a representation of the form $(0, y)$, so let us consider only proto-representations which are triples.

For a triple (ψ, x, y) to be a representation of an element of $L_\alpha(\mathbb{R})$ in Rep_α, it is necessary and sufficient that the following two conditions hold:

1. $x \prec x_\alpha$,

2. y codes a representation $(\hat{\psi}, x_\gamma, x_c)$ and $x_\gamma \prec x$.

[5]Strictly speaking, we should allow x_b to code finite tuples of representations, but we shall generally abuse notation by assuming that these tuples have length 1.

Given $a \in \mathbb{R}$, we define a game $G(a)$ on reals with payoff in $\Delta^1_1(x_\alpha)$. In this game, Player I claims that a codes a representation, and Player II claims otherwise.

If a does not code a triple, the game ends immediately and Player I loses. Otherwise, suppose a codes a triple (ψ, x, y) as above. Player II must begin by challenging one of the two conditions (1) or (2) above. If Player II decides to challenge condition (1), then they must play the game with payoff in $\Delta^1_1(x, x_\alpha)$ determining whether

$$x \prec x_\alpha;$$

if so, the winner of this subgame is the winner of G. If Player II decides to challenge condition (2), then Player II must provide one of the following three reasons:

1. y does not code a triple nor a pair $(0, \hat{y})$, where $\hat{y} \in \mathbb{R}$;
2. y codes a triple $(\hat{\psi}, x_\gamma, x_c)$, but $x \preceq x_\gamma$;
3. y codes a triple $(\hat{\psi}, x_\gamma, x_c)$, but x_c is not a representation.

If Player II claims that y does not code a triple or a pair $(0, \hat{y})$, then the game ends and Player I wins if, and only if, y codes a triple or a pair $(0, \hat{y})$. If Player II claims that $x \preceq x_\gamma$, then, as above, they must play the game with payoff in $\Delta(x, x_\gamma)$ determining whether

$$x_\gamma \prec x.$$

Finally, if Player II claims that y codes a triple $(\hat{\psi}, x_\gamma, x_c)$, but x_c is not a representation, then she again must provide a reason, and so on. If Player II correctly identifies the problems with Player I's purported representations throughout the game, then she must initiate a subgame after finitely many turns (because $\prec$ is wellfounded). Hence, we add as a rule that if the game finishes after infinitely many turns without Player II having initiated a subgame, she loses. We note that, although different subgames can be initiated by Player II, they are all Borel, and in fact they are so in a uniform way, i.e., they are all instances of the same game (namely, the one for determining whether $u \prec v$), and only their parameters vary. This uniformity is necessary for ensuring that the payoff of $G(a)$ is also Borel.

Now, Player I wins a run p of the game if, and only if, one of the following holds:

1. One of the players violates a rule and the first one to do so is Player II;

2. The players obey the rules and for all n, Player II does not initiate a subgame in the nth turn;

3. The players obey the rules, there is a least n such that Player II initiates a subgame in the nth turn, and $p \restriction (n, \infty)$ is a win for Player I in this game.

The first condition is $\Sigma^0_1(a, p, x_\alpha)$, the second one is $\Pi^0_1(a, p)$, and the third one is $\Delta^1_1(a, p)$. Thus, the winning condition is in $\Delta^1_1(a, p, x_\alpha)$. Moreover, if (ψ, x, y) is a representation of an element of $L_\alpha(\mathbb{R})$, then any challenge raised by Player II can easily be overcome by Player I, so she must have a winning strategy. Similarly, if (ψ, x, y) is not a representation of an element of $L_\alpha(\mathbb{R})$, then Player II can identify the reason why this is the case and pose a challenge Player I cannot overcome. Therefore, Player I has a winning strategy in this game if, and only if, a indeed codes a representation of an element of $L_\alpha(\mathbb{R})$ in Rep_α. We have shown that the set of codes of representations of elements of $L_\alpha(\mathbb{R})$ is the set of all a for which Player I has a winning strategy in the Borel game $G(a)$, as was to be shown. □

Remark 9.4.3. The fact that all auxilary games were Borel in a uniform way was crucial in the proof of Lemma 9.4.2. In fact, the set $\bigcup_{\alpha<\gamma^1_1} Rep_\alpha$ is not in $\Im^{\mathbb{R}}\mathbf{\Delta}^1_1$. □

Definition 9.4.4. The *order* of a proto-representation is defined inductively: the order of a proto-representation of the form $(0, x)$ is 0; the order of a proto-representation of the form (ϕ, x_β, x_b) is the maximum of $|x_\beta|_\prec$ and the order of x_b.

Thus, if $x \in \mathrm{Rep}_\alpha$, then the order of x is strictly less than α.

We now proceed to the definability game and, with it, to the proof of Lemma 9.3.1. The idea of the game is very simple: two players argue whether a formula holds of some sets in a given level of the $L(\mathbb{R})$-hierarchy. However, its description is lengthy. This is due to two complications that arise: the first one is that we need the payoff of the game to be $\mathbf{\Delta}^1_1$. This requires us speaking about representations of sets instead of sets directly. The second one is that many rules need to be put into place to ensure that the players are honest in their moves.

Fix some $n \in \mathbb{N}$, a parameter $x_A \in L_\alpha(\mathbb{R})$, and a set

$$A \in \Sigma_n(L_\alpha(\mathbb{R}), x_A),$$

say, $A = \{y \in L_\alpha(\mathbb{R}) : L_\alpha(\mathbb{R}) \models \phi(y, x_A)\}$. In the definability game, Player I attempts to show that a given $y \in L_{\gamma_1^1}(\mathbb{R})$ belongs to the set A above, i.e., that

$$L_\alpha(\mathbb{R}) \models \phi(y, x_A) \tag{9.5}$$

We assume that the scopes of all negations in ϕ (and in all formulae appearing below) are atomic formulae. We also assume that no implications appear in ϕ. The game will involve Players I and II playing purported representations of sets in $L_\alpha(\mathbb{R})$. Fix some representation (ϕ, x_α, x_a) for A and some real x^* coding it. Let $a \in \mathbb{R}$; we define the game $D_A(a)$. If a does not code a pair $(0, y)$ or a triple (ϕ_y, x_y, x_c), the game ends immediately and Player I loses. Otherwise, the game begins. We will describe the rules in the case that a codes a triple; the rules in the case that a codes a pair $(0, y)$ are analogous; soon, we will restrict our attention to reals a of order $<\alpha$ (as the reader will soon realize, this restriction postpones a certain complication in the definition of the game). Player I's goal is to convince Player II that the triple coded by a is a representation for some y such that equation (9.5) is satisfied. The game is defined from x^* and a (as well as the parameters defining $\preceq$, which we assumed for simplicity to be non-existent).

The rules of the game give the players opportunities to initiate subgames in which they challenge the other's moves. If this happens at any point, the definability game will end and the winner will be declared to be the winner of the subgame. We do not need to speak about winning strategies for the subgame—we can simply ask the players to play it. All subgames will have Borel payoff and real moves; this allows the players to check during the game whether a given $\Im^{\mathbb{R}}\mathbf{\Delta}^1_1$ fact holds.

At the beginning of of $D_A(a)$, Player II has the oportunity to claim that

$$(\phi_y, x_y, x_c) \text{ is not a representation of an element of } L_\alpha(\mathbb{R}),$$

i.e., that $(\phi_y, x_y, x_c) \notin \mathrm{Rep}_\alpha$. If this happens, then the game is over and Player I wins if, and only if, the challenged object is indeed a representation; otherwise Player II wins. If Player II does not claim either of the two statements above, the game proceeds.

At the beginning of turn k, Players I and II have defined a proto-representation $(\phi_k, x_{\beta^k}, x_{b^k})$ for some set B^k and finitely many proto-representations

$(\psi_{w_i^k}, x_{w_i^k}, x_{d_i^k})$ for some sets w_i^k. Player I claims that $\vec{w}^k \in B^k$, i.e., that

$$L_{\beta^k}(\mathbb{R}) \models \phi_k(\vec{w}^k, b^k);$$

Player II claims otherwise. Below, we write $\vec{w} = \vec{w}^k$ and $b = b^k$ for notational simplicity; as before, we might abuse notation by assuming $\vec{w}$ is a tuple of length 1 and simply writing w. We define the *order* of the game at turn k to be the maximum of the orders of w and b. Let us describe the rules of the game by cases:

1. If ϕ_k is of the form $\psi \vee \chi$, then Player I selects either ψ or χ. We set ϕ_{k+1} equal to that choice and leave the rest of the objects unchanged.

2. If ϕ_k is of the form $\psi \wedge \chi$, then Player II selects either ψ or χ. We set ϕ_{k+1} equal to that choice and leave the rest of the objects unchanged.

3. If $\phi_k(w, b)$ is of the form $\exists z\, \psi(w, b, z)$, then Player I must play a proto-representation x_z which is allegedly a representation in Rep_{β^k} of some $z \in L_{\beta^k}(\mathbb{R})$ such that

   $$L_{\beta^k}(\mathbb{R}) \models \psi(w, b, z).$$

   At this point, Player II has the opportunity to object to the fact that x_z belongs to Rep_{β^k}. If so, the game ends and Player I wins if, and only if, $x_z \in \mathrm{Rep}_{\beta^k}$. If Player II does not challenge Player I's move, then the game continues with $\phi_{k+1} = \psi$ and z added to $\vec{w}$; the rest of the objects remain unchanged.

4. If $\phi_k(w, b)$ is of the form $\forall z\, \psi(w, b, z)$, then Player II must play a proto-representation x_z which is allegedly a representation in Rep_{β^k} of of some $z \in L_{\beta^k}(\mathbb{R})$ such that

   $$L_{\beta^k}(\mathbb{R}) \not\models \psi(w, b, z).$$

   As before, Player I has the opportunity to object to the fact that x_z belongs to Rep_{β^k}. If so, the game ends and Player I wins if, and only if, $x_z \notin \mathrm{Rep}_{\beta^k}$. If Player I does not challenge Player II's move, then the game continues with $\phi_{k+1} = \psi$ and z added to $\vec{w}$; the rest of the objects remain unchanged.

5. If ϕ_k is a non-negated atomic formula, then it is of the form $u \in v$, where u and v are any of b or some w_i; both u and v have associated proto-representations that have been played in one of the previous turns, or given by the initial data. There are three subcases that we need to distinguish:

(a) Suppose that the proto-representation associated to v is of the form $(0, x)$, for some $x \in \mathbb{R}$, and the proto-representation associated to u is of the form $(0, y)$, for some $y \in \mathbb{R}$; then the game ends. Player I wins if, and only if, $y \in x$; otherwise, Player II wins.

(b) Suppose that the proto-representation associated to v is of the form $(0, x)$, for some $x \in \mathbb{R}$, but the one associated to u is of the form

$$(\phi_u, x_{\eta_u}, x_u).$$

The problem is that—in principle—$(\phi_u, x_{\eta_u}, x_u)$ might be a complicated (true) representation of a simple set.[6] The game proceeds as follows: since the proto-representation associated to v is of the form $(0, x)$, it is a true representation of some real number. Player I must thus play a natural number $n \in x$ and claim that $(\phi_u, x_{\eta_u}, x_u)$ is a representation of n, i.e., that, there is $u' \in L_{\eta_u}(\mathbb{R})$ represented by x_u such that

$$L_{\eta_u}(\mathbb{R}) \models \phi_u(u', m) \text{ if, and only if, } m < n.$$

Player II must object by playing some $m_0 \in \mathbb{N}$ such that one of the following holds:

i. $m_0 < n$ but $L_{\eta_u}(\mathbb{R}) \not\models \phi_u(u', m_0)$; or
ii. $n < m_0$ and $L_{\eta_u}(\mathbb{R}) \models \phi_u(u', m_0)$.

During the remainder of the game, the two players must determine whether

$$L_{\eta_u}(\mathbb{R}) \models \phi_u(u', m_0).$$

We know how to do this: namely, we set B^{k+1} equal to

$$\{m \in \mathbb{N} : L_{\eta_u}(\mathbb{R}) \models \phi_u(u', m)\}$$

or

$$\{m \in \mathbb{N} : L_{\eta_u}(\mathbb{R}) \models \neg\phi_u(u', m)\},$$

according to which objection was raised by Player II, and work with the canonical representation for B^{k+1}; and $\vec{w}^{k+1} = (0, m_0)$.

(c) Suppose that v has an associated proto-representation of the form

$$(\chi, x_\eta, x_{a_v}).$$

[6] Originally introduced into the game by either player.

This case is similar to the previous one: Player I claims $u \in v$, i.e.,

$$L_\eta(\mathbb{R}) \models \chi(u, a_v).$$

The problem (as before) is that the proto-representation

$$(\phi_u, x_{\eta_u}, x_u)$$

associated to u might be unnecessarily complicated. We ask Player I to decide whether $x_{\eta_u} \prec x_\eta$. To ensure that Player I tells the truth, we then ask Player II to decide whether $x_{\eta_u} \prec x_\eta$. If the two players disagree on this, the game ends, with the winner being declared as the winner of the subgame that determines whether $x_{\eta_u} \prec x_\eta$.

Otherwise, suppose first that the players both believe that $x_{\eta_u} \prec x_\eta$. At this point, Player II can claim that the proto-representation $(\phi_u, x_{\eta_u}, x_u)$ associated to u does not belong to Rep_η. If so, then—as above—the game ends, and the winner of the game is the winner of the game given by Lemma 9.4.2 applied to x_η and the proto-representation associated to u.[7] If Player II chooses not to make that claim, the game continues with $\phi_{k+1} = \chi$, $x_{\beta^{k+1}} = x_\eta$, $x_{b^{k+1}} = x_{a_v}$, and $\vec{w}^{k+1} = u$.

If the players both believe that $x_\eta \preceq x_{\eta_u}$, then we need to argue as above. If u truly is an element of v, then u must have a representation of order strictly less than η. Player I must play such a representation, say, $(\phi_{u'}, x_{\eta_{u'}}, x_{u'})$. As usual, Player II may object by claiming that $(\phi_{u'}, x_{\eta_{u'}}, x_{u'}) \notin \mathrm{Rep}_\eta$, in which case the game ends and the winner is declared according as whether $(\phi_{u'}, x_{\eta_{u'}}, x_{u'}) \in \mathrm{Rep}_\eta$ or not; or by claiming that $(\phi_{u'}, x_{\eta_{u'}}, x_{u'})$ is not in fact a representation of the same object as $(\phi_u, x_{\eta_u}, x_u)$. Let u' be the object represented by $(\phi_{u'}, x_{\eta_{u'}}, x_{u'})$ (if any). This is now treated much like the case (5b). Player II must claim one of the following:

i. $u' \not\subset u$; or
ii. $u \not\subset u'$.

Assume without loss of generality that it is the first alternative that is claimed. Player II must play a proto-representation $(\phi_{v'}, x_{\eta_{v'}}, x_{v'})$ for a set v' witnessing that $u' \neq u$. Player I can

[7] Note that this proto-representation could have been played by Player II at an earlier turn; this does not matter.

make the usual objection that $(\phi_{v'}, x_{\eta_{v'}}, x_{v'}) \notin \mathrm{Rep}_{\eta'_u}$, in which case the game proceeds as usual. Otherwise, Player I may claim that $v' \notin u'$ or that $v' \in u$. In the former case, the game proceeds with the proto-representation $(\phi_{u'}, x_{\eta_{u'}}, x_{u'})$ for $u' = B^{k+1}$ and w^{k+1} equal to the proto-representation $(\phi_{v'}, x_{\eta_{v'}}, x_{v'})$. The other case is similar.

6. Finally, if ϕ_k is a negated atomic formula, then it is of the form $u \notin v$, where u and v are any of b or some w_i; both u and v have associated proto-representations that have been played in one of the previous turns. One distinguishes three subcases and proceeds as in case (5).

The hope is that both players play only real representations when they should play proto-representations and that the initial triples are also real representations. Let us say that a (partial) play in which these conditions are satisfied is *honest*.

A key remark is that, if a play p is honest, then the order of p decreases after each time the players find themselves in one of situations (5) or (6) (assuming the game does not end). This implies that an honest play in which both players accept each other's moves will end after finitely many turns by one of the winning conditions in clauses (5)–(6) and the winner will be defined then.

If our expectations on the honesty of the players are not realized, the game will end in one of two possible ways: by means of a subgame initiated by a player challenging the other player's choice, in which case the winner of the definability game will be the winner of the subgame; or perhaps after infinitely many turns, if a player made improper moves and was not challenged by the opponent (or perhaps if one of the initial triples was not a real representation and this was not noticed by Player II). In this latter case, we declare Player I as the winner.[8]

There are three types of subgames that can be initiated: the one given by Lemma 9.4.2, its dual—that in which the roles of Player I and Player II are reversed—, and the game associated to the prewellordering $\prec$. Games of each of those three types are uniformly in $\mathbf{\Delta}^1_1$, and defined using the initial parameters, as well as parameters played during the (main) game.

Thus, Player I wins a run p of the game if, and only if, a codes a triple and

[8]This choice will not be very consequential. Such an outcome will not occur in any case of interest, but it needs to be considered to make the game zero-sum.

1. There is some k such that a player breaks a rule during turn k and the first player to do so is Player II;

2. There is some k such that a player initiates a challenge during turn k and $p \restriction (k, \infty)$ is a win for Player I in the subgame;

3. The game ends at a finite stage k by Player I fulfilling the condition in case (5) or case (6) above; or

4. After infinitely many steps, no player has won or initiated a challenge.

The first condition is $\Sigma^0_1(x^*, a, p)$. The second condition is $\Delta^1_1(x^*, p)$. Lastly, the third condition is $\Sigma^0_1(p)$ and the fourth condition is $\Pi^0_1(p)$. Hence, the payoff set is in $\Delta^1_1(x^*, a, p)$.

Lemma 9.4.5. *Player I has a winning strategy in $D_A(a)$ if, and only if, a codes a representation in Rep_α of some set $y \in L_\alpha(\mathbb{R})$ such that*

$$L_\alpha(\mathbb{R}) \models \phi(y, x_A). \tag{9.6}$$

Proof. Clearly it is necessary that a code a representation in Rep_α for some set $y \in L_\alpha(\mathbb{R})$ in order for Player I to have a winning strategy. Suppose it does and that (9.6) holds. The strategy for Player I is simple—essentially, she will always tell the truth. This will ensure that the intuitive interpretation of the game in terms of Player I claiming that a formula holds in an initial segment of $L(\mathbb{R})$ and Player II claiming otherwise is accurate.

If Player I's assertions are ever challenged by Player II, the subgame can be won as in Lemma 9.4.2. Otherwise, at a given turn k, if $(\phi_k, x_{\beta^k}, x_{b^k})$ and $(\psi_{w^k}, x_{w^k}, x_{d^k})$ are as in the definition of the game, then they will be true representations. If ϕ_k is a disjunction, then one of the disjuncts will hold in $L_{\beta^k}(\mathbb{R})$ and Player I will choose it. If ϕ_k is a conjunction, then both conjuncts will hold in $L_{\beta^k}(\mathbb{R})$ and so it will not matter which one Player II chooses. If ϕ_k is of the form $\exists z\, \psi$, then there really must be a witness to ψ in $L_{\beta^k}(\mathbb{R})$, and Player I will play a true representation of this witness, say, of minimal order. If ϕ_k is of the form $\forall z\, \psi$ and Player II plays a proto-representation which is not a representation of an element of $L_{\beta^k}(\mathbb{R})$, or has order higher than that permissible by the rules of the game, then Player I will challenge Player II's move and win the game as in Lemma 9.4.2. Otherwise, Player II will play a representation of an element of $L_{\beta^k}(\mathbb{R})$ and ψ will hold of this element. If ϕ_k is atomic or negated atomic, then it will be true, so that either the game will end and be won by Player I or continue according to the form of the representations relevant to the turn of the game

the players are in. As remarked earlier, the order of every honest play of the game decreases after each time the players find themselves in one of situations (5) or (6). Hence, the strategy just described ensures that either a player will initiate a challenge that will end by Player I winning, or that the game will end after finitely many turns and be won by Player I.

Similarly, if Player I has a winning strategy, then (9.6) must hold, for otherwise essentially the same strategy for Player II will be a winning strategy. The point is that if Player II plays properly, every play will end either after finitely many turns or with a subgame. □

Thus, given a set of reals $A \in L_{\alpha+1}(\mathbb{R})$ and a code x^* of a representation for A (which exists by Lemma 9.4.1), one has

$$A = \Big\{ y \in \mathbb{R} : \text{ Player I has a winning strategy in } D_A(\langle 0, y\rangle) \Big\}.$$

Since $D_A(\langle 0, y\rangle)$ is in $\Delta^1_1(x^*, \langle 0, y\rangle)$, we have

$$\mathcal{P}(\mathbb{R}) \cap L_{\alpha+1}(\mathbb{R}) \subset \mathfrak{D}^{\mathbb{R}}\mathbf{\Delta}^1_1.$$

Therefore, the proof of Lemma 9.3.1 is complete.

9.5 Derived model theorems

In this section, we prove two lemmata that tie determinacy in $L_\vartheta(\mathbb{R})$ to the existence of countably iterable extender models of Zermelo set theory with infinitely many Woodin cardinals. The first lemma is proved just like its analogue for $L(\mathbb{R})$. The second lemma is also similar to its analogue for $L(\mathbb{R})$ but has some differences. We will assume some familiarity with the theory of extender models, e.g., as presented in Mitchell-Steel [MS94] or Steel [Ste10].

We note that every interval of the form $[(\delta^2_1)^{L_{\vartheta_\alpha}(\mathbb{R})}, \vartheta_\alpha]$ is a gap (cf. the proof of Lemma 9.3.10 above).

Lemma 9.5.1. *Suppose that for every $\alpha < \omega_1$, there is a countable, countably iterable extender model of*

$$\mathsf{Z} + \text{“}V_{\lambda+\alpha} \text{ exists and } \lambda \text{ is a limit of Woodin cardinals.”}$$

Then $L_\vartheta(\mathbb{R}) \models \mathsf{AD}$.

Proof. Let $\alpha < \omega_1$ be large. We show that

$$L_{\vartheta_\alpha}(\mathbb{R}) \models \mathsf{AD}. \tag{9.7}$$

Let M be an extender model as in the statement and let $\{\delta_i : i \in \mathbb{N}\}$ enumerate the first infinitely many Woodin cardinals of M. As usual, M is a model of the Axiom of Choice. Since M is a model of Powerset, we have

$$M \models \text{“for every } \kappa,\ \kappa^+ \text{ exists.”}$$

A consequence of this is that $M \models \mathsf{KP}$; thus M can define the class $(L(A))^M$ for all $A \in M$. Assume without loss of generality that M is minimal, in the sense that

$$M \models V = L(V_\lambda).$$

Since $M \models \mathsf{ZC}$, it can form ultrapowers of itself in the usual (non-fine-structural) sense and realize them e.g., as the union of all ultrapowers of initial segments of its cumulative hierarchy. One can carry out the usual proofs of basic facts about the stationary tower $\mathbb{Q}_{<\delta}$, as well as the proof of Woodin's derived model theorem (e.g., the one in [Steb]) within M.[9] Suppose $g^* \subset \mathrm{Coll}(\omega, \lambda)$ is M-generic. Since $M \models \mathsf{ZC} + \mathsf{KP}$, then

$$M[g^*] \models \mathsf{ZC} + \mathsf{KP},$$

(this follows, e.g., from work of Mathias [Mat15]) so that if $\mathbb{R}^*$ is the set of reals of a symmetric collapse at λ, then

$$L(\mathbb{R}^*)^M \models \mathsf{Z} + \mathsf{KP} \tag{9.8}$$

(for the proof of this in the case of L, see Gostanian [Gos80, Theorem 1.6].) Since λ is a limit of Woodin cardinals in M,

$$L(\mathbb{R}^*)^M \models \mathsf{AD} + \mathsf{DC}. \tag{9.9}$$

Now let $g \subset \mathrm{Coll}(\omega, \mathbb{R})$ be V-generic. In $V[g]$ let $\{x_i : i \in \mathbb{R}\}$ enumerate $\mathbb{R}$ and find a sequence M_i of extender models such that for each $i \in \mathbb{N}$,

1. M_{i+1} is the direct limit of an iteration tree on M_i of length ω by extenders with critical point above the ith Woodin cardinal of M_i and length below the $i+1$th Woodin cardinal of M_{i+1};

[9] This is where we use the assumption that α is large. For instance, the proof in [Steb] involves building conditions for the stationary tower $\mathbb{Q}_{<\delta}$ starting from elementary substructures of $V_{\lambda+\omega}$. We have not attempted to prove (9.7) from optimal hypotheses; it seems plausible that a more careful argument shows e.g., that $L_{\vartheta_1}(\mathbb{R}) \models \mathsf{AD}$ if there is a countable, ω_1-iterable extender model of ZF - Powerset with a limit of Woodin cardinals.

2. If $j_i : M_0 \to M_{i+1}$ is the embedding, then $j_i \in V$ and there is an M_{i+1}-generic $g_i \subset \mathrm{Coll}(\omega, j(\delta_i))$ such that $x_i \in M_{i+1}[g_i]$.

If so, then, letting M_∞ be the direct limit of $\{M_i : i \in \mathbb{N}\}$, it follows by the restriction on the extenders allowed in the trees that M_∞ is wellfounded. If j is the embedding, then $j(\lambda) = \omega_1^V$. By (9.9) and the homogeneity of the symmetric collapse,

$$M \models \text{“}\Vdash \text{every set in } L(\dot{\mathbb{R}}) \text{ is determined and } L(\dot{\mathbb{R}}) \models \mathsf{DC},\text{”}$$

where $\dot{\mathbb{R}}$ is a name for the set of reals of the symmetric collapse. By elementarity,

$$L(\mathbb{R}^*)^{M_\infty} \models \mathsf{AD} + \mathsf{DC}. \tag{9.10}$$

The symmetric collapse can be chosen (in $V[g]$) in such a way that it absorbs the generic g_i for each $i \in \mathbb{N}$, in which case $(\mathbb{R}^*)^{M_\infty} = \mathbb{R}^V$, so $L(\mathbb{R}^*)^{M_\infty}$ is of the form $L_\xi(\mathbb{R})$. Suppose $h \subset \mathrm{Coll}(\omega, \lambda)$ is M_∞-generic and chosen so that the symmetric collapse induced by h absorbs g_i for each $i \in \mathbb{N}$. By the remark before equation (9.8), we have

$$M_\infty[h] \models \mathsf{ZC}.$$

Since $\mathrm{Coll}(\omega, \lambda)$ has the λ^+-chain condition,

$$M_\infty[h] \models \text{“}V_\alpha \text{ exists.”} \tag{9.11}$$

Now, clearly, we have

$$L_\xi(\mathbb{R}) \subset M_\infty[h],$$

so $L_\xi(\mathbb{R})$ satisfies AD by (9.10); it satisfies Z by (9.8), and it satisfies “V_α exists” by (9.11). By minimality, $\vartheta_\alpha < \xi$, which completes the proof. □

Lemma 9.5.2. *Suppose that $L_\vartheta(\mathbb{R}) \models \mathsf{AD}$. Then, for each $\alpha < \omega_1$, there is an ω_1-iterable extender model of*

$$\mathsf{Z} + \text{“}V_{\lambda+\alpha} \text{ exists and } \lambda \text{ is a limit of Woodin cardinals.”}$$

Proof. Suppose that $L_\vartheta(\mathbb{R}) \models \mathsf{AD}$. Let $\alpha < \omega_1$ be large and find some $\gamma < \vartheta$ such that

1. $L_\gamma(\mathbb{R})$ is a model of Z +“V_α exists,”
2. if $x, y \in \mathbb{R}$ and x is definable in $L_\gamma(\mathbb{R})$ from y and ordinal parameters, then there is $\gamma' < \gamma$ such that x is definable in $L_{\gamma'}(\mathbb{R})$ from y and ordinal parameters.

For example, one could take γ to be least such that

$$V_{\alpha+\omega}^{L_{\vartheta_{\alpha+\omega}}(\mathbb{R})} \subset L_\gamma(\mathbb{R}).$$

Clearly $L_\gamma(\mathbb{R})$ is a model of "V_α exists." Moreover, if $a \in L_\beta(\mathbb{R})$, with $\beta < \gamma$, then—by minimality—there is $n \in \mathbb{N}$ such that

$$V_{\alpha+n}^{L_{\vartheta_{\alpha+\omega}}(\mathbb{R})} \not\subset L_\beta(\mathbb{R}).$$

Thus, there is a stage $\beta' + 1 \in (\beta, \gamma)$ at which a new element of $V_{\alpha+n}$ is constructed and, by Proposition 9.2.2, there is a surjection from $V_{\alpha+n} \cap J_{\beta'}(\mathbb{R})$ to $J_{\beta'}(\mathbb{R})$ in $J_{\beta'+1}(\mathbb{R})$. In particular, there is a surjection from $V_{\alpha+n} \cap J_{\beta'}(\mathbb{R})$ to a. Hence, $\mathcal{P}(a)^{L_{\vartheta_{\alpha+\omega}}(\mathbb{R})} \subset V_{\alpha+n+1}^{L_{\vartheta_{\alpha+\omega}}(\mathbb{R})}$ and so

$$\mathcal{P}(a)^{L_{\vartheta_{\alpha+\omega}}(\mathbb{R})} = \mathcal{P}(a)^{L_\gamma(\mathbb{R})} \in L_\gamma(\mathbb{R}),$$

so we have that $L_\gamma(\mathbb{R}) \models \mathsf{Z}$. Lastly, the definition implies that γ is a cardinal (in fact, the largest cardinal) of $L_{\vartheta_{\alpha+\omega}}(\mathbb{R})$ greater than $\Theta^{L_{\vartheta_{\alpha+\omega}}(\mathbb{R})}$, so that (by the Löwenheim-Skolem theorem), we have

$$L_\gamma(\mathbb{R}) \prec_1 L_{\vartheta_{\alpha+\omega}}(\mathbb{R}).$$

Hence, if $x, y \in \mathbb{R}$ and x is definable in $L_\gamma(\mathbb{R})$ from y and ordinal parameters, then

$$L_\gamma(\mathbb{R}) \models \exists \gamma' \, (x \text{ is definable in } L_{\gamma'}(\mathbb{R}) \text{ from } y \text{ and ordinal parameters}),$$

so γ is as desired.

The first thing to verify is that

$$L_\gamma(\mathbb{R}) \models \text{"Mouse Capturing."}$$

Recall that "Mouse Capturing" is the statement that for every pair of reals (x, y), x is definable from y and ordinal parameters if, and only if, x belongs to a countably iterable extender model over y. In Schindler-Steel [SS14, Theorem 3.4.6], it is shown that for all reals x, y, the following are equivalent:

1. x is definable from y and ordinal parameters in $L_\beta(\mathbb{R})$ for some $\beta < \gamma$,
2. x belongs to an extender model over y which is countably iterable by an iteration strategy in $L_\gamma(\mathbb{R})$.

The equivalence is stated modulo an inductive hypothesis W^*_α, but this hypothesis is proved (see the argument in pp. 157-158 of [SS14]) under the assumption of AD (in $L(\mathbb{R})$, but the argument goes through in $L_\gamma(\mathbb{R})$). By our choice of γ, this shows that $L_\gamma(\mathbb{R})$ satisfies Mouse Capturing.

The second thing to verify is that

$$L_\gamma(\mathbb{R}) \models \mathsf{AD}^+;$$

this follows from the usual proof for $L(\mathbb{R})$: Ordinal Determinacy follows from the facts that by Kechris-Kleinberg-Moschovakis-Woodin [KKMW08, Theorem 1.1], $(\delta^2_1)^{L_\gamma(\mathbb{R})}$ is a limit of cardinals with the strong partition property, and that by a theorem due independently to Moschovakis and Woodin (see Larson [Lar17] for a proof), this implies $<(\delta^2_1)^{L_\gamma(\mathbb{R})}$-determinacy. Since

$$L_{(\delta^2_1)^{L_\gamma(\mathbb{R})}}(\mathbb{R}) \prec_1 L_\gamma(\mathbb{R}),$$

this implies Ordinal Determinacy in $L_\gamma(\mathbb{R})$.

That every set of reals is ∞-Borel in $L_\gamma(\mathbb{R})$ follows from the argument in [Lar17]. (It uses the fact that $(\Sigma^2_1)^{L_\gamma(\mathbb{R})}$ has the scale property, which is proved by following the argument of [MS08] as in Lemma 9.3.10).

We have checked that $L_\gamma(\mathbb{R})$ satisfies AD^+ and Mouse Capturing. Now, let $\beta < \alpha$ be arbitrary and let x_β be a real such that β is recursive in x_β. Define the theory

$$S \equiv \mathsf{Z} + \mathsf{KP} + \mathsf{AD}^+ + \mathsf{DC} + \text{“Mouse Capturing”} + \text{“}V_\beta \text{ exists.”}$$

Let κ be least such that $J_\kappa(\mathbb{R}) \models S$; clearly $\kappa < \gamma$ and, in fact, $\kappa < (\delta^2_1)^{L_\gamma(\mathbb{R})}$. We now run the proof of Steel-Woodin [SW16, Theorem 7.2] for the theory S within $L_\gamma(\mathbb{R})$ (so that κ here takes the role of the ordinal denoted by γ therein), except that we consider the pointclass $(\Sigma^2_1(x_\beta))^{J_\kappa(\mathbb{R})}$, rather than $(\Sigma^2_1)^{J_\kappa(\mathbb{R})}$. Similarly, one replaces the $L[E]$-construction on p. 325 of [SW16] by an $L[E](x_\beta)$-construction. Letting M, N, and ξ be as in [SW16], and M_0 be set of all elements of $M|\xi$ which are definable *from* x_β in $M|\xi$, the argument shows that M_0 has the form $J_\zeta(N_0)$, for some ζ and some extender model N_0 over x_β and, moreover, the following hold:

1. N_0 satisfies “there are infinitely many Woodin cardinals”;
2. no initial segment of M_0 projects to N_0;
3. M_0 has an (ω_1, ω_1)-iteration strategy Σ_0;
4. the derived model of M_0 at its limit of Woodin cardinals satisfies S.

Now, let λ be the limit of Woodin cardinals of M_0. Then, an ordinal $\eta > \lambda$ is a cardinal of M_0 if, and only if, it is a cardinal of a generic extension of M_0 by $\mathrm{Coll}(\omega, \lambda)$ if, and only if, it is a cardinal of the derived model of M_0. Since this derived model satisfies S, it follows that M_0 has β-many cardinals above λ and, since M_0 is a model of the Generalized Continuum Hypothesis, we have $M_0 \models$ "V_β exists." Since $\beta < \alpha$ was arbitrary and α was arbitrary, the result follows. □

Theorem 9.1.3 is immediate from Theorem 9.3.4, Lemma 9.5.2 and Lemma 9.5.1.

10

Long F_σ Determinacy

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10.1 Introduction

In this chapter, we study the consistency strength of F_σ-determinacy for games of length ω^2. We see that the situation is—in a way—very similar to that of short games. For example, over the base theory

$$\mathsf{KP} + \mathsf{DC} + \text{“}\mathbb{R} \text{ exists,”}$$

F_σ-determinacy for games of length ω^2 is much stronger than AD; however, it is much weaker than $\mathsf{AD} + \Sigma_1$-separation. Over ZFC, AD is stronger, naturally. There, F_σ-determinacy for games of length ω^2 is stronger than the existence of a transitive model of $\mathsf{KP} + \mathsf{AD}$ containing all reals; yet weaker

than the existence of a transitive model of $\mathsf{KP} + \mathsf{AD} + \Sigma_1$-separation containing all reals. Much like in the case of short F_σ games, long games are tied to reflection for set-theoretic formulae of a certain complexity; except that—rather than in L—one needs to consider reflection in the $L(\mathbb{R})$-hierarchy. Given an admissible set, let A^+ denote the *next admissible set*, in the sense of Barwise-Gandy-Moschovakis [BGM71]; i.e., A^+ is the smallest admissible set containing A. By convention, we shall assume that all admissible sets are transitive.

Definition 10.1.1. An admissible set A is Π^+_1*-reflecting* if for every Π_1 formula ϕ with parameters in A and one free variable, if $A^+ \models \phi(A)$, then there is some $B \in A$ with all parameters in ϕ such that $B^+ \models \phi(B)$.

Our main theorem is:

Theorem 10.1.2. *The following are equivalent:*

1. $\mathbf{\Sigma}^0_2$*-determinacy for games of length* ω^2*;*
2. *There is a* Π^+_1*-reflecting model of* AD *containing* $\mathbb{R}$*.*

Precursors to this work include Blass' [Bla75] theorem that determinacy for all games of length ω^2 is equivalent to $\mathsf{AD}_\mathbb{R}$, Trang's [Tra13] work on analytic games of additively indecomposable length, as well as previous work on games of length ω^2 that are open (see [Agub]), clopen [Aguf], Borel [Agud], or projective [AM].

10.2 Preliminaries

In this section, we collect some preliminary definitions and results.

10.2.1 Games of transfinite length

We study two-player, perfect information games of length ω^2 which are F_σ, i.e., $\mathbf{\Sigma}^0_2$-definable. These are games in which, given $A \in \mathcal{P}(\mathbb{R}) \cap \mathbf{\Sigma}^0_2$, two players, Player I and Player II, alternate ω^2-many turns playing natural numbers, thus producing a sequence $x \in \mathbb{N}^{\omega^2}$. Since the spaces $\mathbb{R} \setminus \mathbb{Q}$ and $\mathbb{N}^{\omega^2}$ are recursively homeomorphic, the sequence x may be identified with a(n irrational) real number. Player I wins if $x \in A$; otherwise, Player II wins.

As above, we will often identify sequences of natural numbers of length ω^2 with ω-sequences of reals and with single reals. We will denote by

$$x = \langle x_0, x_1, \ldots \rangle$$

the *single* real number coding the sequence $(x_0, x_1, \ldots)$, via some fixed recursive coding. The precise coding used will be immaterial, except for the continuity property that the first n digits x should only depend on $x_0, x_1, \ldots, x_n$.

10.2.2 Reflection in $L(\mathbb{R})$

We recall the definition of the $L(\mathbb{R})$ hierarchy: $L_0(\mathbb{R})$ is defined to be $V_{\omega+1}$, the collection of all sets all of whose elements are hereditarily finite. $L_{\alpha+1}(\mathbb{R})$ is the set of all subsets of $L_\alpha(\mathbb{R})$ definable over $L_\alpha(\mathbb{R})$ with parameters. At limit stages, $L_\lambda(\mathbb{R})$ is the union of all $L_\alpha(\mathbb{R})$, for $\alpha < \lambda$. The function

$$\alpha \mapsto L_\alpha(\mathbb{R})$$

is Δ_1 with $\mathbb{R}$ as a parameter. The first step towards proving Theorem 10.1.2 is the observation that it suffices to restrict to admissible sets of the form $L_\alpha(\mathbb{R})$.

We will use the following non-standard notation:

Definition 10.2.1. Let α be an ordinal. We denote by

$$\alpha^+$$

the least ordinal β such that $\alpha < \beta$ and $L_\beta(\mathbb{R})$ is admissible.

Definition 10.2.2. An ordinal α is Π^+_1*-reflecting* if $L_\alpha(\mathbb{R})$ is Π^+_1-reflecting.

Lemma 10.2.3. *Suppose A is a Π^+_1-reflecting set and let $\alpha = \mathsf{Ord} \cap A$. Then, α is Π^+_1-reflecting.*

Proof. Let ϕ be a Π_1 formula with parameters in $L_\alpha(\mathbb{R})$ such that

$$(L_\alpha(\mathbb{R}))^+ \models \phi(L_\alpha(\mathbb{R})).$$

Clearly, $(L_\alpha(\mathbb{R}))^+ = L_{\alpha^+}(\mathbb{R})$. Moreover, A^+ is an admissible set containing L_α, so $L_{\alpha^+}(\mathbb{R}) \subset A^+$. Let ψ be the formula in the language of set theory with parameters in $A \cup \{A\}$ asserting that for all β, if no $\gamma \in (\alpha, \beta)$ is admissible, then $L_\beta(\mathbb{R}) \models \phi(L_\alpha(\mathbb{R}))$. Being admissible is expressible internally, α is Δ_1-definable from A, and—as remarked earlier—$L_\beta(\mathbb{R})$ is Δ_1-definable from β and $\mathbb{R}$ (which belongs to $L_\alpha(\mathbb{R})$. Hence, ψ is Π_1. Moreover, $A^+ \models \psi(A)$,

so, by reflection, there is $B \in A$ such that $B^+ \models \psi(B)$ and B contains all parameters in ψ. In particular, $\mathbb{R} \in B$. Let $\beta = \mathsf{Ord} \cap B$; since $B \in A$, $\beta \in A$, so, in particular, $\beta < \alpha$. $L_{\beta^+}(\mathbb{R}) \subset B^+$. By choice of ψ, we obtain that $L_{\beta^+}(\mathbb{R}) \models \phi(L_\beta(\mathbb{R})$, as desired. □

Corollary 10.2.4. *The following are equivalent:*

1. *There is a Π^+_1-reflecting model of AD containing $\mathbb{R}$; and*
2. *There is a Π^+_1-reflecting ordinal α such that $L_\alpha(\mathbb{R}) \models \mathsf{AD}$.*

Proof. Suppose that A is a Π^+_1-reflecting model of AD containing $\mathbb{R}$ and let $\alpha = \mathsf{Ord} \cap A$. By Lemma 10.2.3, $L_\alpha(\mathbb{R})$ is Π^+_1-reflecting. Moreover, it is a subset of A containing all reals, and thus all possible strategies for games; thus $L_\alpha(\mathbb{R}) \models \mathsf{AD}$. The converse is immediate. □

This motivates the following definition:

Definition 10.2.5. We denote by $\sigma_\mathbb{R}$ the least Π^+_1-reflecting ordinal.

$\sigma_\mathbb{R}$ is the analog of the ordinal σ^1_1 for $L(\mathbb{R})$. Unlike σ^1_1, the set-theoretic properties of $\sigma^\mathbb{R}$ are not decidable within ZFC. For instance, under AD, $\sigma_\mathbb{R}$ is big—it is a limit of weakly inaccessible cardinals. In contrast, if $V = L$, then $\sigma_\mathbb{R} < \omega_2$.

Definition 10.2.6. Let $A \subset \mathbb{R} \times \mathbb{R}$ and write $A_x = \{y \in \mathbb{R} : (x, y) \in \mathbb{R}\}$. We write $\supset^\mathbb{R} A$ for the set of all $x \in \mathbb{R}$ such that Player I has a winning strategy for the game on $\mathbb{R}$ with payoff A_x. We define

$$\supset^\mathbb{R} \mathbf{\Sigma}^0_2 = \{\supset^\mathbb{R} A : A \in \mathbf{\Sigma}^0_2\}.$$

To prove Theorem 10.1.2, it suffices to prove that determinacy for F_σ games of length ω^2 is equivalent to the fact that $L_{\sigma_\mathbb{R}}(\mathbb{R}) \models \mathsf{AD}$. Following previous proofs of determinacy for games of length ω^2, our first goal is to locate winning strategies for $\mathbf{\Sigma}^0_2$ games on $\mathbb{R}$. We shall show that

$$\mathcal{P}(\mathbb{R}) \cap \Sigma^{L_{\sigma_\mathbb{R}}(\mathbb{R})}_1 = \supset^\mathbb{R} \mathbf{\Sigma}^0_2.$$

The main tool for this is the theory of inductive definitions on $\mathbb{R}$, the basics of which we recall next.

10.2.3 Inductive definitions

Suppose that $\phi : \mathcal{P}(\mathbb{R}^m) \to \mathcal{P}(\mathbb{R}^m)$ is an operator. We define sets $\phi^\lambda \subset \mathbb{R}^m$ inductively by

$$\begin{aligned} \phi^0 &= \varnothing, \\ \phi^{<\lambda} &= \bigcup_{\mu<\lambda} \phi(\phi^{<\mu}), \\ \phi^\lambda &= \phi^{<\lambda} \cup \phi(\phi^{<\lambda}), \\ \phi^\infty &= \bigcup_{\lambda \in \mathsf{Ord}} \phi^\lambda. \end{aligned}$$

The least κ such that $\phi^\infty = \phi^\kappa$ is called the *closure ordinal* of ϕ and denoted $|\phi|$. If Γ is a class of operators, we write

$$|\Gamma| = \sup\{|\phi| : \phi \in \Gamma\}.$$

Definition 10.2.7. Let Γ be a class of operators

$$\phi : \mathcal{P}(\mathbb{R}^m) \to \mathcal{P}(\mathbb{R}^m).$$

We say that $R \subset \mathbb{R}^n$ is Γ-*inductive* if there is $b \in \mathbb{R}^{m-n}$ such that for all $a \in \mathbb{R}^n$,

$$a \in R \text{ if, and only if, } (a, b) \in \phi^\infty.$$

An operator ϕ is defined by a formula $\psi(x, X)$ if, and only if,

$$\phi(X) = \{a \in \mathbb{R}^m : \psi(a, X)\}.$$

Thus, it is natural to consider classes of operators specified in terms of definability. Let Γ be a pointclass; we say that an operator ϕ is in Γ if it is defined by a formula in Γ with an additional predicate symbol X. We say that an operator is *positive* if this additional predicate symbol appears only positively, i.e., not in the scope of any negations (or in the antecedent of implications). An operator is *monotone* if $X \subset Y \subset \mathbb{R}$ implies $\phi(X) \subset \phi(Y)$. Every positive operator is monotone, of course. We may also speak of a formula defining an operator being *positive* under the obvious circumstances.

Definition 10.2.8. Let Γ be a pointclass.

1. $\Gamma{-}\mathsf{IND}$ is the pointclass of all Γ-inductive sets.

2. $\Gamma^{pos}{-}\mathsf{IND}$ is the pointclass of all (positive-Γ)-inductive sets.

3. $\Gamma^{mon}-\mathsf{IND}$ is the pointclass of all (monotone-Γ)-inductive sets.

Definition 10.2.9. A subset of $\mathbb{R}$ is *inductive* if it is positive analytical inductive, i.e., letting IND be the class of all inductive sets, we have

$$\mathsf{IND} = \bigcup_{n\in\mathbb{N}} (\mathbf{\Sigma}^{1,pos}_n - \mathsf{IND}).$$

A subset of $\mathbb{R}$ is *co-inductive* if its complement is inductive. We write coIND for the class of co-inductive sets.

Pointclasses of the form $\mathbf{\Sigma}^1_n$ (or of other forms) can be relativized by allowing sets of reals as parameters. This leads to a relativized version of the inductive sets:

Definition 10.2.10. Let $X \subset \mathbb{R}$. A subset of $\mathbb{R}$ is *inductive on* X if it is positive analytical-on-X inductive, i.e., letting $\mathsf{IND}(X)$ be the class of all sets inductive on X, we have

$$\mathsf{IND}(X) = \bigcup_{n\in\mathbb{N}} (\mathbf{\Sigma}^{1,pos}_n(X) - \mathsf{IND}).$$

In the definition above, "positive" refers to the variable on which the induction is carried out; the parameter X need not appear positively in it.

Let κ be the closure ordinal of the inductive sets, i.e., the supremum of closure ordinals of positive analytical inductive definitions. Then $L_\kappa(\mathbb{R})$ is the smallest model of KP containing $\mathbb{R}$ and

$$\mathsf{IND} = \mathcal{P}(\mathbb{R}) \cap \Sigma_1^{L_\kappa(\mathbb{R})},$$

where Σ_1-definability in the equation above allows for parameters. Subsets of $\mathbb{R}$ that are both inductive and co-inductive are called *hyperprojective*. By Δ_1-separation, these are the sets of reals in $L_\kappa(\mathbb{R})$.

In general, let α be an ordinal and suppose that α is smaller than the least non-$\mathbb{R}$-projectible ordinal, i.e., that there is a surjection

$$\rho : \mathbb{R} \to L_\alpha(\mathbb{R})$$

which is Σ_1-definable over $L_\alpha(\mathbb{R})$ (this restriction can be relaxed, but all ordinals relevant to this chapter will be of this form). Let X be a set of reals coding $L_\alpha(\mathbb{R})$ in this way. Then, if κ_X is the closure ordinal of the X-inductive sets, we have that $L_{\kappa_X}(\mathbb{R})$ is the smallest model of KP containing $L_\alpha(\mathbb{R})$ and

$$\mathsf{IND}(X) = \mathcal{P}(\mathbb{R}) \cap \Sigma_1^{L_{\kappa_X}(\mathbb{R})}.$$

There is yet another characterization of inductive sets that we will use. It is probably folklore; a proof can be obtained easily from the main theorem of Moschovakis [Mos72] and can be found in in the previous chapter of this thesis:

Theorem 10.2.11. $\mathsf{IND}(X) = \mathfrak{D}^{\mathbb{R}}\mathbf{\Sigma}^0_1(X)$.

We mention some results on the relation between inductive definitions, monotone inductive definitions, and positive inductive definitions.

Theorem 10.2.12 (Harrington-Kechris [HK76])**.** *Let $\mathcal{F}$ be a collection of operators containing Π^1_1 and closed under $\wedge, \vee, \exists^{\mathbb{R}}$ and recursive substitutions. Suppose that*

1. *$WF \in \breve{\mathcal{F}}$, and*
2. *$\breve{\mathcal{F}} \in \mathcal{F}^{mon}\text{–}\mathsf{IND}$.*

Then $\mathcal{F}^{mon}\text{–}\mathsf{IND} = \mathcal{F}\text{–}\mathsf{IND}$.

In a later section, we will use the preceding theorem in the case $\mathcal{F} = \mathsf{coIND}$. In this case, one can obtain a stronger result (cf. [HK76] or [HK] for an even more general theorem).[1]

Theorem 10.2.13 (Harrington-Moschovakis, 1974)**.**

$$\mathsf{coIND}\text{–}\mathsf{IND} = \mathsf{coIND}^{pos}\text{–}\mathsf{IND}.$$

Harrington and Moschovakis in fact proved a more general theorem that implies both Theorem 10.2.13 and the following classical result:

Theorem 10.2.14 (Grilliot)**.** $\Sigma^1_1\text{–}\mathsf{IND} = \Sigma^{1,pos}_1\text{–}\mathsf{IND}$.

Definition 10.2.15. A pointclass Γ is called a *Spector class on* $\mathbb{R}^2$ if the following conditions hold:

1. Γ is closed under $\wedge, \vee, \exists^{\mathbb{R}}, \forall^{\mathbb{R}}$;
2. Γ contains all analytical relations;
3. Γ is parametrized by $\mathbb{R}$;
4. Γ has the prewellordering property.

[1] We would like to thank A. S. Kechris for sharing the notes [HK] with us.

[2] This differs from the notion of "Spector *pointclass*" in Moschovakis [Mos09].

10.2.4 Short F_σ games

In order to locate winning strategies for $\mathbf{\Sigma}^0_2$ games on $\mathbb{R}$ within the $L(\mathbb{R})$-hierarchy, it might be helpful to recall the location of winning strategies for $\mathbf{\Sigma}^0_2$ games on $\mathbb{N}$ in the L-hierarchy. This is a result of Solovay and is obtained by combining three theorems in recursion theory.

The first one of them is also due to Solovay:

Theorem 10.2.16 (Solovay)**.** $\Sigma^{1,pos}_1\text{−}\mathsf{IND} = \partial\Sigma^0_2$.

Solovay's theorem was not originally published, but a proof can be found in [Kec16] or in [Mos09]. It will adapt to prove the analog for games on $\mathbb{R}$ later, as well as the fact that $\partial^{\mathbb{R}}\mathbf{\Sigma}^0_2$-determinacy implies determinacy for $\mathbf{\Sigma}^0_2$ games of length ω^2. The second theorem is Grilliot's Theorem 10.2.14 which, together with Theorem 10.2.16 implies that

$$\Sigma^1_1\text{−}\mathsf{IND} = \partial\Sigma^0_2.$$

The final theorem is due to Aczel and Richter. Recall that σ^1_1 denotes the least Σ^1_1-reflecting ordinal.

Theorem 10.2.17 (Aczel-Richter [AR74])**.** $|\Sigma^1_1| = \sigma^1_1$*, where* $|\Sigma^1_1|$ *refers to closure under inductive operators on* $\mathbb{N}$*.*

Corollary 10.2.18. *Suppose Player I has a winning strategy in a* Σ^0_2 *game. Then Player I has a winning strategy for that game in* $L_{\sigma^1_1}$*.*

Our strategy for locating Player I's winning strategies for $\mathbf{\Sigma}^0_2$ games on $\mathbb{R}$ will be to follow the same steps as in the argument for games on $\mathbb{N}$.

10.3 Coinductive operators

Theorem 10.3.1. $\partial^{\mathbb{R}}\mathbf{\Sigma}^0_2 = \mathsf{coIND}^{pos}\text{−}\mathsf{IND}$.

Proof. We begin by quoting Wolfe's proof of Σ^0_2-determinacy and Solovay's proof of Theorem 10.2.16 to prove that

$$\partial^{\mathbb{R}}\mathbf{\Sigma}^0_2 \subset \mathsf{coIND}^{pos}\text{−}\mathsf{IND}. \tag{10.1}$$

Although the argument for this inclusion is very similar to the one for games on $\mathbb{N}$, a variation of it will be used below in the proof of Lemma 10.5.1 below, so we prefer to include all the details.

Let $A \subset \mathbb{R}^2$ be a Σ^0_2 set (where we assume no parameter is needed; the general result follows by relativization). Say A is given by

$$(x, y) \in A \leftrightarrow \exists n \forall m \, (n, x \upharpoonright m, y \upharpoonright m) \in P,$$

where P is recursive. The proof proceeds by showing that winning a certain game $G^*(\langle\rangle)$ equivalent to A_x is equivalent to the membership of x in a set in $\mathsf{coIND}^{pos}-\mathsf{IND}$. For s a finite sequence of real numbers of even length, define $G^*(s)$ to be the following game:

1. Players I and II alternate turns playing real numbers α_n.
2. After infinitely many rounds have taken place, Player I wins if, letting

   $$t(m) = s^\frown\langle\alpha_0, \alpha_1, \ldots, \alpha_m\rangle$$

   (i.e., letting t be the real coding the result of concatenating s with the first m moves of the play $(\alpha_0, \alpha_1, \ldots)$), we have

   $$\exists n \forall m \forall m' < m \, (n, x \upharpoonright m', t(m) \upharpoonright m') \in P.$$

Note that by our conventions on coding sequences of reals by single reals, we have

$$s^\frown\langle\alpha_0, \alpha_1, \ldots\rangle \upharpoonright m = t(m) \upharpoonright m.$$

This implies that Player I has a winning strategy for the game on A if, and only if, she has one for the game on $G^*(\langle\rangle)$. The game $G^*(s)$ is like $G^*(\langle\rangle)$, except that we assume that s has already been played. We shall show that Player I having a winning strategy is in $\mathsf{coIND}^{pos}-\mathsf{IND}$.

Let s be a finite sequence of real numbers of even length and X be a set of reals. Consider the following game, $G(X, s)$:

1. Players I and II alternate turns playing real numbers α_n.
2. After infinitely many rounds have taken place, Player I wins if, and only if, letting

   $$t(2m) = s^\frown\langle\alpha_0, \ldots, \alpha_{2m}\rangle,$$

   one of the following holds for each $m \in \mathbb{N}$:

   (a) there is $n \leq \text{LTH}(s)$ such that we have

   $$\forall m' < m \, (n, x \upharpoonright m', t(2m) \upharpoonright m') \in P.$$

(b) $t(2m) \in X$.

The formula

$$\phi(s, X) = \text{“}s \text{ has even length and Player I has a winning strategy in } G(X, s)\text{”} \tag{10.2}$$

is clearly in $\Im^{\mathbb{R}}\Pi^0_1$ with an additional predicate for X and X appears positively in it. Since Π^0_1 games on $\mathbb{R}$ are determined by the Gale-Stewart theorem [GM53], the dual pointclass of $\Im^{\mathbb{R}}\Pi^0_1$ is $\Im^{\mathbb{R}}\Sigma^0_1$. By Theorem 10.2.11, (10.2) is in coIND^{pos}. The claim is now that

$$s \in \phi^\infty \leftrightarrow \text{“Player I has a winning strategy in } G^*(s)\text{”}$$

As in Solovay's proof, it is first shown that if $s \in \phi^\xi$, then Player I has a winning strategy in $G^*(s)$, by induction on ξ. Suppose that this holds for $\phi^{<\xi}$ and that $s \in \phi^\xi$, so that Player I has a winning strategy in $G(\phi^{<\xi}, s)$, say, σ. A winning strategy for $G^*(s)$ is obtained as follows: Player I begins by playing $G^*(s)$ by σ so long as the first winning condition of $G(s, X)$ is satisfied, i.e., so long as after round m, letting $t(m) = s^\frown\langle\alpha_0, \alpha_1, \ldots, \alpha_m\rangle$, we have

$$\exists n \leq \text{LTH}(s)\, \forall m' < m\, (n, x \restriction m', t(2m) \restriction m') \in P.$$

If after some round this condition is not satisfied, then (since σ is a winning strategy), we must have

$$\exists \zeta < \xi \left(s^\frown\langle\alpha_0, \ldots, \alpha_{2m}\rangle \in \phi^\zeta\right),$$

in which case the induction hypothesis yields a winning strategy for the game $G^*(s^\frown\langle\alpha_0, \ldots, \alpha_{2m}\rangle)$ which should now be followed.

Conversely, if Player I has a winning strategy σ in $G^*(s)$ then s must belong to ϕ^∞, for otherwise (by monotonicity of ϕ and determinacy of closed games) Player II has a winning strategy τ in $G(s, \phi^\infty)$. If so, we could face off the strategies σ and τ against each other. Since τ is winning for Player II, after finitely many rounds, one will reach a partial play $t(2m_1) = s^\frown\langle\alpha_0, \ldots, \alpha_{2m_1}\rangle$ such that the following hold:

1. $\forall n \leq \text{LTH}(s)\, \exists m' < m_1\, (n, x \restriction m', t(2m_1) \restriction m') \notin P$; and
2. $t(2m_1) \notin \phi^\infty$.

The second condition yields a new strategy τ_1 for Player II that can now be played against σ in $G(t(2m_1), \phi^\infty)$ until some stage m_2 at which the two

conditions above hold again, etc. Continuing this process infinitely often yields a play

$$t = s^\frown\langle \alpha_0, \alpha_1, \ldots\rangle$$

such that for each k,

$$\forall n \leq \text{LTH}(t(2m_k))\, \exists m' < m_k\, (n, x \restriction m', t(2m_{k+1}) \restriction m') \notin P,$$

so that

$$\forall n\, \exists m' (n, x \restriction m', t \restriction m') \notin P,$$

contradicting the fact that σ was a winning strategy for Player I in $G(s)$. This completes the proof of (10.1).

The second step consists in showing that

$$\mathsf{coIND}^{pos} - \mathsf{IND} \subset \mathfrak{D}^{\mathbb{R}}\mathbf{\Sigma}^0_2. \tag{10.3}$$

Let $\phi(s, X)$ be a positive coIND operator.

By a theorem of Kechris and Moschovakis (cf. [Kec16, Theorem 2.18]) and since $\mathbf{\Sigma}^0_2$ is parametrized by $\mathbb{R}$ and has the prewellordering property, $\mathfrak{D}^{\mathbb{R}}\mathbf{\Sigma}^0_2$ is a Spector class on $\mathbb{R}$. By the "Main Lemma" of Moschovakis [Mos74] (cf. also [Kec16, Theorem 1.7]), in order to see that $\phi^\infty \in \mathfrak{D}^{\mathbb{R}}\mathbf{\Sigma}^0_2$, it suffices to show that $\mathfrak{D}^{\mathbb{R}}\mathbf{\Sigma}^0_2$ is closed under ϕ, i.e., that for all $A \in \mathfrak{D}^{\mathbb{R}}\mathbf{\Sigma}^0_2$, the set

$$A_\phi = \Big\{(x, z) : \phi\Big(x, \{y : (y, z) \in A\}\Big)\Big\}$$

is in $\mathfrak{D}^{\mathbb{R}}\mathbf{\Sigma}^0_2$. Let us put

$$A_z = \Big\{y \in \mathbb{R} : (y, z) \in A\Big\}$$

and verify that

$$\Big\{(x, z) : \phi(x, A_z)\Big\} \in \mathfrak{D}^{\mathbb{R}}\mathbf{\Sigma}^0_2.$$

Since ϕ is in coIND, it follows from Moschovakis [Mos72] that there is a $\mathbf{\Pi}^1_1$ formula ψ such that

$$\phi(x, X) \leftrightarrow \exists y_0\, \forall y_1\, \ldots\, \forall n\, \psi(x, \langle y_0, \ldots, y_n\rangle, X),$$

say,

$$\phi(x, X) \leftrightarrow \exists y_0\, \forall y_1\, \ldots\, \forall n\, \forall w\, \psi_0(w, x, \langle y_0, \ldots, y_n\rangle, X),$$

for some arithmetical ψ_0.

To verify that A_ϕ is in $\mathfrak{D}^{\mathbb{R}}\mathbf{\Sigma}^0_2$, fix x and z; we play the natural game given by the equation above:

1. Players I and II begin by playing real numbers $y_0, y_1, \ldots$ until Player II decides to move on to the next stage after turn n. If this never happens, Player I wins.

2. Player II plays $w \in \mathbb{R}$. Let $\theta_0 = \psi_0(w, x, \langle y_0, \ldots, y_n\rangle, A_z)$ and assume without loss of generality that θ_0 has been rewritten without implications and without negations whose scope is not only an atomic formula.

3. If θ_k has been defined and is a formula in the language of second-order arithmetic with an additional predicate A_z, we proceed by cases:

   (a) If the outermost logical connective of θ_k is a disjunction, then Player I chooses one of the disjuncts; we set θ_{k+1} equal to this choice.

   (b) If the outermost logical connective of θ_k is a conjunction, then Player II chooses one of the conjuncts; we set θ_{k+1} equal to this choice.

   (c) If θ_k is atomic then either it is an atomic formula not involving X, in which case the game ends and Player I wins if, and only if, θ_k holds; or it is of the form $a \in A_z$ (since ϕ is positive). In the latter case, the game continues.

   (d) Let $B \in \mathbf{\Sigma}^0_2$ be such that $A_z = \mathcal{D}^{\mathbb{R}}B$. Players I and II alternate infinitely many turns playing reals $w_0, w_1, w_2, \ldots$. At the end, Player I wins if, and only if, $(\langle w_0, w_1, \ldots\rangle, w, a, z) \in B$.

Clearly the winning condition for the game is $\mathbf{\Sigma}^0_2$. It is easy to verify that Player I has a winning strategy in this game if, and only if $(x, z) \in A_\phi$. This proves (10.3). □

Together with Harrington and Moschovakis's Theorem 10.2.13, the previous result yields:

Corollary 10.3.2. $\mathsf{coIND}{-}\mathsf{IND} = \mathcal{D}^{\mathbb{R}}\mathbf{\Sigma}^0_2$.

10.4 Coinductive operators and reflection

Let $\preceq$ be a binary relation. As a convention, we will use $\prec$ to denote the strict part of $\preceq$ (i.e., $x \prec y$ whenever $x \preceq y$ and $y \not\preceq x$) and $\equiv$ to denote $\preceq$-equivalence (i.e., $x \equiv y$ whenever $x \preceq y$ and $y \preceq x$). Similar conventions

shall apply to variations of $\preceq$, e.g., by subscripts. Notice that any two of $\preceq$, $\prec$, and $\equiv$ determine the other one.

Given a binary relation $\preceq$ and an equivalence relation E with the same field, $\preceq$ is said to commute with E if $x \preceq y$, xEa, and yEb imply $a \preceq b$, and similarly for the strict part $\prec$. A binary relation $\preceq$ *commutes with itself* if it commutes with $\equiv$; this follows from transitivity. An induction on the complexity of formulae shows that if $\preceq$ is a binary relation that commutes with itself, then any first-order formula in the vocabulary $\{\prec, =\}$ holds of some tuple $(x_1, \ldots, x_n)$ of objects in field$(\preceq)$ if, and only if, it holds of any tuple $(y_1, \ldots, y_n)$ such that x_i is $\equiv$-equivalent to y_i for each i, so long as equality is interpreted as $\equiv$.

Definition 10.4.1. Let α be an ordinal. A *coding* of $L_\alpha(\mathbb{R})$ is a preorder $\preceq$ on a subset of $\mathbb{R}$ such that

$$(L_\alpha(\mathbb{R}), \in, =) \cong (\text{field}(\preceq), \prec, \equiv)/\equiv .$$

The right-hand side of the displayed equation is the quotient structure of the structure $(\text{field}(\preceq), \prec, \equiv)$ by $\equiv$. The relation $\equiv$ becomes equality in this quotient. If $\preceq$ is a coding of some $L_\alpha(\mathbb{R})$, then $\prec$ induces a wellfounded transitive relation on $\equiv$-equivalence classes and there is a surjection ρ from field$(\preceq)$ to $L_\alpha(\mathbb{R})$ such that $x \prec y$ if, and only if, $\rho(x) \in \rho(y)$. If so then *in this section only*, let us denote $\rho(x)$ by $|x|_\preceq$.

Theorem 10.4.2. $|\mathsf{coIND}| = \sigma_\mathbb{R}$.

The proof of Theorem 10.4.2 will be divided into two lemmata:

Lemma 10.4.3. $|\mathsf{coIND}|$ *is* Π^+_1*-reflecting.*

Proof. The lemma will be proved by an argument similar to the one for [AR74, Theorem 10.7], but one deals with recursion on $\mathbb{R}$ in our context. Another difference is that, since we are only interested in proving Lemma 10.4.3 (and are not analizing, e.g., ordinals of the form $|\Sigma^1_n|$), we do not need to be very careful with issues of definability, which greatly simplifies the situation.

Given an inductive definition ϕ, there is a natural way of associating to it a prewellordering on $\mathbb{R}$, namely, letting

$$w(x) = \begin{cases} \text{least } \xi \text{ such that } x \in \phi^{\xi+1}, & \text{if it exists,} \\ \infty, & \text{otherwise;} \end{cases}$$

we put

$$x \preceq_\phi y \text{ if, and only if, } w(x) \leq x(y).$$

Let ϕ be a universal coinductive operator, i.e., a coinductive operator such that whenever ψ is a coinductive operator, we have

$$\psi(X) = \{x \in \mathbb{R} : (x, a) \in \phi(X)\}$$

for some $a \in \mathbb{R}$. (See Moschovakis [Mos74] for a proof of the existence of universal coinductive sets; they can be defined uniformly on X.) Below, we will define a coinductive operator Θ which simultaneously applies ϕ, defines the prewellordering associated to ϕ, and codes initial segments of $L_\alpha(\mathbb{R})$ along the prewellordering given by ϕ. Given $X \subset \mathbb{R}$, write

$$(i, X) = \{(i, x) : x \in \mathbb{R}\}.$$

Conversely, we write

$$X_i = \{x \in \mathbb{R} : (i, x) \in X\}.$$

Let $X \subset \mathbb{R}$ and $\preceq$ be a binary relation on X. Suppose moreover that U is a set of real numbers and that X consists only of tuples of real numbers none of whose first coordinate belongs to U. We define a set $\mathrm{Def}(X, \preceq)$ and a binary relation $\preceq^+$ such that if $\preceq$ has field X and codes $L_\alpha(\mathbb{R})$, then $\preceq^+$ is a relation on $X \oplus \mathrm{Def}(X, \preceq)$ which codes $L_{\alpha+1}(\mathbb{R})$. The reason for mentioning U at all will become clear soon; roughly, it contains indicators which we will use to identify which reals belong to $\mathrm{Def}(X, \preceq)$ and which do not.

$\mathrm{Def}(X, \preceq)$ is the set of all tuples $(x_0, \varphi, \vec{b})$, where $x_0 \in U$, φ is a formula of arity $\mathrm{LTH}(\vec{b})+1$ in the vocabulary $\{=, \prec\}$ and $\vec{b}$ is a finite tuple of elements of X. For $x, y \in X \oplus \mathrm{Def}(X, \preceq)$, we put $x \prec^* y$ if, and only if, one of the following holds:

1. $x, y \in X$ and $x \prec y$;
2. $y \in \mathrm{Def}(X, \preceq)$ is of the form $(x_0, \varphi, \vec{b})$ and $(X, \prec, \equiv) \models \varphi(\vec{b}, x)$.

Afterwards, let xEy if, and only if x and y have exactly the same $\prec^*$-predecessors, and put $x \preceq^+ y$ if, and only if, xEy or there are aEx and bEy such that $a \prec^* b$. Hence, the relation $\prec^*$ is a first approximation to $\prec^+$. It extends $\prec$ by specifying which of the elements of X are smaller than which of the new elements. It may thus be that an element x of $\mathrm{Def}(X, \preceq)$ turns out to have exactly the same $\prec^*$-predecessors as some other element y of $\mathrm{Def}(X, \preceq)$

or of X. We would then like to force these x and y to be $\equiv^+$-equivalent, which is what we do. Note that this definition depended on U, so when in need of precision, we may write $\mathrm{Def}(X, \preceq, U)$ for $\mathrm{Def}(X, \preceq)$ and $\preceq^{+U}$ for $\preceq^+$. By inspecting the construction, one sees that both $\mathrm{Def}(X, \preceq, U)$ and $\preceq^+$ are, say, hyperprojective (in fact, much simpler) in X, U, and $\preceq$. Another feature of this definition is that if $\{\preceq_\iota\}_\iota$ is an increasing family of relations obtained this way, and $\preceq_0$ is a coding of some $L_\alpha(\mathbb{R})$ then $\bigcup_\iota \preceq_\iota$ is a coding of some $L_\beta(\mathbb{R})$ (this is where the set U in the definition is used).

We now define the operator Θ. Given $X \subset \mathbb{R}$, Θ will map X to a set Y with four parts, Y_0, Y_1, Y_2, and Y_3 defined in terms of X_0, X_1, X_2, and X_3. If X is of the right form, Y_1 will be a prewellordering of some length $\alpha + 1$ and Y_3 will be a coding of $L_{\alpha+1}(\mathbb{R})$ with field Y_2.

1. $Y_0 = \phi(X_0)$;
2. $Y_1 = X_1 \cup \Big\{(x, y) : x \in Y_0 \cup \mathrm{field}(X_1) \text{ and } y \in Y_0 \setminus \mathrm{field}(X_1)\Big\}$;
3. $Y_2 = X_2 \cup \mathrm{Def}(X_2, X_3, U)$, where $U = Y_0 \setminus \mathrm{field}(X_1)$;
4. $Y_3 = X_3^{+U}$, where $U = Y_0 \setminus \mathrm{field}(X_1)$.

The class of coinductive operators is closed under Boolean connectives and real quantifiers, so Θ is coinductive. The operator generates the inductive definition ϕ^∞ on its first component, while simultaneously coding $\preceq_\phi$ in its second component. At each stage ξ, all new elements added to the field of $\preceq_\phi$, i.e., those of $\preceq_\phi$-rank $\xi + 1$, are used to define the new elements of the set $(\Theta^\xi)_2$ which acts as the field of a relation $(\Theta^\xi)_3$ that codes $L_{\xi+1}(\mathbb{R})$.

Any fixed point of Θ will contain a fixed point of ϕ in its first component. Because ϕ is universal coinductive, an argument as in [AR74, Theorem 8.5] (but using continuous reducibility in place of many-one reducibility[3]) shows that

$$|\Theta| = |\phi| = |\mathsf{coIND}|.$$

Hence, we need to show that $|\Theta|$ is Π^+_1-reflecting (although, to show that there is a Π^+_1-reflecting ordinal $\leq |\mathsf{coIND}|$—which is really the point of this lemma—, we do not need the argument from [AR74, Theorem 8.5]). Recall that, given a coding $\preceq$ of some $L_\alpha(\mathbb{R})$, in this section we write $|x|_\preceq$ for the element of $L_\alpha(\mathbb{R})$ coded by x, if any.

[3] A proof of the Kleene's Recursion Theorem in this context can be found e.g., in [KW10, Theorem 3.1].

Sublemma 10.4.4. *Let ψ be a Π_1 formula in the language of set theory. Then, there is a coinductive operator Ψ, such that whenever $\lambda \leq |\Theta|$,*

1. *for all $c_1, \dots, c_l \in (\Theta^\lambda)_2$,*

$$L_{\lambda^+}(\mathbb{R}) \models \psi(|c_1|_{(\Theta^\lambda)_3}, \dots, |c_l|_{(\Theta^\lambda)_3}) \leftrightarrow (c_1, \dots, c_l) \in \Psi(\Theta^\lambda).$$

2. *for all $c_1, \dots, c_l \in \mathbb{R}$, if there is some $i < l+1$ such that $c_i \notin (\Theta^\lambda)_2$, then*

$$(c_1, \dots, c_l) \notin \Psi(\Theta^\lambda).$$

Proof. The main observation is that by Barwise-Gandy-Moschovakis [BGM71, Lemma 2.9], for every admissible set A, a relation P on A is coinductive on A with parameters if, and only if, it is Π_1 over A^+, with parameters in $A \cup \{A\}$. Moreover, by inspecting the proof, one sees that this correspondence is uniform, in that the definition of the coinductive relation depends only on the Π_1 formula and its parameters, and not on the admissible set A (as long as it contains the parameters).[4]

Hence, given a Π_1-formula ψ, the set of all $a_1, \dots, a_l \in L_\lambda(\mathbb{R})$ such that

$$L_{\lambda^+}(\mathbb{R}) \models \psi(a_1, \dots, a_l)$$

is a coinductive subset of $L_\lambda(\mathbb{R})$ (and thus of $L_{\lambda+1}(\mathbb{R})$). Since $(\Theta^\lambda)_3$ is a coding of $L_{\lambda+1}(\mathbb{R})$, there is a surjection

$$\rho : (\Theta^\lambda)_2 \to L_{\lambda+1}(\mathbb{R}),$$

such that a pair (x, y) belongs to the strict part of $(\Theta^\lambda)_3$ if, and only if, $\rho(x) \in \rho(y)$. It follows that the preimage of a coinductive subset of $L_{\lambda+1}(\mathbb{R})$ under ρ is coinductive on $\mathbb{R}$ from the parameters $(\Theta^\lambda)_2$ and $(\Theta^\lambda)_3$. Hence, there is a coinductive operator Ψ' such that

1. for all $c_1, \dots, c_l \in (\Theta^\lambda)_2$,

$$L_{\lambda^+}(\mathbb{R}) \models \psi(|c_1|_{(\Theta^\lambda)_3}, \dots, |c_l|_{(\Theta^\lambda)_3}) \leftrightarrow (c_1, \dots, c_l) \in \Psi'((\Theta^\lambda)_2 \oplus (\Theta^\lambda)_3).$$

[4] This can also be shown directly for admissible sets of the form $L_\alpha(\mathbb{R})$: given such a set, one defines an operator, elementary in $L_\alpha(\mathbb{R})$, that successively outputs extensions of $L_\alpha(\mathbb{R})$, like Θ does. To ensure that the operator is elementary, one can have it e.g., add only Σ_1-definable sets at each stage, instead of all first-order–definable sets. One can ask the operator to also check at each limit stage whether the Π_1 fact with parameters in $L_\alpha(\mathbb{R})$ holds of the structure it outputs. Like we did with Θ, this operator can be ensured to have closure ordinal α^+ by asking it to generate a universal $\Sigma_1(L_\alpha(\mathbb{R}))$ inductive definition in parallel. The operator for that can be obtained e.g., from the universal $\Sigma_1(P)$ subset of $\mathbb{R}$, where P is a subset of $\mathbb{R}$ coding $L_\alpha(\mathbb{R})$. The set P can be obtained uniformly in a $\Sigma_1(L_\alpha(\mathbb{R}))$ way (for multiplicatively indecomposable α; see Steel [Ste08c, Lemma 1.4]).

2. for all $c_1, \ldots, c_l \in \mathbb{R}$, if there is some $i < l+1$ such that $c_i \notin (\Theta^\lambda)_2$, then
$$(c_1, \ldots, c_l) \notin \Psi'((\Theta^\lambda)_2 \oplus (\Theta^\lambda)_3).$$

Now, both $(\Theta^\lambda)_2$ and $(\Theta^\lambda)_3$ reduce to Θ^λ continuously, and this reduction is uniform in λ, so there is an operator Ψ as desired. □

Let ψ be a Π_1 sentence and let Ψ be given by the sublemma, so that for all $\lambda \leq |\Theta|$ and all $c_1, \ldots, c_l \in (\Theta^\lambda)_2$, we have

$$L_{\lambda^+}(\mathbb{R}) \models \psi(|c_1|_{(\Theta^\lambda)_3}, \ldots, |c_l|_{(\Theta^\lambda)_3}) \leftrightarrow (c_1, \ldots, c_l) \in \Psi(\Theta^\lambda).$$

Since ϕ was chosen to be a universal coinductive operator, $\Theta(X)$ is complete coinductive for all X, so there is a continuous function

$$g : \mathbb{R}^l \mapsto \mathbb{R}$$

such that for all $c_1, \ldots, c_l \in \mathbb{R}$ and all $X \subset \mathbb{R}$ do we have

$$(c_1, \ldots, c_l) \in \Psi(X) \leftrightarrow g(c_1, \ldots, c_l) \in \Theta(X).$$

Hence,

$$L_{\lambda^+}(\mathbb{R}) \models \psi(|c_1|_{(\Theta^\lambda)_3}, \ldots, |c_l|_{(\Theta^\lambda)_3}) \leftrightarrow g(c_1, \ldots, c_l) \in \Theta(\Theta^\lambda). \tag{10.4}$$

Suppose then that there are elements $\gamma_1, \ldots, \gamma_l$ of $L_\lambda(\mathbb{R})$ such that

$$L_{|\Theta|^+}(\mathbb{R}) \models \psi(\gamma_1, \ldots, \gamma_l).$$

By construction, $(\Theta^{<\lambda})_3$ is a coding of $L_\lambda(\mathbb{R})$ for each λ, so $(\Theta^\infty)_3$ is a coding of $L_{|\Theta|}(\mathbb{R})$ with field $(\Theta^\infty)_2$. Thus, there is a surjection

$$\rho : (\Theta^\infty)_2 \to L_{|\Theta|}(\mathbb{R})$$

such that the pair (x, y) belongs to the strict part of $(\Theta^\infty)_3$ if, and only if, $\rho(x) \in \rho(y)$. Pick $c_1, \ldots, c_l$ such that for each $0 < i < l+1$, $\rho(c_i) = \gamma_i$. Then,

$$L_{|\Theta|^+}(\mathbb{R}) \models \psi(|c_1|_{(\Theta^\infty)_3}, \ldots, |c_l|_{(\Theta^\infty)_3}).$$

By (10.4),

$$g(c_1, \ldots, c_l) \in \Theta(\Theta^\infty) \subset \Theta^\infty.$$

$|\Theta|$ is a limit ordinal, so there is some $\lambda < |\Theta|$ such that

$$g(c_1, \ldots, c_l) \in \Theta(\Theta^\lambda),$$

so that, by (10.4),

$$L_{\lambda^+}(\mathbb{R}) \models \psi(|c_1|_{(\Theta^\lambda)_3}, \ldots, |c_l|_{(\Theta^\lambda)_3}).$$

Notice that the sequence of sets $\{(\Theta^\lambda)_2 : \lambda < |\Theta|\}$, as well as the sequence of isomorphisms witnessing that each $(\Theta^\lambda)_3$ is a coding of $L_{\lambda+1}(\mathbb{R})$ are strictly increasing, and this implies that

$$|c_i|_{(\Theta^\lambda)_3} = |c_i|_{(\Theta^\infty)_3}$$

for each $0 < i < l + 1$, which yields the result. This proves the lemma. □

Lemma 10.4.5. *If κ is Π^+_1-reflecting, then $|\mathsf{coIND}| \leq \kappa$.*

Proof. This proof is like that of [AR74, Lemma 10.1]. Suppose κ is Π^+_1-reflecting and let ϕ be a coinductive operator. Since κ is Π^+_1-reflecting, it is $\mathbb{R}$-recursively inaccessible. Because coinductive relations are universal relations on the next admissible set, it follows that if $A \in \mathcal{P}(\mathbb{R}) \cap L_\alpha(\mathbb{R})$, then $L_{\alpha^+ +1}(\mathbb{R})$ can compute coinductive relations on A. Now, $L_\kappa(\mathbb{R})$ can compute the sequence $\{\phi^\lambda : \lambda < \kappa\}$ in a Π_1 way (with parameters): x is the λth element of this sequence if every transitive set of the form $L_\alpha(\mathbb{R})$ containing an increasing $(\lambda + 1)$-sequence of $\mathbb{R}$-admissible sets believes that x is the λth element of the sequence. Moreover, this definition is uniform on $\mathbb{R}$-recursively inaccessible ordinals. Thus, letting ψ be a Π_1 formula such that

$$a \in \phi^{<\kappa} \leftrightarrow L_\kappa(\mathbb{R}) \models \exists \lambda\, \psi(a, \lambda),$$

the question of whether a belongs to $\phi(\phi^{<\kappa})$ is coinductive over $\phi^{<\kappa}$, hence over $L_\kappa(\mathbb{R})$, hence universal over $L_{\kappa^+}(\mathbb{R})$, so, by Π^+_1-reflection, $\phi(\phi^{<\kappa}) \subset \phi^{<\kappa}$, as desired. □

This concludes the proof of Theorem 10.4.2. The main consequence of interest to us is:

Corollary 10.4.6. $\mathcal{P}(\mathbb{R}) \cap \Sigma^{L_{\sigma_\mathbb{R}}(\mathbb{R})}_1 = \mathfrak{D}^\mathbb{R}\mathbf{\Sigma}^0_2$.

Proof. By Theorems 10.3.1 and 10.4.2, we need to show that

$$\mathcal{P}(\mathbb{R}) \cap \Sigma^{L_{|\mathsf{coIND}|}(\mathbb{R})}_1 = \mathsf{coIND} \cap \mathsf{IND}.$$

That the class on the right-hand side is contained in the one on the left is clear. The converse follows from the proof of Lemma 10.4.3: one can use a coinductive operator to inductively generate codes for initial segments of $L(\mathbb{R})$ below $|\mathsf{coIND}|$, so an existential statement about $L_{|\mathsf{coIND}|}(\mathbb{R})$ can be rephrased in terms of membership in the least fixed point of a coinductive operator. □

10.5 F_σ-determinacy

Lemma 10.5.1. *Suppose that all sets in $\Im^{\mathbb{R}}\mathbf{\Sigma}^0_2$ are determined. Then, all $\mathbf{\Sigma}^0_2$ games of length ω^2 are determined.*

Proof. Note that $\mathbf{\Sigma}^0_2$ games of length ω with moves in $\mathbb{R}$ are determined. Suppose that all sets in $\Im^{\mathbb{R}}\mathbf{\Sigma}^0_2$ are determined and let $A \in \mathbf{\Sigma}^0_2$. We show that the game of length ω^2 with payoff A is determined. Let us refer to this game as G. The proof begins very much like that of Theorem 10.3.1: we begin by replacing G with a game $G^*(\langle\rangle)$. Without loss of generality we shall assume that there is a recursive set P such that A is given by

$$x \in A \leftrightarrow \exists n \, \forall m \, (n, x \restriction m) \in P,$$

For s a finite sequence of real numbers of even length, define $G^*(s)$ to be the following game:

1. Players I and II alternate ω^2-many turns playing natural numbers to produce a countable sequence $\{\alpha_n : n \in \mathbb{N}\}$ of real numbers.

2. After ω^2-many rounds have taken place, Player I wins if, letting

   $$t(m) = s^\frown\langle \alpha_0, \alpha_1, \ldots, \alpha_m \rangle,$$

   we have

   $$\exists n \, \forall m \, \forall m' < m \, (n, t(m) \restriction m') \in P.$$

As before, Player I has a winning strategy for G if, and only if, she has one for $G^*(\langle\rangle)$. Moreover, Player II has a winning strategy for G if, and only if, she has one for $G^*(\langle\rangle)$. The game $G^*(s)$ is like $G^*(\langle\rangle)$, except that we assume that s has already been played.

Let n_s be a natural number, s be a sequence of natural numbers of even length $\omega \cdot n_s$ and X be a set of sequences of natural numbers of length $<\omega^2$. Consider the following game, $G(X, s)$:

1. Players I and II alternate ω^2-many turns playing natural numbers to produce digits corresponding to a countable sequence $\{\alpha_n : n \in \mathbb{N}\}$ of real numbers.

2. After ω^2-many rounds have taken place, Player I wins if, and only if, letting

   $$t(m) = s^\frown\langle \alpha_0, \ldots, \alpha_m \rangle,$$

   one of the following holds for each $m \in \mathbb{N}$:

(a) there is $n \leq n_s$ such that we have

$$\forall m' < m \, (n, t(m) \restriction m') \in P.$$

(b) $t(m) \in X$.

Thus $G(X, s)$ is a game on $\mathbb{N}$ of length ω^2. Write

$$\phi(s, X) = \text{“Player I has a winning strategy in } G(X, s).\text{”} \tag{10.5}$$

The argument from Theorem 10.3.1 shows that if $s \in \phi^\infty$, then Player I has a winning strategy in $G^*(s)$. To complete the proof, it remains to show that if $s \notin \phi^\infty$, then Player II has a winning strategy in $G^*(s)$.

Notice that $G(X, s)$ is a game of length ω^2 with moves in $\mathbb{N}$ and payoff in Π^0_1 (using X as a second-order oracle). We consider the following auxiliary game, which we shall denote by $H(X, s)$:

$$\begin{array}{c|cccc} I & & \sigma_0 & & \sigma_1 & \dots \\ II & \tau_0 & & \tau_1 & & \dots \end{array}$$

Here, σ_i and τ_i are real numbers coding strategies for games on $\mathbb{N}$ of length ω. Player I wins the game if, and only if, the sequence

$$(s, \sigma_0 * \tau_0 *, \sigma_1 * \tau_1, \dots)$$

satisfies the winning condition of $G(X, s)$. Here, $\sigma * \tau$ denotes the result of facing off the strategies σ and τ against each other. Thus, $H(X, s)$ is a game of length ω with moves in $\mathbb{R}$ and payoff in $\Pi^0_1(X)$. We will need a sublemma:

Sublemma 10.5.2. *Suppose that all subsets of $\mathbb{R}$ in $\Im^{\mathbb{R}}\mathbf{\Sigma}^0_1(X)$ are determined. Then, the following are equivalent:*

1. *Player I has a winning strategy in $H(X, s)$; and*
2. *Player I has a winning strategy in $G(X, s)$.*

Moreover, the games are determined.

Proof Sketch. The equivalence is proved by arguing as in Chapter 7. This is possible because the game $H(s, X)$ is $\Pi^0_1(X)$. The proof of Chapter 7 uses determinacy of sets in $\Im^{\mathbb{R}}\mathbf{\Sigma}^0_1$; in this case, we need determinacy of sets in $\Im^{\mathbb{R}}\mathbf{\Sigma}^0_1(X)$, which we are assuming. $H(X, s)$ is closed and thus determined by the Gale-Stewart theorem. The proof of Chapter 7 also shows that if Player II has a winning strategy in $H(X, s)$, then she has one in $G(X, s)$, so that this game is also determined. □

Remark 10.5.3. Sublemma 10.5.2 essentially states that Lemma 10.5.1 holds if one replaces $\mathbf{\Sigma}^0_2$ by $\mathbf{\Pi}^0_1$. The part of the proof of Chapter 7 used in the proof of Sublemma 10.5.2 goes through for $\mathbf{\Pi}^0_1$ games. It would also go through for $\mathbf{\Sigma}^0_1$ games if one switched the order in which the players move in the definition of $H(X, s)$. The proof, however, does not go through for more complicated games. By different arguments (involving the theory of scales in $L(\mathbb{R})$), one can prove analogues of Lemma 10.5.1 for more complicated pointclasses, up to $\mathbf{\Delta}^1_1$. The result of replacing $\mathbf{\Sigma}^0_2$ by $\mathbf{\Pi}^1_1$ in the statement of Lemma 10.5.1 is not provable in ZFC.

It follows from Corollary 10.2.11 that if $X \in L_\alpha(\mathbb{R})$, then all winning strategies for games on $\mathbb{R}$ with payoff in $\mathbf{\Sigma}^0_1(X)$ are definable over $L_{\alpha^+}(\mathbb{R})$, so $\mathcal{D}^{\mathbb{R}}\mathbf{\Sigma}^0_1(X)$ belongs e.g., to $L_{\alpha^{++}}(\mathbb{R})$.

Sublemma 10.5.4. *$\mathcal{D}^{\mathbb{R}}\mathbf{\Sigma}^0_2$ is closed under the open-game-on-$\mathbb{R}$ quantifier; i.e., if $X \in \mathcal{D}^{\mathbb{R}}\mathbf{\Sigma}^0_2$, then $\mathcal{D}^{\mathbb{R}}\mathbf{\Sigma}^0_1(X) \subset \mathcal{D}^{\mathbb{R}}\mathbf{\Sigma}^0_2$.*

Proof. This can be proved directly by an argument like the one of (10.3) in Theorem 10.3.1. Alternatively, one can also appeal to the "Main Lemma" of Moschovakis [Mos74] (as in the proof of Theorem 10.3.1): since operators definable from X by an open game quantifier are X-inductive (by Corollary 10.2.11) and $\mathcal{D}^{\mathbb{R}}\mathbf{\Sigma}^0_2$ is a Spector class, it suffices to show that $\mathcal{D}^{\mathbb{R}}\mathbf{\Sigma}^0_2$ is closed under the inductive step. But the inductive step is positive analyical on X and a real parameter, so clearly $\mathcal{D}^{\mathbb{R}}\mathbf{\Sigma}^0_2$ is closed under it. □

Write

$$\phi'(s, X) = \text{"Player I has a winning strategy in } H(X, s)\text{."}$$

Clearly, $\phi'(s, X)$ is an operator in $\mathcal{D}^{\mathbb{R}}\mathbf{\Pi}^0_1$ and thus in coIND (by Corollary 10.2.11). By Theorem 10.4.2, there is some $\eta \leq \sigma_{\mathbb{R}}$ such that $\phi'^\infty = \phi'^\eta$. By Corollary 10.4.6 and the hypothesis of the lemma,

$$L_{\sigma_{\mathbb{R}}}(\mathbb{R}) \models \mathsf{AD}.$$

Since $\sigma_{\mathbb{R}}$ is recursively $\mathbb{R}$-inaccessible, i.e., $\mathbb{R}$-admissible and a limit of $\mathbb{R}$-admissibles, it follows that for every $\zeta < \eta$, every subset of $\mathbb{R}$ in $\mathcal{D}^{\mathbb{R}}\mathbf{\Sigma}^0_1(\phi'^\zeta)$ is determined. Hence, an induction along ζ shows that

$$\phi'^\zeta = \phi^\zeta$$

for all $\zeta < \eta$. It follows that

$$\phi^\infty = \phi'^\infty \in \Sigma^{L_{\sigma_{\mathbb{R}}}(\mathbb{R})}_1 = \mathcal{D}^{\mathbb{R}}\mathbf{\Sigma}^0_2.$$

(In case $\eta = \sigma^{\mathbb{R}}$, the last step of the induction follows from Sublemma 10.5.4.)

Now, suppose that $s \notin \phi^\infty$. By definition, Player I does not have a winning strategy in $G(\phi^\infty, s)$. By the two sublemmata, $G(\phi^\infty, s)$ is determined. Thus, Player II has a winning strategy in $G(\phi^\infty, s)$. But then, an argument as in the proof of Theorem 10.3.1 shows how to turn this into a winning strategy for $G^*(s)$. This concludes the proof of the lemma. □

We are now led to the main theorem of the chapter.

Theorem 10.5.5. *The following are equivalent:*

1. $\mathbf{\Sigma}^0_2$*-determinacy for games of length* ω^2*;*
2. *There is a* Π^+_1*-reflecting model of* AD *containing* $\mathbb{R}$*.*

Proof. If $\mathbf{\Sigma}^0_2$ games of length ω^2 are determined, then clearly sets in $\mathfrak{I}^{\mathbb{R}}\mathbf{\Sigma}^0_2$ are determined, so that $L_{\sigma_{\mathbb{R}}}(\mathbb{R}) \models \mathsf{AD}$, by Corollary 10.4.6.

Conversely, suppose that there is a Π^+_1-reflecting model of AD containing $\mathbb{R}$. By Corollary 10.2.4, $L_{\sigma_{\mathbb{R}}}(\mathbb{R}) \models \mathsf{AD}$. Since $\sigma_{\mathbb{R}}$ is $\mathbb{R}$-admissible,

$$\mathcal{P}(\mathbb{R}) \cap L_{\sigma_{\mathbb{R}}}(\mathbb{R}) = \mathcal{P}(\mathbb{R}) \cap \Delta_1^{L_{\sigma_{\mathbb{R}}}(\mathbb{R})}.$$

Let Γ be the pointclass of all sets of reals which are Σ_1-definable over $L_{\sigma_{\mathbb{R}}}(\mathbb{R})$ with parameters. Using that $L_{\sigma_{\mathbb{R}}}(\mathbb{R}) \models \mathsf{AD}$, Steel [Ste08c, Theorem 2.1] implies that Γ has the scale property. Since $\sigma_{\mathbb{R}}$ is $\mathbb{R}$-admissible, Γ is closed under existential and universal real quantification and thus by Moschovakis [Mos71], it has the uniformization property. Hence, the hypotheses for the Kechris-Woodin determinacy transfer theorem [KW83] are satisfied, and we may apply it to conclude that all sets in $\Sigma_1^{L_{\sigma_{\mathbb{R}}}(\mathbb{R})}$ are determined. By Corollary 10.4.6, all sets in $\mathfrak{I}^{\mathbb{R}}\mathbf{\Sigma}^0_2$ are determined. By Lemma 10.5.1, $\mathbf{\Sigma}^0_2$-games of length ω^2 are determined. □

10.6 Concluding Remarks

A classical theorem of Kripke states that, in L, the least α such that L_α is a model of $\mathsf{KP} + \Sigma_1$-separation is the union of a chain of Σ_1-elementary substructures. The situation is analogous in $L(\mathbb{R})$. This and Theorem 10.1.2 imply the consistency results mentioned in the introduction.

Theorem 10.6.1. *The following theories are strictly increasing in consistency strength:*

1. $\mathsf{KP} + \mathsf{DC} +$ "$\mathbb{R}$ *exists*" + AD*;*
2. $\mathsf{KP} + \mathsf{DC} +$ "$\mathbb{R}$ *exists*" + $\mathbf{\Sigma}^0_2$*-determinacy for games of length* ω^2*;*
3. $\mathsf{KP} + \mathsf{DC} +$ "$\mathbb{R}$ *exists*" + AD + Σ_1*-separation.*

Proof. Working in $\mathsf{KP} + \mathsf{DC} +$ "$\mathbb{R}$ exists," assume that $\mathbf{\Sigma}^0_2$ games of length ω^2 are determined. We show that there is a model of

$$\mathsf{KP} + \mathsf{DC} + \text{“}\mathbb{R} \text{ exists”} + \mathsf{AD}.$$

Consider the following game of length ω^2:

1. Player I begins by choosing a $\mathbf{\Sigma}^0_1$ game of length ω^2;
2. Player II decides which role she wants to have in the game;
3. Players I and II play the game chosen by Player I, taking the roles specified by Player II.

This game is perhaps not $\mathbf{\Sigma}^0_1$, but it is certainly, say, Δ^0_2. It clearly cannot be won by Player I, and any winning strategy for Player II easily reduces continuously to a winning strategy for some player for any prescribed $\mathbf{\Sigma}^0_1$ game of length ω^2. Since $\mathbb{R}$ exists, one may use collection to conclude that there is a set A containing winning strategies for all $\mathbf{\Sigma}^0_1$ games of length ω^2. From A and $\mathbb{R}$ one may use Σ_0-separation to conclude that there is a universal $\mathcal{C}^{\mathbb{R}}\mathbf{\Sigma}^0_1$ set. Hence, there is a universal IND set, from which one can easily compute a prewellordering of $\mathbb{R}$ whose length is an $\mathbb{R}$-admissible ordinal κ. By collection, this ordinal exists, so $L_\kappa(\mathbb{R}) \models \mathsf{KP} + \mathsf{DC}$ and

$$\mathsf{IND} = \mathcal{P}(\mathbb{R}) \cap \Sigma_1^{L_\kappa(\mathbb{R})}.$$

Since all $\mathbf{\Sigma}^0_2$ games of length ω^2 are determined, all $\mathcal{C}^{\mathbb{R}}\mathbf{\Sigma}^0_1$ games of length ω are determined, and so

$$L_\kappa(\mathbb{R}) \models \mathsf{AD}.$$

For the remaining implication, suppose that AD holds and Σ_1-separation holds. It follows that

$$L(\mathbb{R}) \models \mathsf{AD} + \Sigma_1\text{-Separation}.$$

To see this, we assume without loss of generality that for no ordinal η do we have

$$L_\eta(\mathbb{R}) \models \mathsf{AD} + \Sigma_1\text{-Separation},$$

i.e., that the least non-$\mathbb{R}$-projectible ordinal does not exist. Thus, it suffices to show that every subset of $\mathbb{R}$ which is Σ_1-definable (with parameters) over $L(\mathbb{R})$ belongs to $L(\mathbb{R})$. If not, then by Steel [Ste08c, Lemma 1.14], there is a partial surjection from $\mathbb{R}$ onto $L(\mathbb{R})$ which is Σ_1-definable over $L(\mathbb{R})$ with parameters. However, $L(\mathbb{R})$ is a Σ_1-definable class and Σ_1-separation holds (in V), so this surjection is actually a set. But this is impossible, for its restriction to the ordinals is also a surjection. Thus,

$$L(\mathbb{R}) \models \mathsf{AD} + \Sigma_1\text{-Separation}$$

as claimed. Working in $L(\mathbb{R})$, there are arbitrarily large $\mathbb{R}$-stable ordinals. But it is immediate from the definition that if

$$L_\alpha(\mathbb{R}) \prec_1 L_{\alpha^+ + 1}(\mathbb{R}),$$

(recall the definition of α^+ in this chapter) then $\sigma_{\mathbb{R}} < \alpha$, so there is a (set) model of KP in which all games of length ω^2 are determined (and AD holds in this model, additionally). □

By a slightly more elaborate argument involving Inner Model Theory (one can e.g., use the results in Part II of Müller [Uhl16]), one can replace KP by ZFC in the statement of Theorem 10.6.1(2) (and, in fact, by much stronger theories). Similarly, and using Theorem **??**, one can replace Theorem 10.6.1(1) by the theory $\mathsf{ZFC} + \mathbf{\Sigma}^0_1$-determinacy for games of length ω^2. We omit the details.

11

A Model of $\mathsf{KP} + \mathsf{AD}_{\mathbb{R}}$

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11.1 Introduction

Given sets X and $A \subset X^{\mathbb{N}}$, the Gale-Stewart game of length ω on A is defined as follows: two players, I and II, alternate turns playing elements of X infinitely many times, producing a sequence $x \in X^{\mathbb{N}}$. Player I wins if $x \in A$; otherwise, Player II wins. A is *determined* if one of the players has a winning strategy for this game.

The Axiom of Determinacy (AD) states that all Gale-Stewart games of length ω on (subsets of) $\mathbb{N}$ are determined. Similarly, the Axiom of Real Determinacy ($\mathsf{AD}_{\mathbb{R}}$) states that all Gale-Stewart games of length ω on $\mathbb{R}$ are determined. Both axioms are inconsistent with the Axiom of Choice, but are consistent with Zermelo-Fraenkel set theory (ZF) without the Axiom of Choice, although this fact is not itself provable in ZFC. To do this, one needs to make use of *large cardinal axioms*, strengthenings of the axiom of infinity. A theorem of Woodin (see Koellner-Woodin [KW10] and Steel [Steb]) states that $\mathsf{ZF} + \mathsf{AD}$ is consistent if, and only if, ZFC is consistent with the existence of infinitely many Woodin cardinals. Two unpublished theorems of Steel and Woodin show that $\mathsf{ZF} + \mathsf{AD}_{\mathbb{R}}$ is consistent if, and only if, ZFC is consistent with the existence of a cardinal λ which is a limit of Woodin cardinals and of cardinals which are $<\lambda$-strong. In particular, it is

stronger than ZFC with a proper class of Woodin cardinals.

Kripke-Platek set theory (KP) is a weak set theory studied for its recursion-theoretic properties. Many results about recursion on $\mathbb{N}$ generalize to arbitrary sets which satisfy all axioms of KP; these are called *admissible sets*. A natural question is that of the strength of the theories $\mathsf{KP} + \mathsf{AD}$ and $\mathsf{KP} + \mathsf{AD}_{\mathbb{R}}$ (here, AD and $\mathsf{AD}_{\mathbb{R}}$ are assumed to include a clause stating that the real numbers form a set—this is not a consequence of KP). Clearly, $\mathsf{ZF} + \mathsf{AD}$ proves the consistency of $\mathsf{KP} + \mathsf{AD}$, although the latter theory is not significantly weaker than the former; e.g., $\mathsf{KP} + \mathsf{AD}$ proves the consistency of ZFC with all finite amounts of Woodin cardinals, and much more (see the remark at the end of this section). One may be led to conjecture that $\mathsf{KP} + \mathsf{AD}_{\mathbb{R}}$ has similar consistency strength to that of $\mathsf{ZF} + \mathsf{AD}_{\mathbb{R}}$. Here, we observe that this is not the case at all:

Theorem 11.1.1. *Suppose there are ω^2 Woodin cardinals and a measurable cardinal above them. Then, there is a transitive model of $\mathsf{KP} + \mathsf{AD}_{\mathbb{R}}$ containing all reals.*

The discrepancy between Steel and Woodin's theorem on $\mathsf{AD}_{\mathbb{R}}$ and Theorem 11.1.1 can be explained by recalling the theorem (due independently to Martin and Woodin, both unpublished) whereby $\mathsf{AD}_{\mathbb{R}}$ is equivalent (over, say, $\mathsf{ZF} + \mathsf{AD} + \mathsf{DC}$) to the assertion that all sets of reals are Suslin. The model M considered in the proof below will satisfy that all sets are Suslin. However, even if one assumes that it can be extended to a transitive model of $\mathsf{ZF} + \mathsf{AD}$, the least such extension will have many sets that are not Suslin.

For all undefined notions in what follows, we refer the reader to Kechris [Kec16]. Let Q be a quantifier on $\mathbb{R}$ and consider the language $L(Q)$ of second-order arithmetic augmented with a symbol for Q as a quantifier and constants for each element of $\mathbb{R}$. If one further extends the language by an additional predicate symbol, the notion of a formula being *positive* is defined in the natural way. We write $\mathsf{IND}(Q)$ for the pointclass of all sets that are inductively definable by a positive operator which is definable in $L(Q)$. We also say:

$$A \text{ is } Q\text{-hyperprojective} \longleftrightarrow A \in \mathsf{IND}(Q) \wedge \mathbb{R} \setminus A \in \mathsf{IND}(Q).$$

As is customary, we identify the real numbers $\mathbb{R}$ and the Baire space $\mathbb{N}^{\mathbb{N}}$ in what follows.

Proof of the theorem. Given a set $A \subset \mathbb{R}^2$, write $A_x = \{y \in \mathbb{R} : (x, y) \in A\}$ and define $\mathfrak{I}^{\mathbb{R}}_{\omega^2} A$ to be the set of all $x \in \mathbb{R}$ such that Player I has a winning

strategy in the game of length ω^2 on $\mathbb{R}$ with payoff A_x.[1] Write

$$\partial^{\mathbb{R}}_{\omega^2}\mathbf{\Sigma}^0_1 = \{\partial^{\mathbb{R}}_{\omega^2}A : A \text{ is open}\}.$$

We state two lemmata and postpone their proofs until the next section.

Lemma 11.1.2. $\mathsf{IND}(\partial^{\mathbb{R}}) \subset \partial^{\mathbb{R}}_{\omega^2}\mathbf{\Sigma}^0_1$.

Lemma 11.1.3. *Suppose that $\partial^{\mathbb{R}}$-hyperprojective games of length ω^2 on $\mathbb{N}$ are determined. Then every $\partial^{\mathbb{R}}$-hyperprojective game of length ω on $\mathbb{R}$ has a $\partial^{\mathbb{R}}$-hyperprojective winning strategy.*

By a theorem of Aczel [Acz75], $\mathsf{IND}(\partial^{\mathbb{R}})$ is a Spector class on $\mathbb{R}$. A theorem of Moschovakis [Mos74, Theorem 9E.1] then implies that there is an admissible set M whose sets of reals are precisely the $\partial^{\mathbb{R}}$-hyperprojective sets.

To finish the proof of the theorem, suppose that there are ω^2 Woodin cardinals and a measurable cardinal above them. It follows from a theorem of Steel [Ste93] that the extender model $M^{\sharp}_{\omega^2}$ exists and is countably iterable. By work of Neeman [Nee04, Section 2], this implies that analytic games of length ω^3 on $\mathbb{N}$ are determined. It follows that games of length ω^2 on $\mathbb{N}$ with payoff in $\partial^{\mathbb{R}}_{\omega^2}\mathbf{\Sigma}^0_1$ are also determined (this only needs *open* determinacy for games of length ω^3; see the remark below). By Lemma 11.1.2, $\partial^{\mathbb{R}}$-hyperprojective games of length ω^2 on $\mathbb{N}$ are determined. By Lemma 11.1.3, every $\partial^{\mathbb{R}}$-hyperprojective game of length ω on $\mathbb{R}$ has a $\partial^{\mathbb{R}}$-hyperprojective winning strategy, so M satisfies $\mathsf{AD}_{\mathbb{R}}$. □

Remark 11.1.4.

1. The converse of Lemma 11.1.2 is true, but was not used in the proof. Similarly, determinacy for games of length ω^2 on $\mathbb{N}$ with payoff in $\partial^{\mathbb{R}}_{\omega^2}\mathbf{\Sigma}^0_1$ implies open determinacy for games of length ω^3, but this implication was also not used in the proof; it can be shown by the argument of [Aguf, Section 3] (having Player II be the closed player).

2. The proof of Theorem 11.1.1 shows that if open games of length ω^3 are determined, then there is a transitive model of $\mathsf{KP} + \mathsf{AD}_{\mathbb{R}}$ containing all reals.

[1] Fix some recursive wellordering of $\mathbb{N}$ in length ω^3. This wellordering induces a bijection between $\mathbb{N}^{\mathbb{N}}$ and $\mathbb{N}^{\omega^3}$. The space $\mathbb{N}^{\omega^3} \cong (\mathbb{N}^{\mathbb{N}})^{\omega^2}$ is considered with the topology induced by this bijection.

3. Woodin has shown that, over $\mathsf{ZF} + \mathsf{DC}$, $\mathsf{AD}_{\mathbb{R}}$ implies that every game of fixed countable length with moves in $\mathbb{N}$ is determined. As the interested reader will verify using the model M above, ZF cannot be replaced by KP in the statement of Woodin's theorem.

11.2 Proof of the lemmata

Given two strategies σ and τ, we denote by $\sigma * \tau$ the result of facing them off against each other. Given a strategy σ for a player and a sequence of moves x for the opponent, we denote by $\sigma(x)$ the unique play given by x which is consistent with σ.

Proof of Lemma 11.1.2. We emphasize that the lemma is stated with no determinacy assumptions. For every quantifier Q on $\mathbb{R}$, a theorem of Aczel [Acz75] shows that

$$\mathsf{IND}(Q) = \{Q^+ u\, \phi(u, \vec{x}) : \phi \text{ is projective}\},$$

where Q^+ denotes the *next quantifier* of Q, given by

$$Q^+ u\, R(u) \leftrightarrow Q x_0\, \breve{Q} x_1\, \exists x_2\, \forall x_3\, Q x_4\, \ldots\, \exists n\, R(\langle x_0, x_1, \ldots, x_n\rangle).$$

To prove the lemma, suppose A belongs to $\mathsf{IND}(\partial^{\mathbb{R}})$. By Aczel's theorem, there is an analytical ϕ and some sequence $\vec{a}$ of parameters in $\mathbb{R}$ such that

$$x \in A \leftrightarrow (\partial^{\mathbb{R}})^+ u\, \phi(u, x, \vec{a}).$$

By definition of $(\partial^{\mathbb{R}})^+$,

$$x \in A \leftrightarrow \partial^{\mathbb{R}} x_0\, \breve{\partial}^{\mathbb{R}} x_1\, \ldots\, \exists n\, \phi(\langle x_0, \ldots, x_n\rangle, x, \vec{a}).$$

This means that $x \in A$ if, and only if, Player I has a winning strategy in the following game G of length ω:

1. During a turn k of one of the forms $4n$ or $4n + 3$, Player I plays a strategy σ_k for a game of length ω on $\mathbb{R}$;

2. During a turn k of one of the forms $4n + 1$ or $4n + 2$, Player II plays a strategy σ_k for a game of length ω on $\mathbb{R}$.

Letting x_k be the real coding $\sigma_{2k} * \sigma_{2k+1}$ (which is an infinite sequence of reals), Player I wins if, and only if there is n such that $\phi(\langle x_0, \ldots, x_n\rangle, x, \vec{a})$ holds.

For simplicity, let us assume that ϕ is $\mathbf{\Sigma}^1_1$, so that

$$\phi(u, x, \vec{a}) \leftrightarrow \exists y \in \mathbb{R}\, \forall m\, R(u \restriction m, x \restriction m, \vec{a} \restriction m, y \restriction m),$$

for some recursive relation R.

Let us define an open game H of length ω^2 with moves on $\mathbb{R}$ and open payoff which determines membership in A. The game consists of infinitely many blocks, each of length ω, and has two stages.

1. During turns $(\omega \cdot n, \omega \cdot (n+1))$, if the game is in the first stage, then players I and II alternate turns playing real numbers $x^n_0, x^n_1, \ldots$, etc. If n is even, then Player I begins; if n is odd, then Player II begins. Denote by x^n the real coding the infinite sequence $(x^n_0, x^n_1, \ldots)$.

2. During turn $\omega \cdot n$, player I may decide to advance the game to the second stage, if she has not done so before.

3. If so, then Player I must play some $y \in \mathbb{R}$.

4. If so, Player II must respond with some $m \in \mathbb{N}$.

Player I wins the game if, and only if, there is some n such that the game advances to the second stage on turn $\omega \cdot n$, and if $y \in \mathbb{R}$ and $m \in \mathbb{N}$ are the two numbers played immediately afterwards, then

$$R(\langle x^0, x^1, \ldots, x^n \rangle \restriction m, x \restriction m, \vec{a} \restriction m, y \restriction m),$$

where $x^i = (x^i_0, x^i_1, \ldots)$.

Now, suppose Player I has a winning strategy Σ in H. Then she has one in G: during a turn k of one of the forms $4n$ or $4n+3$, Player I plays the restriction of Σ to the next ω-many moves of the game. Player II's move during turn $4n+1$ or $4n+2$ is a strategy which, when applied to Σ, yields a real which we will denote by x^{2n} or x^{2n+1}, according to the parity of k. Since Σ is winning for Player I, whenever she plays by the strategy just described, there will be an $n \in \mathbb{N}$ such that Player I can play some $y \in \mathbb{R}$ which ensures that

$$\forall m\, R(\langle x^0, x^1, \ldots, x^n \rangle \restriction m, x \restriction m, \vec{a} \restriction m, y \restriction m),$$

so the strategy described ensures winning G.

Conversely, if Player I has a winning strategy Σ in G, then she has one in H, obtained from using, rather than playing, the strategies provided by Σ. Since Σ is a winning strategy, there can be no infinite play $\langle x^0, x^i, \ldots \rangle$ such

that $\phi(\langle x_0, \dots, x_n\rangle, x, \vec{a})$ holds for no $n \in \mathbb{N}$. Thus, for any play obtained in this way, there will be a least n for which $\phi(\langle x_0, \dots, x_n\rangle, x, \vec{a})$ holds. Player I can then decide to advance the game to the second stage during turn $\omega \cdot n$, after which she can play a real number witnessing ϕ. This proves the lemma. □

The proof of Lemma 11.1.3 requires some preliminary observations. The first one we isolate as Lemma 11.2.1 below. Lemmata 11.1.3 and 11.2.1 are both easy consequences of the proofs of known theorems, but not of the statements. We refer the reader to Moschovakis [Mos09] for the definitions of all undefined notions in what follows.

Lemma 11.2.1. *Suppose games of length ω^2 on $\mathbb{N}$ with $\mathfrak{D}^{\mathbb{R}}$-hyperprojective payoff are determined. Then $\mathsf{IND}(\mathfrak{D}^{\mathbb{R}})$ has the scale property.*

Proof. The proof is somewhat standard, so we only sketch it. The key result which we need is one of Martin [Mar08b, Corollary 7.2] by which if a pointclass Γ has the scale property and games of length ω^2 on Γ are determined, then $\mathfrak{D}^{\mathbb{R}}\Gamma$ has the scale property.

Afterwards, the proof is much like the argument of Moschovakis [Mos08] for showing that inductive sets admit inductive scales: one verifies that for every operator $\phi(X)$ in which X appears positively and which is definable in the language $L(\mathfrak{D}^{\mathbb{R}})$, every stage ϕ^λ of the inductive definition on ϕ admits a scale which is $\mathfrak{D}^{\mathbb{R}}$-hyperprojective. As in [Mos08], this transfinite recursion can be carried out effectively, using the recursion theorem and the fact that the theorem on the transfer of scales from a set A to $\mathfrak{D}^{\mathbb{R}}A$ has an effective proof (this is the case because the definitions of the scales are written down in [Mar08b]), to show that each scale constructed this way is in fact $\mathfrak{D}^{\mathbb{R}}$-hyperprojective. Finally the same construction, together with the Stage Comparison Theorem, yields a scale on ϕ^∞ which is in $\mathsf{IND}(\mathfrak{D}^{\mathbb{R}})$. This proves the lemma. □

A consequence of the preceding argument is (much like in the case of IND) that if games of length ω^2 on $\mathbb{N}$ with payoff in $\mathsf{IND}(\mathfrak{D}^{\mathbb{R}})$ are determined, then $\mathfrak{D}^{\mathbb{R}}$-hyperprojective sets have $\mathfrak{D}^{\mathbb{R}}$-hyperprojective scales.

Recall that, by our convention for dealing with games of length ω on $\mathbb{R}$, a run of one these games (i.e., an infinite sequence of reals) is identified with the (single) real coding it. We will also assume that if x is a real coding an n-tuple $(x_0, \dots, x_{n-1})$ and $k < n$, then the first k digits of x depend only on $x_0, \dots, x_{k-1}$; similarly for codes of infinite sequences. This has the

consequence that, if $(x_0, x_1, \ldots)$ is an infinite sequence, then, letting y_k code $(x_0, \ldots, x_{k-1})$ for all $k \in \mathbb{N}$ and y code $(x_0, x_1, \ldots)$, we have

$$\lim_{k\to\infty} y_k = y.$$

Proof of Lemma 11.1.3. One way of proving this is to adapt the argument for the Third Periodicity Theorem of Moschovakis [Mos09, Theorem 6E.1] to games on $\mathbb{R}$. Let A be a $\mathfrak{D}^{\mathbb{R}}$-hyperprojective set and identify it with the game on $\mathbb{R}$ it induces. Assume without loss of generality that it is Player I who has a winning strategy.

Following Moschovakis' proof, define W_k to be the collection of all $(k+1)$-tuples of reals from which Player I has a winning strategy in A. Let ψ_k be the norm defined as in [Mos09, Theorem 6E.1] and call a sequence of reals of odd length $(a_0, \ldots, a_k) \in W_k$ *minimal* if

$$(a_0, \ldots, a_{k-1}, a_k) \leq_{\psi_k} (a_0, \ldots, a_{k-1}, b)$$

for all $b \in \mathbb{R}$. By inspecting the definition and appealing to Lemma 11.2.1, it is easy to see that ψ_k can be taken to be $\mathfrak{D}^{\mathbb{R}}$-hyperprojective. The point of this norm is that it is shown as part of the proof of [Mos09, Theorem 6E.1] that if $a \in \mathbb{R}^{\mathbb{N}}$ is an infinite play such that $a \restriction 2k+1$ is minimal for all k, then a is a win for Player I in A (this uses our remark on the convergence of codes before the beginning of this proof). Since ψ_k is $\mathfrak{D}^{\mathbb{R}}$-hyperprojective, the collection of all minimal (codes for finite) tuples of reals is $\mathfrak{D}^{\mathbb{R}}$-hyperprojective.

In [Mos09], where games on $\mathbb{N}$ are being played, the definable winning strategy is obtained by playing the least number which is minimal in each turn. When playing games on $\mathbb{R}$, one cannot do that, but the problem can be fixed in the obvious way:

Since $\mathfrak{D}^{\mathbb{R}}$-hyperprojective sets are (of course) closed under $\forall^{\mathbb{R}}$, a theorem of Moschovakis [Mos71] implies that every $\mathfrak{D}^{\mathbb{R}}$-hyperprojective set can be uniformized by a $\mathfrak{D}^{\mathbb{R}}$-hyperprojective function. Hence, there is a $\mathfrak{D}^{\mathbb{R}}$-hyperprojective function f such that if u is a tuple of reals of even length in W_k, then $f(u) \in \mathbb{R}$ is minimal. The winning strategy for Player I is to play $f(u)$ whenever it is defined. This completes the proof of the lemma. □

Part III

Proofs in Systems with Unsound Rules

12

Unsound Rules in Finite Proof Systems

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12.1 Introduction

Consider the following argument:

1. That Kurt Gödel is Austrian entails that Kurt Gödel is Austrian.
2. Hence, that Kurt Gödel is Austrian entails that everyone is Austrian.
3. That is, if Kurt Gödel is Austrian, then all people are Austrian.
4. Therefore, there exists a person such that, if that person is Austrian, then all people are Austrian.

This chapter was written jointly with M. Baaz.

The argument can be formalized as follows:

$$\dfrac{\dfrac{\dfrac{A(a) \vdash A(a)}{A(a) \vdash \forall y\, A(y)}}{\vdash A(a) \to \forall y\, A(y)}}{\vdash \exists x\, (A(x) \to \forall y\, A(y))} \tag{12.1}$$

The study of formal proofs is motivated by our desire for deductive reasoning to be *correct*—we wish that the procedures involved derive only true conclusions. The traditional way of ensuring this involves restricting proofs to those satisfying the two following properties:

- Inferences are *sound*, i.e., only true conclusions result from true premises.
- Derivations are *hereditary*, i.e., initial segments of proofs are proofs themselves.

In particular, one might impose certain *characteristic-variable conditions* that restrict the circumstances under which quantifiers can be inferred. One can use unsound rules based on weak characteristic-variable conditions to define enhanced (correct) logical calculi wherein derivations such as (12.1) are allowed.

In our current treatment, the correctness of proofs can only be gauged by considering them in their entirety, as forfeiting soundness while maintaining correctness necessarily violates hereditariness. This is not unlike other aspects of reasoning that are frequently not reflected in formal derivations. For example, hereditariness is not compatible with proofs by contradiction—one would not take as correct an initial segment of a proof obtained by interrupting it after an assumption directed towards a contradiction had been made, but before the contradiction had been reached.

To illustrate this, we note that the following subproof of the opening example should not be allowed as a proof in the system:

1. That Kurt Gödel is Austrian entails that Kurt Gödel is Austrian.
2. Hence, that Kurt Gödel is Austrian entails that everyone is Austrian.

The *strong* inference from 1. to 2. is the only unsound one:

1. That Kurt Gödel is Austrian entails that Kurt Gödel is Austrian.

3′ That is, if Kurt Gödel is Austrian, then Kurt Gödel is Austrian.

4′ Therefore, there exists a person such that, if that person is Austrian, then Kurt Gödel is Austrian.

Here, the name 'Kurt Gödel' is simply manipulated syntactically, so its meaning plays no role. Moreover, the name does not even appear in sentence 4. It could well have been replaced by any other throughout the derivation with no effect—we call (the result of formalizing) this condition *substitutability*. In our example, it is what allows us to mend the proof after a 'lie' has been introduced in 2.

Formal proofs can often be taken without loss of generality to be *regular*, i.e., any variable which is the *characteristic variable*[1] of a quantifier inference can be assumed to appear only before then. Forfeiting this condition allows us to accept the following proof:

$$\dfrac{\dfrac{\dfrac{\dfrac{A(a) \vdash A(a)}{A(a) \vdash \forall x\, A(x)} \quad \dfrac{A(f(a)) \vdash A(f(a))}{\forall x\, A(x) \vdash A(f(a))}}{A(a) \vdash A(f(a))}}{\vdash A(a) \to A(f(a))}}{\vdash \exists x\, (A(x) \to A(f(x)))} \tag{12.2}$$

In this example, a is the characteristic variable of only one quantifier inference. We call this *weak regularity*. Our main result is that a sequent calculus augmented with unsound quantifier inferences satisfying substitutability, weak regularity (or even a further weakening thereof), and a technical *side-variable condition* is correct. This means that, although inferences therein may not be sound, the three (global) conditions are enough to guarantee that any lie introduced during the proof is eventually cleared. Moreover, the calculus yields non-elementarily–shorter[2] cut-free proofs than standard proof systems. This is proved in Section 12.3. Some consequences of the speed-up theorem are explained in Section 12.4.

The three characteristic-variable conditions are motivated by the rules governing possible inferences in Hilbert's *ε-calculus*, whose language contains no quantifiers (see Section 12.5). As a consequence, proofs therein are less restrictive. The weakened conditions thus result from attempting to pull back some extra freedom from the ε-calculus to first-order logic. They can also be thought of as a first attempt in tracking down and covering the spots where first-order logic proofs might break down without the usual quantifier-inference restrictions. In particular, it seems plausible that the conditions

[1] See Section 12.2 for preliminaries and definitions.

[2] Size can be measured, e.g., by counting the number of inferences in the proof.

here can be further weakened, and the corresponding calculi can result even faster.

Weakened characteristic-variable conditions essentially identical to these have been considered by Takeuti [Tak80] in the context of infinitary proof systems. These proof systems will be explored in the second part of this chapter.

12.2 Quantifier inferences

Let π be any derivation. We say b is a *side variable* of a in π (written $a <_\pi b$) if π contains a strong-quantifier inference of the form:

$$\frac{\Gamma \vdash \Delta, A(a, b, \vec{c})}{\Gamma \vdash \Delta, \forall x\, A(x, b, \vec{c})}$$

or of the form:

$$\frac{A(a, b, \vec{c}), \Gamma \vdash \Delta}{\exists x\, A(x, b, \vec{c}), \Gamma \vdash \Delta}$$

The Skolemization of a first-order formula A is defined by replacing every strongly quantified variable y with a new function symbol $f_y^A(x_1, \ldots, x_n)$, where $x_1, \ldots, x_n$ are the weakly quantified variables such that Qy appears in the scope of their quantifiers, and removing the quantifier Qy. We write $sk(A)$ for the Skolemization of A. Recall that whether a quantifier is weak or strong depends on the side of the sequent on which it is. Thus, $sk(A)$ means different things depending on whether A appears in the antecedent or the succedent. The Skolemization of a sequent

$$A_0, \ldots, A_m \vdash A_{m+1}, \ldots, A_{m+n}$$

is defined as:

$$sk(A_0), \ldots, sk(A_m) \vdash sk(A_{m+1}), \ldots, sk(A_{m+n}).$$

In classical logic, a sequent is derivable if and only if its Skolemization is.

Definition 12.2.1 (Suitable quantifier inference)**.** We say a quantifier inference is *suitable for a proof* π if either it is a weak-quantifier inference, or the following three conditions are satisfied:

- (substitutability) the characteristic variable does not appear in the conclusion of π.

- (side-variable condition) the relation $<_\pi$ is acyclic.

- (weak regularity) the characteristic variable is not the characteristic variable of another strong-quantifier inference in π.

In particular, if all the quantifier inferences in a proof π are suitable for it, then every free variable a in π is the characteristic variable of at most one inference, in whose principal formula a does not appear (since $a <_\pi a$ cannot hold).

We will work with various sequent calculi extending *Propositional* LK, the sequent calculus whose only axioms are $\bot \vdash$ and $A \vdash A$ (with A atomic) and whose rules are:

Structural rules:

$$\frac{\Gamma \vdash \Delta}{\Gamma, A \vdash \Delta}\, WL \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, A}\, WR$$

$$\frac{\Gamma, A, A \vdash \Delta}{\Gamma, A \vdash \Delta}\, CL \qquad \frac{\Gamma \vdash \Delta, A, A}{\Gamma \vdash \Delta, A}\, CR$$

$$\frac{\Gamma_1, A, B, \Gamma_2 \vdash \Delta}{\Gamma_1, B, A, \Gamma_2 \vdash \Delta}\, EL \qquad \frac{\Gamma \vdash \Delta_1, A, B, \Delta_2}{\Gamma \vdash \Delta_1, B, A, \Delta_2}\, ER$$

$$\frac{\Gamma \vdash \Delta, A \quad A, \Gamma \vdash \Delta}{\Gamma \vdash \Delta}\, Cut$$

Logical rules:

$$\frac{\Gamma, A \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}\, \wedge_1 L \qquad \frac{\Gamma \vdash \Delta, A}{\Gamma \vdash \Delta, A \vee B}\, \vee_1 R$$

$$\frac{\Gamma, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}\, \wedge_2 L \qquad \frac{\Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \vee B}\, \vee_2 R$$

$$\frac{\Gamma, A \vdash \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \vee B \vdash \Delta}\, \vee L \qquad \frac{\Gamma \vdash \Delta, A \quad \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \wedge B}\, \wedge R$$

$$\frac{\Gamma \vdash \Delta, A \quad \Gamma, B \vdash \Delta}{\Gamma, A \to B \vdash \Delta}\, \to L \qquad \frac{\Gamma, A \vdash \Delta, B}{\Gamma \vdash \Delta, A \to B}\, \to R$$

Negation $\neg A$ is defined as $A \to \bot$. The names of the structural rules stand, respectively, for 'weakening,' 'contraction,' and 'exchange.' First-order LK is

the extension of Propositional LK obtained by adding quantifier inferences:

$$\frac{\Gamma, A(t) \vdash \Delta}{\Gamma, \forall x\, A(x) \vdash \Delta}\ \forall L \qquad \frac{\Gamma \vdash \Delta, A(t)}{\Gamma \vdash \Delta, \exists x\, A(x)}\ \exists R$$

$$\frac{\Gamma, A(a) \vdash \Delta}{\Gamma, \exists x\, A(x) \vdash \Delta}\ \exists L \qquad \frac{\Gamma \vdash \Delta, A(a)}{\Gamma \vdash \Delta, \forall x\, A(x)}\ \forall R$$

with the only restriction that in the inferences $\forall R$ and $\exists L$, the variable a is not allowed to appear in the conclusion. In particular, note that every quantifier inference is suitable for every regular LK-proof. Gentzen's famous *Cut-Elimination Theorem* states that the cut rule is redundant in both propositional and first-order LK.

Definition 12.2.2 (LK^+)**.** The calculus LK^+ is defined like LK, except that one drops the restriction preventing the characteristic variable of an inference from appearing in its conclusion, and one adds the restriction that a proof may only contain quantifier inferences that are suitable for it.

A further weakening of the characteristic-variable conditions gives rise to the notion of weak suitability:

Definition 12.2.3 (Weakly suitable quantifier inference)**.** A quantifier inference is *weakly suitable for a proof* π if either it is a weak-quantifier inference or it satisfies substitutability, the side-variable condition, and:

- (very weak regularity) whenever the characteristic variable is also the characteristic variable of another strong-quantifier inference in π, then it has the same principal formula.

Definition 12.2.4 (LK^{++})**.** The calculus LK^{++} is the extension of LK^+ that results from allowing all weakly suitable quantifier inferences.

It is easy to find examples of sequents that are more easily provable in LK^+ than in LK:

Example 12.2.5. The sequent

$$\forall x\, A(x) \to B \vdash \exists x\, (A(x) \to B)$$

is provable in LK:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A(a) \vdash A(a)}{A(a) \vdash A(a), B}}{\vdash A(a), A(a) \to B}}{\vdash A(a), \exists x\,(A(x) \to B)}}{\vdash \exists x\,(A(x) \to B), A(a)}}{\vdash \exists x\,(A(x) \to B), \forall x\,A(x)} \qquad B \vdash B}}{\forall x\,A(x) \to B \vdash \exists x\,(A(x) \to B), B}}{\forall x\,A(x) \to B, A(b) \vdash \exists x\,(A(x) \to B), B}}{\forall x\,A(x) \to B \vdash \exists x\,(A(x) \to B), A(b) \to B}}{\forall x\,A(x) \to B \vdash \exists x\,(A(x) \to B), \exists x\,(A(x) \to B)}}{\forall x\,A(x) \to B \vdash \exists x\,(A(x) \to B)}$$

However, one can find a shorter LK^+-proof:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A(a) \vdash A(a)}{A(a) \vdash \forall x\,A(x)} \qquad B \vdash B}}{A(a), \forall x\,A(x) \to B \vdash B}}{\forall x\,A(x) \to B, A(a) \vdash B}}{\forall x\,A(x) \to B \vdash A(a) \to B}}{\forall x\,A(x) \to B \vdash \exists x\,(A(x) \to B)}$$

□

Recall that a function on the natural numbers is *elementary* if it can be defined by a quantifier-free formula from $+$, $\cdot$, and the function $x \mapsto 2^x$. By independent results of R. Statman [Sta79] (which were formalized in [BL94]) and of V. P. Orevkov [Ore82], the sizes of the smallest cut-free LK-proofs of sequents of length n are not bounded by any elementary function on n. Our main theorem is that cut-free LK^+-proofs are non-elementarily shorter than cut-free LK-proofs:

Theorem 12.2.6. *There is no elementary function bounding the length of the shortest cut-free* LK*-proof of a formula in terms of its shortest cut-free* LK^+*-proof.*

An immediate consequence is the following:

Corollary 12.2.7. *There is no elementary function bounding the length of the shortest cut-free* LK*-proof of a formula in terms of its shortest cut-free* LK^{++}*-proof.*

We prove Theorem 12.2.6 in Section 12.3. First, we consider the question of the correctness of LK^{++}. The proof of Theorem 12.2.8 that we present here is due to the referee:

Theorem 12.2.8. *If a sequent is* LK^{++}*-derivable, then it is already* LK*-derivable.*

Proof. Let π be an LK^{++}-proof. Replace every unsound universal quantifier inference by a $\to L$ inference:

$$\frac{\Gamma \vdash \Delta, A(a) \qquad \forall x\, A(x) \vdash \forall x\, A(x)}{\Gamma, A(a) \to \forall x\, A(x) \vdash \Delta, \forall x\, A(x)}$$

Similarly replace every unsound existential quantifier by an $\to L$ inference

$$\frac{\exists x\, A(x) \vdash \exists x\, A(x) \qquad A(a), \Gamma \vdash \Delta}{\Gamma, \exists x\, A(x), \exists x\, A(x) \to A(a) \vdash \Delta}$$

By doing this, we obtain a proof of the desired sequent, together with many formulae of the form $A(a) \to \forall x\, A(x)$ or $\exists x\, A(x) \to A(a)$ on the left-hand side. However, we can eliminate each of them by adding an existential quantifier inference and cutting with formulae of the form

$$\vdash \exists y\, (A(y) \to \forall x\, A(x))$$

or of the form

$$\vdash \exists y\, (\exists x\, A(x) \to A(y)),$$

both of which are easily derivable. Note that the existential quantifier inferences can be carried out in a way that is permissible by LK because the initial proof satisfied substitutability, weak regularity and the side-variable condition. □

Corollary 12.2.9. *If a sequent is derivable in* LK^{+}*, then it is already derivable in* LK*.*

12.3 Non-Elementary Speed-Up

Our strategy for proving Theorem 12.2.6 is to show that LK^{+} simulates a strong calculus that is already non-elementarily faster than LK. It is related to *quantifier shifts*, formulae such as:

$$(\forall x\, A \to B) \to \exists x\, (A \to B).$$

The main feature of the calculus is the addition of certain rules for quickly shifting quantifiers. Below, if A is a subformula of B, we say that B is a *context* of A. We generally write $\kappa[A]$ to emphasize that A occurs in a context $\kappa[\cdot]$; they should be thought as syntactical operators $A \mapsto \kappa[A]$.

Definition 12.3.1. The calculus $\mathrm{LK}_{\mathtt{shift}}$ is obtained by extending LK with the following rules:

$$\frac{\Gamma, \kappa[Qx\, A \lhd B] \vdash \Delta}{\Gamma, \kappa[Q'x\, (A \lhd B)] \vdash \Delta} \qquad \frac{\Gamma, \kappa[A \lhd Qx\, B] \vdash \Delta}{\Gamma, \kappa[Q'x\, (A \lhd B)] \vdash \Delta}$$

$$\frac{\Gamma \vdash \Delta, \kappa[Qx\, A \lhd B]}{\Gamma \vdash \Delta, \kappa[Q'x\, (A \lhd B)]} \qquad \frac{\Gamma \vdash \Delta, \kappa[A \lhd Qx\, B]}{\Gamma \vdash \Delta, \kappa[Q'x\, (A \lhd B)]}$$

where $\kappa[\cdot]$ is a context, $\lhd \in \{\wedge, \vee, \to\}$, x does not appear in B, and $Q' = Q$, except if $\lhd$ is $\to$ and Q is taken from the antecedent, in which case Q' is opposite. We refer to these rules as *deep quantifier shifts*.

Proposition 12.3.2. *Cut-free* LK^+ *simulates cut-free* $\mathrm{LK}_{\mathtt{shift}}$ *double-exponentially, i.e., every* $\mathrm{LK}_{\mathtt{shift}}$*-provable sequent is* LK^+*-provable and there is a double exponential function that bounds the length of the least cut-free* LK^+*-proof of a* LK^+*-provable sequent in terms of its least cut-free* $\mathrm{LK}_{\mathtt{shift}}$*-proof.*

Proof. Let π be a cut-free $\mathrm{LK}_{\mathtt{shift}}$-proof of size n. We transform it into a cut-free LK^+-proof. Assume by induction that there is only one application of a quantifier shift and this is the last inference. We proceed by cases. For simplicity, we assume κ does not change the polarity of the quantifiers. We also assume $\lhd$ is $\to$; the other connectives are treated similarly.

Case I. The last inference is:

$$\frac{\Gamma, \kappa[\forall x\, A(x) \to B] \vdash \Delta}{\Gamma, \kappa[\exists x\, (A(x) \to B)] \vdash \Delta}$$

so that the proof has the following structure:

$$\dfrac{\dfrac{\dfrac{\vdots\,\sigma_2}{\Gamma'' \vdash \Delta'', A(a)}}{\Gamma'' \vdash \Delta'', \forall x\, A(x)}\,(*)}{\dfrac{\dfrac{\vdots\,\sigma_1}{\Gamma' \vdash \Delta', \forall x\, A(x)} \qquad \dfrac{\vdots\,\sigma_3}{\Gamma', B \vdash \Delta'}}{\Gamma', \forall x\, A(x) \to B \vdash \Delta'}}{\dfrac{\dfrac{\vdots\,\sigma_0}{\Gamma, \kappa[\forall x\, A(x) \to B] \vdash \Delta}}{\Gamma, \kappa[\exists x\, (A(x) \to B)] \vdash \Delta}}$$

where the σ_i denote subproofs. The subproof σ_0 might split into several branches not indicated in the diagram, each of which could potentially include its own copy of $\forall x\, A(x)$ being inferred and its own subproof σ_3' with conclusion of the form $\Pi, B \vdash \Lambda$. Each branch can be dealt with the same way and so we will only focus on the indicated parts of π.

Similarly, the subproof σ_1 might split into several branches not indicated in the diagram, each of which could potentially include its own copy of $\forall x\, A(x)$ being inferred. This we need to bear in mind. Note that each of those copies of $\forall x\, A(x)$ is necessarily inferred from a different characteristic variable. In fact, the variable a does not appear below $(*)$, by the regularity of $\mathrm{LK}_{\mathtt{shift}}$.

We modify the proof. Our approach is as follows: we would like to merge the subproofs σ_1 and σ_2 simply by postponing the inference $(*)$ as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdots\,\sigma_2 + \sigma_1}{\Gamma' \vdash \Delta', A(a)} \qquad \dfrac{\vdots\,\sigma_3}{\Gamma', B \vdash \Delta'}}{\Gamma', A(a) \to B \vdash \Delta'}}{\Gamma', \exists x\, (A(x) \to B) \vdash \Delta'}\,(**)}{\dfrac{\vdots\,\sigma_0}{\Gamma, \kappa[\exists x\, (A(x) \to B)] \vdash \Delta}}$$

The problem that might arise is that some occurrence of $\forall x\, A(x)$ that would be contracted in σ_1 with the indicated occurrence is unable to be contracted. We describe how to deal with each of them.

Notice that the problematic occurrence of $\forall x\, A(x)$ must originate from a strong-quantifier inference. Omit that inference and drag the unquantified

formula $A(b)$ until after $(**)$, so that at that point we have a derivation of

$$\Gamma_0', A(b), \Gamma_1', \exists x\,(A(x) \to B), \vdash \Delta',$$

where $\Gamma' = \Gamma_0', \Gamma_1'$. Add to that some exchanges to obtain:

$$\Gamma', \exists x\,(A(x) \to B), A(b) \vdash \Delta',$$

and a subproof

$$\frac{\vdots\,\sigma_3}{\Gamma', B \vdash \Delta'} \tag{12.3}$$

in order to infer

$$\Gamma', \exists x\,(A(x) \to B), A(b) \to B \vdash \Delta'.$$

with an application of $\exists L$ and a contraction, we are left again with

$$\Gamma', \exists x\,(A(x) \to B) \vdash \Delta',$$

to which we can apply σ_0 as desired. The proof grows by adding a copy of (12.3)—of size at most n—for each problematic occurrence of $\forall x\,A(x)$ (of which there are at most n). Hence, it grows quadratically.

Case II. The last inference is:

$$\frac{\Gamma, \kappa[\exists x\,A(x) \to B] \vdash \Delta}{\Gamma, \kappa[\forall x\,(A(x) \to B)] \vdash \Delta}$$

The proof has the following structure:

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdots\,\sigma_2}{\Gamma'' \vdash \Delta'', A(t)}}{\Gamma'' \vdash \Delta'', \exists x\,A(x)}}{\dfrac{\dfrac{\vdots\,\sigma_1}{\Gamma' \vdash \Delta', \exists x\,A(x)} \quad \dfrac{\vdots\,\sigma_3}{\Gamma', B \vdash \Delta'}}{\Gamma', \exists x\,A(x) \to B \vdash \Delta'}}}{\dfrac{\dfrac{\vdots\,\sigma_0}{\Gamma, \kappa[\exists x\,A(x) \to B] \vdash \Delta}}{\Gamma, \kappa[\forall x\,(A(x) \to B)] \vdash \Delta}}$$

As in Case I, the subproof σ_0 could branch off into several subtrees each of which is taken care of in the same way, and so we only focus on the indicated parts of π. We would like to merge the subproofs σ_1 and σ_2 as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdots\,\sigma_2 + \sigma_1}{\Gamma' \vdash \Delta', A(t)} \quad \dfrac{\vdots\,\sigma_3}{\Gamma', B \vdash \Delta'}}{\Gamma', A(t) \to B \vdash \Delta'}}{\Gamma', \forall x\,(A(x) \to B) \vdash \Delta'}}{\dfrac{\vdots\,\sigma_0}{\Gamma, \kappa[\forall x\,(A(x) \to B)] \vdash \Delta}}$$

As before, we face the problem of circumventing a contraction of (possibly several occurrences of) $\exists x\,A(x)$ in σ_1 and solve it in the same way. The proof grows quadratically again. We note the following: it might happen that the term t contains some free variable a. Hence, the proof of this case does not necessarily go through in LK, as a could be the characteristic variable of a strong-quantifier inference in σ_1. Substitutability holds clearly. Since no free variable in B is a characteristic variable in the proof, the side-variable condition is satisfied.

Case III. The last inference is:

$$\dfrac{\Gamma, \kappa[A \to \forall x\,B(x)] \vdash \Delta}{\Gamma, \kappa[\forall x\,(A \to B(x))] \vdash \Delta}$$

This is analogous to Case II.

Case IV. The last inference is:

$$\dfrac{\Gamma, \kappa[A \to \exists x\,B(x)] \vdash \Delta}{\Gamma, \kappa[\exists x\,(A \to B(x))] \vdash \Delta}$$

This is analogous to Case I.

Case V. The last inference is:

$$\dfrac{\Gamma \vdash \Delta, \kappa[\forall x\,A(x) \to B]}{\Gamma \vdash \Delta, \kappa[\exists x\,(A(x) \to B)]}$$

The proof has the following form, modulo qualifications as in Cases I and II:

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdots\,\sigma_2}{\Gamma'', A(t) \vdash \Delta''}}{\Gamma'', \forall x\, A(x) \vdash \Delta''}\,(*)}{\Gamma', \forall x\, A(x) \vdash \Delta', B}}{\Gamma' \vdash \Delta', \forall x\, A(x) \to B}$$

$$\dfrac{\dfrac{\vdots\,\sigma_0}{\Gamma \vdash \Delta, \kappa[\forall x\, A(x) \to B]}}{\Gamma \vdash \Delta, \kappa[\exists x\, (A(x) \to B)]}$$

We would like to postpone the inference $(*)$, thus merging σ_1 and σ_2:

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdots\,\sigma_2 + \sigma_1}{\Gamma', A(t) \vdash \Delta', B}}{\Gamma' \vdash \Delta', A(t) \to B}\,(**)}{\Gamma' \vdash \Delta', \exists x\, (A(x) \to B)}}{\Gamma \vdash \Delta, \kappa[\exists x\, (A(x) \to B)]}$$

However, we might have—similarly to the previous cases—an occurrence of $\forall x\, A(x)$ that would be contracted on the left-hand side as part of σ_1. For simplicity, assume there is only one. This occurrence is inferred from a formula $A(s)$. Postpone this occurrence until after $(**)$ so that at that point we have a derivation of

$$\Gamma'_0, A(s), \Gamma'_1 \vdash \Delta', \exists x\, (A(x) \to B),$$

where $\Gamma' = \Gamma'_0, \Gamma'_1$. Add to that some exchanges to obtain:

$$\Gamma', A(s) \vdash \Delta', \exists x\, (A(x) \to B),$$

and a weakening:

$$\dfrac{\Gamma', A(s) \vdash \Delta', \exists x\, (A(x) \to B)}{\Gamma', A(s) \vdash \Delta', \exists x\, (A(x) \to B), B}\; WR \tag{12.4}$$

in order to infer

$$\Gamma' \vdash \Delta', \exists x\, (A(x) \to B), A(s) \to B.$$

with an application of $\exists R$ and a contraction, we are left again with

$$\Gamma' \vdash \Delta', \exists x\,(A(x) \to B),$$

to which we can apply the rest of σ_0 as desired. In this case the proof only grows linearly. As in Case II, the proof of this case would not necessarily work for LK, but it is correct from the point of view of LK^+, as follows from the regularity of $\mathrm{LK}_{\mathtt{shift}}$ and the fact that the conclusion of the proof is π.

Case VI. The last inference is:

$$\frac{\Gamma \vdash \Delta, \kappa[\exists x\, A(x) \to B]}{\Gamma \vdash \Delta, \kappa[\forall x\,(A(x) \to B)]}$$

The proof has the following form, modulo qualifications as in Cases I and II:

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdots\,\sigma_2}{\Gamma'', A(a) \vdash \Delta''}}{\Gamma'', \exists x\, A(x) \vdash \Delta''}\,(*)}{\Gamma', \exists x\, A(x) \vdash \Delta', B}}{\dfrac{\Gamma' \vdash \Delta', \exists x\, A(x) \to B}{\dfrac{\dfrac{\vdots\,\sigma_0}{\Gamma \vdash \Delta, \kappa[\exists x\, A(x) \to B]}}{\Gamma \vdash \Delta, \kappa[\forall x\,(A(x) \to B)]}}}$$

As before, we would like to postpone the inference $(*)$, thus merging σ_1 and σ_2:

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdots\,\sigma_2 + \sigma_1}{\Gamma', A(a) \vdash \Delta', B}}{\Gamma' \vdash \Delta', A(a) \to B}}{\Gamma' \vdash \Delta', \forall x\,(A(x) \to B)}}{\dfrac{\vdots\,\sigma_0}{\Gamma \vdash \Delta, \kappa[\forall x\,(A(x) \to B)]}}$$

We deal with contractions of $\exists x\, A(x)$ on the left-hand side as in the previous case. The proof grows linearly.

Case VII. The last inference is:

$$\frac{\Gamma \vdash \Delta, \kappa[A \to \forall x\, B(x)]}{\Gamma \vdash \Delta, \kappa[\forall x\, (A \to B(x))]}$$

This is analogous to Case VI.

Case VIII. The last inference is:

$$\frac{\Gamma \vdash \Delta, \kappa[A \to \exists x\, B(x)]}{\Gamma \vdash \Delta, \kappa[\exists x\, (A \to B(x))]}$$

This is analogous to Case V.

Hence, we have dealt with all the cases. Note that for each quantifier shifted, the proof grows at most quadratically. Since there are at most n deep quantifier shifts, the size of the resulting proof is bounded by

$$n^{2^n} = 2^{2^n \cdot \log(n)} \approx 2^{2^n}.$$

This finishes the proof. □

Theorem 12.2.6 is a consequence of Proposition 12.3.2 and the following result:

Theorem 12.3.3. *There is no elementary function bounding the length of the shortest cut-free* LK*-proof of a formula in terms of its shortest cut-free* LK_{shift}*-proof.*

Proof. We will make use of a very specific family of sequents $\{S_i\}_{i<\omega}$ described in [BL94] and due to Statman [Sta79], and specific LK-proofs thereof. The sequents and the proofs themselves are not important for our proof. What is relevant is that they have the following properties:

1. the size of S_i is polynomial in i;
2. there is no bound on the size of their smallest cut-free LK-proofs that is elementary in i;
3. the size of these proofs (with cuts), however, is polynomially bounded in i;

4. all cut formulae are closed; we can also assume they are prenex by, e.g., [BL99, Theorem 3.3].

Let $\Gamma_i \vdash \Delta_i$ be one of the sequents. We modify the proof as follows: first, replace each cut

$$\frac{\Gamma \vdash \Delta, A \quad A, \Gamma \vdash \Delta}{\Gamma \vdash \Delta}\ Cut$$

with an application of $\to L$:

$$\frac{\Gamma \vdash \Delta, A \quad \Gamma, A \vdash \Delta}{\Gamma, A \to A \vdash \Delta} \to L$$

We are left with a cut free proof π_0 whose end sequent is of the form:

$$A_0 \to A_0, \ldots, A_m \to A_m, \Gamma_i \vdash \Delta_i. \tag{12.5}$$

Choose any occurrence of a quantifier in A_0 that is not in the scope of another quantifier. Since it appears once in the antecedent of $A_0 \to A_0$ and once in the consequent, it appears once with each polarity. Because A_0 is assumed to be prenex, we can apply two deep quantifier shifts, so that we are left with a proof π_0' of the sequent:

$$\forall x_0^0\, \exists x_1^0\, (A_0' \to A_0'), \ldots, A_m \to A_m, \Gamma_i \vdash \Delta_i, \tag{12.6}$$

Continue choosing any occurrence of a quantifier in A_0' that is not in the scope of another quantifier (within A_0') and applying deep quantifier shifts in pairs until we obtain a proof π_1 of the sequent:

$$\forall x_0^0\, \exists x_1^0\, \ldots (\hat{A}_0 \to \hat{A}_0), \ldots, A_m \to A_m, \Gamma_i \vdash \Delta_i,$$

where $\hat{A}_0$ is quantifier-free and the prefix of $\hat{A}_0 \to \hat{A}_0$ consists of alternating quantifiers. Repeat this procedure for each of the A_i to obtain a proof π_m where each formula $A_i \to A_i$ is replaced by an expression of the form:

$$\forall x_0^i\, \exists x_1^i\, \ldots (\hat{A}_i \to \hat{A}_i),$$

where $\hat{A}_i$ is quantifier free and the quantifier prefix is alternating. Let $\hat{\Gamma}^i \vdash \hat{\Delta}^i$ be the sequent:

$$\forall x_0^0\, \exists x_1^0\, \ldots (\hat{A}_0 \to \hat{A}_0), \ldots, \forall x_0^m\, \exists x_1^m\, \ldots (\hat{A}_m \to \hat{A}_m), \Gamma_i \vdash \Delta_i. \tag{12.7}$$

Note that, since the size of the initial proof was polynomial in i, there were polynomially many quantifiers in the proof; hence, we added only polynomially many deep quantifier shifts, and so the size of the resulting proof π_m of (12.7) is bounded polynomially in i. Moreover, it is cut-free. Consequently, it suffices to show:

Claim 12.3.4. *There is no elementary function bounding the size of the smallest cut-free* LK*-proofs of* (12.7).

Proof. Let $\{\sigma_i\}_{i<\omega}$ be a sequence of such proofs. First, we transform it into a sequence of proofs of the Skolemizations of the sequents (12.7), so that for each sequent, each of the implications $\hat{A}_i \to \hat{A}_i$ remains only universally quantified. Then, by Herbrand's theorem, there are propositional proofs $\{\theta_i\}_{i<\omega}$ of the Herbrand sequents of (12.7), each of which is of the form:

$$\bigwedge_k B_k^0, \ldots, \bigwedge_k B_k^m, \Gamma' \vdash \Delta'. \tag{12.8}$$

(Here Γ' and Δ' are the Skolemizations of Γ_i and Δ_i.) Moreover, the lengths of the Herbrand sequents are bounded exponentially by the lengths of $\{\theta_i\}_{i<\omega}$ (see [BL94, Theorem 4.3]). Each conjunct B_k^p is a quantifier-free implication of the form:

$$A(t_1, \ldots, t_l) \to A(s_1, \ldots, s_l), \tag{12.9}$$

for some terms $t_1, \ldots, t_l, s_1, \ldots, s_l$. The key point is that, in the process of Skolemizing the sequents (12.7), exactly one term in each pair (t_j, s_j) was weakly quantified and the other (which was under the scope of the strong quantifier) was replaced by a Skolem function. Hence, either t_j is of the form $f(s_j, \vec{r})$ or s_j is of the form $f(t_j, \vec{r})$, for some tuple of terms $\vec{r}$.

Transform the sequent (12.8) as follows: pick a pair (t_j, s_j) of some disjunct B_k^p such that the element thereof that was replaced by a Skolem term—say, $f(s_j, \vec{r}) = t_j$—does not have any other of $t_1, \ldots, t_l, s_1, \ldots, s_l$ as an argument. Such a term of course exists—it is the first term whose quantifier was shifted in (12.6). Substitute s_j for t_j throughout the sequent.

By repeating this process sufficiently-many times, each formula (12.9) is transformed into a propositional tautology of the form $A(\vec{t}) \to A(\vec{t})$. This means that the sequent

$$\vdash \bigwedge_k B_k^0, \ldots, \bigwedge_k B_k^m.$$

is also a propositional tautology, whereby so too is $\Gamma' \vdash \Delta'$, from which follows that it is LK-provable. By [BHW12, Theorem 2], there is an LK-proof of the unskolemized sequent of length exponential in that of $\Gamma' \vdash \Delta'$. But this is impossible, since—by assumption—there exist no short proofs of the unskolemized sequent. □

This establishes Theorem 12.3.3 and thus Theorem 12.2.6. □

12.4 Properties of the calculi

We discuss some properties of the calculi augmented with unsound inferences. Some of the arguments are only sketched—we leave the details to the interested reader.

12.4.1 LJ^+ and LJ^{++}

Consider LJ^+ and LJ^{++}, the analogs of the extended calculi LK^+ and LK^{++} for intuitionistic logic. Specifically, these are the calculi obtained by restricting possible sequents to those with at most one formula on the right-hand side. The calculi are not sound for intuitionistic logic, as shown by any of the examples in the introduction. A consequence of this that they do not admit cut elimination. The argument makes use of an important consequence of cut elimination, *subformula property*—every formula appearing in a cut-free proof of a sequent S is a subformula of a formula in S.

Proposition 12.4.1. LJ^+ *and* LJ^{++} *do not admit cut elimination.*

Proof. Consider example (12.2). Since only one formula appears on the right-hand side of each sequent, this is an LJ^+-proof. Suppose towards a contradiction that there were a cut-free proof of

$$\vdash \exists x \, (A(x) \to A(f(x))). \tag{12.10}$$

By the subformula property, this derivation would consist entirely of subformulae of (12.10). In particular, it would contain no strong quantifier inferences. Therefore, it would already be an LJ-proof. But this is impossible, as (12.10) is not intuitionistically valid. □

Proposition 12.4.1 has the consequence that LK^+ and LK^{++} do not admit cut elimination by an algorithm that resembles Gentzen's. We give a rough definition of what we mean by this, but the reader may consult the appendix of [BH11] for further details.

Definition 12.4.2. We say that a cut-eliminating procedure is *Gentzen-style* if it is a transformation of proofs consisting of permutation of rules, substitution of free variables, reduction of cuts of a formula to cuts of its outermost subformulae, and absorption of axioms, i.e., elimination of cuts

$$\frac{\Gamma \vdash \Delta, A \quad A \vdash A}{\Gamma \vdash \Delta, A} \, Cut$$

by deleting the indicated occurrence of $A \vdash A$.

The point in considering Gentzen-style cut elimination algorithms is that they transform intuitionistic proofs into cut-free intuitionistic proofs. Thus, we obtain:

Corollary 12.4.3. LK^{++} *and* LK^{+} *do not admit any Gentzen-style cut elimination.*

As is well known, the following three formulae are the only quantifier shifts not derivable in LJ:

1. $\forall x\,(A \vee B(x)) \vdash A \vee \forall x\,B(x)$;
2. $(\forall x\,A(x) \to B) \vdash \exists x\,(A(x) \to B)$;
3. $(A \to \exists x\,B(x)) \vdash \exists x\,(A \to B(x))$.

They give rise to a characterization of LJ^{++}:

Proposition 12.4.4. *A sequent is provable in* LJ^{++} *if and only if it is provable in* LJ *with all quantifier shifts added as axioms.*

Proof. All three quantifier shifts 1–3 are provable in LJ^{++} (see, e.g., Example 12.2.5). We show the converse. Clearly, the sequents $\vdash \forall y\,A(y) \to \forall x\,A(x)$ and $\vdash \exists y\,A(y) \to \exists x\,A(x)$ are provable. Hence, using the quantifier shifts 2 and 3 one derives $\vdash \exists y\,(A(y) \to \forall x\,A(x))$ and $\vdash \exists y\,(\exists x\,A(x) \to A(y))$. Therefore, we can apply the proof of Theorem 12.2.8 to obtain the desired result. □

12.4.2 Skolemization

The following are applications of (the proof of) Theorem 12.2.6:

Proposition 12.4.5. *If* $\Gamma \vdash \Delta$ *is* LK^{+}*-derivable (resp.* LK^{++}*-derivable), then its Skolemization is* LK^{+}*-derivable (resp.* LK^{++}*-derivable) in quadratically many steps using additional cuts.*

Proof. Let $\Gamma \vdash \Delta$ be any LK^{++}-derivable sequent. For each formula A in Δ, the sequent $A \vdash sk(A)$ is already LK-derivable, it follows that, by using one cut per formula, we can derive $\Gamma \vdash sk(\Delta)$ in LK^{++}. Similarly, $sk(A) \vdash A$ is already LK-derivable. Hence, by also using one cut for each formula in Γ, we can derive $sk(\Gamma) \vdash sk(\Delta)$. □

However, as we see, the additional cuts are necessary:

Proposition 12.4.6. *Let S be a sequent. Then, there is no elementary bound on the length of the smallest cut-free* LK^{++}*-proof of the Skolemization of S in terms of the smallest cut-free* LK^{+}*-proof of S.*

Proof. We argue as in the proof of Claim 12.3.4: if the Skolemization of $\Gamma \vdash \Delta$ were cut-free LK^{++}-derivable in elementary many steps, the cut-free LK^{++}-proof would already be an LK-proof, whence by [BHW12, Theorem 2], there would be an LK-proof of $\Gamma \vdash \Delta$ of elementary length with respect to the LK^{+}-proof. □

Note, in contrast, that a cut-free proof can be Skolemized in LK at no extra cost.

Proposition 12.4.7. *There is no elementary bound on the length of the Herbrand sequent of a sequent in terms of its smallest cut-free* LK^{+}*-proof (resp.* LK^{++}*-proof).*

Proof. This follows from Proposition 12.4.6, as a cut-free LK^{++}-proof of the Skolemization of a sequent contains no strong quantifier and is hence already an LK-proof, whence we can apply Herbrand's theorem. □

Although deskolemization of cut-free proofs is exponential in LK, it is linear in LK^{++}:

Proposition 12.4.8. *If the Skolemization of $\Gamma \vdash \Delta$ is cut-free* LK^{++}*-derivable, then $\Gamma \vdash \Delta$ is cut-free* LK^{++}*-derivable in linearly many steps.*

Proof. Let π be a cut-free LK^{++}-proof of the Skolemization of $\Gamma \vdash \Delta$. By definition, the sets of Skolem functions assigned to each formula in $\Gamma \vdash \Delta$ are disjoint. Modify π into a proof of $\Gamma \vdash \Delta$ as follows:

First, replace each Skolem term t by a new free variable a_t (recall that the term t is of the form $f_y(x_0, x_1, \ldots, x_n)$, for some variables y and $x_0, \ldots, x_n$). Then, for each such variable a_t, add to the proof a strong quantifier inference with characteristic variable a_t below each topmost sequent in which (for t as above) a_t appears and each variable among $x_0, \ldots x_n$ is bound by a (weak) quantifier having a_t in its scope. This results in an LK^{++}-proof having $\Gamma \vdash \Delta$ as its end-sequent. Substitutability clearly holds, as does weak regularity. The side-variable condition must also hold, since all weak quantifier inferences are obtained from the new free variables indexed by Skolem terms, and the subterm relation of (Skolem) terms is acyclic. □

Remark 12.4.9. The proof of Proposition 12.4.8 goes through for LJ^{++}.

12.5 Connections to the epsilon calculus

We start by recalling the sequent-calculus reformulation of Hilbert's *ε-calculus* (see [**?**]).

Definition 12.5.1. The quantifier-free language $\mathcal{L}_\varepsilon$ is obtained by removing symbols $\exists$ and $\forall$ from the language of first-order logic and adding symbols ε and τ. Formulae and terms are simultaneously defined by induction in a way that the following clauses are satisfied. We leave the precise recursion to the reader.

1. constants and free variables are terms;
2. if $t_1, \ldots, t_n$ are terms and f is an n-ary function symbol, then $f(t_1, \ldots, t_n)$ is a term;
3. if $A(t)$ is a formula, t is a term, and x is a bound variable, then $\varepsilon_x A(x)$ and $\tau_x A(x)$ are terms.
4. if $t_0, \ldots, t_n$ are terms and A is an n-ary predicate symbol, then $A(t_0, \ldots, t_n)$ is a formula;
5. if A and B are formulae, then $A \wedge B$, $A \vee B$, $\neg A$, and $A \to B$ are formulae.

The *ε-sequent calculus* LK^ε is obtained by adding to Propositional LK the rules:

$$\frac{\Gamma, A(t) \vdash \Delta}{\Gamma, A(\tau_x A(x)) \vdash \Delta}\ \tau \qquad\qquad \frac{\Gamma \vdash \Delta, A(t)}{\Gamma \vdash \Delta, A(\varepsilon_x A(x))}\ \varepsilon$$

The term $\varepsilon_x A(x)$ is to be understood as a 'generic' object satisfying property A. The dual term $\tau_x A(x)$ is added for symmetry; it is in fact redundant—one can define $A(\tau_x A) = A(\varepsilon_x(\neg A))$. Terms of the form $\varepsilon_x A(x)$ are called *ε-terms* and terms of the form $\tau_x A(x)$ are called *τ-terms*. The notion of an ε- or a τ-term being *weak* or *strong* is defined in analogy with quantifiers in first-order logic.

The *standard translation* of first-order logic into epsilon calculus is defined recursively by mapping atomic formulae to themselves, respecting Booleans, and dealing with quantified formulae as follows: suppose A' is the standard translation of A, then we map:

$$\exists x\, A(x) \mapsto A'(\varepsilon_x A'(x)), \qquad\qquad \forall x\, A(x) \mapsto A'(\tau_x A'(x)).$$

Proposition 12.5.2. *A sequent is* LK*-derivable if and only if the standard translation of its Skolemization is* $\mathrm{LK}^{\varepsilon}$*-derivable.*

In the $\mathrm{LK}^{\varepsilon}$-proof of the standard translation of a sequent, ε- and τ-rules play the role of weak quantifier inferences, while the substitution of variables for ε- and τ-terms throughout the proof plays the role of strong quantifier inferences. The converse can be proved by appealing to the *Extended First Epsilon Theorem* (cf. [MZ06, Theorem 16]) and the elimination of Skolem functions as in [BHW12]. It can also be proved by a semantic argument.

The characteristic-variable conditions of LK^{++} ensure the following:

Proposition 12.5.3. *If a sequent is has an* LK^{++}*-proof of length* k*, then its standard translation has an* $\mathrm{LK}^{\varepsilon}$*-proof of length* $\leq k$*.*

The proof proceeds by replacing weak-quantifier inferences with ε- or τ-rules and omitting strong quantifier inferences—instead, we substitute the appropriate ε- or τ-term for the characteristic variable throughout the proof.

Proposition 12.5.3 readily yields an alternative proof of Theorem 12.2.8: let $\Gamma \vdash \Delta$ be an LK^{++}-provable sequent. Then its Skolemization is LK^{++}-provable by Proposition 12.4.5. But the Skolemization is common to LK and preserves validity, so by Propositions 12.5.2 and 12.5.3, $\Gamma \vdash \Delta$ is LK-provable.

One can state the relationship between the ε-calculus and LJ^{++}.

Theorem 12.5.4. *A sequent is* LJ^{++}*-provable if and only if its standard translation is* $\mathrm{LJ}^{\varepsilon}$*-provable.*

Proof. Suppose $\Gamma \vdash D$ is LJ^{++}-provable. By Proposition 12.4.4 it is then LJ-provable by adding quantifier shifts as axioms. As in Proposition 12.5.3, the standard translation of $\Gamma \vdash D$ is $\mathrm{LJ}^{\varepsilon}$-provable by adding the standard translations of quantifier shifts as axioms. However, the standard translations of quantifier shifts are already $\mathrm{LJ}^{\varepsilon}$-provable.

Conversely, suppose the standard translation of $\Gamma \vdash D$ is $\mathrm{LJ}^{\varepsilon}$-provable, say by a proof π. We want to transform π into an LJ^{++}-proof. One of the problems is that π might have inferences arranged the following way:

$$\dfrac{\dfrac{\dfrac{\vdots}{\Pi \vdash B(s)}}{\Pi \vdash B(\varepsilon_x B(x))}}{\dfrac{\dfrac{\vdots}{\Pi' \vdash A(\varepsilon_x B(x))}}{\Pi' \vdash A(\varepsilon_x A(x))}}$$

If so, let us refer to

$$\dfrac{\Pi \vdash B(s)}{\Pi \vdash B(\varepsilon_x B(x))}$$

as an *inner inference*.

Claim 12.5.5. *Suppose that π is an LJ^ε-proof of the standard translation of a sequent $\Gamma \vdash D$ in which no strong quantifiers appear. Then there is a LJ^ε-proof of the same sequent in whose axioms appear no ε- or τ-terms.*

Proof. Suppose

$$A(\varepsilon_x B, \vec{b}) \vdash A(\varepsilon_x B, \vec{b})$$

is an axiom in the proof. Replace $\varepsilon_x B$ by a new variable $x_{\varepsilon_x B}$ throughout the proof. Below each inference of the form

$$\dfrac{\Pi \vdash C}{\Pi \vdash B(x_{\varepsilon_x B})}$$

such that $C \neq B(x_{\varepsilon_x B})$, add an inference

$$\dfrac{\Pi \vdash B(x_{\varepsilon_x B})}{\Pi \vdash B(\varepsilon_x B)}$$

and proceed similarly for τ-terms. To see that the resulting proof has the same conclusion, notice that since the conclusion of π contains no strong ε- or τ-terms, any strong ε- or τ-terms must be cut or (weakly) modified by an ε- or τ-rule during the proof, and both those transformations go through after carrying out the substitution of free variables for terms. □

Claim 12.5.6. *Suppose that π is an LJ^ε-proof of the standard translation of a sequent $\Gamma \vdash D$ in which no strong quantifiers appear. Then there is a LJ^ε-proof π' such that*

1. *π' is cut-free;*

2. *π' has as conclusion the standard translation of $\bar{\Gamma}, \Gamma \vdash \Delta$, where $\bar{\Gamma}$ consists of formulae of one of the following forms:*

   (a) $\forall \vec{x}\,(A(\vec{x}) \to A(\vec{x}))$,

   (b) $\forall \vec{x} \forall \vec{y}\,(A(\vec{x}, \vec{y}) \to A(\vec{x}, f(\vec{x}))$, *or*

   (c) $\forall \vec{x} \forall \vec{y}\,(A(\vec{x}, g(\vec{x})) \to A(\vec{x}, \vec{y}))$,

   where f and g are Skolem functions;

3. *π' has no inner inferences.*

Proof. The proof is obtained by replacing occurrences of $A(\varepsilon_x A)$ by $A(f_A(x))$, for Skolem functions f_A. Suppose $A(t)$ is a cut formula in π and t is an ε- or τ-term obtained via an inner inference; e.g., $t = \varepsilon_x B(x)$ below:

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{\Pi \vdash B(s)}}{\Pi \vdash B(\varepsilon_x B(x))}}{\dfrac{\vdots}{\Pi' \vdash A(\varepsilon_x B(x))} \qquad \dfrac{\pi_0 \ \vdots}{A(\varepsilon_x B(x)), \Pi' \vdash E'}}}{\dfrac{\Pi' \vdash E'}{\vdots}}$$

(Here, let us assume that no other terms appear in B, to simplify notation.) Modify π_0 into a proof π_1 as follows: the indicated occurrence of $\varepsilon_x B(x)$ must have been either introduced by weakening—in which case we modify the weakening so that a Skolem constant $c_{B(s)}$ is introduced instead—or by an axiom:

$$\dfrac{\dfrac{C(\varepsilon_x B(x)) \vdash C(\varepsilon_x B(x))}{\vdots}}{A(\varepsilon_x B(x)), \Pi' \vdash E'}$$

in which case we replace it by:

$$\dfrac{\dfrac{C(c_{B(s)}) \vdash C(c_{B(s)}) \qquad C(\varepsilon_x B(x)) \vdash C(\varepsilon_x B(x))}{C(c_{B(s)}) \to C(\varepsilon_x B(x)), C(c_{B(s)}) \vdash C(\varepsilon_x B(x))}}{\dfrac{\vdots}{A(c_{B(s)}), \Pi'' \vdash E'}}$$

where Π'' has the form $C(c_{B(s)}) \to C(\varepsilon_x B(x)), \Pi'$. (In the general case, $c_{B(s)}$ is of the form $f_{B(s)}(\vec{z})$.) We modify π—say, in the latter case—as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{\Pi \vdash B(s)} \quad \dfrac{\vdots}{B(c_{B(s)}) \vdash B(c_{B(s)})}}{B(s) \to B(c_{B(s)}), \Pi \vdash B(c_{B(s)})}}{\dfrac{\vdots}{B(s) \to B(c_{B(s)}), \Pi' \vdash A(c_{B(s)})} \quad \dfrac{\pi_1 \ \vdots}{A(c_{B(s)}), \Pi'' \vdash E'}}}{C(c_{B(s)}) \to C(\varepsilon_x B(x)), B(s) \to B(c_{B(s)}), \Pi' \vdash E'}}{\vdots}$$

Repeat this for every weak ε- and τ-term inferred in each cut formula. We then make a similar modification to the proof for each inner inference. For example,

$$\dfrac{\dfrac{\dfrac{\vdots}{\Pi, \vdash B(s)}}{\Pi, \vdash B(\varepsilon_x A(x))}}{\dfrac{\dfrac{\vdots}{\Pi', \vdash A(\varepsilon_x B(x))}}{\Pi', \vdash A(\varepsilon_x A(x))}}$$

becomes

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{\Pi, \vdash B(s)} \quad B(c_{B(s)}) \vdash B(c_{B(s)})}{B(s) \to B(c_{B(s)}), \Pi, \vdash B(c_{B(s)})}}{\dfrac{\vdots}{B(s) \to B(c_{B(s)}), \Pi', \vdash A(c_{B(s)})}}}{B(s) \to B(c_{B(s)}), \Pi', \vdash A(\varepsilon_x A(x))}$$

After this, replace all cuts with $\to L$ rules as in the proof of Theorem 12.3.3 and extend the proof with one τ-rule applied to each Skolem term and to each ε- and τ-term in each of the introduced implications (in any order). Thus one is left with a proof satisfying all desired conditions. $\square$

To prove the theorem, let π be a proof in LJ^ε of the standard translation of a sequent

$$\Gamma \vdash D.$$

First, find a proof π_0 in LJ^ε of the standard translation of $sk(\Gamma) \vdash sk(D)$ as in Proposition 12.4.5. By the first claim, all axioms in π_0 can be assumed not to contain ε- or τ-terms. Apply the second claim to obtain a cut-free proof π_1 with no inner inferences and with conclusion $\hat{\Gamma}, \Gamma_0 \vdash D_0$, where

1. $\Gamma_0 \vdash D_0$ is the standard translation of the Skolemization of $\Gamma \vdash D$;
2. formulae in $\hat{\Gamma}$ are standard translations of formulae of one of the following forms:
   (a) $\forall \vec{x}\,(A(\vec{x}) \to A(\vec{x}))$,
   (b) $\forall \vec{x} \forall \vec{y}\,(A(\vec{x}, \vec{y}) \to A(\vec{x}, f(\vec{x}))$, or
   (c) $\forall \vec{x} \forall \vec{y}\,(A(\vec{x}, g(\vec{x})) \to A(\vec{x}, \vec{y}))$,

   where f and g are Skolem functions.

Since the proof has no strong terms in its conclusion, is cut-free, and has no inner inferences, it can be directly translated into a proof in LJ. By the argument in Proposition 12.4.8, there is an LJ^{++}-proof of the deskolemized sequent

$$\bar{\Gamma}, \Gamma \vdash D,$$

where $\bar{\Gamma}$ consists of formulae of one of the following forms:

1. $\forall \vec{x}\,(A(\vec{x}) \to A(\vec{x}))$,
2. $\forall \vec{x}\, \exists \vec{z} \forall \vec{y}\,(A(\vec{x}, \vec{y}) \to A(\vec{x}, \vec{z})$, or
3. $\forall \vec{x}\, \exists \vec{z} \forall \vec{y}\,(A(\vec{x}, \vec{z}) \to A(\vec{x}, \vec{y}))$.

By Proposition 12.4.4, these formulae are derivable in LJ^{++}, so they can be cut to obtain a proof of

$$\Gamma \vdash D,$$

as desired. □

13

Unsound Rules in Infinite Proof Systems

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13.1 Introduction

Theorem 13.1.1 (Takeuti)**.** *The following are equivalent for every transitive model M of* $\mathsf{ZF}+\mathsf{DC}$ *containing* $\mathbb{R}$*:*

1. $M \models$ *"* DL *satisfies the cut-elimination theorem,"*
2. $M \models \mathsf{AD}$.

The proof of Theorem 13.1.1 is unusual in that it uses the Axiom of Choice to prove that a statement inconsistent with it holds in an inner model. Using a simple forcing argument, Takeuti's theorem can be rephrased without making reference to inner models.

Corollary 13.1.2. *Assume* $\mathsf{ZF}+\mathsf{DC}$*. The following are equivalent:*

1. DL *satisfies the cut-elimination theorem,*

2. AD.

In the second part of Chapter III, we prove variants of Theorem 13.1.2. The langauge of Takeuti's system DL is uncountable. Its formulae are countable, but sequents need not be. Let "weak cut elimination" stand for the assertion that every sequent having a proof with no atomic formulae of infinite arity has a cut-free proof, and let "σ-projective determinacy" stand for the assertion that every game of length ω whose payoff set is coded by a set of reals in the smallest σ-algebra containing the open sets and closed under continuous images is determined. (Again, cf. Section **??**.) We show:

Theorem 13.1.3. *Assume* $\mathsf{ZF} + \mathsf{DC}$. *The following are equivalent:*

1. DL *satisfies weak cut elimination.*

2. σ*-projective determinacy.*

These statements are consistent with the Axiom of Choice and, using the proof of a theorem of Martin and Steel [MS89], one can see that they are provable in the theory ZFC + "there are infinitely many Woodin cardinals."

Cut elimination for countable sequents in DL is consistent with the Axiom of Choice, provided an additional axiom schema `fin` (defined below) is added to DL. More precisely, let "countable cut elimination" be the assertion that every countable sequent provable in DL has a proof without the cut rule, perhaps making use of instances of `fin`.

Theorem 13.1.4. *Assume* $\mathsf{ZF} + \mathsf{DC}$. *The following are equivalent:*

1. DL *satisfies countable cut elimination.*

2. σ*-projective determinacy.*

Theorem 13.1.3 is in fact seen to be a particular case of Theorem 13.1.4. Theorem 13.1.4 is proved following Takeuti's proof of Theorem 13.1.1, except that one needs to prove soundness and completeness theorems for a semantics slightly different from the intended one. The language of DL involves a (perhaps unnatural) restriction with regard to the length of strings of quantifiers one is allowed to form. Removing this restriction gives rise to a proof system we call $\mathsf{DL}_{\mathbb{R}}$. Applying essentially the proof of Theorem 13.1.2 to $\mathsf{DL}_{\mathbb{R}}$ yields:

Theorem 13.1.5. *Assume* $\mathsf{ZF} + \mathsf{DC}$. *Then, the following are equivalent:*

1. $\mathsf{DL}_{\mathbb{R}}$ *satisfies the cut-elimination theorem.*
2. $\mathsf{AD}_{\mathbb{R}}$.

Similarly, applying the proof of Theorems 13.1.3 and 13.1.4 to $\mathsf{DL}_{\mathbb{R}}$ yields:

Theorem 13.1.6. *Assume* $\mathsf{ZF} + \mathsf{DC}$. *Then, the following are equivalent:*

1. $\mathsf{DL}_{\mathbb{R}}$ *satisfies countable cut elimination.*
2. $\mathsf{DL}_{\mathbb{R}}$ *satisfies weak cut elimination.*
3. *Every game of (fixed) countable length whose payoff set is coded by a projective set of reals is determined.*

These three statements are again consistent with the Axiom of Choice. By a theorem of Neeman [Nee04], they are provable in the theory ZFC + "there are uncountably many Woodin cardinals."

13.2 Determinate logic

For our purposes, the language of $\mathsf{DL}_{\mathbb{R}}$ consists of:

1. An uncountable set of free variables, say, a_i, indexed by $\mathbb{R}$.
2. An uncountable set of bound variables, say x_i, indexed by $\mathbb{R}$.
3. A countable collection of constants, which we identify with the set of natural numbers.
4. Predicate symbols. Each predicate symbol has either a finite number or the set $\mathbb{N}$ associated to it. If n is associated to a symbol A, we say that A has arity n. Otherwise, we say A has infinite arity or arity $\mathbb{N}$.
5. Logical symbols: equality $=$; negation $\neg$; finite and infinite disjunctions $\bigvee_{i\in\alpha}$, for $\alpha \in \mathbb{N}\cup\{\mathbb{N}\}$; and quantifier chains ∂_α, for α a countable ordinal.

If $\vec{x} = \{x_\iota : \iota < \alpha\}$, then the intended interpretation of $\partial_\alpha \vec{x}$ is the quantifier chain

$$\exists x_0\, \forall x_1\, \exists x_2\, \forall x_3\, \ldots\, \exists x_\omega\, \forall x_{\omega+1}\, \ldots,$$

of length α.

We have chosen this specific language to simplify certain arguments. We remark that all results here would remain true after undertaking any (or all) of the following changes:

1. adding predicate symbols of arbitrary countable arity;
2. adding constant symbols to the language, as long as there is a surjection from $\mathbb{R}$ to the set of all constant symbols;
3. adding function symbols of any countable arity;
4. adding implication or conjunctions;
5. adding disjunctions of transfinite, but countable, length;
6. adding chains of quantifiers following any given binary pattern (or arbitrary patterns).

The notion of a formula is defined as usual by induction: if A is a predicate symbol of arity α and $\vec{t} = \{t_\iota : \iota < \alpha\}$ is a string of symbols each of which is either a constant or a free variable, then $A(\vec{t})$ is a formula. We sometimes write $A(t_0, t_1, \ldots)$ or $A(t_\iota)_{\iota<\alpha}$ for $A(\vec{t})$. Inductively, if A is a formula, then $\neg A$ is a formula; if A_ι is a formula for each $\iota < \alpha$ and $\alpha \in \mathbb{N} \cup \{\mathbb{N}\}$, then $\bigvee_{\iota<\alpha} A_\iota$ is a formula. If A is a formula and $\vec{t} = \{t_\iota : \iota < \alpha\}$ is a string of constants or variables appearing in (subformulae) of A, we may write $A(\vec{t})$ for A to indicate this fact. If so, and $\vec{s} = \{s_\iota : \iota < \alpha\}$ is a string of constants or variables of the same length, we write $A(\vec{s})$ for the result of substituting each s_ι for the corresponding t_ι.

If $A(a_\iota)_{\iota<\alpha}$ is a formula and each of the a_ι is a free variable, then

$$\eth_\alpha(x_\iota)_{\iota<\alpha} A(\vec{x})$$

is a formula, provided each of the x_ι does not appear in $A(a_\iota)_{\iota<\alpha}$ and all the x_ι are distinct. Formulae such as this one allow us to express usual quantifiers e.g., by writing:

$$\forall y\, A(y) := \eth_2(x, y) A(y).$$

In fact, we may frequently abuse notation by writing, e.g.,

$$\exists (x_{2k})_{k\in\mathbb{N}}\, A(x_{2k})_{k\in\mathbb{N}} := \eth_\omega (x_k)_{k\in\mathbb{N}}\, A(x_{2k})_{k\in\mathbb{N}}.$$

If A is a predicate symbol of arity $\xi + \eta$ and $\vec{t}$ is a sequence of terms of length ξ, we may regard $A(\vec{t}, \cdot)$ as a predicate of arity η by fixing $\vec{t}$. Similarly, if $A(t_\iota)_{\iota<\alpha}$ is a formula, we may regard it as having arity α if we intend for $(t_\iota)_{\iota<\alpha}$ to vary, and if $A(\cdot, \cdot)$ is a formula of arity $\xi + \eta$ and $\vec{t}$ is a sequence of terms of length ξ, $A(\vec{t}, \cdot)$ may be regarded as a formula of arity η. An example of this is the formula $A(a_n)_{n\in\mathbb{N}}$ given by

$$\bigvee_{n\in\mathbb{N}} a_n = 0.$$

Definition 13.2.1. The language of DL is the restriction of the language of $\mathsf{DL}_{\mathbb{R}}$ in which chains of quantifiers have length at most ω.

A *sequent* is an expression

$$\Gamma \vdash \Delta$$

such that Γ and Δ are sets of formulae and there is a surjection

$$\rho : \mathbb{R} \twoheadrightarrow \Gamma \cup \Delta.$$

The expression $\Gamma \vdash \Delta$ is interpreted as "if all the formulae in Γ hold, then some formula in Δ holds." We will be somewhat liberal in denoting sequents – for instance, the following is (or denotes) a sequent:

$$\{A(n)\}_{n\in\mathbb{N}} \vdash A(4), B(a), \Pi.$$

Ideally, what is meant will always be clear from the context.

A sequent calculus

We reproduce Takeuti's definition of the sequent calculus for DL. It also yields a sequent calculus for $\mathsf{DL}_{\mathbb{R}}$ if one considers the corresponding language. The main feature of the calculus is that, although proofs (viewed as trees) might be infinite (and uncountable), they are wellfounded, in the sense that every branch is finite. Because of this, one needs to allow for many inferences to be performed simultaneously. Consider the following example in which we simultaneously derive $\bigvee_{j\in\mathbb{N}} i = j$ from $\{i = j\}_{j\in\mathbb{N}}$ for each natural number $i \in \mathbb{N}$:

$$\frac{\vdash \{i = j\}_{i,j\in\mathbb{N}}}{\vdash \{\bigvee_{j\in\mathbb{N}} i = j\}_{i\in\mathbb{N}}}$$

Here, the bottom sequent $\{\bigvee_{j\in\mathbb{N}} i = j\}_{i\in\mathbb{N}}$ is a wellordered set of formulae, but it need not be. In general, we will denote sets of formulae indexed by some set I by

$$\{A_\iota\}_{\iota\in I}.$$

By definition, any such I needs to be a surjective image of $\mathbb{R}$ if $\{A_\iota\}_{\iota\in I}$ is part of a sequent. Nonetheless, if I is not relevant, we may simply write $\{A_\iota\}_\iota$. One would expect the dual of the inference above to have many premises:

$$\frac{\{i = f(i)\}_{i\in\mathbb{N}} \vdash \qquad \text{for all functions } f : \mathbb{N} \to \mathbb{N}}{\{\bigvee_{j\in\mathbb{N}} i = j\}_{i\in\mathbb{N}} \vdash}$$

Before defining the calculus, let us introduce some more abuse of notation. Formally, formulae do not have transfinite arity, yet quantifiers might have order type greater than ω. Given an atomic formula $A(x_i)_{i\in\mathbb{N}}$ and a bijection

$$\rho : \mathbb{N} \to \theta,$$

we sometimes write $A(x_\iota)_{\iota<\theta}$ to mean $A(x_{\rho(i)})_{i\in\mathbb{N}}$. We also write

$$\widetilde{\Im}(x_k)_{k\in\mathbb{N}}\, A(x_k)_{k\in\mathbb{N}}$$

for

$$\Im(y, x_0, x_1, x_2, \ldots)\, A(x_0, x_1, x_2, \ldots),$$

where y is some bound variable not appearing in $A(x_0, x_1, x_2, \ldots)$ or anywhere else in the context of discourse.

The calculus includes axioms and rules for Booleans, equality, and quantifiers, as well as a structural rule. The cut rule can be defined either as usual:

$$\frac{\Gamma \vdash \Delta, A \qquad A, \Gamma \vdash \Delta}{\Gamma \vdash \Delta}$$

or in more general ways, e.g.,

$$\frac{\{\Gamma \vdash \Delta, A_i\}_{i\in A} \qquad \{A_i : i \in I\}, \Gamma \vdash \Delta}{\Gamma \vdash \Delta}$$

A *derivation tree* is a tree of inferences of $\mathsf{DL}_{\mathbb{R}}$ with the obvious restrictions. The axioms are:

$$A \vdash A; \qquad \vdash \{t = i\}_{i\in\mathbb{N}}; \qquad \vdash t = t; \qquad i = j \vdash;$$

where A is atomic, t is any term, and $i \neq j \in \mathbb{N}$. We define the inference rules.

1. The structural rule is:

$$\frac{\Gamma \vdash \Delta}{\Gamma' \vdash \Delta'}$$

where every formula occurring in Γ occurs in Γ' and every formula occurring in Δ occurs in Δ'.

2. The first equality rule is:

$$\frac{\Gamma \vdash \Delta}{\{t_\iota = s_\iota\}_{\iota \in I}, \Gamma' \vdash \Delta'} =_1$$

where Γ' and Δ' are respectively obtained from Γ and Δ by arbitrarily exchanging any t_ι and the corresponding s_ι.

3. The second equality rule is as follows: let Σ consist of all formulae $a = b$, for a and b in some set of variables. Let $\widetilde{\Sigma}$ be the set of all pairs (Σ_1, Σ_2) that partition Σ. The rule is:

$$\frac{\Sigma_1, \Gamma \vdash \Delta, \Sigma_2 \text{ for all } (\Sigma_1, \Sigma_2) \in \widetilde{\Sigma}}{\Gamma \vdash \Delta} =_2$$

4. One can negate several formulae at once:

$$\frac{\{A_\iota\}_{\iota \in I}, \Gamma \vdash \Delta}{\Gamma \vdash \Delta, \{\neg A_\iota\}_{\iota \in I}} \neg{:}R \qquad \frac{\Gamma \vdash \Delta, \{A_\iota\}_{\iota \in I}}{\{\neg A_\iota\}_{\iota \in I}, \Gamma \vdash \Delta} \neg{:}L$$

5. Let I be a set and β_ι be a sequence of countable ordinals indexed by I. The rules for disjunction allow us to infer a collection of disjunctions indexed by elements ι of I, with each disjunction of length β_ι:

$$\frac{\Gamma \vdash \Delta, \{A_{\iota,\mu}\}_{\mu < \beta_\iota, \iota \in I}}{\Gamma \vdash \Delta, \{\bigvee_{\mu < \beta_\iota} A_\iota\}_{\iota \in I}} \bigvee{:}R$$

$$\frac{\{A_{\iota,\mu_\iota}\}_{\iota \in I}, \Gamma \vdash \Delta \text{ for all sequences } \{\mu_\iota\}_{\iota \in I} \in \prod_{\iota \in I} \beta_\iota}{\{\bigvee_{\mu < \beta_\iota} A_\iota\}_{\iota \in I}, \Gamma \vdash \Delta} \bigvee{:}L$$

6. The right-quantifier rule is:

$$\frac{\Gamma \vdash \Delta, \{A_\iota(a_{\iota,\xi})_{\xi < \lambda_\iota}\}_{\iota < \theta}}{\Gamma \vdash \Delta, \{\eth_{\lambda_\iota}(x_{\iota,\xi})_{\xi < \lambda_\iota} A_\iota(x_{\iota,\xi})_{\xi < \lambda_\iota}\}_{\iota < \theta}}$$

7. The left-quantifier rule is:

$$\frac{\{A_\iota(a_{\iota,\xi})_{\xi < \lambda_\iota}\}_{\iota < \theta}, \Gamma \vdash \Delta}{\{\eth_{\lambda_\iota}(x_{\iota,\xi})_{\xi < \lambda_\iota} A_\iota(x_{\iota,\xi})_{\xi < \lambda_\iota}\}_{\iota < \theta}, \Gamma \vdash \Delta}$$

To quantifier inferences is associated a restriction – the *characteristic-variable conditions*. These are given as follows: suppose

$$\frac{\Gamma \vdash \Delta, \{A_\iota(a_{\iota,\xi})_{\xi<\lambda_\iota}\}_{\iota<\theta}}{\Gamma \vdash \Delta, \{\Im_{\lambda_\iota}(x_{\iota,\xi})_{\xi<\lambda_\iota}\, A_\iota(x_{\iota,\xi})_{\xi<\lambda_\iota}\}_{\iota<\theta}}$$

is a quantifier inference. Then, for each $\iota < \theta$ and each even $\xi < \lambda_\iota$, the variable $a_{\iota,\xi}$ is called the ξth *characteristic variable* of the inference (and of A_ι). Moreover, for each $\iota < \theta$, $A_\iota(a_{\iota,\xi})_{\xi<\lambda_\iota}$ is called an *auxiliary formula* of the inference (and of each of its characteristic variables, and of $\Im_{\lambda_\iota}(x_{\iota,\xi})_{\xi<\lambda_\iota}\, A_\iota(x_{\iota,\xi})_{\xi<\lambda_\iota}$) and $\Im_{\lambda_\iota}(x_{\iota,\xi})_{\xi<\lambda_\iota}\, A_\iota(x_{\iota,\xi})_{\xi<\lambda_\iota}$ is called a *principal formula* of the inference (and of each of its characteristic variables, and of $A_\iota(a_{\iota,\xi})_{\xi<\lambda_\iota}$). If π is a derivation tree in which the above inference occurs, we write $b <_\pi a$ if one of the following holds:

1. $a = a_{\iota,\xi}$, $b = a_{\iota,\zeta}$, $\zeta < \xi$, and $a_{\iota,\xi}$ is a characteristic variable of A_ι;
2. a is a characteristic variable of A_ι and b appears in the principal formula of a.

Similarly, if

$$\frac{\{A_\iota(a_{\iota,\xi})_{\xi<\lambda_\iota}\}_{\iota<\theta}, \Gamma \vdash \Delta}{\{\Im_{\lambda_\iota}(x_{\iota,\xi})_{\xi<\lambda_\iota}\, A_\iota(x_{\iota,\xi})_{\xi<\lambda_\iota}\}_{\iota<\theta}, \Gamma \vdash \Delta}$$

is a quantifier inference, then, for each ι and each odd ξ, the variable $a_{\iota,\xi}$ is a characteristic variable of the inference and of A_ι. The notions of "auxiliary formula," "principal formula," as well as the fragment of the relation $<_\pi$ induced by this inference, are defined similarly.

Definition 13.2.2. A quantifier inference I in a derivation tree π is *π-suitable* if it satisfies the following three conditions:

- (substitutability) none of its characteristic variables appear in the conclusion of π;
- (side-variable condition) the relation $<_\pi$ is wellfounded;
- (very weak regularity) if its ξth characteristic variable is a characteristic variable of a quantifier inference J in π, and A is the corresponding principal formula in J, then A is the principal formula of I and, letting $\{a^I_\iota\}_\iota$ and $\{a^J_\iota\}_\iota$ be the quantified variables in both inferences in that order, we have $a^I_\iota = a^J_\iota$ for all $\iota \leq \xi$.

(These are Takeuti's conditions; the terminology comes from [**?**].)

Definition 13.2.3. A derivation tree π is a *proof in* $\mathsf{DL}_{\mathbb{R}}$ if all quantifier inferences in π are π-suitable. A derivation tree π is a *proof in* DL if it is a proof in $\mathsf{DL}_{\mathbb{R}}$ and every formula in π belongs to the language of DL.

Without loss of generality we assume that for every proof π, there are continuum-many free and bound variables not appearing in it. The power of the calculus for Determinate Logic lies in the following fact. The analog for DL also holds.

Lemma 13.2.4. *Let θ be a countable ordinal and $A(x_\iota)_{\iota<\theta}$ be a formula. Then the sequent*

$$\vdash \Im_\theta(x_\iota)_{\iota<\theta}\, A(x_\iota)_{\iota<\theta}, \widetilde{\Im}_\theta(x_\iota)_{\iota<\theta}\, \neg A(x_\iota)_{\iota<\theta}$$

is provable in $\mathsf{DL}_{\mathbb{R}}$.

Proof. A simple inductive argument shows that compound axioms are derivable, i.e., for any formula B, one can find a proof of

$$B \vdash B$$

in $\mathsf{DL}_{\mathbb{R}}$. Letting $(a_\iota)_{\iota<\theta}$ be a sequence of distinct free variables and π be a proof of

$$A(a_\iota)_{\iota<\theta} \vdash A(a_\iota)_{\iota<\theta},$$

it is not hard to see that the following is a $\mathsf{DL}_{\mathbb{R}}$-proof:

$$\dfrac{\dfrac{\dfrac{\vdots\ \pi}{A(a_\iota)_{\iota<\theta} \vdash A(a_\iota)_{\iota<\theta}}}{\vdash A(a_\iota)_{\iota<\theta}, \neg A(a_\iota)_{\iota<\theta}}}{\vdash \Im_\theta(x_\iota)_{\iota<\theta}\, A(x_\iota)_{\iota<\theta}, \widetilde{\Im}_\theta(x_\iota)_{\iota<\theta}\, \neg A(x_\iota)_{\iota<\theta}}$$

□

Canonical sequents

The proof of Theorem 13.1.1 involves representing sets $X \subset \mathbb{R}$ with sequents in the language of DL (or even in $L_{\omega_1,0}$). A slight variant of Takeuti's representation will be used later on – we call this the *canonical-sequent* representation of X. This sequent will be of the form

$$\Gamma_X \vdash \Delta_X.$$

The variant makes use of the fact that simple sets are representable in simple ways. We start by defining a formula A_X for each $X \subset \mathbb{R}$. Henceforth we will abuse notation by writing $A(t_0, t_1, \ldots, t_k)$ for $A(t_0, t_1, \ldots, t_{\mathrm{ar}(A)})$, where $\mathrm{ar}(A) < k$ and $\mathrm{ar}(A)$ denotes the arity of A. Similarly, if A is of finite arity, we may abuse notation by writing $A(t_k)_{k\in\mathbb{N}}$ for $A(t_0, \ldots, t_{\mathrm{ar(A)}})$.

Suppose $X \subset \mathbb{R}$ is a basic open set, i.e., there exist $n \in \mathbb{N}$ and $s \in \mathbb{N}^n$ such that

$$x \in X \text{ if, and only if, } x \restriction n = s.$$

We define A_X to be an arbitrarily chosen (but fixed) predicate symbol of arity n. Without loss of generality we assume that if X and Y are different basic open sets, then A_X and A_Y are different atomic formulae. The function from $\bigcup_{n\in\mathbb{N}} \mathbb{N}^n$ to the language given by

$$s \mapsto A_{O(s)}$$

exists by the Principle of Dependent Choice.

Suppose $X = \bigcup_{k\in\mathbb{N}} X_k$ and that A_{X_k} has been defined for each $k \in \mathbb{N}$. Then we set

$$A_X = \bigvee_{k\in\mathbb{N}} A_{X_k}.$$

Note already that A_X may have infinite arity.

Suppose $X = \mathbb{R} \setminus Y$ and that A_Y has been defined. Then we set $A_X = \neg A_Y$. Suppose that

$$X = \{x \in \mathbb{R} : \exists y \in \mathbb{R}\, (x, y) \in Y\}$$

and that $A_{Y'}$ has been defined, where

$$Y' = \{y \in \mathbb{R} : \big((y_{2n})_{n\in\mathbb{N}}, (y_{2n+1})_{n\in\mathbb{N}}\big) \in Y\}.$$

We define

$$A_X(x_0, x_1, x_2, \ldots) = \exists (y_k)_{k\in\mathbb{N}}\, A_{Y'}(x_0, y_0, x_1, y_1, x_2 \ldots).$$

Definition 13.2.5. Suppose $X \subset \mathbb{R}$. If X is σ-projective, then A_X is defined to be the formula above. Otherwise, A_X is an arbitrary predicate symbol of infinite arity.

We do not have enough symbols in the language to assume that if X and Y are different non-σ-projective sets, then the symbols A_X and A_Y are distinct; however, at no point will this be an issue, since we will not need to consider more than one canonical sequent at once. In principle, one needs to show:

Lemma 13.2.6. *Suppose $X \subset \mathbb{R}$ is σ-projective, then A_X is defined.*

Proof. One can inductively define a hierarchy $\{S_\alpha : \alpha < \omega_1\}$ of subsets of $\mathbb{R}$ by letting S_0 consist of all basic open sets; taking unions at limit stages; and alternating between closing under countable unions, complements, and continuous images at successor stages. Inductively one shows that every σ-algebra containing the open sets and closed under continuous images contains each S_α. Moreover, the least such σ-algebra is $\bigcup_{\alpha<\omega_1} S_\alpha$ since, if $\{X_n : n \in \mathbb{N}\}$ is a countable set of subsets of $\mathbb{R}$ in $\bigcup_{\alpha<\omega_1} S_\alpha$, then one can use DC to choose a sequence $\{\xi_n : n \in \mathbb{N}\}$ of countable ordinals such that $X_n \in S_{\xi_n}$ for all $n \in \mathbb{N}$. If so, then, letting $\xi = \bigcup_{n\in\mathbb{N}} \xi_n$, we have $\bigcup_{n\in\mathbb{N}} X_n \in S_{\xi+\omega}$, by definition; so $\bigcup_{\alpha<\omega_1} S_\alpha$ is, in fact, a σ-algebra.

Afterwards, a simple induction shows that A_X is defined for every $X \in S_\alpha$ and every $\alpha < \omega_1$. □

Given $X \subset \mathbb{R}$, one uses the formula A_X to define the canonical sequent for X. Recall that natural numbers are constants in the language.

Definition 13.2.7. Let $X \subset \mathbb{R}$. The *canonical sequent* $\Gamma_X \vdash \Delta_X$ is defined as follows:

1. If X is σ-projective, then let $A_{X_0}, A_{X_1}, A_{X_2}, \ldots$, list all atomic formulae appearing in A_X and let $X_0, X_1, X_2, \ldots$, be the corresponding basic open sets in $\mathbb{R}$. Then,
   (a) $\Gamma_X = \{A_{X_n}(i_0, i_1, \ldots, i_{k-1}) : \text{each } i_m \text{ is a natural number, } n \in \mathbb{N}, k \text{ is the arity of } A_{X_n}, \text{ and } X_n = O(i_0, i_1, \ldots, i_{k-1})\}$.
   (b) $\Delta_X = \{A_{X_n}(i_0, i_1, \ldots, i_{k-1}) : \text{each } i_m \text{ is a natural number, } n \in \mathbb{N}, k \text{ is the arity of } A_{X_n}, \text{ and } X_n \neq O(i_0, i_1, \ldots, i_{k-1})\}$.
2. Otherwise,
   (a) $\Gamma_X = \{A_X(i_k)_{k\in\mathbb{N}} : \text{each } i_m \text{ is a natural number and } (i_k)_{k\in\mathbb{N}} \in X\}$.
   (b) $\Delta_X = \{A_X(i_k)_{k\in\mathbb{N}} : \text{each } i_m \text{ is a natural number and } (i_k)_{k\in\mathbb{N}} \notin X\}$.

The semantics for Determinate Logic is defined in the next section, but we remark now that it will agree with the usual semantics of L_{ω_1,ω_1} for formulae in L_{ω_1,ω_1}. It follows that if $\mathfrak{A}$ is a structure with universe $\mathbb{N}$, then the canonical sequent $\Gamma_X \vdash \Delta_X$ is satisfied in $\mathfrak{A}$ if, and only if, A_X is not interpreted as X.

13.3 Standard semantics and restricted semantics

Definition 13.3.1. An $\mathbb{N}$-*model* for Determinate Logic is a structure $\mathfrak{A}$ with universe $\mathbb{N}$ and a set $X_A \subset \mathbb{N}^\alpha$ for each predicate symbol A of arity α.

In particular, atomic formulae of infinite arity are interpreted as subsets of $\mathbb{R}$. Recall that given an atomic formula $A(x_i)_{i\in\mathbb{N}}$ and a bijection

$$\rho : \mathbb{N} \to \theta,$$

we sometimes write $A(x_\iota)_{\iota<\theta}$ to mean $A(x_{\rho(i)})_{i\in\mathbb{N}}$. If so, and given an interpretation of A as a subset of $\mathbb{R}$, we can also interpret A as the corresponding subset of $\mathbb{N}^\theta$.

Given an $\mathbb{N}$-model for Determinate Logic, the satisfaction relation is defined in such a way that it commutes with disjunctions and negations. If $A(t_\iota)_{\iota<\theta}$ is a formula, then the satisfaction relation is extended to $\Im_\theta(x_\iota)_{\iota<\theta}\, A(x_\iota)_{\iota<\theta}$ as follows:

$$\mathfrak{A} \models \Im_\theta(x_\iota)_{\iota<\theta}\, A(x_\iota)_{\iota<\theta}$$

if, and only if, Player I has a winning strategy in the following game. Players I and II take turns playing natural numbers for θ-many rounds:

I	n_0		$\dots$	n_ι	$\dots$	$(\iota<\theta)$
II		n_1	$\dots$		$\dots$	$(\iota<\theta)$

Player I wins if, and only if, $(n_\iota)_{\iota<\theta}$ belongs to the interpretation of A. Inductively, the interpretation of A as a subset of $\mathbb{N}^\theta$ has been defined, so this makes sense.

A sequent $\Gamma \vdash \Delta$ is *satisfied* in an $\mathbb{N}$-model $\mathfrak{A}$ if one of the following holds:

1. There is a formula $A \in \Gamma$ such that $\mathfrak{A} \not\models A$; or
2. There is a formula $A \in \Delta$ such that $\mathfrak{A} \models A$.

The following is established by a simple induction:

Lemma 13.3.2. *Let $\Gamma \vdash \Delta$ be a sequent and suppose that $\Gamma \vdash \Delta$ has a proof in which no quantifiers appear. Then $\Gamma \vdash \Delta$ is satisfied in every $\mathbb{N}$-model.*

Lemma 13.3.3. *Let $X \subset \mathbb{R}$. Then $\Gamma_X \vdash \Delta_X$ is not cut-free provable.*

Proof. This is immediate from the previous lemma, as any cut-free proof of the canonical sequent $\Gamma_X \vdash \Delta_X$ has no quantifiers and $\Gamma_X \vdash \Delta_X$ is satisfied in any $\mathbb{N}$-model in which A_X is not interpreted as X. □

We will consider six soundness lemmata in total – four of them are stated in this section, and two more in the next one. The first lemma is due to Takeuti and the rest are obtained through simple modifications of the proof. The first (new) lemma we prove fully, for self-containedness; the other four we simply sketch.

Lemma 13.3.4 (Takeuti)**.** *Suppose* ZFC *holds and let M be a transitive model of* ZF + DC + AD *containing* $\mathbb{R}$*. Then $M \models$ "every provable sequent is satisfied in every $\mathbb{N}$-model."*

The use of the Axiom of Choice in the preceding lemma and in Lemma 13.3.7 can be avoided by arguing as in the proof of Theorem 13.1.5 on page 356.

Lemma 13.3.5. *Assume that σ-projective sets of reals are determined and that every atomic formula in the language of* DL *has finite arity. Suppose that a sequent $\Gamma_0 \vdash \Delta_0$ is provable in* DL*. Then it is satisfied in every $\mathbb{N}$-model.*

Proof. Let π be a proof of a sequent $\Gamma_0 \vdash \Delta_0$ and let $\mathfrak{A}$ be an $\mathbb{N}$-model. We construct a new proof with conclusion

$$\Gamma_0 \vdash \Delta_0$$

each of whose sequents, including the conclusion, is satisfied in $\mathfrak{A}$. The modified proof will differ from the original proof only with regard to its characteristic variables, and so it will have

$$\Gamma_0 \vdash \Delta_0$$

as end sequent, by substitutability. It will not, strictly speaking, be a DL-proof, since it will contain new symbols, but it will be enough for the argument. For each quantifier inference in the proof, e.g.,

$$\frac{A(a_\iota)_{\iota<\theta}, \Gamma \vdash \Delta}{\exists(x_\iota)_{\iota<\theta}\, A(x_\iota)_{\iota<\theta}, \Gamma \vdash \Delta}$$

we will define inductively, using the Principle of Dependent Choice, new symbols u_ι^A, for even $\iota < \theta$, thought of as free variables, together with their interpretation in $\mathfrak{A}$. The reason why we define u_ι^A for even ι is that those are the indices of the characteristic variables among $(a_\iota)_{\iota<\theta}$; quantifier inferences on the right-hand side will be treated dually.

The new proof will be obtained by induction along $<_\pi$ by substituting these new symbols for the characteristic variables in π. It will be such that if

$$\frac{A(u_\iota)_{\iota<\theta}, \Gamma \vdash \Delta}{\eth(x_\iota)_{\iota<\theta}\, A(x_\iota)_{\iota<\theta}, \Gamma \vdash \Delta}$$

is an inference in the new proof, then

$$\mathfrak{A} \models \eth(x_\iota)_{\iota<\theta}\, A(x_\iota)_{\iota<\theta} \to A(u_\iota)_{\iota<\theta}; \tag{13.1}$$

and if

$$\frac{\Gamma \vdash \Delta, A(u_\iota)_{\iota<\theta}}{\Gamma \vdash \Delta, \eth(x_\iota)_{\iota<\theta}\, A(x_\iota)_{\iota<\theta}}$$

is an inference in the new proof, then

$$\mathfrak{A} \models A(u_\iota)_{\iota<\theta} \to \eth(x_\iota)_{\iota<\theta}\, A(x_\iota)_{\iota<\theta}. \tag{13.2}$$

Note that the result easily follows from equations (13.1) and (13.2).

The construction is as follows: at any given even stage, consider the collection of all characteristic variables a such that every free variable b with $b <_\pi a$ has been modified. Each a corresponds to an inference, say,

$$\frac{A(a_\iota)_{\iota<\theta}, \Gamma \vdash \Delta}{\eth(x_\iota)_{\iota<\theta}\, A(x_\iota)_{\iota<\theta}, \Gamma \vdash \Delta}$$

For each such a, say $a = a_\xi$, we proceed as follows. Since a is a characteristic variable of the inference, ξ is even. Every b with $b <_\pi a$ has been modified, so it follows in particular that, letting $(a_\iota)_{\iota<\theta}$ be the above sequence, then a_ι, for even $\iota < \xi$, has been replaced with a new variable u_ι^A and the interpretation of u_ι^A in $\mathfrak{A}$ has been defined. Furthermore, some a_ι with $\iota < \xi$ odd has perhaps been modified; let us write u_ι^A for the term appearing in place of a_ι also in this case.

Consider the two-player game of length $-\xi + \theta$ given by:

I	n_ξ	$\dots$	$n_{\xi+\eta}$	$\dots$	$(\eta < -\xi + \theta)$
II	$n_{\xi+1}$	$\dots$		$\dots$	$(\eta < -\xi + \theta)$

Player I wins if, and only if,

$$(n_{\xi+\eta})_{\eta<-\xi+\theta} \in \Big\{(x_{\xi+\eta})_{\eta<-\xi+\theta} : \mathfrak{A} \models A'\Big((u_\iota^A)_{\iota<\xi}, (x_{\xi+\eta})_{\eta<-\xi+\theta}\Big)\Big\}.$$

Here, the formula A' is A, except that free variables b with $b <_\pi a$ appearing in A, but not within the quantified variables $(a_\iota)_{\iota<\theta}$, have been replaced inductively. This game can be viewed as the game A' (of length θ), except that we assume that the plays $(u_\iota^A)_{\iota<\xi}$ have already been made.

If Player I has a winning strategy in A', then we would like to choose one such strategy σ and let the new variable u_ξ^A be interpreted in $\mathfrak{A}$ according to σ. Moreover, we would like to assume that the previous plays $(u_\iota^A)_{\iota<\xi}$ are consistent with σ, so that it remains a winning strategy. Below, let us abuse notation by identifying formulae in π with their interpretations in $\mathfrak{A}$.

Sublemma 13.3.6. *There is a function f assigning a winning strategy (for the winning player) to each formula A in the language of* DL *(as well as in its extension by the new symbols), if any exist.*

Let us postpone the proof of the sublemma temporarily and assume for now that it holds. Notice that it follows immediately from the Axiom of Choice, so the reader who is content with working in ZFC may deem the sublemma as proved. If the Axiom of Choice does not hold, one can argue as in the proof of Theorem 13.1.5 on page 356. Nonetheless, we give an alternate proof of Sublemma 13.3.6 below which does not involve forcing.

Hence, if Player I has a winning strategy, let u_ξ^A be interpreted in $\mathfrak{A}$ according to $f(A')$. This meets our earlier criteria. Otherwise, if Player I does not have a winning strategy, let u_ξ^A be interpreted as 0.

At any given odd stage of the construction, consider the collection of all characteristic variables a, say, corresponding to an inference

$$\frac{\Gamma \vdash \Delta, A(a_\iota)_{\iota<\theta}}{\Gamma \vdash \Delta, \Im(x_\iota)_{\iota<\theta}\, A(x_\iota)_{\iota<\theta}}$$

such that every free variable b with $b <_\pi a$ has been modified. Using notation as before (in this case ξ must be odd), consider the two-player game of length $-\xi+\theta$ given by:

I		$n_{\xi+1}$	$\dots$	$\dots$	$(\eta < -\xi+\theta)$
II	n_ξ	$\dots$	$n_{\xi+\eta}$	$\dots$	$(\eta < -\xi+\theta)$

Again, Player I wins if, and only if,

$$(n_{\xi+\eta})_{\eta<-\xi+\theta} \in \Big\{(x_{\xi+\eta})_{\eta<-\xi+\theta} : \mathfrak{A} \models A'\Big((u_\iota^A)_{\iota<\xi}, (x_{\xi+\eta})_{\eta<-\xi+\theta}\Big)\Big\},$$

where A' is as before. If Player II has a winning strategy in this game, let the new variable u_ξ^A be interpreted in $\mathfrak{A}$ according to $f(A')$. Otherwise, let it be interpreted as 0.

By construction, the new proof has the following properties:

1. If A is the principal formula of a left quantifier inference, then equation (13.1) holds.

2. If A is the principal formula of a right quantifier inference, then

$$\mathfrak{A} \models \widetilde{\exists}(x_\iota)_{\iota<\theta}\, \neg A(x_\iota)_{\iota<\theta} \to \neg A(u_\iota)_{\iota<\theta}.$$

Therefore, it suffices to show

$$\mathfrak{A} \models \neg\exists(x_\iota)_{\iota<\theta}\, A(x_\iota)_{\iota<\theta} \to \widetilde{\exists}(x_\iota)_{\iota<\theta}\, \neg A(x_\iota)_{\iota<\theta}. \tag{13.3}$$

That is, we need to show that, letting $\mathcal{G}$ be the game given by A as interpreted in the structure $\mathfrak{A}$, if Player I does not have a winning strategy in $\mathcal{G}$, then Player II does. By the definition of DL, we must have $\theta \leq \omega$. If $\theta \in \mathbb{N}$, then there is nothing to show, so we assume $\theta = \omega$. Thus, it is sufficient to verify that the set

$$\{(x_i)_{i\in\mathbb{N}} : \mathfrak{A} \models A(x_i)_{i\in\mathbb{N}}\}$$

is σ-projective; this is done by induction.

Specifically, let B be a subformula of A. In B, as well as in A, there might be constants and free variables; say $B(x_i)_{i\in\mathbb{N}}$ is of the form $D((x_i)_{i\in\mathbb{N}}, \vec{b})$ (here, as in the previous section, we might be abusing notation by writing, e.g., $B(x_i)_{i\in\mathbb{N}}$ for $B(x_0, \ldots, x_{\mathrm{ar}(B)})$, if B is of finite arity). Let

$$B_{\mathfrak{A}} = \{(x_i)_{i\in\mathbb{N}} : \mathfrak{A} \models D((x_i)_{i\in\mathbb{N}}, \vec{b})\}.$$

We proceed by cases according to the form of B. If B is atomic, then it is easy to see that $B_{\mathfrak{A}}$ is an open subset of $\mathbb{R}$. If $B = \neg C$ and $C_{\mathfrak{A}}$ is σ-projective, then so too is B. A similar argument takes care of the case that $B = \bigvee_\alpha B_\alpha$. Suppose

$$B(x_i)_{i\in\mathbb{N}} = \exists(y_i)_{i\in\mathbb{N}}\, C\big((x_i)_{i\in\mathbb{N}}, (y_i)_{i\in\mathbb{N}}\big)$$

and that the induction hypothesis holds for $C_{\mathfrak{A}}$. Then, $(x_i)_{i\in\mathbb{N}} \in B_{\mathfrak{A}}$ if, and only if, there exists a real $x_\sigma \in \mathbb{R}$ coding a strategy

$$\sigma : \bigcup_{n\in\mathbb{N}} \mathbb{N}^n \to \mathbb{N}$$

such that it is not the case that there exist a real y_τ coding a strategy

$$\tau : \bigcup_{n\in\mathbb{N}} \mathbb{N}^n \to \mathbb{N}$$

and a real $z \in \mathbb{R}$ such that $z(2i) = \sigma(z \restriction 2i)$ and $z(2i+1) = \tau(z \restriction 2i+1)$ for all $i \in \mathbb{N}$ for which

$$\mathfrak{A} \models C\big((x_i)_{i\in\mathbb{N}}, (z(i))_{i\in\mathbb{N}}\big)$$

holds. It follows that $B_{\mathfrak{A}}$ can be obtained in finitely many steps by applying continuous images and complements to $C_{\mathfrak{A}}$ and so is as desired. This shows that equation (13.3) holds and therefore the lemma is proved, granted the sublemma.

Proof of the Sublemma. The function f can be defined by appealing to σ-projective determinacy. Since $\mathfrak{A}$ is an $\mathbb{N}$-model, the interpretation of every formula (in the generalized sense of page 334) in the language of DL is a σ-projective subset of $\mathbb{R}$, by the argument given above.

If $A \subset \mathbb{R}$ is σ-projective, then it is obtained from open sets by means of applying complements, continuous images, and countable unions; and the open sets needed to do this, together with the "record" of all operations applied to them, can be coded by some $x_A \in \mathbb{R}$. The point is that the existence of a winning strategy for a given σ-projective game A is σ-projective, and—in fact—projective in A. By appealing to [**?**, Lemma 2.8] and a theorem of Moschovakis [Mos71], every set projective in A has a uniformizing function projective in A. It might have many uniformizing functions, but the proofs of [**?**, Lemma 2.8] and Moschovakis [Mos71] give a way of constructing such a function uniformly from x_A (using x_A as the x in the statement of [**?**, Lemma 2.8]). Again, there might be many "records" x_A for A, but we do not need to choose one, as it is already given to us by π and $\mathfrak{A}$. Hence, using π and $\mathfrak{A}$, there is a uniform way of assigning winning strategies to the games associated to the interpretation of formulae in π. □

This completes the proof of the lemma. □

We now state and prove the remaining soundness lemmata for $\mathbb{N}$-models.

Lemma 13.3.7. *Assume that* ZFC *holds and let* M *be a transitive inner model of* $\mathsf{AD}_{\mathbb{R}} + \mathsf{DC}$ *containing* $\mathbb{R}$*. Suppose that, in* M*, a sequent* $\Gamma_0 \vdash \Delta_0$ *is* $\mathsf{DL}_{\mathbb{R}}$*-provable. Then it is satisfied in every* $\mathbb{N}$*-model in* M*.*

Proof. Given a sequent $\Gamma_0 \vdash \Delta_0 \in M$ in the langauge of $\mathsf{DL}_{\mathbb{R}}$ (from the point of view of M), a proof $\pi \in M$ of $\Gamma_0 \vdash \Delta_0$, and a structure $\mathfrak{A} \in M$, one constructs a new proof π' of $\Gamma_0 \vdash \Delta_0$ obtained by replacing new symbols for characteristic variables as in the proof of Lemma 13.3.5. The construction of π' takes place in V, where there are enough choice functions for the analog of Sublemma 13.3.6 in this context to hold, and so π' need not belong to M. Together with π', one defines an extension $\mathfrak{A}'$ of $\mathfrak{A}$ in the language of $\mathsf{DL}_{\mathbb{R}}$ enhanced with all the new symbols of the form u^A_ξ. Again, $\mathfrak{A}'$ need not belong to M, but Theorem **??** and the fact that $\mathbb{R} \in M$ imply that equation (13.3) holds in this context, so inductively one shows in V that

every sequent in π' is satisfied in $\mathfrak{A}'$. This implies that the end-sequent of π', namely, $\Gamma_0 \vdash \Delta_0$, is satisfied in $\mathfrak{A}'$, and thus it too is in $\mathfrak{A}$. □

Lemma 13.3.8. *Assume that all projective games of (fixed) countable length are determined and that every atomic formula in the language of* $\mathsf{DL}_{\mathbb{R}}$ *has finite arity. Suppose that a sequent* $\Gamma_0 \vdash \Delta_0$ *is provable in* $\mathsf{DL}_{\mathbb{R}}$*. Then it is satisfied in every* $\mathbb{N}$*-model.*

Proof. Very much like in Lemma 13.3.5, one verifies that equation (13.3) holds by showing that the game given by

$$\{(x_\iota)_{\iota<\theta} : \mathfrak{A} \models A(x_\iota)_{\iota<\theta}\}$$

is (reducible to a) projective game of (fixed) countable length.

This is shown by induction on A by an argument similar to that of [AMS]. Note that "projective" could be replaced by "Borel" without any problem. This is because the extra quantifiers in the definition of the payoff set can be absorbed by additional moves in the game. If A is atomic, say, then A codes an open set, so the result follows. Suppose $A = \bigvee_{i\in\mathbb{N}} A_i(\vec{t})$, and that the game $\mathcal{G}_i$ given by each A_i is coded by a Borel set of reals. Consider the following game:

I	n_0		$\ldots$	n_α	$\ldots$	$(\alpha<\theta)$	i
II		n_1	$\ldots$		$\ldots$	$(\alpha<\theta)$	

Here the players play natural numbers as usual. At the end, player I chooses some $i \in \mathbb{N}$ and I wins if the play is winning, according to the rules of $\mathcal{G}_i$; otherwise, II wins. Clearly, I has a winning strategy in this game if and only if I has a winning strategy in the game given by A. Similarly, II has a winning strategy in this game if and only if II has a winning strategy in the game given by A. Thus the game is as desired.

Finally, suppose A is of the form $\exists(y_\iota)_{\iota<\alpha} B((x_\iota)_{\iota<\theta}, (y_\iota)_{\iota<\alpha})$. Then the result easily follows by the induction hypothesis, as the extra moves given by $\exists(y_\iota)_{\iota<\alpha}$ can be absorbed into a longer game with Borel payoff.

This proves equation (13.3) in this context. The proof of Sublemma 13.3.6 in this case is similar; the difference is that the interpretation in $\mathfrak{A}$ of formulae in the language of $\mathsf{DL}_{\mathbb{R}}$ need not be σ-projective. Rather, they are sets of reals definable by the existence of winning strategies for games of (fixed) countable length with Borel payoff (this follows from the argument just given). To obtain uniformizing functions for these sets as in the proof of Sublemma 13.3.6, one utilizes the constructions from Moschovakis [Mos71] and Martin [Mar08a]. □

13.3.1 Restricted models

The soundness lemmata above will be used to prove the results related to the fragments of the calculi in which atomic formulae have finite arity. By Lemma 13.2.4 and Theorem **??**, the calculi are not sound if there is an atomic formula with infinite arity in the language and the Axiom of Choice holds. Hence, we will need to consider a different semantics to prove countable cut elimination.

Definition 13.3.9. A *restricted* $\mathbb{N}$*-model* for Determinate Logic is a pair $(\mathfrak{A}, e)$, where

1. $\mathfrak{A}$ is a model such that for every predicate symbol of infinite arity A in the language, the set

$$\{x \in \mathbb{R} : \mathfrak{A} \models A(x(0), x(1), x(2), \ldots)\}$$

   is countable, and

2. e is a function assigning to each atomic formula in the language an enumeration of its interpretation.

This smaller class of models will turn out to be sufficient for the arguments to follow. The proofs of Lemma 13.3.5 and Lemma 13.3.8 easily adapt to yield the following. The point is that countable sets of reals are F_σ. The function e can be used in the proof of Sublemma 13.3.6 in this context to obtain the "records" x_A for formulae A (this way one need not choose enumerations of the interpretations of atomic formulae appearing in A).

Lemma 13.3.10. *Assume that all sets of reals in the smallest σ-algebra containing the open sets and closed under continuous images are determined. Suppose that $\Gamma_0 \vdash \Delta_0$ is provable in* DL*. Then it holds in every restricted* $\mathbb{N}$*-model.*

Lemma 13.3.11. *Assume that all projective games of (fixed) countable length are determined. Suppose that $\Gamma_0 \vdash \Delta_0$ is provable in* $\mathsf{DL}_{\mathbb{R}}$*. Then it holds in every restricted* $\mathbb{N}$*-model.*

13.4 Completeness

As companions to the soundness lemmata, we prove (in $\mathsf{ZF} + \mathsf{DC}$, as usual) a pair of completeness results in this section.

Lemma 13.4.1. *Suppose $\Gamma \vdash \Delta$ is a sequent in the language of $\mathsf{DL}_{\mathbb{R}}$ which is satisfied in every $\mathbb{N}$-model. Then it is is cut-free provable in $\mathsf{DL}_{\mathbb{R}}$. Moreover, if it belongs to the language of DL, then it is cut-free provable in DL.*

The proof of Lemma 13.4.1 is included in that of Theorem 13.1.1 and is almost the same as that of Lemma 13.4.2 below. We will need to consider an additional axiom schema for DL to make it complete with respect to restricted $\mathbb{N}$-models. The axiom schema `fin` is given by

$$\{F(\vec{x}) : \vec{x} \in I\} \vdash,$$

whenever F is a predicate symbol of infinite arity and $I \subset \mathbb{N}^{\mathrm{ar}(F)}$ is an uncountable set. We denote the enhancement of DL with `fin` by $\mathsf{DL} + \mathtt{fin}$.

Lemma 13.4.2. *Suppose $\Gamma \vdash \Delta$ is a countable sequent in the language of $\mathsf{DL}_{\mathbb{R}}$ which is satisfied in every restricted $\mathbb{N}$-model. Then it is is cut-free provable in $\mathsf{DL}_{\mathbb{R}} + \mathtt{fin}$. Moreover, if it belongs to the language of DL, then it is cut-free provable in $\mathsf{DL} + \mathtt{fin}$.*

Proof. Assume that $\Gamma \vdash \Delta$ is countable and satisfied in every restricted $\mathbb{N}$-model. Recall that we assume that no conjunctions or implications appear in $\Gamma \vdash \Delta$, i.e., every formula has its conjunctions and implications rewritten in terms of negations and disjunctions. We will consider a set of free variables including those in $\Gamma \vdash \Delta$, as well as some new variables of the form $u^A(\vec{t})$.

Specifically, let D be the smallest set containing

$$\mathbb{N} \cup \{t : t \text{ is a variable appearing in } \Gamma \vdash \Delta\}.$$

and such that whenever A is a subformula of arity α (in the sense of page 334) of a formula in $\Gamma \vdash \Delta$, perhaps with some of its terms replaced by other terms in D, and $\vec{t}$ is a sequence of natural numbers of length less than α, then $u^A(\vec{t})$ belongs to D. The object $u^A(\vec{t})$ here is regarded as a new free variable (similarly to the proof of Lemma 13.3.7), so $u^A(\cdot)$ is essentially a function mapping sequences of natural numbers to new free variables. Note that there is a surjection from $\mathbb{R}$ to D.

We define a tree T by induction. Specifically, we define an increasing sequence of trees $\{T_n\}_n$ such that $T_{n+1} \restriction lh(T_n) = T_n$ and set $T = \bigcup_n T_n$. T_0 is defined to be the one-node tree consisting of

$$R := \forall x \bigvee_{i \in \mathbb{N}} x = i, \Gamma \vdash \Delta.$$

The root R is clearly valid for restricted $\mathbb{N}$-models. T_1 is defined to be the root R together with a first level consisting solely of the node

$$\Big\{ \bigvee_{i\in\mathbb{N}} t = i \Big\}_{t\in D}, \Gamma \vdash \Delta.$$

T_2 is obtained by adjoining to T_1 all nodes of the form

$$\{t = i(t)\}_{t\in D}, \Gamma \vdash \Delta,$$

for all functions $i : D \to \mathbb{N}$.

For $2 \leq n$, T_{n+1} is defined inductively.

1. If $n \equiv 0 \mod 4$, then we add all expressions of the form $\{A_\iota\}_\iota, \Gamma' \vdash \Delta', \{B_\iota\}_\iota$, where $\{\neg B_\iota\}_\iota, \Gamma' \vdash \Delta', \{\neg A_\iota\}_\iota$ is a node in the nth level and no formula in $\Gamma' \cup \Delta'$ has a negation as its outermost connective.

2. If $n \equiv 1 \mod 4$, then we add all expressions of the form $\{A_{\iota_\lambda,\lambda}\}_\lambda, \Gamma' \vdash \Delta', \{B_{\iota_\mu,\mu}\}_{\iota_\mu,\mu}$, where $\{\iota_\lambda\}_\lambda$ is a sequence of the appropriate length and $\{\bigvee_{\iota_\lambda} A_{\iota_\lambda,\lambda}\}_\lambda, \Gamma' \vdash \Delta', \{\bigvee_{\iota_\mu} B_{\iota_\mu,\mu}\}_\mu$ is a node in the nth level and no formula in $\Gamma' \cup \Delta'$ has a disjunction as its outermost connective.

3. If $n \equiv 2 \mod 4$, then one adds for each node in T_n of the form

   $$\{\Im_\alpha(x_\iota)_{\iota<\alpha} A(x_\iota)_{\iota<\alpha}\}_A, \Gamma' \vdash \Delta',$$

   where no formula in Γ' has a quantifier as outermost symbol, and for each sequence $\vec{s}$ of natural number indexed by odd ordinals $< \alpha$, a node of the form

   $$\{A(t_\iota)_{\iota<\alpha}\}_A, \Gamma' \vdash \Delta',$$

   as successor, where the odd elements of $(t_\iota)_{\iota<\alpha}$ are $\vec{s}$ and if $\beta < \alpha$ is even, then t_β is a new (Skolem) variable of the form $u^A(\vec{s} \restriction \beta)$.

4. If $n \equiv 3 \mod 4$, then one adds for each node in T_n of the form

   $$\Gamma' \vdash \Delta', \{\Im_\alpha(x_\iota)_{\iota<\alpha} A(x_\iota)_{\iota<\alpha}\}_A,$$

   where no formula in Δ' has a quantifier as outermost symbol, and for each sequence $\vec{s}$ of natural number indexed by even ordinals $< \alpha$, a node of the form

   $$\Gamma' \vdash \Delta', \{A(t_\iota)_{\iota<\alpha}\}_A$$

   as successor, where the even elements of $(t_\iota)_{\iota<\alpha}$ are $\vec{s}$ and if $\beta < \alpha$ is odd, then t_β is a new (Skolem) variable of the form $u^A(\vec{s} \restriction \beta)$.

The ordering on the tree after the third level is defined in the obvious way, e.g., if $\{A_\iota\}_\iota, \Gamma' \vdash \Delta', \{B_\iota\}_\iota$ is a node on the $n+1$th level and $n+1 \equiv 0 \mod 4$, then it is a successor of $\{\neg B_\iota\}_\iota, \Gamma' \vdash \Delta', \{\neg A_\iota\}_\iota$.

Claim 13.4.3. *Let B be a branch through T and let $\equiv$ be the equivalence relation induced by setting $s \equiv t$ for two terms s and t if, and only if, $s = t$ appears in the antecedent of a sequent in B. Then, there is a sequent in B of one of the following forms:*

1. *$\Gamma_1, A, \Gamma_2 \vdash \Delta_1, A', \Delta_2$, where A and A' are the same formula, except that terms in A might be replaced by $\equiv$-equivalent terms in A';*
2. *$s = t, \Gamma_1 \vdash \Delta_1$, where s and t are $\equiv$-equivalent to different natural numbers;*
3. *$\{F(\vec{t}) : \vec{t} \in I\}, \Gamma_1 \vdash \Delta_1$, where F is a predicate symbol of infinite arity, I is a set of sequences of terms and I', the set of sequences of natural numbers induced by I and $\equiv$, is uncountable.*

Proof. We sketch the proof. Suppose towards a contradiction that some branch B through T does not contain any sequent of the required form. We define a restricted $\mathbb{N}$-model in which R is not satisfied. It follows from the definition of T_1 and the assumption on B that $D/\equiv$ is isomorphic to $\mathbb{N}$, and thus qualifies as a universe for a model. [By the definition of T_1, every term is $\equiv$-equivalent to a natural number; by (2), no two natural numbers are $\equiv$-equivalent.]

In the model, each variable a is interpreted as $[a]_\equiv$. For each predicate symbol F, $F(\vec{s})$ is true if, and only if, $F(\vec{s})$ appears in the antecedent of a sequent in B (modulo $\equiv$). By construction, atomic formulae in a node of the tree appear in all its successors. It follows from the assumption (3) on B that

$$\{x \in \mathbb{R} : F(x(0), x(1), \ldots) \text{ belongs to the antecedent of a sequent in } B, \text{ modulo } \equiv\}$$

is countable for every predicate symbol F of infinite arity. Let $\mathfrak{A}$ denote the $\mathbb{N}$-model thus defined. Since $\Gamma \vdash \Delta$ is countable, only countably many predicate symbols have nonempty extension in $\mathfrak{A}$, and the extension of each of those formulae is countable. Using DC, one can choose enumerations thereof. It follows that if e is the choice function, then $(\mathfrak{A}, e)$ is a restricted $\mathbb{N}$-model.

It remains to check that R is not satisfied in $\mathfrak{A}$. To do this, one verifies by induction on the complexity that formulae in antecedents of sequents of

B are true in $\mathfrak{A}$ and formulae in the succedents of sequents of B are false $\mathfrak{A}$. (Assumption (1) is used here.)

We only consider the nontrivial cases, which are those of formulae A of the form $\Im\vec{x}\, B(\vec{x})$. Suppose A is true in $\mathfrak{A}$ and appears in the succedent of a sequent in the branch. This means that Player I has a winning strategy in the game given by A. Thus, the free (Skolem) variables of the form $u^A(\vec{s} \restriction \beta)$ in the T-successors of A in the branch B, regarded as a continuous $\mathbb{N}$-valued function, code a non-winning strategy for Player II. We inductively obtain a substitution instance $B(\vec{s})$ of $B(\vec{x})$ by letting the even elements of $\vec{s}$ be given by Player I's winning strategy and letting the odd elements of $\vec{s}$ be given by the interpretation in $\mathfrak{A}$ of the free variables of the form $u^A(\vec{s} \restriction \beta)$. This formula is satisfied in $\mathfrak{A}$ because Player I played according to a winning strategy, but it is (by the construction of T) equivalent in $\mathfrak{A}$ to a formula in the succedent of a sequent in B, contrary to the induction hypothesis.

For the other case, suppose A appears in the antecedent. Then the function $u^A(\cdot)$ must describe a winning strategy for Player I for the game given by A since by construction, all the possible plays of Player II for the game appear in the antecedent and thus are true by induction hypothesis. This proves the claim. □

It follows from the claim that R is cut-free provable: call a sequent S in the tree *good* if it has a cut-free proof π all of whose variables appear in T. Note that – by construction – if all the T-successors of S are good, then S is good. If R is not cut-free provable, then it is not good, whence, by DC, there is an infinite branch of non-good sequents through the tree, contradicting Claim 13.4.3.

Hence, R is cut-free provable. In fact the tree T already yields a proof of R and it has the following form:

$$\dfrac{\dfrac{\dfrac{\vdots}{\{t = i(t)\}_{t\in D}, \Gamma \vdash \Delta} \qquad \text{all functions } i : D \to \mathbb{N}}{\left\{\bigvee_{i\in\mathbb{N}} t = i\right\}_{t\in D}, \Gamma \vdash \Delta}}{\forall x \bigvee_{i\in\mathbb{N}} x = i, \Gamma \vdash \Delta}$$

where each $\{t = i(t)\}_{t\in D}, \Gamma \vdash \Delta$, for $i \in \mathbb{N}^D$, is derivable by a cut-free proof π_i. Now, for any partition of the set

$$P := \{t = n : n \in \mathbb{N}, t \in D\}$$

into two parts, P_1 and P_2, one of the following holds:

1. For each $t \in D$, there is an identity of the form $t = n$ ($n \in \mathbb{N}$) in P_1;
2. for some $t \in D$, all identities of the form $t = n$ ($n \in \mathbb{N}$) are in P_2.

In the former case, let $i : D \to \mathbb{N}$ be defined by

$$i(t) = \text{least } n \in \mathbb{N} \text{ such that } t = n \text{ belongs to } P_1$$

and let $\pi_{P_1,P_2} = \pi_i$. In the latter case, let π_{P_1,P_2} be the proof

$$\frac{\vdash t = 0, t = 1, \dots}{\Gamma, \vdash \Delta, t = 0, t = 1, \dots}$$

and consider the following proof:

$$\frac{\dfrac{\vdots}{\pi_{P_1,P_2}} \qquad \text{all partitions } (P_1, P_2) \text{ of } P}{\Gamma \vdash \Delta} =_2$$

Therefore $\Gamma \vdash \Delta$ is cut-free provable. This proves Lemma 13.4.2. □

13.5 Determinacy and cut elimination

The theorems promised in the introduction follow easily from the results in the previous sections. We continue working in $\mathsf{ZF} + \mathsf{DC}$.

Theorem 13.5.1. *The following are equivalent:*

1. DL *satisfies countable cut elimination.*
2. σ*-projective determinacy.*

Proof. If every σ-projective game of length ω is determined, then DL is sound for restricted $\mathbb{N}$-models by Lemma 13.3.10, so if a countable sequent $\Gamma \vdash \Delta$ is provable, then it is satisfied in every restricted $\mathbb{N}$-model. But then it is cut-free provable if one allows the use of `fin`, by Lemma 13.4.2.

Conversely, suppose $X \subset \mathbb{R}$ is a non-determined σ-projective set of reals. Then, letting A_X and

$$\Gamma_X \vdash \Delta_X$$

denote the canonical formula and the canonical sequent for X, define

$$F_1 = \Im(x_i)_{i\in\mathbb{N}}\, A_X(x_i)_{i\in\mathbb{N}}$$

and

$$F_2 = \widetilde{\Im}(x_i)_{i\in\mathbb{N}}\, \neg A_X(x_i)_{i\in\mathbb{N}}.$$

Since X is not determined and σ-projective, the sequents

1. $F_1, \Gamma_X \vdash \Delta_X$, and
2. $F_2, \Gamma_X \vdash \Delta_X$,

are valid and countable, so by the completeness lemma they are provable in $\mathsf{DL} + \mathtt{fin}$, say, by proofs π_1 and π_2. By Lemma 13.2.4, the sequent $\vdash F_1, F_2$ is provable in DL, say by a proof σ. But then the following derivation tree is a DL-proof:

$$\dfrac{\dfrac{\dfrac{\vdots\ \sigma}{\vdash F_1, F_2} \qquad \dfrac{\pi_2\ \vdots}{F_2, \Gamma_X \vdash \Delta_X}}{\Gamma_X \vdash \Delta_X, F_1} \qquad \dfrac{\pi_1\ \vdots}{F_1, \Gamma_X \vdash \Delta_X}}{\Gamma_X \vdash \Delta_X}$$

However, $\Gamma_X \vdash \Delta_X$ is not cut-free provable in DL by Lemma 13.3.3 and no cut-free proof of $\Gamma_X \vdash \Delta_X$ can include instances of $\mathtt{fin}$. □

Theorem 13.5.2. *The following are equivalent:*

1. DL *satisfies weak cut elimination.*
2. σ*-projective determinacy.*

Proof. If every σ-projective game of length ω is determined and every atomic formula has finite arity, then DL is sound for $\mathbb{N}$-models by Lemma 13.3.5, so if a sequent $\Gamma \vdash \Delta$ is provable in DL using only atomic formulae of finite arity, then it is satisfied in every $\mathbb{N}$-model. But then it is cut-free provable by Lemma 13.4.1. The converse is shown as above. □

Theorem 13.5.3. *The following are equivalent:*

1. $\mathsf{DL}_{\mathbb{R}}$ *satisfies weak cut elimination.*
2. *Every projective game of (fixed) countable length is determined.*

Proof. As above, using Lemma 13.3.8 and Lemma 13.4.1. □

Theorem 13.5.4. *The following are equivalent:*

1. $\mathsf{DL}_{\mathbb{R}}$ *satisfies countable cut elimination.*
2. *Every projective game of (fixed) countable length is determined.*

Proof. As above, using Lemma 13.3.11 and Lemma 13.4.2. □

The remaining proof requires a simple forcing argument. The proof of Theorem 13.1.2 which we omitted earlier is similar.

Theorem 13.5.5. *The following are equivalent:*

1. $\mathsf{DL}_{\mathbb{R}}$ *satisfies the cut-elimination theorem.*
2. $\mathsf{AD}_{\mathbb{R}}$.

Proof. Suppose that $\mathsf{ZF} + \mathsf{DC}$ holds. By inspecting the definition, one can see that proofs in $\mathsf{DL}_{\mathbb{R}}$ are essentially $\mathcal{P}(\mathbb{R})$-indexed, $\mathcal{P}(\mathbb{R})$-branching, well-founded trees. Hence, the collection of all proofs in $\mathsf{DL}_{\mathbb{R}}$ can be coded by a subset of $\mathcal{P}(\mathcal{P}(\mathbb{R}))$. Let g_1 be a generic wellordering of $\mathbb{R}$ in order-type ω_1, i.e., a generic filter for $\mathrm{Coll}(\omega_1, \mathbb{R})$, the partial order consisting of countable partial functions from ω_1 to $\mathbb{R}$ ordered by reverse inclusion. Similarly, let g_2 be a $V[g_1]$-generic wellordering of $\mathcal{P}(\mathbb{R})^{V[g_1]}$ in order-type $\omega_2^{V[g_1]}$, and g_3 be a $V[g_1][g_2]$-generic wellordering of $\mathcal{P}(\mathcal{P}(\mathbb{R}))^{V[g_1][g_2]}$ in order-type $\omega_3^{V[g_1][g_2]}$. Letting $h \in \mathcal{P}(\omega_3)^{V[g_1][g_2][g_3]}$ code $\mathcal{P}(\mathcal{P}(\mathbb{R}))^V$ and $N = L[h]$, we have $N \models \mathsf{ZFC}$ and $\mathcal{P}(\mathcal{P}(\mathbb{R}))^V \in N$.

Since DC holds in V, $\aleph_1$ is a regular cardinal. In particular, every countable set of countable ordinals has a countable supremum, which implies that the partial order for generically wellordering $\mathbb{R}$ with length ω_1 is countably closed. By DC, countably closed partial orders add no new real numbers, so $\mathbb{R}^V = \mathbb{R}^{V[g_1]}$. Again using DC, one can show that $V[g_1] \models \mathsf{ZF} + \mathsf{DC}$. [To see this, let $\dot{T}$ be a name for a tree in $V[g_1]$ with no terminal nodes and consider the tree S in V whose elements at level $n \in \mathbb{N}$ are pairs $(p, \dot{x})$ where $p \in \mathrm{Coll}(\omega_1, \mathbb{R})$ forces $\dot{x}$ to be an element of $\dot{T}$ at level n and such that $(p, \dot{x}) < (q, \dot{y})$ if, and only if, q extends p and q forces $\dot{y}$ to extend $\dot{x}$ in $\dot{T}$. Then this tree has no terminal nodes by choice of $\dot{T}$. By DC there is a branch through S the union of whose first coordinates is a condition (by countable completeness) which forces the set of second coordinates to be a branch through $\dot{T}$.]

Thus, $V[g_1] \models \mathsf{ZF} + \mathsf{DC}$, so it follows that $\omega_2^{V[g_1]}$ has uncountable cofinality in $V[g_1]$ and, in particular, that every countable union of ordinals of cardinality $\aleph_1$ in $V[g_1]$ also has cardinality $\aleph_1$ in $V[g_1]$. Thus, the partial order for generically wellordering $\mathcal{P}(\mathbb{R})^{V[g_1]}$ with length $\omega_2^{V[g_1]}$ is also countably closed. One can thus argue as before to show that

$$\mathbb{R}^V = \mathbb{R}^{V[g_1][g_2][g_3]} = \mathbb{R}^N.$$

Letting $M = L(\mathcal{P}(\mathcal{P}(\mathbb{R}))^V)$, it follows that, from the point of view of N, M is a transitive model of ZF containing the set of all real numbers.

Moreover,

$$M \models \mathsf{DC}.$$

To see this, notice that $\mathsf{DC}_{\mathbb{R}}$, the Principle of Dependent Choice restricted to $\mathbb{R}$-splitting trees, is witnessed in V by elements of $\mathbb{R}^V$, and these belong to $L(\mathcal{P}(\mathcal{P}(\mathbb{R}))^V)$; similarly for $\mathcal{P}(\mathbb{R})$- and $\mathcal{P}(\mathcal{P}(\mathbb{R}))$-splitting trees, so

$$M \models \mathsf{DC}_{\mathcal{P}(\mathcal{P}(\mathbb{R}))^V},$$

where $\mathsf{DC}_{\mathcal{P}(\mathcal{P}(\mathbb{R}))^V}$ denotes the principle of Dependent Choice restricted to $\mathcal{P}(\mathcal{P}(\mathbb{R}))^V$-splitting trees. Furthermore, every element of M is definable in M from an element of $\mathcal{P}(\mathcal{P}(\mathbb{R}))^V$ and ordinal parameters. Hence, arbitrary trees in M with no terminal nodes can be reduced to trees on $\mathcal{P}(\mathcal{P}(\mathbb{R}))^V \times \mathsf{Ord}$, and M can certainly find branches through these trees. Thus, $M \models \mathsf{DC}$ as claimed.

Working in N, a model of ZFC, it follows from Lemma 13.3.7, Lemma 13.4.1, and the argument of Lemma 13.5.1 that the following are equivalent:

1. $M \models$ " $\mathsf{DL}_{\mathbb{R}}$ satisfies the cut-elimination theorem."
2. $M \models \mathsf{AD}_{\mathbb{R}}$.

By definition of M,

$$\mathcal{P}(\mathcal{P}(\mathbb{R}))^M = \mathcal{P}(\mathcal{P}(\mathbb{R}))^V,$$

so M can carry out the definition of DL and $\mathsf{DL}_{\mathbb{R}}$ and determine whether a given $A \in \mathcal{P}(\mathcal{P}(\mathbb{R}))^V$ codes a proof, so the cut-elimination theorem for $\mathsf{DL}_{\mathbb{R}}$ holds in M if, and only if, it holds in V. Since

$$\mathcal{P}(\mathbb{R})^M = \mathcal{P}(\mathbb{R})^V,$$

M and V contain exactly the same strategies for games on $\mathbb{R}$ and the same games on $\mathbb{R}$. Therefore $\mathsf{AD}_{\mathbb{R}}$ holds in M if, and only if, it holds in V, so the theorem follows. □

Part IV

Topological Interpretations of Provability

14

Strong Completeness of Provability Logic for Ordinal Spaces

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14.1 Introduction

The study of formalized provability can be traced back to the seminal 1931 paper by Kurt Gödel [Gö31], where he first showed how any mathematical theory that is able to describe the arithmetic of natural numbers, such as Peano Arithmetic (PA), also possesses the expressive power to reason about provability within said theory. For this, he first noted that sentences *about*

This chapter was written jointly with D. Fernández-Duque.

numbers can be encoded using numbers themselves. Gödel then showed that we can define in the language of arithmetic a formula $B(n)$ that holds true if, and only if, n encodes a formula that is provable in the theory.

He also noted that many statements about provability may be captured in the language $\mathcal{L}_\Box$ of modal logic. Formulae in $\mathcal{L}_\Box$ are built from a countable set of propositional variables $\mathbb{P}$ by applying Boolean connectives and the operator $\Box$, together with its dual $\Diamond$. The provability logic GL has as axioms the schemata

$$\textbf{K:}\quad \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi) \qquad \textbf{Löb:}\quad \Box(\Box\varphi \to \varphi) \to \Box\varphi$$

and the inference rules *modus ponens* and *necessitation*: $\frac{\varphi}{\Box\varphi}$.

The intended interpretation for provability logic is to read $\Box\varphi$ as "φ is provable in T" for some formal theory T. However, other interpretations for GL are possible. As was proved by K. Segerberg [Seg71], GL is complete with respect to its Kripke semantics, albeit not *strongly* complete. L. Esakia [Esa81] showed that GL is also complete with respect to its topological interpretation, and indeed it is strongly complete. M. Abashidze [Aba85] and A. Blass [Bla90] improved Esakia's theorem by showing that GL is also complete with respect to a single space; namely, an ordinal number with what is known as the *order topology*. It follows from the results of L. Beklemishev and D. Gabelaia [BG13] and the second author [FD14] that GL is also complete for a large collection of spaces introduced by T. Icard [Ica11] and some of its generalizations [FDJ11]. These topologies find natural interpretations in the construction of models of the polymodal logics GLP_Λ with Λ modalities, although technical difficulties have stopped these results from being extended to uncountable Λ. This work partially solves these issues.

Our goal is to show how one can start with any topological space $\mathfrak{X} = (X, \tau)$ that validates the axioms of GL and modify it slightly so as to obtain a class of spaces of the form $\mathfrak{X}_{\uparrow\lambda} = (X, \tau_{\uparrow\lambda})$ with λ an ordinal, such that GL is strongly complete with respect to $\mathfrak{X}_{\uparrow\lambda}$ for each $\lambda < \Lambda$, where Λ depends on $\mathfrak{X}$ (although it may well be arbitrarily large). We call the spaces $\mathfrak{X}_{\uparrow\lambda}$ thus obtained *generalized Icard spaces.* We expect the present work to be of interest, as strong completeness of provability logic for single spaces is novel, and generalized Icard spaces give a wide class of spaces which provide natural and easily described topological models for GL.

Outline

The chapter consists of six main sections. In Section 14.2, we review some facts about ordinal arithmetic and transfinite iterations of normal and initial

functions. Section 14.3 reviews the topological semantics of modal logic. In Section 14.4, we introduce a class of models for provability logic called ω-*bouquets* and prove the strong completeness of GL with respect to this class. In Section 14.5, we introduce *generalized Icard spaces* and our main result, strong completeness of GL, is proved in Section 14.6. We then finish with some concluding remarks.

14.2 Ordinal numbers

We will be dealing extensively with ordinal numbers and so we assume knowledge of elementary ordinal arithmetic, although we will review some basic facts, as well as some operations that will be of great importance. For a thorough review, we refer the reader to, for example, [Jec03a].

Lemma 14.2.1.

1. *If $\alpha < \beta$, there exists a unique ordinal γ such that $\alpha + \gamma = \beta$. We denote this γ by $-\alpha + \beta$.*

2. *For all γ, if $\alpha > 0$, there exist unique β and unique $\rho < \alpha$ such that $\gamma < \alpha \cdot \beta + \rho$.*

3. *For all nonzero ξ, there exist ordinals α and β such that $\xi = \alpha + \omega^\beta$. Such a β is unique. We denote it by $\ell\xi$ and call it the* end logarithm *of ξ.*

By a λ-*sequence* of ordinals, we mean an ordinal-valued function with domain λ. As usual, if λ is a limit ordinal and $(\xi_\eta)_{\eta<\lambda}$ is a λ-sequence, we define $\xi = \lim_{\eta\to\lambda} \xi_\eta$ if for every $\zeta < \xi$ there is $\delta < \lambda$ such that $\xi_\eta \in (\zeta, \xi]$ whenever $\eta > \delta$. In particular, if $(\xi_\eta)_{\eta<\lambda}$ is non-decreasing, then $\lim_{\eta\to\lambda} \xi_\eta = \bigcup\{\xi_\eta \colon \eta \in \lambda\}$. A (class) function f on the ordinals is *continuous* if $f(\lambda) = \lim_{\eta\to\lambda} f(\eta)$ for each limit λ. A function that is both continuous and increasing is called *normal*. An important example is given as follows:

Definition 14.2.2 (Exponential function)**.** The *exponential function* is the normal function $e \colon \mathsf{Ord} \to \mathsf{Ord}$ given by $\xi \mapsto -1 + \omega^\xi$.

When $f \colon X \to X$ is a function, it is natural and often useful to ask whether f has *fixed points,* i.e., solutions to the equation $x = f(x)$. In particular, normal functions have many fixed points:

Proposition 14.2.3. *Every normal function on* Ord *has arbitrarily large fixed points.*

The first ordinal α such that $\alpha = \omega^\alpha$ is the limit of the ω-sequence $(\omega, \omega^\omega, \omega^{\omega^\omega}, ...)$, and is usually called ε_0. In general, we call an *epsilon number* any nonzero fixed point of the exponential function, and *epsilon function* the function given by $\alpha \mapsto \varepsilon_\alpha$, assumed to be increasing and to have as range all epsilon numbers. It is easily proved that the epsilon function is normal and $\varepsilon_{\alpha+1} = \sup\{\omega^{\varepsilon_\alpha+1}, \omega^{\omega^{\varepsilon_\alpha+1}}, ...\}$ for each α.

Recall that a non-zero ordinal number ξ is said to be *additively indecomposable* if $\beta + \gamma < \xi$ for all $\beta, \gamma < \xi$. Also, ξ is said to be *multiplicatively indecomposable* if $\beta\gamma < \xi$ for all $\beta, \gamma < \xi$. The following facts are well-known:

Lemma 14.2.4.

1. *An ordinal is additively indecomposable if, and only if, it is of the form* ω^ρ.

2. *An ordinal is multiplicatively indecomposable if, and only if, it is of the form* ω^{ω^ρ}.

In [FDJ13a], the second author and J. Joosten introduced families of transfinite iterations of functions called *hyperations* and *cohyperations*. Although we will not provide a detailed discussion, we consider two examples which have remarkable applications to provability logic, as will be shown later on.

Definition 14.2.5 (Hyperexponential functions)**.** The *hyperexponentials* $(e^\zeta)_{\zeta\in\mathsf{Ord}}$ are the unique family of normal functions that satisfy

1. $e^1 = e$,

2. $e^{\alpha+\beta} = e^\alpha \circ e^\beta$ for all α and β, and

3. if $(f^\xi)_{\xi\in\mathsf{Ord}}$ is a family of functions satisfying 1 and 2, then for all $\alpha, \beta \in \mathsf{Ord}$, $e^\alpha\beta \leq f^\alpha\beta$.

This definition is proved to be correct in [FDJ13a]. From now on, to ease notation, we will sometimes omit the symbol '$\circ$' for composition of functions, as well as parentheses. In order to compute hyperexponentials, we use the following lemma:

Lemma 14.2.6 (Recursive hyperexponentiation)**.**

1. e^0 *is the identity;*

2. $e^\xi 0 = 0$ *for all* ξ*;*

3. *for any ξ and any limit λ, $e^\xi\lambda = \lim_{\eta\to\lambda} e^\xi\eta$;*

4. *for any ξ and any limit λ, $e^\lambda(\xi+1) = \lim_{\eta\to\lambda} e^\eta(e^\lambda(\xi)+1)$.*

Corollary 14.2.7. *If ξ is a limit ordinal, then $cf(e^\lambda\xi) = cf(\xi)$. Otherwise, $cf(e^\lambda\xi) = \max(cf(\lambda), \aleph_0)$.*

As a different application of Lemma 14.2.6, the reader might want to check that the function $\alpha \mapsto e^\omega(1+\alpha)$ coincides with the epsilon function. Of course, Lemma 14.2.6.4 is not the only way to compute $e^\lambda(\xi+1)$. Two useful alternatives are given as follows:

Lemma 14.2.8. *Let ξ be an ordinal, λ be an additively indecomposable limit, and f be a nonzero-valued function on* Ord.

1. *If $0 < f\eta < e^\lambda(\xi+1)$ for all $\eta < \lambda$, then $\lim_{\eta\to\lambda} e^\eta(e^\lambda\xi + f\eta) = e^\lambda(\xi+1)$.*

2. *If $\gamma < \lambda$ is a limit and $0 < f\eta < e^\omega(e^\lambda\xi+1)$ for all $\eta < \gamma$, then $\lim_{\eta\to\gamma} e^\eta(e^\lambda\xi + f\eta) = e^\gamma(e^\lambda\xi+1)$.*

Proof. For the first claim, we clearly have

$$\lim_{\eta\to\lambda} e^\eta(e^\lambda\xi + f\eta) \geq \lim_{\eta\to\lambda} e^\eta(e^\lambda\xi+1) = e^\lambda(\xi+1).$$

For the other inequality, fix $\eta < \lambda$ and take $\lambda_\eta \in (0,\lambda)$ such that $f\eta < e^{\lambda_\eta}(e^\lambda\xi+1)$. Since $e^{\lambda_\eta}(e^\lambda\xi+1)$ is additively indecomposable, we have that

$$e^\lambda\xi + f\eta < e^{\lambda_\eta}(e^\lambda\xi+1),$$

whence

$$\begin{aligned} e^\eta(e^\lambda\xi + f\eta) &\leq e^{\eta+\lambda_\eta}(e^\lambda\xi+1) \\ &\leq \lim_{\zeta\to\lambda} e^\zeta(e^\lambda\xi+1) \\ &= e^\lambda(\xi+1). \end{aligned}$$

Since η was arbitrary, the claim follows.

For the second claim we have that, if $\eta < \gamma$, then

$$e^\eta(e^\lambda\xi + f\eta) \geq e^\eta(e^\lambda\xi+1),$$

and since

$$\lim_{\eta\to\gamma} e^\eta(e^\lambda\xi+1) = \lim_{\eta\to\gamma} e^\eta(e^\gamma e^\lambda\xi+1) = e^\gamma(e^\lambda\xi+1),$$

it follows that

$$\lim_{\eta\to\gamma} e^{\eta}(e^{\lambda}\xi + f\eta) \geq e^{\gamma}(e^{\lambda}\xi + 1).$$

In addition, for any $\eta < \gamma$ and any $n < \omega$ such that $f\eta < e^n(e^{\lambda}\xi + 1)$, we have that

$$e^{\eta}(e^{\lambda}\xi + f\eta) \leq e^{\eta+n}(e^{\lambda}\xi + 1) < e^{\gamma}(e^{\lambda}\xi + 1),$$

so that equality holds. □

Lemma 14.2.9 (Hyperexponential normal form)**.** *For every ordinal $\xi > 0$, there exist unique ordinals α, β such that β is 1 or additively decomposable and $\xi = e^{\alpha}\beta$.*

Proof. Let α be the supremum of the set $A = \{\zeta : e^{\zeta}1 \leq \xi\}$. Note that this collection is indeed a set since it is a subset of $\xi + 1$, while it is non-empty since $0 \in A$. Moreover, α belongs to A, since the function $\zeta \mapsto e^{\zeta}1$ is normal and thus A is closed.

It follows that $\xi = e^{\alpha}\beta$ for some unique β. But β is not in the range of e, since if we had $\beta = e\gamma$ then $\xi = e^{\alpha+1}\gamma$, contradicting the assumption that α is an upper bound for A. It follows from the definition of e and Lemma 14.2.4.1 that β is either 1 or additively decomposable, as claimed. □

We call α above the *degree of indecomposability* of ξ; in particular, if ξ is already additively decomposable, then $\alpha = 0$. Note that by writing β as a sum of indecomposables we may iterate this lemma and thus write any ordinal in terms of $e, +, 0$ and 1. This form is unique if we do not allow sums of the form $\xi + \eta$ for $\xi < \eta$. We next turn our attention to *hyperlogarithms,* which are iterations of *initial,* rather than normal, functions.

Definition 14.2.10 (Initial function)**.** A function f on the ordinals is said to be *initial* if it maps initial segments to initial segments.

Definition 14.2.11 (Hyperlogarithms)**.** We define the *hyperlogarithms* $(\ell^{\xi})_{\xi\in\mathsf{Ord}}$ to be the unique family of initial functions that satisfy:

1. $\ell^1 = \ell$,
2. $\ell^{\alpha+\beta} = \ell^{\beta}\ell^{\alpha}$ and
3. if $(f^{\xi})_{\xi\in\mathsf{Ord}}$ is a family of functions satisfying 1 and 2, then for all $\alpha, \beta \in \mathsf{Ord}$, $\ell^{\alpha}\beta \geq f^{\alpha}\beta$.

As can be seen from the definition, hyperlogarithms are not right-additive but rather left-additive. This is to allow for surjectivity and the possibility that they have a right-inverse. Note, however, that they are not invertible since they are not injective (for example, $\ell 1 = 0 = \ell(\omega + 1)$); we will, however, be interested in taking preimages under hyperlogarithms, in which case for a set of ordinals A we write $\ell^{-\xi}(A)$ instead of $(\ell^\xi)^{-1}(A)$.

Hyperlogarithms have some nice properties (see [FDJ13a]):

Lemma 14.2.12 (Properties of hyperlogarithms)**.**

1. *$\ell\xi \leq \xi$ for all ξ.*
2. *If ξ and δ are nonzero, then $\ell^\xi(\gamma + \delta) = \ell^\xi\delta$.*
3. *For any γ, the sequence $(\ell^\xi\gamma)_{\xi\in\mathsf{Ord}}$ is non-increasing.*
4. *If $\lambda = \alpha + \omega^\beta$ is a limit ordinal and ξ is any ordinal, then there exists $\sigma < \lambda$ such that $\ell^\zeta\xi = e^{\omega^\beta}\ell^\lambda\xi$ for all $\zeta \in [\sigma, \lambda)$.*

Part 3 of the above lemma states that hyperlogarithms are non-increasing in the exponent. They are not, however, monotone in their arguments: $\ell\omega = 1$ while $\ell(\omega + 1) = 0$.

It is easy to see that ℓ functions as a left inverse for e. Moreover, [FDJ13a, Theorem 9.1] implies that ℓ^ξ is also a left inverse for e^ξ for all ξ. Hence, if $\xi < \zeta$, then by rewriting e^ζ as $e^\xi e^{-\xi+\zeta}$, we obtain the following:

Lemma 14.2.13. *If $\xi < \zeta$, then $\ell^\xi e^\zeta = e^{-\xi+\zeta}$ and $\ell^\zeta e^\xi = \ell^{-\xi+\zeta}$. Furthermore, if $\alpha < e^\xi\beta$, then $\ell^\xi\alpha < \beta$, and if $\alpha < \ell^\xi\beta$, then $e^\xi\alpha < \beta$.*

We conclude this section with two examples, with aims at illustrating the properties and recursive computations of hyperexponentials and hyperlogarithms:

Example 14.2.14. We calculate $e^{\omega_1}1$:

$$e^{\omega_1}1 = \lim_{\eta\to\omega_1} e^\eta(e^{\omega_1}0 + 1) = \lim_{\eta\to\omega_1} e^\eta(1).$$

On the one hand, each term $e^\eta 1$ is smaller than ω_1; on the other hand, for any $\eta < \omega_1$, we have $\eta \leq e^\eta 1$. Therefore, the sequence must converge to ω_1, whence $e^{\omega_1}1 = \omega_1$.

Example 14.2.15. Consider the ordinal $\gamma = \omega_{\varepsilon_0} + \varepsilon_{\varepsilon_{\omega_1}+\varepsilon_{\omega^{2\cdot 3}}}$. We compute the sequence $(\ell^\xi\gamma)_{\xi\in\mathsf{Ord}}$.

Recall that e^ω equals the epsilon function. Thus, for $0 < \xi < \omega$:

$$\ell^\xi\gamma = \ell^\xi e^\omega(\varepsilon_{\omega_1} + \varepsilon_{\omega^{2\cdot 3}}) = e^{-\xi+\omega}(\varepsilon_{\omega_1} + \varepsilon_{\omega^{2\cdot 3}}) = e^\omega(\varepsilon_{\omega_1} + \varepsilon_{\omega^{2\cdot 3}}) = \varepsilon_{\varepsilon_{\omega_1}+\varepsilon_{\omega^{2\cdot 3}}}.$$

When $\xi = \omega$, the exponential and the logarithm are cancelled: $\ell^{\xi}\gamma = \varepsilon_{\omega_1} + \varepsilon_{\omega^2 \cdot 3}$. For $\omega < \xi < \omega + \omega$, with $0 < \zeta$, we have

$$\ell^{\xi}\gamma = \ell^{-\omega+\xi}(\varepsilon_{\omega_1} + \varepsilon_{\omega^2 \cdot 3}) = \ell^{-\omega+\xi} e^{\omega}(\omega^2 \cdot 3) = \varepsilon_{\omega^2 \cdot 3}.$$

As above, if $\xi = \omega \cdot 2$, then $\ell^{\xi}\gamma = \omega^2 \cdot 3$. $\omega^2 \cdot 3$ can be written as $\omega^2 \cdot 2 + \omega^2$, which implies that $\ell^{\omega \cdot 2+1} = \ell\omega^2 = \ell(e2) = 2$ and $\ell^{\omega \cdot 2+2} = \ell 2 = 0$. For any ξ greater than $\omega \cdot 2 + 2$, we also have $\ell^{\xi}\gamma = 0$.

Hyperexponential ordinal notations will be crucial in the description of topological semantics in later sections, while hyperlogarithms will play an important role in the description of the topologies that we will use.

14.3 Topological interpretation of provability logic

Recall that a *topological space* is a pair (X, τ), where X is a set and τ is a family of subsets of X containing X and $\varnothing$ that is closed under arbitrary unions and finite intersections. An open set U containing a point x is a *neighborhood* of x. The set $U \setminus \{x\}$ is then called a *punctured neighborhood* of x. For any $A \subset X$, we say that x is a *limit point* of A if A intersects every punctured neighborhood of x. We call the set of limit points of A the *derived set* of A and denote it by dA. We will also denote it by $d_\tau A$ if we want to emphasize the topology with respect to which we take the limit points. A point in A that is not a limit point is *isolated* in A. Equivalently, a point x is isolated in A if, and only if, $\{x\}$ is open in A under the inherited topology (recall that for any topological space X and any subset S, the *inherited topology* is formed by all sets $U \cap S$, where $U \in \tau$). We can define topological semantics for the language $\mathcal{L}_\Box$ by assigning subsets of X to each propositional variable, in the following way:

Definition 14.3.1 (Topological semantics)**.** Let $\mathfrak{X} = (X, \tau)$ be a topological space. A *valuation* is a function $[\![\cdot]\!] : L_\Box \to \wp(X)$ such that for any φ, ψ, $[\![\bot]\!] = \varnothing$, $[\![\neg\varphi]\!] = X \setminus [\![\varphi]\!]$, $[\![\varphi \wedge \psi]\!] = [\![\varphi]\!] \cap [\![\psi]\!]$ and $[\![\Diamond\varphi]\!] = d[\![\varphi]\!]$.

A *model* $\mathfrak{M} = (\mathfrak{X}, [\![\cdot]\!])$ is a topological space together with a valuation. If $\mathfrak{M}$ is a model, we may write $\mathfrak{M}, x \models \varphi$ instead of $x \in [\![\varphi]\!]$; in this case, we say φ is *true* at x (in $\mathfrak{M}$). We say that φ is *satisfied* in $\mathfrak{M}$ if $[\![\varphi]\!]$ is nonempty, that φ is *true* in $\mathfrak{M}$ and write $\mathfrak{M} \models \varphi$ if $[\![\varphi]\!] = W$, and we say φ is *valid* in a space $\mathfrak{X}$ and write $\mathfrak{X} \models \varphi$ if $[\![\varphi]\!] = X$ for any model $(\mathfrak{X}, [\![\cdot]\!])$.

Interpreting the modal diamond as the derived set operator is sometimes called the *d-interpretation* of modal logic [BM10]. An important example is given as follows:

Definition 14.3.2 (Relational structure)**.** For our purposes, a *relational structure* is a set T together with a transitive, irreflexive relation R on T. Given a relational structure (T, R), we assign a topology τ_R, called the *upset topology,* to T, whose open sets are those $U \subset W$ such that, whenever $x \in U$ and xRy, it follows that $y \in U$.

For those familiar with Kripke semantics, it is straightforward to check that for these structures, the usual Kripke interpretation based on R and the d-interpretation based on τ_R coincide.

Definition 14.3.3 (Soundness and completeness)**.** Let $\mathcal{X}$ be a class of topological spaces and L be a logic over $\mathcal{L}_\Box$. We say L is *sound* with respect to $\mathcal{X}$ if $\mathfrak{X} \models \varphi$ for all $\mathfrak{X} \in \mathcal{X}$ whenever $\mathsf{L} \vdash \varphi$. Conversely, we say L is *complete* with respect to $\mathcal{X}$ if whenever $\mathfrak{X} \models \varphi$ for all $\mathfrak{X} \in \mathcal{X}$, then $\mathsf{L} \vdash \varphi$. Moreover, L is *strongly complete* with respect to $\mathcal{X}$ if for any set Γ of $\mathcal{L}_\Box$-formulae that is consistent with respect to L, there is some $\mathfrak{X} \in \mathcal{X}$ such that Γ is satisfied in $\mathfrak{X}$.

It is easy to verify that strong completeness implies simple completeness.

Definition 14.3.4 (Scattered space)**.** We say that a topological space (X, τ) is *scattered* if each nonempty subset of X has an isolated point.

Example 14.3.5. Recall that a relation R on a set T is *converse well-founded* if every nonempty subset of T has an R-maximal element. Then, if R is converse well-founded, (T, τ_R) is a scattered space.

Lemma 14.3.6 ([Esa81])**.** *Löb's axiom is valid in a topological space $\mathfrak{X}$ if and only if $\mathfrak{X}$ is scattered.*

It is straightforward to check that the axiom K is valid and the rules of necessitation and modus ponens preserve validity in all topological spaces (see, for example, [vBB06]). With this and the preceding discussion we may formulate Esakia's theorem [Esa81]:

Theorem 14.3.7. *A formula φ is a theorem of GL if, and only if, it is valid in all scattered spaces.*

There are several possible improvements to Esakia's theorem. Recall that a *tree* is a relational structure (T, R) (which by our definition is transitive), such that each R-predecessor set is well-ordered, and T contains a unique R-minimal element, its *root.* The following was originally proved by K. Segerberg [Seg71]:

Theorem 14.3.8. GL *is sound and complete with respect to the class of finite trees.*

Nonetheless, GL is not strongly complete with respect to this semantics, as is well known:

Example 14.3.9. Consider the set $\Gamma = \{\Diamond p_0\} \cup \{\Box(p_i \to \Diamond p_{i+1}) \colon i < \omega\}$. It is easy to see that any finite subset of Γ is satisfiable in a finite tree, whence Γ is consistent with GL by soundness. Moreover, all of Γ is not satisfiable in any converse well-founded tree. We leave the details to the reader.

Some more examples of scattered spaces are provided by natural topologies on ordinal numbers.

Example 14.3.10.

1. Let Θ be an ordinal number and τ consist of all sets $[0, \beta]$. Then (Θ, τ) is a scattered space: the least element of each set is isolated. We call this topology the *left topology* and denote it $\mathcal{I}_0$. Whenever an ordinal is equipped with the left topology, we may write Θ_0 instead of $(\Theta, \mathcal{I}_0)$.

2. Let Θ be an ordinal number and τ consist of all sets $(\alpha, \beta]$ and all sets $[0, \beta]$. Then (Θ, τ) is a scattered space: each successor ordinal is isolated. We call this topology the *interval topology* and denote it $\mathcal{I}_1$. Whenever an ordinal is equipped with the interval topology, we may write Θ_1 instead of $(\Theta, \mathcal{I}_1)$.

3. Let Θ be an ordinal number. We define the *club topology*, τ_c to be the unique topology such that U contains a neighborhood of $\xi \in U$ if, and only if, U contains a club (i.e., $\mathcal{I}_1$-closed and unbounded set) in ξ or $cf(\xi) < \aleph_1$. (Θ, τ_c) is a scattered space, for if $\aleph_1 \leq cf(\xi)$ and A is a club on ξ, then A contains a point ζ such that $cf(\zeta) < \aleph_1$.

In the following sections, we will introduce further improvements to Esakia's theorem and in fact exhibit single topological spaces with respect to which GL is *strongly* complete. In order to do this, we must study scattered spaces in more detail. If (X, τ) is a topological space, we define transfinite iterations $(d^\xi)_{\xi \in \mathsf{Ord}}$ of the derived-set operator by $d^0 A = A$, $d^{\xi+1} A = d d^\xi A$ and $d^\lambda A = \bigcap_{\xi < \lambda} d^\xi A$ for limit λ.

The derived set operator and its iterations have the following properties, which are easy to prove (see, for example, [BM10]):

Lemma 14.3.11. *Let (X, τ) be a scattered space.*

1. *If A is closed in X, then dA is closed in X;*

2. *$d^\alpha X$ is closed for each α;*

3. *if $\alpha \leq \beta$, then $d^\beta X \subset d^\alpha X$.*

By Lemma 14.3.11.3, iterations of the derived-set operator stabilize at some stage below the successor of $|X|$. The following was famously noted by Cantor:

Proposition 14.3.12. *A topological space (X, τ) is scattered if and only if $d^\xi X = \varnothing$ for some ξ.*

This leads naturally to a notion of the "size" of a scattered space— its rank:

Definition 14.3.13 (Rank)**.** Let $\mathfrak{X} = (X, \tau)$ be a scattered space. For any $x \in X$, we define $\rho_\tau x$, the *rank* of x, to be the least ordinal ξ such that $x \notin d^{\xi+1}X$. We may also write it as $\rho_{\mathfrak{X}} x$, or even $\rho_X x$ or ρx if there is no risk of confusion. We also define the rank of $\mathfrak{X}$ to be $\rho\mathfrak{X} = \sup_{x \in X}(\rho_\tau x + 1)$. We may also write it as $\rho_\tau X$ or even ρX when there is no risk of confusion.

One useful fact about the rank function is that near any point x, we can always find points of every smaller rank:

Lemma 14.3.14. *If V is a neighborhood of x in a scattered space, then $[0, \rho x) \subset \rho(V \setminus \{x\})$. Furthermore, there is a neighborhood of x for which equality holds.*

As way of illustration, let us list the rank functions in some important scattered spaces.

Proposition 14.3.15.

1. *The rank function on any ordinal space Θ_0 is the identity: $\rho_0(\xi) = \xi$.*

2. *The rank function on Θ_1 is the end-logarithm: $\rho_1(\xi) = \ell\xi$.*

3. *The rank function on (Θ, τ_c) is given as follows: let $(\Omega_\alpha)_{\alpha \in \mathsf{Ord}}$ enumerate all infinite regular cardinals. Then, for nonzero α, $\rho_{\tau_c}(\xi) = \alpha$ if, and only if, $cf(\xi) = \Omega_\alpha$.*

Proof. Items 1 and 2 are instances of Lemma 14.5.2 below. To prove item 3, we show by induction that if $0 < \alpha$, then $\xi \in d^\alpha_{\tau_c}\Theta$ if and only if $cf(\xi) \geq \Omega_\alpha$. We proceed by induction on α. The claim is vacuously true when $\alpha = 0$, and readily follows from the induction hypothesis when α is a limit since in this case $d^\alpha_{\tau_c}\Theta = \bigcap_{\beta<\alpha} d^\beta_{\tau_c}\Theta$.

Thus we may assume $\alpha = \beta + 1$ and $\xi \in d^{\beta}_{\tau_c}\Theta$. By induction hypothesis, $\Omega_\beta \leq cf(\xi)$. If the inequality is strict, then ξ is a limit point of $d^{\beta}_{\tau_c}\Theta$, as is it easy to check that $\{\zeta < \xi\colon cf(\zeta) = \Omega_\beta\}$ is stationary in ξ (i.e., it intersects every club). By [BG14, Theorem 11.5], τ_c is generated by $\mathcal{I}_1$ and sets $\{d_{\mathcal{I}_1}A\colon A \subset \Theta\}$, so that if, on the contrary, $\Omega_\beta = cf(\xi)$, then $\{\xi\} = d^{\beta}_{\tau_c}\Theta \cap d_{\mathcal{I}_1}A$, where A is any set cofinal in ξ of order-type Ω_β and so ξ is isolated in $d^{\beta}_{\tau_c}\Theta$. □

It is crucial to define appropriate structure-preserving mappings between scattered spaces. Obviously, homeomorphisms preserve all the relevant structure but, as it turns out, a weaker condition will do for our purposes. Below, a function between topological spaces is *pointwise discrete* if the preimage of any singleton is a discrete subspace.

Definition 14.3.16 (d-map)**.** Let X and Y be scattered spaces. Then $f : X \to Y$ is a *d-map* if it is continuous, open, and pointwise discrete.

An important example is one that we introduced above, as shown in [BG13]:

Lemma 14.3.17. *For any scattered space $\mathfrak{X} = (X, \tau)$ and any ordinal $\Theta \geq \rho\mathfrak{X}$, the rank function is the unique d-map $f : \mathfrak{X} \to \Theta_0$.*

Obviously, any homeomorphism is a d-map and so, since the composition of d-maps is a d-map, they can be thought of as morphisms in the category of scattered spaces. Two key facts about d-maps is that they preserve ranks and the validity of formulae between topological spaces if they are surjective (see [BG13] for item 1 and [BMM03] for item 2).

Lemma 14.3.18.

1. *If $f : \mathfrak{X} \to \mathfrak{Y}$ is a d-map, then $\rho_{\mathfrak{X}} = \rho_{\mathfrak{Y}} f$.*
2. *Suppose $\mathfrak{X}$ and $\mathfrak{Y}$ are scattered spaces such that a surjective d-map $f : \mathfrak{X} \to \mathfrak{Y}$ exists. Then for any $\mathcal{L}_\Box$-formula φ, $\mathfrak{X} \models \varphi$ implies $\mathfrak{Y} \models \varphi$.*

14.4 ω-bouquets

As we have seen, GL is not strongly Kripke-complete. However, it turns out that this situation may be remedied with only a minor modification of its relational semantics.

Definition 14.4.1 (ω-bouquet)**.** Let (T, R) be a countable, converse well-founded tree, and let $\rho\colon T \to \mathsf{Ord}$ be the rank function on T with respect to the upset topology.

We define a new topology, σ_R, to be the least topology extending τ_R such that if $w \in T$ is such that $\rho(w)$ is a limit ordinal, $\{v_i\}_{i<\omega}$ enumerates all daughters of w exactly once, and $n < \omega$, then $\{w\} \cup \bigcup_{n<i} (\{v_i\} \cup R(v_i)) \in \sigma_R$.

We say a topological space (T, σ) is an *ω-bouquet* if there exists a binary relation R on T such that (T, R) is a countable, converse well-founded tree and $\sigma = \sigma_R$.

Thus, points of limit rank have new punctured neighbourhoods, consisting of the union of upsets of all but finitely many immediate successors. An easy exercise is to check that σ_R is a rank-preserving extension of τ_R. This apparently innocuous change will give us a class of structures for which GL is strongly complete. A similar construction was used by V. Shehtman [**?**] to prove the strong completeness of many modal logics for their topological semantics, from which we adopt the term *bouquets*.

In this section, we will prove that GL is strongly complete for the class of ω-bouquets (i.e., any consistent set of formulae Γ is satisfiable on an ω-bouquet). Without loss of generality, we will assume Γ to be maximal consistent. Recall that we are asuming that the set of propositional variables is countable, so that Γ will be countable as well. We introduce some auxiliary notation:

Definition 14.4.2. If Γ is a set of GL formulae, define $\Gamma^{\Box} = \{\varphi : \Box\varphi \in \Gamma\}$ and $\Box\Gamma = \{\Box\varphi : \varphi \in \Gamma\}$.

Definition 14.4.3. If Γ is a maximally consistent set of GL formulae, we define the *characteristic* of Γ as the supremum of $\{n < \omega : \Diamond^n \top \in \Gamma\}$.

Note that the characteristic of Γ may be either finite or ω. In our proof, we will consider each of these two cases separately. First, we will consider the case where Γ has finite characteristic. We will need the following lemma:

Lemma 14.4.4. *Suppose that Γ is maximally consistent with characteristic $n + 1 < \omega$, and $\Diamond\varphi \in \Gamma$. Then, $\Delta = \{\varphi, \Box^{n+1}\bot\} \cup \Gamma^{\Box} \cup \Box\Gamma^{\Box}$ is consistent.*

Proof. Towards a contradiction, assume that Δ is inconsistent, so that there are $\psi_0, ..., \psi_{m-1} \in \Gamma^{\Box}$ such that

$$\vdash \Box^{n+1}\bot \wedge \bigwedge_{i<m} (\psi_i \wedge \Box\psi_i) \to \neg\varphi.$$

Reasoning in GL, it follows that

$$\vdash \Box^{n+2}\bot \wedge \bigwedge_{i<m} \Box(\psi_i \wedge \Box\psi_i) \to \Box\neg\varphi.$$

But $\Box(\psi_i \wedge \Box\psi_i)$ is equivalent to $\Box\psi_i$, so that

$$\vdash \Box^{n+2}\bot \wedge \bigwedge_{i<m} \Box\psi_i \to \Box\neg\varphi.$$

Since Γ has characteristic $n+1$ and is maximal consistent, we have that $\Box^{n+2}\bot \in \Gamma$, while also each $\Box\psi_i \in \Gamma$. Thus $\Gamma \vdash \Box\neg\varphi$, together with $\Diamond\varphi \in \Gamma$, contradicts the consistency of Γ. □

As an immediate consequence, we obtain:

Lemma 14.4.5. *Suppose that Γ is maximally consistent with characteristic $n+1 < \omega$, and $\Diamond\varphi \in \Gamma$. Then, there is a maximal consistent set Γ_φ of some characteristic $m \leq n$ such that $\Gamma_\varphi \supset \{\varphi\} \cup \Gamma^\Box \cup \Box\Gamma^\Box$.*

With this, we may prove our strong consistency result in the case of finite characteristic.

Lemma 14.4.6. *Let Γ be maximal consistent and of finite characteristic n. Then, Γ is satisfiable in a model based on an ω-bouquet of rank $n+1$.*

Proof. By induction on n. Assume by induction hypothesis that the lemma holds for any Γ' with characteristic $m < n$. Thus, for each $\Diamond\varphi \in \Gamma$, there is a model $(T_\varphi, R_\varphi, [\![\cdot]\!]_\varphi)$ of rank $n_\varphi < n$ satisfying Γ_φ, defined as in Lemma 14.4.5. Moreover, $n_\varphi = n-1$ if $\varphi = \Diamond^{n-1}\top \in \Gamma$.

Define (T, R) by taking the disjoint union of all (T_φ, R_φ) and adding a fresh root r. We consider a valuation $[\![\cdot]\!]$ over T induced by the valuations $[\![\cdot]\!]_\varphi$ described as follows. Define $[\![\cdot]\!]_0$ by setting $r \in [\![p]\!]_0$ if and only if $p \in \Gamma$, and then set $[\![p]\!] = [\![p]\!]_0 \cup \bigcup_{\Diamond\varphi\in\Gamma} [\![p]\!]_\varphi$ for each propositional variable p. It is then straightforward to check that $r \in [\![\varphi]\!]$ if and only if $\varphi \in \Gamma$. □

It remains to exhibit models that satisfy sets of characteristic ω. We will do so by "decomposing" Γ into smaller pieces and applying Theorem 14.3.8 to each of them.

Definition 14.4.7. Assume that Γ has characteristic ω. Let $(\psi_i)_{i<\omega}$ enumerate all formulae such that $\Diamond\psi_i \in \Gamma$, in a way such that each ψ_i occurs infinitely often; let $(\phi_i)_{i<\omega}$ enumerate all formulae such that $\Box\phi_i \in \Gamma$ (in any way). For $i < \omega$, define $\Gamma(i) = \{\psi_i\} \cup \{\phi_j \wedge \Box\phi_j\}_{j<i}$.

Lemma 14.4.8. *If Γ is consistent, then so is each $\Gamma(i)$. Moreover, each $\Gamma(i)$ is satisfied in a finite model $(T_i, R_i, [\![\cdot]\!]_i)$.*

Proof. It is easy to see that $\Gamma \vdash \Diamond \bigwedge \Gamma(i)$ for all i, so the latter must be consistent. Since $\Gamma(i)$ is finite, we can apply Theorem 14.3.8. □

With this, we are ready to prove the main result of this section.

Theorem 14.4.9. GL *is strongly complete for the class of ω-bouquets.*

Proof. Let Γ be a consistent set of formulae; without loss of generality, assume Γ to be maximal. If Γ has finite rank, then apply Lemma 14.4.6.

Otherwise, Γ has rank ω. By Lemma 14.4.8, each $\Gamma(i)$ is satisfied in a finite model $(T_i, R_i, [\![\cdot]\!]_i)$.

As in the proof of Lemma 14.4.6, take the disjoint union of all $(T_i, R_i, [\![\cdot]\!]_i)$, and add a root r so that, for all variables p, $r \in [\![p]\!]$ if and only if $p \in \Gamma$; call the resulting model $\mathfrak{M}$. It is then easy to check that $\mathfrak{M}, r \models \Gamma$. □

Theorem 14.4.9 is interesting on its own right, but our focus is on ordinal spaces. Our strategy will be to 'lift' this result using d-maps through Lemma 15.2.13.2. Observe that our construction uses only very specific ω-bouquets; they are either of finite rank, or of rank $\omega + 1$, consisting of a root added to countably many finite subtrees. However, it will be just as easy to construct d-maps onto an arbitrary ω-bouquet as onto one of the above forms.

Let us move on to define the main scattered spaces in which we will be interested.

14.5 Generalized Icard spaces

In this section, we will discuss *generalized Icard spaces based on a scattered space $\mathfrak{X}$,* which are an increasing sequence of topologies $(\mathfrak{X}_{\uparrow\lambda})_{\lambda\in\mathsf{Ord}}$. These topologies were introduced with $\mathfrak{X}$ of the form $(\Theta, \mathcal{I}_0)$ to give semantics for the variable-free fragment of GLP_ω in [Ica11]. They were then defined for arbitrary GLP_Λ in [FDJ11] and subsequently studied in [FD14]. We generalize them further by letting $\mathfrak{X}$ be an arbitrary scattered space. As a notational convention, we will sometimes write ordinal intervals $[0, \alpha]$ as $(-1, \alpha]$.

Definition 14.5.1 (Icard topology)**.** Let $\mathfrak{X} = (X, \tau)$ be a scattered space of rank Θ. Given an ordinal λ, we define a topology $\tau_{\uparrow\lambda}$ to be the least topology containing τ and all sets of the form

$$(\alpha, \beta]^{\mathfrak{X}}_\xi = \{x \in X : \alpha < \ell^\xi \rho_\tau x \leq \beta\},$$

for some $-1 \leq \alpha < \beta \leq \Theta$ and some $\xi < \lambda$. These are called the *generalized Icard topologies based on* $\mathfrak{X}$. We call spaces of the form $\mathfrak{X}_{\uparrow\lambda} = (X, \tau_{\uparrow\lambda})$ for some τ *generalized Icard spaces.*

If $\mathfrak{X} = (X, \tau)$, we may write $(\alpha, \beta]^{\tau}_{\xi}$ instead of $(\alpha, \beta]^{\mathfrak{X}}_{\xi}$. We may also index properties by an ordinal λ if they refer to the topology $\tau_{\uparrow\lambda}$. In this way, $\rho_{\tau_{\uparrow\lambda}}$ becomes the λ-rank ρ_λ.

In the case of ordinal spaces, hyperlogarithms allow us to compute λ-ranks. Recall that we denote the left topology on the ordinals by $\mathcal{I}_0$. Thus, $(\mathcal{I}_0)_{\uparrow 0} = \mathcal{I}_0$ and $(\mathcal{I}_0)_{\uparrow 1} = \mathcal{I}_1$. In this case, we will write simply $\mathcal{I}_\lambda$ instead of $\mathcal{I}_{0\uparrow\lambda}$ and define $\Theta_\lambda = (\Theta, \mathcal{I}_\lambda)$. The following is proved in [FD14]:

Lemma 14.5.2. *If* Θ, λ, *and* $\xi < \Theta$ *are ordinals, then* $\rho_\lambda \xi = \ell^\lambda \xi$. *Moreover, if* μ *is any ordinal, then* $\ell^\lambda : \Theta_{\lambda+\mu} \to \Theta_\mu$ *is a d-map*

Let us now prove some additional properties of generalized Icard spaces:

Lemma 14.5.3. *Let* $\mathfrak{X}$ *and* $\mathfrak{Y}$ *be scattered spaces and* $f : \mathfrak{X} \to \mathfrak{Y}$ *be a d-map. Then,* $f : \mathfrak{X}_{\uparrow\lambda} \to \mathfrak{Y}_{\uparrow\lambda}$ *is a d-map for any ordinal* λ.

Proof. By Lemma 15.2.13.1, $\rho_{\mathfrak{X}} = \rho_{\mathfrak{Y}} f$, so that for any $-1 \leq \alpha < \beta$, any $\xi < \lambda$, and all τ-open A and σ-open B,

$$f\big(A \cap (\alpha, \beta]^{\mathfrak{X}}_{\xi}\big) = f(A) \cap (\alpha, \beta]^{\mathfrak{Y}}_{\xi}$$

and

$$f^{-1}(B) \cap (\alpha, \beta]^{\mathfrak{X}}_{\xi} = f^{-1}(B \cap (\alpha, \beta]^{\mathfrak{Y}}_{\xi}).$$

From this it follows that f is open and continuous. Furthermore, f is pointwise discrete because $\tau \subset \tau_{\uparrow\lambda}$. □

Corollary 14.5.4. *Let* $\mathfrak{X}$ *be a scattered space of rank* Θ. *Then for any ordinal* λ, *we have that* $\rho_{\mathfrak{X}} : \mathfrak{X}_{\uparrow\lambda} \to \Theta_\lambda$ *is a surjective d-map.*

Proof. Immediate from Lemmas 1 and 3. □

Corollary 14.5.4 provides us with a nice way of determining the rank function of Icard topologies:

Lemma 14.5.5. *If* $\mathfrak{X}$ *is a scattered space, then* $\rho_\lambda = \ell^\lambda \rho_{\mathfrak{X}}$.

Proof. By Corollary 14.5.4, $\rho_{\mathfrak{X}} : \mathfrak{X}_{\uparrow\lambda} \to (\rho\mathfrak{X})_\lambda$ is a d-map, and the topology in $(\rho\mathfrak{X})_\lambda$ is precisely $\mathcal{I}_\lambda$. Since, by lemma 14.5.2, ℓ^λ is the rank function on $(\rho\mathfrak{X})_\lambda$, we have that $\ell^\lambda \rho_{\mathfrak{X}} : \mathfrak{X}_{\uparrow\lambda} \to (\rho\mathfrak{X})_0$ is a d-map and coincides with the rank function by Lemma 1. □

Our choice of notation is explained by the following result:

Lemma 14.5.6. *For any scattered space (X, τ), and any ordinals λ and μ:*

$$(\tau_{\uparrow\lambda})_{\uparrow\mu} = \tau_{\uparrow(\lambda+\mu)}.$$

Proof. $(\tau_{\uparrow\lambda})_{\uparrow\mu}$ is the topology generated by $\tau_{\uparrow\lambda}$ and all sets $(\alpha, \beta]_\xi^{\tau_{\uparrow\lambda}}$, for $\xi < \mu$. $\tau_{\uparrow\lambda}$ is generated by τ and all sets $(\alpha, \beta]_\xi^\tau$, for $\xi < \lambda$; and for any α, β, and $\xi < \mu$:

$$\begin{aligned}(\alpha, \beta]_\xi^{\tau_{\uparrow\lambda}} &= \{x \in X : \alpha < \ell^\xi \rho_{\tau_{\uparrow\lambda}} x \leq \beta\} = \{x \in X : \alpha < \ell^\xi \ell^\lambda \rho_\tau x \leq \beta\} \\ &= \{x \in X : \alpha < \ell^{\lambda+\xi} \rho_\tau x \leq \beta\} = (\alpha, \beta]_{\lambda+\xi}^\tau,\end{aligned}$$

so that $(\tau_{\uparrow\lambda})_{\uparrow\mu}$ is generated by τ and all sets $(\alpha, \beta]_\xi^\tau$, for $\xi < \lambda + \mu$, and hence is equal to $\tau_{\uparrow(\lambda+\mu)}$. □

Another example of d-maps, which we present without proof, is given by ordinal subtraction:

Lemma 14.5.7. *Let ζ, λ, Θ be ordinals. Then, $(-\zeta + \cdot) : [\zeta, \zeta + \Theta]_\lambda \to [0, \Theta]_\lambda$, i.e., the function given by $\xi \to -\zeta + \xi$, is a homeomorphism.*

The following result shows that each point can be separated from points of equal rank in Icard topologies:

Lemma 14.5.8. *Let (X, τ) be a scattered space and λ be an ordinal. Any x in X has a λ-neighborhood U such that whenever $x \neq y \in U$, $\ell^\lambda \rho_0 y < \ell^\lambda \rho_0 x$.*

Proof. This is immediate from Lemma 14.3.14, using the fact that $\rho_\lambda = \ell^\lambda \rho_0$ by Lemma 2. □

Recall that for any topological space (X, τ), a *neighborhood base* for x is a collection of open sets $\mathcal{N}$ such that for any open set U containing x, there exists an open set $V \in \mathcal{N}$, also containing x, such that $V \subset U$. We will now provide neighborhood bases for Icard topologies in two useful ways.

Definition 14.5.9. A *simple function* is a function $r \colon A \to \mathsf{Ord}$ with $A \subset \mathsf{Ord}$ a finite set.

If $\mathfrak{X}$ is a scattered space, r is a simple function and the rank of $x \in X$ is an ordinal α such that $r(\xi) < \ell^\xi \alpha$ for all $\xi \in \mathrm{dom}(r)$, define $B_r^{\mathfrak{X}}(x)$ to be the set $\bigcap_{\xi \in dom(r)} (r(\xi), \ell^\xi \alpha]_\xi^{\mathfrak{X}}$.

As usual, if $\mathfrak{X} = (X, \tau)$, we may write $B_r^\tau(x)$ instead of $B_r^{\mathfrak{X}}(x)$. Note that any set $B_r^\tau(\alpha)$ is an open set of $\tau_{\uparrow\lambda}$ if $\max(\mathrm{dom}(r)) < \lambda$. In fact:

Lemma 14.5.10. *Let $\mathfrak{X} = (X, \tau)$ be a scattered space, $0 < \lambda$, $0 < \ell^\lambda \xi$, and $x \in X$ have rank ξ. Then the sets of the form $U \cap B_r^\tau(\xi)$ with $U \in \tau$ and $dom(r) \subset \lambda$ form a neighborhood base for x in $\tau_{\uparrow\lambda}$. Moreover, if $\mathfrak{X}$ is an ordinal with the left topology, we can take $U = X$.*

Proof. By definition, every neighborhood of ξ in $\tau_{\uparrow\lambda}$ contains another neighborhood of the form $V = U \cap \bigcap_{k<K} (\alpha_k, \beta_k]^{\mathfrak{X}}_{\sigma_k}$ for some $U \in \tau$, where all $\sigma_k < \lambda$.

We may assume that all α_k are greater than -1 since $0 < \ell^\lambda \xi \leq \ell^{\sigma_k} \xi$ for any k and that all σ_k are distinct because $\sigma_j = \sigma_k$ implies

$$(\alpha_k, \beta_k]^{\mathfrak{X}}_{\sigma_k} \cap (\alpha_j, \beta_j]^{\mathfrak{X}}_{\sigma_k} = (\max\{\alpha_k, \alpha_j\}, \min\{\beta_k, \beta_j\}]^{\mathfrak{X}}_{\sigma_k}.$$

To obtain the desired result, define r to have as domain the set of all σ_k, so that $r(\sigma_k) = \alpha_k$ and note that $U \cap B_r^{\mathfrak{X}}(x) \subset V$.

Finally, if $\mathfrak{X}$ is an ordinal with the left topology, then by definition $(0, \xi]_0 \subset U$. If we define s so that $\mathrm{dom}(s) = \{0\} \cup \mathrm{dom}(r)$, $s(\eta) = r(\eta)$ when defined, and $s(0) = 0$ if $r(0)$ is undefined, then we also obtain $X \cap B_s^{\mathfrak{X}}(x) = B_s^{\mathfrak{X}}(x) \subset V$, as claimed. □

Lemma 14.5.11. *Let $\mathfrak{X} = (X, \tau)$ be a scattered space, $\lambda = \alpha + \omega^\beta$ be an ordinal, and $x \in X$ have rank $e^\lambda \Theta$ with $\Theta > 0$. Then, sets of the form $U \cap (\eta, e^{\omega^\beta}\Theta]^{\mathfrak{X}}_\gamma$, where $U \in \tau$, $\eta < e^{\omega^\beta}\Theta$, and $\gamma < \lambda$, form a $\tau_{\uparrow\lambda}$-neighborhood base for x. If $\mathfrak{X}$ is an ordinal with the left topology, then we may take $U = [0, e^\lambda\Theta]_0$.*

Proof. Let V be a $\tau_{\uparrow\lambda}$-neighborhood of x. By Lemma 14.5.10, V contains a set of the form $U \cap B_r^\tau(e^\lambda\Theta)$ for some simple function r with domain contained in λ. We may assume using Lemma 14.3.14 that

$$U \setminus \{x\} \subset [0, e^\lambda\Theta)^\tau_0.$$

Define $\gamma = \max(\{\alpha\} \cup \mathrm{dom}(r))$. Before defining η, observe that for all $\xi \in \mathrm{dom}(r)$,

$$r(\xi) < \ell^\xi e^\lambda \Theta,$$

so that by Lemma 2,

$$e^\xi r(\xi) < e^\lambda \Theta = e^\gamma e^{\omega^\beta}\Theta.$$

Thus by normality of e^γ and the fact that $e^{\omega^\beta}\Theta$ is a limit, for $\eta < e^{\omega^\beta}\Theta$ large enough we have that $e^\gamma \eta > e^\xi r(\xi)$ for all $\xi \in \mathrm{dom}(r)$. We claim that

$$U \cap (\eta, e^{\omega^\beta}\Theta]^\tau_\gamma \subset B_r^\tau(x).$$

To see this, assume that $x \neq y \in U \cap (\eta, e^{\omega^\beta}\Theta]^\tau_\gamma$ and let $\theta = \rho_\tau y$, so that $\theta \in (\eta, e^{\omega^\beta}\Theta]_\gamma$. Then, for $\xi \in \mathrm{dom}(r)$ we see that $\ell^\gamma\theta = \ell^{-\xi+\gamma}\ell^\xi\theta > \eta$ and hence $\ell^\xi\theta > e^{-\xi+\gamma}\eta$ by Lemma 2. But

$$e^\xi e^{-\xi+\gamma}\eta = e^\gamma\eta > e^\xi r(\xi),$$

so by normality of e^ξ, $e^{-\xi+\gamma}\eta > r(\xi)$ and thus $\ell^\xi\theta > r(\xi)$.

Meanwhile, for any $\xi < \lambda$, since $\theta < e^\lambda\Theta = e^\xi e^{-\xi+\lambda}\Theta$ by the way we chose U, it follows once again from Lemma 2 that

$$\ell^\xi\theta \leq e^{-\xi+\lambda}\Theta = \ell^\xi e^\lambda\Theta.$$

Hence, for all $\xi \in \mathrm{dom}(r)$, we have

$$r(\xi) < \ell^\xi\theta < \ell^\xi e^\lambda\Theta,$$

that is,

$$\rho y = \theta \in (r(\xi), \ell^\xi e^\lambda\Theta]_\xi,$$

and thus $y \in B^\tau_r(x)$

We conclude that $U \cap (\eta, e^{\omega^\beta}\Theta]^\tau_\gamma \subset B^\tau_r(x)$, and thus $U \cap (\eta, e^{\omega^\beta}\Theta]^\tau_\gamma \subset V$, as needed. The modification for $\mathfrak{X}$ an ordinal with the left topology follows immediately as a special case. □

The following particular cases will be used later on:

Corollary 14.5.12. *Let λ be additively indecomposable. Then for any neighborhood V of $e^\lambda\Theta$ in $[1, e^\lambda\Theta]_\lambda$ there exist ordinals $\eta < e^\lambda\Theta$ and $\zeta < \lambda$ such that V contains the set $(\eta, e^\lambda\Theta]_\zeta$.*

Below, if (X, τ) is a scattered space and $(x_\xi)_{\xi<\mu}$ is a sequence of points in X, we write $x_\xi \overset{\lambda}{\to} y$ if x_ξ converges to y in $\tau_{\uparrow\lambda}$; that is, if for every λ-neighborhood U of y, there is $\delta < \mu$ such that $x_\xi \in U$ whenever $\xi > \delta$.

Lemma 14.5.13. *Let $\lambda > 0$, Θ be any ordinal and $\mu = \max(cf\lambda, \omega)$. Then, there exists a sequence $(\theta_\xi)_{\xi<\mu}$ such that $\theta_\xi \overset{\lambda}{\to} e^\lambda(\Theta + 1)$.*

Proof. Write $\lambda = \alpha + \omega^\beta$. We will define θ_ξ for $\xi < \mu$ considering two cases. If $\beta = 0$, then $\mu = \omega$. For $n < \omega$ define $\delta_n = \omega^\Theta \cdot n$, so that $(\delta_n)_{n<\omega}$ is cofinal in $\omega^{\Theta+1}$. Then, define $\theta_n = e^\alpha\delta_n$, which is cofinal in

$$e^\lambda(\Theta + 1) = e^{\alpha+1}(\Theta + 1) = e^\alpha\omega^{\Theta+1}$$

by normality of e^α.

If $\beta > 0$, choose a sequence $(\gamma_\xi)_{\xi<\mu}$ which is cofinal in ω^β. Then, set $\eta_\xi = e^{\gamma_\xi}(e^{\omega^\beta}\Theta + 1)$, so that the sequence $(\eta_\xi)_{\xi<\mu}$ is cofinal in $e^{\omega^\beta}(\Theta+1)$ by Lemma 14.2.6.4. Finally, define $\theta_\xi = e^\alpha \eta_\xi$.

In either case, it is straightforward to check using Lemma 15.2.24 that $\theta_\xi \xrightarrow{\lambda} e^\lambda(\Theta+1)$, as desired. □

Lemma 15.2.24 and Corollary 15.2.25 will be particularly useful for describing neighborhoods around points of high ranks in the completeness proof to follow.

14.6 Completeness for generalized Icard spaces

In this section, we will prove the strong completeness of GL with respect to its topological semantics. As we previously mentioned, we intend to construct d-maps from ordinal spaces onto ω-bouquets. We start with a simple case:

Lemma 14.6.1. *For all nonzero additively indecomposable λ and any ω-bouquet generated by (T, R) whose root r has rank Θ, there exists a surjective d-map $f : ([0, e\Theta], \mathcal{I}_1) \to T$ such that $f(e\Theta)$ is the root of the tree.*

Proof. By induction on Θ. Denote by r the root of the tree. If $\Theta = 0$, then the mapping $0 \mapsto r$ is clearly a d-map. Otherwise, suppose the result holds for all ordinals $\theta < \Theta$. We fix an enumeration $(v_i)_{i<\omega}$ of all daughters of r with different properties, according as to whether Θ is a successor ordinal or a limit. If Θ is a limit ordinal, we let $(v_i)_{i<\omega}$ enumerate each daughter exactly once. If Θ is a successor ordinal, let $(v_i)_{i<\omega}$ be any enumeration where each daughter appears infinitely often. To each v_i we associate its rank θ_i and its generated subbouquet T_i. Define $\alpha_0 = 0$ and, recursively, $\beta_i = \alpha_i + \omega^{\theta_i}$ and $\alpha_{i+1} = \beta_i + 1$.

If $\Theta = \Xi + 1$ is a successor ordinal, then $\theta_i = \Xi$ for infinitely many i. If Θ is a limit, then the sequence $(\theta_i)_{i<\omega}$ is cofinal in Θ. In either case it follows that $\beta_i \xrightarrow{1} e\Theta$ and thus the sets $X_i := [\alpha_i, \beta_i]$ form a partition of $[0, e\Theta)$.

By induction hypothesis, there are surjective d-maps $g_i : [0, e\theta_i] \to T_i$, and thus by Lemma 14.5.7 we obtain surjective d-maps $f_i \colon X_i \to T_i$ given by $f_i(\xi) = g_i(-\alpha_i + \xi)$. Let $f = \bigcup_{i<\omega} f_i \cup \{(e\Theta, r)\}$. As the sets X_i are clopen, f is a d-map when restricted to $e\Theta$. We show it is also a d-map in $[0, e\Theta]$.

Clearly, the function is pointwise discrete and surjective. Moreover, the family $\{[\alpha_i, e\Theta]\}_{i<\omega}$ forms a decreasing neighborhood base around $e\Theta$, and it is easy to check that $f[\alpha_n, e\Theta] = \{r\} \cup \bigcup_{i\geq n} T_i$, which is open, so f is

open. As for continuity, $f^{-1}T = [0, e\Theta]$, so we just need to check that the preimage of basic open sets around r are open when Θ is a limit. But this is also true, as for any such set $A = \{r\} \cup \bigcup_{i \geq n} T_i$, we have $f^{-1}(A) = [\alpha_n, e\Theta]$ by construction. □

The d-maps constructed in Lemma 14.6.1 can be further exploited. As the composition of d-maps is a d-map, it suffices to find d-maps from any space onto ordinals with the interval topology to obtain d-maps onto ω-bouquets. As stated by Lemma 14.5.2, this is sometimes possible by using logarithms $\ell^\xi \colon \Theta_{\xi+1} \to \Theta_1$.

This is, however, impossible for limit ordinal Icard topologies. Instead, we could try to find different mappings. For example, the Λ-*reductive maps* from [FD14] function for limit ordinal Icard topologies as long as the index is countable. Theorem 14.6.2 below will show that it is impossible to go further using this technique.

Theorem 14.6.2. *Let λ be a nonzero ordinal, κ be a limit ordinal, and $\Theta > 0$ be any ordinal. If $\max(cf(\kappa), \aleph_0) \neq \max(cf(\lambda), \aleph_0)$, then there exist no d-maps $f \colon (e^\kappa\Theta + 1)_\kappa \to (e^\lambda\Theta + 1)_\lambda$.*

Proof. Write $\lambda = \alpha + \omega^\beta$ and $\kappa = \gamma + \omega^\delta$, so that $cf\omega^\beta = cf\lambda$ and $cf\omega^\delta = cf\kappa$. It suffices to prove the theorem for $\Theta = 1$ and $\lambda = \omega^\beta$, for if

$$f \colon (e^\kappa\Theta + 1)_\kappa \to (e^\lambda\Theta + 1)_\lambda$$

were a d-map, we would have that $\ell^\lambda f(e^\kappa 1) = 1$ (since d-maps are rank-preserving by Lemma 15.2.13.1), and thus by Lemma 14.2.12.4, there would exist $\sigma \in [\alpha, \lambda)$ such that $\ell^\sigma f(e^\kappa 1) = e^{\omega^\beta} 1$. But then it is straightforward to check that

$$\tilde{f} \colon (e^\kappa 1 + 1)_\kappa \to (e^{\omega^\beta} 1 + 1)_{\omega^\beta},$$

defined by

$$\tilde{f}(\xi) = \begin{cases} \ell^\sigma f(\xi), & \text{if } \xi \in f^{-1}\ell^{-\sigma}[0, e^{\omega^\beta} 1] \\ 0, & \text{otherwise,} \end{cases}$$

would also be a d-map.

Let $\mu = \max(cf(\kappa), \aleph_0)$ and $\nu = \max(cf(\lambda), \aleph_0)$. By Corollary 14.2.7, $\mu = cf(e^\kappa 1)$ and $\nu = cf(e^\lambda 1)$. Towards a contradiction, suppose that $\mu \neq \nu$ and

$$f \colon (e^\kappa 1 + 1)_\kappa \to (e^\lambda 1 + 1)_\lambda$$

is a d-map. First assume that $\mu < \nu$. Use Lemma 14.5.13 to find a sequence $(\theta_\xi)_{\xi<\mu}$ such that $\theta_\xi \stackrel{\kappa}{\to} e^\kappa 1$. By continuity, we should also have $f(\theta_\xi) \stackrel{\lambda}{\to} e^\lambda 1$; but, since f preserves rank, $f(\theta_\xi) < e^\lambda 1$ for all ξ, and since $\mu < \nu = cfe^\lambda 1$, we have that $(f(\theta_\xi))_{\xi<\mu} < \theta_*$ for some $\theta_* < e^\lambda 1$. But this means that $f(\theta_\xi) \notin (\theta_*, e^\lambda 1]_1$ for all $\xi < \mu$, contradicting that $f(\theta_\xi) \stackrel{\lambda}{\to} e^\lambda 1$.

Now assume that $\mu > \nu$. Use Lemma 14.5.13 once again to find a sequence $(\delta_\xi)_{\xi<\nu}$ such that $\delta_\xi \stackrel{\lambda}{\to} e^\lambda 1$. By continuity, $f^{-1}(\delta_\xi, e^\lambda 1]_0$ is λ-open for all $\xi < \nu$, hence by Corollary 15.2.25, there are sequences $(\eta_\xi)_{\xi<\nu} \subset e^\kappa 1$ and $(\gamma_\xi)_{\xi<\nu} \subset \kappa$ such that, for all $\xi < \nu$,

$$[0, e^\kappa 1]_0 \cap (\eta_\xi, e^\kappa 1]_{\gamma_\xi} \subset f^{-1}(\delta_\xi, e^\lambda 1]_0.$$

Since $\nu < \mu = cfe^\kappa 1$, there are $\eta_* < e^\kappa 1$ and $\gamma_* < \kappa$ bounding $(\eta_\xi)_{\xi<\nu}$ and $(\gamma_\xi)_{\xi<\nu}$, respectively. Then,

$$U_* = [0, e^\kappa 1]_0 \cap (\eta_*, e^{\omega^\beta} 1]_{\gamma_*}$$

is a κ-neighborhood of $e^\kappa 1$, so that $f(U_*)$ contains a λ-neighborhood V_* of $e^\lambda 1$. But, V_* cannot contain any δ_ξ, contradicting the fact that $\delta_\xi \stackrel{\lambda}{\to} e^\lambda 1$.

In either case, we conclude that there can be no such d-map. □

Our strategy will hence be to instead construct d-maps directly from Icard spaces $\mathfrak{X}_{\uparrow\lambda}$ onto ω-bouquets. The construction will be an adaptation of the one on Lemma 14.6.1, but the general case will be rather more involved. We will first assume that λ is additively indecomposable and $\mathfrak{X}$ is an ordinal with the left topology for, as we shall see, the general case follows quickly from this particular one. As in Lemma 14.6.1, we will proceed by induction on the rank.

14.6.1 The successor stage

For this section, we fix an additively indecomposable ordinal λ and an ω-bouquet (T, σ_R) with root r of rank $\Theta + 1$. Let $(v_i)_{i<\omega}$ list all daughters of r in such a way that each daughter is counted infinitely often, and let T_i be the subbouquet generated by v_i. Denote the rank of T_i by θ_i. We introduce the following auxiliary notation:

Definition 14.6.3. Let $\iota < \lambda$. By Lemma 14.2.1.2, there is a unique $k = k(\iota) < \omega$ such that ι can be written as $\alpha \cdot \omega + k$. For $\iota < \lambda$, define

$$\beta_\iota := e^{\iota+1}(e^\lambda \Theta + 1 + e^\lambda \theta_{k(\iota)}).$$

Then define α_ι by cases:

1. $\alpha_0 := 0$,
2. $\alpha_{\iota+1} := \beta_\iota + 1$, and
3. $\alpha_\iota := e^\iota(e^\lambda\Theta + 1)$ if ι is a limit.

Finally, define $X_\iota = [\alpha_\iota, \beta_\iota]$, $Y_\iota = X_\iota \cap [0, 1 + e^\lambda\Theta]_{\iota+1}$, and $Z_\iota = X_\iota \setminus Y_\iota$.

Lemma 14.6.4. *The sets X_ι, Y_ι, and Z_ι have the following properties:*

1. *for all $\iota < \lambda$, each of the sets X_ι, Y_ι and Z_ι are λ-clopen;*
2. *the collection $\{X_\iota \colon \iota < \lambda\}$ forms a partition of $e^\lambda(\Theta + 1)$;*
3. *if U is a λ-neighborhood of $e^\lambda(\Theta + 1)$, then $Z_\iota \subset U$ for all ι large enough, and*
4. *for all $\iota < \lambda$, $\ell^{\iota+1}(X_\iota) = [0, e^\lambda\Theta + 1 + e^\lambda\theta_{k(\iota)}]$, $\ell^{\iota+1}(Y_\iota) = [0, e^\lambda\Theta]$ and $\ell^{\iota+2}(Z_\iota) = [0, e^\lambda\theta_{k(\iota)}]$.*

Proof. Claim (1) follows immediately from their definitions which use clopen intervals. For claim (2), note that from Lemma 14.2.8.1 it follows that for limit ι: $\lim_{\eta\to\iota} \alpha_\eta = \alpha_\iota$. Hence, the sets X_ι are disjoint and no gaps are left between them. Moreover, from Lemma 14.2.8.2, it also follows that $e^\lambda(\Theta + 1) = \lim_{\iota\to\lambda} \alpha_\iota$.

For claim (3), let U be any λ-neighborhood of $e^\lambda(\Theta + 1)$, so that by Lemma 15.2.25 there are $\eta < e^\lambda(\Theta + 1)$ and $\delta < \lambda$ such that

$$(\eta, e^\lambda(\Theta + 1)]_\delta \subset U.$$

By Lemma 14.2.6.4, for γ large enough we have that $\eta < e^\gamma(e^\lambda\Theta + 1)$. Since λ is additively indecomposable, $\delta + \gamma < \lambda$, hence if $\iota > \delta + \gamma$ we have that for $\zeta \in Z_\iota$,

$$\ell^\gamma\ell^\delta\zeta = \ell^{\delta+\gamma}\zeta \geq \ell^{\iota+1}\zeta \geq e^\lambda\Theta + 1,$$

so that by Lemma 2,

$$\ell^\delta\zeta \geq e^\gamma(e^\lambda\Theta + 1) > \eta,$$

that is,

$$\zeta \in (\eta, e^\lambda(\Theta + 1)]_\delta \subset U$$

and, since ζ was arbitrary, $Z_\iota \subset U$.

To show (4), fix some $\iota < \lambda$. Since λ is an additively indecomposable limit, $\iota + 1 + \lambda = \lambda$. Thus by Lemma 14.5.2,

$$\ell^{\iota+1} \colon (\beta_\iota + 1)_\lambda \to (e^\lambda\Theta + 1 + e^\lambda\theta_{k(\iota)} + 1)_\lambda$$

is a d-map. Also by Lemma 14.5.2, $\ell^{\iota+1}$ is the rank function with respect to $\mathcal{I}_{\iota+1}$, and X_ι is $(\iota+1)$-open (in fact, 1-open). Therefore, since $\beta_\iota \in X_\iota$ and

$$\ell^{\iota+1}\beta_\iota = e^\lambda\Theta + 1 + e^\lambda\theta_{k(\iota)},$$

it follows from Lemma 14.3.14 that

$$[0, e^\lambda\Theta + 1 + e^\lambda\theta_{k(\iota)}] \subset \ell^{\iota+1}(X_\iota).$$

Since

$$Y_\iota = \{\xi \in X_\iota : \ell^{\iota+1}\xi \leq e^\lambda\Theta\},$$

it follows that $\ell^{\iota+1}(Y_\iota) = [0, e^\lambda\Theta]$.

Finally, observe that if $\xi \in Z$, we can write

$$\ell^{\iota+1}\xi = e^\Lambda\Theta + 1 + \zeta$$

for some $\zeta \leq 1 + e^\lambda\theta_{k(\iota)}$, thus by Lemma 14.2.12.2,

$$\ell^{\iota+2}\xi = \ell(e^\Lambda\Theta + 1 + \zeta) = \ell(1+\zeta) \leq e^\lambda\theta_{k(\iota)},$$

but Z is $(\iota+2)$-open, so once again by Lemma 14.3.14, $\ell^{\iota+2}(Z) = [0, e^\lambda\theta_{k(\iota)}]$. □

The induction hypothesis for a successor stage will amount to assuming we are given fragments of a d-map, which we then need to complete. We make this notion precise now:

Definition 14.6.5. We define a *pre-d-map for* (Θ, λ) *over* T to be a collection of d-maps $(g_i)_{i<\omega}$ such that, for every $i < \omega$, $g_i \colon (e^\lambda\theta_i + 1)_\lambda \to T$ and g_i has range T_i.

Lemma 14.6.6. *Assume there exists a pre-d-map* $(g_i)_{i<\omega}$ *for* $(\Theta+1, \lambda)$ *over* T. *Then there is a surjective d-map* $f \colon (e^\lambda(\Theta+1)+1)_\lambda \to T$.

Proof. We may assume without loss of generality that $\theta_0 = \Theta$. As before, we have $k(\iota)$ denote the remainder of ι modulo ω.

Before defining f, we define for $\iota < \lambda$,

$$f_\iota \colon X_\iota \to T_0 \cup T_{k(\iota)}$$

by

$$f_\iota(\xi) = \begin{cases} g_0 \circ \ell^{\iota+1}(\xi) & \text{if } \xi \in Y_\iota, \\ g_{k(\iota)} \circ \ell^{\iota+2}(\xi) & \text{if } \xi \in Z_\iota. \end{cases}$$

We claim that each

$$f_\iota : X_\iota \to T_0 \cup T_{k(\iota)}$$

is a surjective d-map, with $f_\iota(Y_\iota) = T_0$ and $f_\iota(Z_\iota) = T_{k(\iota)}$:

SURJECTIVITY. That $f_\iota(Y_\iota) = T_0$ and $f_\iota(Z_\iota) = T_{k(\iota)}$ follows easily from 14.6.4.4 and the assumption that each g_i is onto T_i, and hence f_ι is surjective.

OPENNESS. Let $U \subset X_\iota$ be open. Define $U_1 = U \cap Y_\iota$, and $U_2 = U \setminus U_1 \subset Z_\iota$, both of which are open. Because g_0, g_ι, $\ell^{\iota+1}$ and $\ell^{\iota+2}$ are all d-maps, $f_\iota(U)$ is the union of the open sets $f_\iota(U_1)$ and $f_\iota(U_2)$, so $f_\iota(U)$ is open; since U was arbitrary, f_ι is an open function.

CONTINUITY. This follows by a similar argument: if $V \subset T_0 \cup T_{k(\iota)}$ is open, then

$$f_\iota^{-1}(V) = (f_\iota^{-1}(V) \cap Y_\iota) \cup (f_\iota^{-1}(V) \cap Z_\iota),$$

which is a union of two open subsets of X_ι.

POINTWISE DISCRETENESS. If $t \in T_0 \cup T_{k(\iota)}$, then once again $f_\iota^{-1}(t) = (f_\iota^{-1}(V) \cap Y_\iota) \cup (f_\iota^{-1}(V) \cap Z_\iota)$. Since $g_0 \circ \ell^{\iota+1}$ is a d-map, $A = f_\iota^{-1}(V) \cap Y_\iota$ is a discrete subset of Y_ι, and similarly $B = f_\iota^{-1}(V) \cap Z_\iota$ is a discrete subset of Z_ι. But both Y_ι and Z_ι are clopen, so $A \cup B$ is also discrete.

This completes the proof of the claim. We define

$$f = \bigcup_{\iota \in \lambda} f_\iota \cup \{(e^\lambda(\Theta + 1), r)\}.$$

It can be quickly verified that f is surjective. We show it is also a d-map:

OPENNESS. If U is open in $[0, e^\lambda(\Theta + 1))$, then

$$f(U) = f(\bigcup_{\iota \in \lambda} (U \cap X_\iota) = \bigcup_{\iota \in \lambda} f_\iota(U \cap X_\iota)$$

is open since each f_ι is open. Otherwise, if U is a neighborhood of $e^\lambda(\Theta+1)$, we claim that $f(U) = T$. Indeed, $r = f(e^\lambda(\Theta + 1)) \in f(U)$, and if $t \neq r$, we have that $t \in T_i$ for some i. By Lemma 14.6.4.3 we have that, for some $\iota_* < \lambda$, $Z_\iota \subset U$ whenever $\iota > \iota_*$; in particular, since every subbouquet is counted infinitely often and λ is a limit, we can take $\iota > \iota_*$ so that $k(\iota) = i$,

and since $T_i = f(Z_\iota)$, we have that $t \in f(Z_\iota) \subset f(U)$. It follows that f is open.

CONTINUITY. Suppose $V \in \sigma_R$ and $\xi \in f^{-1}(V)$. If $f(\xi) = r$, then since the only neighborhood of r is all of T, it follows that $f^{-1}(V) = e^\lambda(\Theta + 1) + 1$. Otherwise, by Lemma 14.6.4.2, $\xi \in X_\iota$ for some unique ι. Then,

$$\xi \in f^{-1}(V) \cap X_\iota \subset f^{-1}(V);$$

but

$$f^{-1}(V) \cap X_\iota = f_\iota^{-1}(V \cap T_{k(\iota)}),$$

which is open since f_ι is continuous. Since $\xi \in f^{-1}(V)$ was arbitrary, we conclude that $f^{-1}(V)$ is open.

POINTWISE DISCRETENESS. Each $f_\iota^{-1}(t)$ is discrete and each X_ι is $(\iota+1)$-clopen and thus λ-clopen, so $f^{-1}(t)$ is discrete for each $t \neq r$, while $f^{-1}(r)$ is a singleton and thus discrete.

Therefore, f is a d-map, as desired. □

Note that the above may already be used to give an inductive construction of d-maps onto any ω-bouquet of finite rank, but we have yet to deal with limit Θ.

14.6.2 The limit stage

For the following, we fix an additively indecomposable ordinal λ and an ω-bouquet (T, σ_R) with root r of countable limit rank Θ. We also fix a sequence $(v_i)_{i<\omega}$ enumerating each daughter of r exactly once and denote by T_i and θ_i, respectively, the generated subbouquet and rank of v_i.

A *dominating subsequence of* $(\theta_i)_{i<\omega}$ is a subsequence $(\theta_{m_i})_{i<\omega}$ of $(\theta_i)_{i<\omega}$ such that $(\theta_{m_i})_{i<\omega}$ is strictly increasing and for all $i < \omega$, $\theta_i < \theta_{m_i}$. Observe that if this is the case, then $\lim_{i\to\omega} \theta_{m_i} \to \Theta$. It is evident that dominating sequences for $(\theta_i)_{i<\omega}$ do exist. We fix one such sequence $(\theta_{m_i})_{i<\omega}$.

Lemma 14.6.7. *If there exists a pre-d-map $(g_i)_{i<\omega}$ for (Θ, λ) over T, then there is a continuous surjective map*

$$h\colon (e^\lambda\Theta + 1)_\lambda \to T$$

such that $h^{-1}(r) = \{e^\lambda\Theta\}$, and $h|_{e^\lambda\Theta}$ is a d-map.

Proof. Let $h\colon e^\lambda\Theta + 1 \to T$ be defined by

$$h(\xi) = \begin{cases} r & \text{if } \xi = e^\lambda\Theta, \\ g_{m_0}(\xi) & \text{if } \xi \leq e^\lambda\theta_{m_0}, \\ g_{m_i}(\xi) & \text{if } e^\lambda\theta_{m_{i-1}} < \xi \leq e^\lambda\theta(m_i) \text{ with } 0 < i. \end{cases}$$

It is straightforward to check that h has the desired properties. □

The map h just defined may well not be a d-map, as it might fail to be open. In the following, we will modify h so as to make it open near $e^\lambda\Theta$. This will require constructing some auxiliary sets, in a fashion similar to Definition 14.6.3.

Definition 14.6.8. Let $k(\iota)$ denote the remainder of ι modulo ω as before. For $j < \omega$ and $\iota < \lambda$, we set

$$\gamma_{j\iota} = e^\lambda\theta_{m_{j+1}} + e^\iota(e^\lambda\theta_{m_j} + 1)$$

and

$$\delta_{j\iota} = \gamma_{j\iota} + e^\iota(e^\lambda\theta_{m_j} + 1 + e^\lambda\theta_j).$$

We also define

$$W_{j\iota} = (\gamma_{j\iota}, \delta_{j\iota}]_0 \cap (e^\lambda\theta_{m_j}, e^\lambda\theta_{m_{j+1}} + 1 + e^\lambda\theta_j]_\iota$$

and

$$W_j = \bigcup_{\iota<\lambda} W_{j\iota}.$$

Lemma 14.6.9. *The sets $W_{j\iota}$ and W_j have the following properties:*

1. *$\ell^{\iota+1}(W_{j\iota}) = e^\lambda\theta_j + 1$ for all $j < \omega$ and $\iota < \lambda$,*
2. *$W_j \subset (e^\lambda\theta_{m_{j+1}}, e^\lambda\theta_{m_{j+2}})$ and $W_{i\kappa} \cap W_{j\iota} = \varnothing$ if $i \neq j$ or $\iota \neq \kappa$,*
3. *the set W_j is closed for all j, and*
4. *if U is any λ-neighborhood of $e^\lambda\Theta$, there exist $n < \omega$ and $\iota < \lambda$ such that $W_{j\iota} \subset U$ for all $j > n$.*

Proof. Claim (1) is similar to Lemma 14.6.4.4. We prove claim (2). Observe that for any $j < \omega$, $\gamma_{j\iota} < \delta_{j\iota} < \gamma_{j\kappa}$ whenever $\iota < \kappa$, from which it follows that $W_{j\kappa} \cap W_{j\iota} = \varnothing$ if $\iota \neq \kappa$. Moreover, if we define

$$\nu_j = e^\lambda\theta_{m_{j+1}} + e^\lambda(\theta_{m_j} + 1) < e^\lambda\theta_{m_{j+2}},$$

by Lemma 14.2.8.1 we have that $\gamma_{j\iota}, \delta_{j\iota} \xrightarrow{1} \nu_j$ as $\iota \to \lambda$, which by the definition of γ_{j0} implies that

$$W_j \subset (e^\lambda\theta_{m_{j+1}}, \nu_j) \subset (e^\lambda\theta_{m_{j+1}}, e^\lambda\theta_{m_{j+2}}).$$

It follows that $W_{i\kappa} \cap W_{j\iota} = \varnothing$ if $i \neq j$.

For claim (3), since each $W_{j\iota}$ is closed, it also follows that $\overline{W}_j \backslash W_j \subset \{\nu_j\}$. However, $[0, \nu_j]_0 \cap (e^\lambda\theta_{m_{j+1}}, \infty)_1$ is a λ-neighborhood of ν_j which does not intersect any of the intervals $(\gamma_{j\iota}, \delta_{j\iota}]$. We conclude that $\overline{W}_j \setminus W_j$, that is, W_j is closed.

Finally, for claim (4), let U be any neighborhood of $e^\lambda\Theta$, so that by Lemma 15.2.25 and the fact that $\theta_{m_i} \to \Theta$, there are $n < \omega$ and $\iota < \lambda$ such that

$$[0, e^\lambda\Theta]_0 \cap (e^\lambda\theta_{m_n}, e^\lambda\Theta]_\iota \subset U.$$

Then, it is immediate that if $j > n$, $W_{j\iota} \subset U$. □

Lemma 14.6.10. *Assume there exists a pre-d-map $(g_i)_{i<\omega}$ for (Θ, λ) over T. Then there is a surjective d-map $f\colon (e^\lambda\Theta + 1)_\lambda \to T$.*

Proof. We use Lemma 14.6.7 to construct a continuous map

$$h\colon (e^\lambda\Theta + 1)_\lambda \to T$$

such that $h^{-1}(r) = \{e^\lambda\Theta\}$, and $h|_{e^\lambda\Theta}$ is a d-map. Then, we define f by

$$f(\xi) = \begin{cases} h(\xi) & \text{if } \xi \notin \bigcup_{j<\omega} W_j, \\ g_j \ell^{\iota+1}\xi & \text{if } \xi \in W_{j\iota}. \end{cases}$$

That f is surjective and pointwise discrete is clear from its construction. We verify openness and continuity.

Openness. Choose $\xi < e^\lambda\Theta$. If $\xi \in W_{j\iota}$ for some j, ι, we use the fact that $W_{j\iota}$ is open and that g_j, $\ell^{\iota+1}$ are d-maps to see that f is open near ξ. Otherwise, $\xi \in (e^\lambda\theta_{m_j}, e^\lambda\theta_{m_{j+1}}]$ for some j, and

$$(e^\lambda\theta_{m_j}, e^\lambda\theta_{m_{j+1}}] \setminus \bigcup_{i<\omega} W_i = (e^\lambda\theta_{m_j}, e^\lambda\theta_{m_{j+1}}] \setminus W_j.$$

But W_j is λ-closed by Lemma 14.6.9.3, whereby

$$(e^\lambda\theta_{m_j}, e^\lambda\theta_{m_{j+1}}] \setminus W_j$$

is λ-open. Since h is open near ξ then so is f. Hence, f is open in $[0, e^\lambda\Theta)$.

Now let V be an open neighborhood of $e^\lambda\Theta$. By Lemma 14.6.9.4, there are $\iota < \lambda$ and $n < \omega$ such that $W_{j\iota} \subset U$ for all $j > n$, and it follows by the surjectivity of g_j that $\{r\} \cup \bigcup_{j>n} T_j \subset f(V)$. Since V was arbitrary, we conclude that f is open near $e^\lambda\Theta$.

Continuity. By a similar argument to the one above, f is continuous in $[0, e^\lambda\Theta)$, and so it remains to show that it is continuous at $e^\lambda\Theta$. Take a neighborhood U of r, so that $U = \{r\} \cup \bigcup_{i>n} T_i$, and choose $k > n$ large enough so that $m_k > n$. It is straightforward to check that

$$(e^\lambda\theta_{m_k+1}, e^\lambda\Theta] \subset f^{-1}(U),$$

so the latter is open near $e^\lambda\Theta$. □

14.6.3 Main result

First, we join Lemmas 14.6.6 and 14.6.10 to build d-maps from Icard spaces onto ω-bouquets.

Theorem 14.6.11. *Let (T, σ) be any ω-bouquet with root r of (countable) rank Θ, and let $\lambda > 0$ be any ordinal. Then, there is a surjective d-map $f\colon (e^\lambda\Theta + 1)_\lambda \to T$.*

Proof. If $\lambda = \alpha + \omega^\beta$ and $f\colon (e^{\omega^\beta}\Theta + 1)_{\omega^\beta} \to T$ is a surjective d-map, then $f \circ \ell^\alpha\colon (e^\lambda\Theta + 1)_\lambda \to T$ is also a surjective d-map. Hence, it suffices to consider additively indecomposable λ. If $\lambda = 1$ the claim becomes Lemma 14.6.1, so we assume that λ is a limit and proceed by induction on Θ.

If $\Theta = 0$, then $T = \{r\}$, and $f\colon e^\lambda 0 + 1 \to T$ given by $f(0) = r$ trivially has all desired properties.

Otherwise, we may enumerate all immediate subbouquets of T by $(T_i)_{i<\omega}$ and denote their ranks by θ_i. By induction on their rank, we find surjective d-maps $g_i\colon (e^\lambda\theta_i + 1)_\lambda \to T_i$, so that $(g_i)_{i<\omega}$ is a pre-d-map. Thus, by Lemma 14.6.6 if Θ is a successor, or Lemma 14.6.10 if Θ is a limit, there is a surjective d-map $f\colon (e^\lambda\Theta + 1)_\lambda \to T$, as needed. □

Corollary 14.6.12. *Let λ be a nonzero ordinal and Θ be countable. For each scattered space $\mathfrak{X}$ of rank $e^\lambda\Theta + 1$ and any ω-bouquet T of rank Θ there exists a surjective d-map $f : \mathfrak{X}_{\uparrow\lambda} \to T$.*

Proof. From Corollary 14.5.4 and Theorem 14.6.11. □

Note that it follows from Lemma 15.2.13.1 that if $\mathfrak{X}$ contains only one point of maximal rank, it is also the preimage of the root. Meanwhile, as a consequence of Theorem 14.6.11, we obtain the following:

Theorem 14.6.13 (Strong completeness)**.** *Let λ be a nonzero ordinal and $\mathfrak{X}$ be a scattered space of rank at least $e^\lambda\omega+1$. Then,* GL *is strongly complete with respect to $\mathfrak{X}_{\uparrow\lambda}$.*

Proof. Suppose Γ is a consistent set of formulae. By Theorem 14.4.9, Γ is satisfiable on some ω-bouquet T of rank $\Theta \leq \omega$, whence by Lemma 15.2.13.2 and Theorem 14.6.11, Γ is satisfiable on $(e^\lambda\Theta+1)_\lambda$, a λ-open set of $\mathfrak{X}_{\uparrow\lambda}$. □

As an immediate consequence of Theorem 14.6.13, we obtain the following result that is of particular importance when λ is uncountable:

Corollary 14.6.14. GL *is strongly complete with respect to an ordinal $(\Theta, \mathcal{I}_\lambda)$ whenever $e^\lambda\omega < \Theta$.*

The instance of Corollary 14.6.14 when $\lambda = 1$ is a strenghtening of the Abashidze-Blass theorem. Another remarkable consequence of Theorem 14.6.13 is what results when applying it to the club topology τ_c, (see Example 14.3.10) as it has been shown (see [Bla90]) that it is consistent with ZFC + "there exists a Mahlo cardinal" that GL be incomplete with respect to τ_c for any ordinal. However, using generalized Icard topologies we obtain the following:

Corollary 14.6.15. GL *is strongly complete with respect to an ordinal $(\Theta, \tau_{c\uparrow\lambda})$ whenever $\aleph_{e^\lambda\omega+1} < \Theta$.*

15

A Topological Completeness Theorem for Transfinite Provability Logic

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15.1 Introduction

The purpose of this chapter is to prove a topological completeness theorem for the transfinite extension of Japaridze's logic GLP. GLP is a provability logic in a propositional language augmented with a possibly transfinite sequence of modal operators; our case of interest is that in which the sequence

is wellfounded. Each of these operators can be interpreted arithmetically as asserting provability within a given theory, and the logic GLP relates these notions of provability to one another. For arithmetical interpretations of GLP, see Beklemishev [Bek04], Fernández-Duque and Joosten [FDJ13b], Cordón-Franco et al. [CFFDJLM15], and others. As a modal logic, GLP has some unusual properties. For example, it is not complete with respect to any class of relational semantics; a natural question is whether it is complete with respect to its neighborhood (i.e., topological) semantics. Beklemishev and Gabelaia [BG14] showed that GLP_ω, the restriction of GLP to ω-many modalities, is complete with respect to a natural topological space on the ordinal ε_0. Because sentences in the language of GLP contain instances of only finitely many modalities, the spaces constructed by Beklemishev and Gabelaia serve as models also for formulas in the language of transfinite GLP; hence it is also topologically complete. However, it is an open problem whether transfinite GLP is complete with respect to a single polytopological space, but there are some very natural candidates. An example of these is what are known as the *canonical topological semantics* for GLP. The question of completeness with respect to these spaces has very interesting connections with stationary reflection and indescribable cardinals (see Bagaria [Bag15], Bagaria [Bag], Bagaria-Magidor-Sakai [BMS15] and Brickhill [Bri16]). Completeness for the two-modality fragment with respect to these topologies was proved by Beklemishev [Bek11]. It is not hard to see that GLP is not strongly complete with respect to its canonical topological semantics; we shall prove this below.

Another example is the natural extension of the spaces from Beklemishev-Gabelaia [BG13]. Completeness with respect to these spaces was proved by Fernández-Duque [FD14] for restrictions of GLP to any countable amount of modalities. It is not known whether this result can be extended to arbitrarily long sequences of modalities, but the results from [AFD17] show that the techniques would need to be very different.

The topological completeness theorem we shall prove here goes in this direction. Roughly, given a sentence consistent with GLP, we construct a topological model for it with finitely many topologies. The new feature is that these topologies in a way correspond to the modalities appearing in the sentence; we call these $\vec{\vartheta}$-polytopologies, for $\vec{\vartheta}$ a finite sequence of ordinals. In particular, one can extend the space with intermediate increasing topologies corresponding to modalities not appearing in the sentence in such a way that each topology results in a model of the unimodal GL. Unfortunately, this extension (with the intermediate topologies) will not be a model of GLP, but we expect that a similar construction can yield models of GLP.

Our main tool is a technical "product lemma." Essentially, given two ordinals κ and λ, we find an ordinal Θ and natural embeddings $\pi_0 : \Theta \to \kappa$ and $\pi_1 : \Theta \to \lambda$ which preserve satisfiability of polymodal formulas, if the ordinals are equipped with the right topologies. This is a generalization of a technical lemma of Beklemishev-Gabelaia [BG13], which corresponds to the case in which the first element of the sequence $\vec{\vartheta}$ is 1. The proof is largely arithmetical and relies heavily on the theory of hyperexponentials and hyperlogarithms of Fernández-Duque and Joosten [FDJ13a].

15.2 Preliminaries

15.2.1 The polymodal logic of provability

For any ordinal number Λ we consider a language $\mathcal{L}_\Lambda$ consisting of a countable set of propositional variables $\mathbb{P}$ together with the constants $\top$, $\bot$; Boolean connectives $\wedge$, $\vee$, $\neg$, $\to$; and a modality $[\xi]$ for each ordinal $\xi < \Lambda$. As usual, we write $\langle\xi\rangle$ as a shorthand for $\neg[\xi]\neg$.

Definition 15.2.1. The logic GLP_Λ is then defined to be the least logic containing all propositional tautologies and the following axiom schemata:

(i) $[\xi](\varphi \to \psi) \to ([\xi]\varphi \to [\xi]\psi)$ for all $\xi < \Lambda$,

(ii) $[\xi]([\xi]\varphi \to \varphi) \to [\xi]\varphi$ for all $\xi < \Lambda$,

(iii) $[\xi]\varphi \to [\zeta]\varphi$ for all $\xi < \zeta < \Lambda$,

(iv) $\langle\xi\rangle\varphi \to [\zeta]\langle\xi\rangle\varphi$ for all $\xi < \zeta < \Lambda$,

and closed under the rules modus ponens and necessitation for each $[\xi]$:

$$\frac{\varphi \to \psi \qquad \varphi}{\psi}\ \text{MP} \qquad\qquad \frac{\varphi}{[\xi]\varphi}\ \text{nec}$$

We will often write simply GLP for GLP_Λ when we do not want to specify a Λ. Note that GLP when restricted to any one modality is simply the well-known logic GL. Modal logics are usually studied by means of relational semantics. A *Kripke Λ-frame* is a structure $(W, \{R_\xi\})_{\xi<\Lambda}$, where each R_ξ is a binary relation on W. We define a *valuation* $[\![\cdot]\!]$ to be a function assigning subsets of W to each $\mathcal{L}_\Lambda$-formula such that $[\![\cdot]\!]$ respects boolean connectives and such that

$$[\![\langle\xi\rangle\varphi]\!] = R_\xi^{-1}[\![\varphi]\!].$$

Proposition 15.2.2 (Segerberg [Seg71])**.** GLP_1 *is complete with respect to the class of finite relational structures* (W, R) *that are conversely wellfounded trees.*

The preceding proposition provides a convenient way to study GLP_1. However, as is well known, GLP_Λ is incomplete with respect to any class of relational structures whenever $1 < \Lambda$. This motivates the search for topological models of the GLP.

Recall that x is a *limit point* of A if A intersects every punctured neighborhood of x. We call the set of limit points of A the *derived set* of A and denote it by dA. We may also denote it by $d_\tau A$ to emphasize the topology we are considering. The derived set operator is iterated transfinitely by setting

1. $d^0 A = A$,
2. $d^{\alpha+1} A = dd^\alpha A$, and
3. $d^\gamma A = \bigcap_{\alpha<\gamma} d^\alpha A$ for limit ordinals.

Since $d^\alpha X \supset d^\beta X$ whenever $\alpha < \beta$, there exists a minimal ordinal $ht(X)$—the *height* or *rank* of X—such that $d^{ht(X)}X = d^{ht(X)+1}X$. For any $x \in X$, we let $\rho_\tau x$, the *rank* of x, be the least ordinal ξ such that $x \notin d^{\xi+1}X$, if it exists.

Throughout this paper, we will speak about *rank-preserving* extensions of topologies.

Lemma 15.2.3 (Beklemishev-Gabelaia [BG13])**.** *A topology* σ *is a rank-preserving extension of a scattered topology* τ *if, and only if,* $\rho_\tau[U]$ *is an ordinal for each* $U \in \sigma$*.*

A point in A that is not a limit point is *isolated*. Thus a point is isolated if and only if it has rank 0. We denote by $iso(A)$ the set of isolated points in A. A topological space is *scattered* if $iso(A) \neq \varnothing$ for each $A \subset X$ (alternatively, if $d^{ht(X)}X = \varnothing$). Not all scattered spaces are T_1 (e.g., $X = \{0, 1\}$ with open sets $\varnothing$, $\{0\}$ and X), however, the examples in which we will focus are.

We study *polytopological spaces*—structures $(X, \{\mathcal{T}_\iota\}_{\iota<\Lambda})$, where X is a set and $\{\mathcal{T}_\iota\}_{\iota<\Lambda}$ is a sequence of topologies of length Λ. Topological semantics for modal logics may be defined by interpreting diamonds as topological derivatives.

Definition 15.2.4 (Topological semantics)**.** Let $\mathfrak{X} = (X, \{\mathcal{T}_\iota\}_{\iota<\Lambda})$ be an polytopological space. A *valuation* is a function $[\![\cdot]\!] : \mathcal{L}_\Lambda \to \mathcal{P}(X)$ such that for any $\mathcal{L}_\Lambda$-formulae φ, ψ:

(i) $[\![\bot]\!] = \varnothing$;

(ii) $[\![\neg\varphi]\!] = X \backslash [\![\varphi]\!]$;

(iii) $[\![\varphi \wedge \psi]\!] = [\![\varphi]\!] \cap [\![\psi]\!]$;

(iv) $[\![\langle\xi\rangle\varphi]\!] = d_{\mathcal{T}_\xi}[\![\varphi]\!]$.

A *model* $\mathfrak{M} = (\mathfrak{X}, [\![\cdot]\!])$ is a polytopological space together with a valuation. We say that φ is *satisfied* in $\mathfrak{M}$ if $[\![\varphi]\!]$ is nonempty and we say φ is *valid* in a space $\mathfrak{X}$ and write $\mathfrak{X} \models \varphi$ if $[\![\varphi]\!] = X$ for any model based on $\mathfrak{X}$.

In order that a space validate the axioms of GLP, we need to impose some regularity conditions (see Beklemishev-Bezhanishvili-Icard [BBI10]). A space $(X, \{\mathcal{T}_\iota\}_{\iota<\Lambda})$ is a GLP_Λ-*space* if $\{\mathcal{T}_\iota\}_{\iota<\Lambda}$ is non-decreasing, scattered, and

$$d_\xi A \in \mathcal{T}_\zeta \text{ for all } \xi < \zeta \text{ and all } A \in \mathcal{P}(X) \tag{15.1}$$

In the situation above, we refer to $\{\mathcal{T}_\iota\}_{\iota<\Lambda}$ as a GLP_Λ-*polytopology*. Clearly, we have:

Lemma 15.2.5. *Any* GLP_Λ*-space validates all theorems of* GLP_Λ.

A natural way of constructing GLP_Λ-polytopologies appears to be to start with any scattered topology and simply add all derived sets at each stage, thus making $\mathcal{A} = \wp(X)$. This results in what has come to be known as the *canonical* GLP*-space generated by* X. In doing so, the topologies quickly become extremely fine. In fact, for the most natural examples, their non-discreteness becomes undecidable within ZFC after two or three iterations.

One way out of this, explored in Beklemishev-Gabelaia[BG13], is to extend the topology at each stage before adding derived sets. Extending the topology reduces the amount of derived sets attainable and makes subsequent topologies coarser. A different approach, introduced in Fernández-Duque [FD14], is to fix increasing topologies from the beginning and restrict the algebra of possible valuations. We will consider the first approach in our completeness proofs.

We will make use of a reduction from GLP_Λ to GLP_ω in the following way: for each $\mathcal{L}_\Lambda$-formula φ where modalities $[\lambda_0], ..., [\lambda_{n-1}]$ appear, define its *condensation* to be the $\mathcal{L}_n$-formula φ^c that results by uniformly substituting $[k]$ for $[\lambda_k]$. The following reduction lemma holds:

Lemma 15.2.6 (see, e.g., Fernández-Duque [FD14])**.** *Let* φ *be a* $\mathcal{L}_\Lambda$*-formula where* n *modalities appear. Then* $\mathsf{GLP}_n \vdash \varphi^c$ *implies* $\mathsf{GLP}_\Lambda \vdash \varphi$.

Let us finish this section with the remark that GLP is not strongly complete with respect to its canonical semantics. By *strong completeness* (with respect to a class of models $\mathcal{X}$), we mean the following assertion: whenever Γ is a set of $\mathcal{L}_\Lambda$-sentences consistent with GLP_Λ, then there is some model $\mathfrak{X} \in \mathcal{X}$ where Γ is satisfied.

Proposition 15.2.7. *Suppose X is a scattered T_1 space in which every G_δ set is open. Then* GL *is not strongly complete with respect to X.*

Proof. This is a generalization of the usual proof that GL is not strongly complete with respect to trees. Let

$$\Gamma = \{\Diamond p_0\} \cup \{\Box(p_i \to \Diamond p_{i+1}) : i < \omega\}.$$

Suppose Γ is satisfied at some x. Then, for each i, there is a punctured neighborhood U_i of x in which any point satisfying p_i is a limit of points satisfying p_{i+1}. Since X is T_1, each U_i is open. It follows that $U = \bigcap_{i<\omega} U_i$ is G_δ and thus open. Since x satisfies $\Diamond p_0$, U contains some x_0 of rank α_0 satisfying p_0. Inductively, for each $i < \omega$, there is some $x_i \in U$ satisfying p_i and, by

$$\Box(p_i \to \Diamond p_{i+1}),$$

U contains some x_{i+1} of rank $\alpha_{i+1} < \alpha_i$ satisfying p_{i+1}. This gives an infinite decreasing sequence of ordinals. □

Recall that if κ is an ordinal of uncountable cofinality, then the intersection of countably many sets which are closed and cofinal in κ is also closed and cofinal in κ. Hence, Proposition 15.2.7 applies to the closed-unbounded topology from Blass [Bla90]. More generally:

Corollary 15.2.8. GL *is not strongly complete with respect to topologies on ordinals given by countably complete filters, such as the closed-unbounded topology.*

The spaces we will consider will instead be built around the *Generalized Icard topologies*.

Definition 15.2.9 (Generalized Icard Topologies)**.** Let (X, τ) be a scattered space of rank Θ. We define a topology $\tau_{\uparrow 1}$ generated by τ and all sets of the form

$$(\alpha, \beta)^\tau := \{x \in X : \alpha < \rho_\tau x < \beta\},$$

for ordinals $\alpha < \beta \leq \Theta + 1$. We iterate this construction by setting

- $\tau_{\uparrow(\iota+1)} = (\tau_{\uparrow\iota})_{\uparrow 1}$, and

- $\tau_{\uparrow\lambda} = \bigcup_{\xi<\lambda} \tau_{\uparrow\xi}$ at limit stages.

These are called the *generalized Icard topologies*.

These topologies were defined differently in [AFD17]. By Lemma 15.2.21.1 below, both definitions coincide. Another equivalent formulation is as follows: $\tau_{\uparrow 1}$ is generated by τ and the family

$$\{d^{\xi}X : \xi \in \mathsf{Ord}\}. \tag{15.2}$$

15.2.2 Arithmetic, I

Definition 15.2.10. We fix some notation related to ordinal arithmetic.

1. Whenever $\alpha < \beta$, we denote by $-\alpha+\beta$ the unique ordinal γ such that $\alpha + \gamma = \beta$.

2. Whenever A is a set of ordinals, we denote by $\alpha+A$ the set $\{\alpha+\beta : \beta \in A\}$. Expressions such as $-\alpha + A$ are defined analogously, if they make sense.

3. For all nonzero ξ, there exist ordinals α and β such that $\xi = \alpha + \omega^{\beta}$. Such a β is unique. We denote it by $\ell\beta$ and call it the *end-logarithm* of ξ.

4. For all nonzero ξ, there exists a unique ordinal η such that ξ can be written as $\omega^{\eta} + \gamma$, with $\gamma < \xi$. We denote this ordinal by $L\xi$ and call it the *initial logarithm* of ξ.

The operations ℓ, L, and $\pounds$ should be regarded as functions on (a sufficiently large subset of) Ord. Nonetheless, in its use and in general whenever we deem it convenient, we will omit the symbol '$\circ$' for function composition, as well as perhaps parentheses.

Our completeness proof will heavily rely upon an analysis of generalized Icard topologies and their structure induced by the arithmetical properties of ordinals. Hence, developing a thorough intuition about them will be crucial. A most useful remark in this direction is the fact that they are to arbitrary topological spaces as the usual order topology is to ordinal numbers. Indeed, define the *initial segment topology* $\mathcal{I}_0$ on an ordinal Θ (or on Ord) to be generated by all initial segments $[0, \alpha]$, for $\alpha < \Theta$. Then $(\Theta, \mathcal{I}_0)$ is a scattered space: a rather trivial scattered space—it carries no further information than the usual ordering on Ord. For instance, we have $\rho_{\mathcal{I}_0}\alpha = \alpha$ for all α and $ht(\Theta, \mathcal{I}_0) = \Theta$.

Lemma 15.2.11. *$\mathcal{I}_1 := \mathcal{I}_{0\uparrow 1}$ is the order topology. We have $\rho_{\mathcal{I}_1}\alpha = \ell\alpha$ for all α, so in particular isolated points are exactly the successor ordinals. Moreover, $ht(\Theta, \mathcal{I}_1) = L(\Theta) + 1$.*

Proof. It is not hard to see that $\mathcal{I}_1$ is the order topology, and that the rank function is ℓ is established by a simple induction. Finally, let $\mathbb{H}$ be the class of additively indecomposable ordinals. It follows that

$$ht(\Theta, \mathcal{I}_1) = \sup_{\xi<\Theta}(\ell\xi + 1) = \sup_{\xi<\Theta\cap\mathbb{H}}(\ell\xi + 1) = \sup_{\xi<\Theta\cap\mathbb{H}}(L\xi + 1) = L(\Theta) + 1,$$

as claimed. □

In what follows, we write simply $\mathcal{I}_\lambda$ instead of $\mathcal{I}_{0\uparrow\lambda}$. These topologies are important because, as we will see, the completeness theorem can quickly be reduced to the case when the underlying space is an ordinal equipped with a topology of the form $\mathcal{I}_\lambda$.

15.2.3 d-maps and J-maps

There is an appropriate notion of structure-preserving mappings between scattered spaces. We say that a function between topological spaces is *pointwise discrete* if the preimage of any singleton is a discrete subspace.

Definition 15.2.12 (d-map)**.** Let X and Y be scattered spaces. A function $f : X \to Y$ is a *d-map* if it is continuous, open, and pointwise discrete.

Clearly, any homeomorphism is a d-map. In particular, ordinal addition and substraction, i.e., functions of the form

$$(-\xi + \cdot)\colon ([\xi, \xi + \Theta], \mathcal{I}_\zeta) \to ([0, \Theta], \mathcal{I}_\zeta),$$

are d-maps. The rank function

$$\rho_\tau\colon (X, \tau) \to ([0, \rho_\tau X], \mathcal{I}_0)$$

is also a d-map. A more interesting example is given by end-logarithms of the form:

$$\ell\colon (\Theta, \mathcal{I}_{1+\zeta}) \to (\Theta, \mathcal{I}_\zeta).$$

A proof of this, and the more general Lemma 15.2.22 below can be found in Fernández-Duque [FD14].

Since the composition of d-maps is a d-map, they can be thought of as morphisms in the category of scattered spaces. We will now state various properties of d-maps.

Lemma 15.2.13. *Let $f : X \to Y$ be a d-map.*

1. *If Y is an ordinal Θ with the initial segment topology, then f is the rank function on X.*
2. *For any $A \subset Y$, $f^{-1}dA = df^{-1}A$.*
3. *$f : (X, \tau_{\uparrow\lambda}) \to (Y, \sigma_{\uparrow\lambda})$ is a d-map for any λ.*
4. *If f is surjective, then for any $\mathcal{L}_\Lambda$-formula φ, $X \models \varphi$ implies $Y \models \varphi$.*

Proof. Items 1 and 2 appear in Beklemishev-Gabelaia [BG13]; item 4 appears in Bezhanishvili-Mines-Morandi [BMM03] in the current formulation. Item 3 is proved in [AFD17], but therein a different definition of $\tau_{\uparrow\lambda}$ is used, and we still have not shown that they are equivalent. Nonetheless, the claim can be proved by an easy induction.

Suppose $f : (X, \tau_{\uparrow\lambda}) \to (Y, \sigma_{\uparrow\lambda})$ is a d-map. Note that 2 implies that d-maps are rank-preserving, i.e.,

$$\rho_{\tau_{\uparrow\lambda}}(x) = \rho_{\sigma_{\uparrow\lambda}}(f(x)) \text{ for every } x \in X.$$

It follows that for any $\tau_{\uparrow\lambda}$-open A,

$$f(A \cap (\alpha, \beta)^{\tau_{\uparrow\lambda}}) = f(A) \cap (\alpha, \beta)^{\sigma_{\uparrow\lambda}} \in \sigma_{\uparrow\lambda+1};$$

and for any $\sigma_{\uparrow\lambda}$-open B,

$$f^{-1}(B \cap (\alpha, \beta)^{\sigma_{\uparrow\lambda}}) = f^{-1}(B) \cap (\alpha, \beta)^{\tau_{\uparrow\lambda}} \in \tau_{\uparrow\lambda+1};$$

so that f is $(\tau_{\uparrow\lambda+1}, \sigma_{\uparrow\lambda+1})$-continuous and open. Clearly it is also pointwise discrete. The case for limit λ follows from Fernández-Duque [FD14, Lemma 5.8]. □

As mentioned in the proof of 15.2.13.3, Lemma 15.2.13.2 implies that d-maps are rank-preserving. Also, it follows from 15.2.13.3 that if the rank of (X, τ) is Θ, then

$$\rho_\tau : (X, \tau_{\uparrow\lambda}) \to (\Theta, \mathcal{I}_\lambda)$$

is a d-map. The main feature of d-maps is as follows:

Lemma 15.2.14. GLP_1 *is complete with respect to a scattered space (X, τ) if, and only if, for any finite, converse-wellfounded tree T, there exists a τ-open subspace S of X and a d-map $f\colon (S, \tau) \to T$.*

Proof. That completeness follows from the existence of d-maps is independently due to Abashidze [Aba85] and Blass [Bla90]. Note that it immediately follows from Proposition 15.2.2 and Lemma 15.2.13.4.

The converse is probably folklore and will not be needed below, but we prove it anyway. Suppose GLP_1 is complete with respect to (X, τ), where τ is scattered. Let $(T, <)$ be a finite, converse wellfounded tree. We define from T a modal formula φ consistent with GLP_1. Let $\{p_t : t \in T\}$ be a set of distinct propositional variables and r be the root of T. Set

$$\begin{aligned}\varphi = p_r \wedge \bigwedge_{s\in T;\, s\neq r} \neg p_s \wedge \left(\bigwedge_{t\in T; t\neq r} \Diamond p_t\right) \wedge \Box\left(\bigvee_{t\in T} p_t\right) \wedge \Box(\neg p_r) \\ \wedge \left(\bigwedge_{s,t\in T;\, s\neq t} \Box(p_s \to \neg p_t)\right) \wedge \left(\bigwedge_{s<t} \Box(p_s \to \Diamond p_t)\right) \\ \wedge \left(\bigwedge_{s\not< t} \Box(p_s \to \neg\Diamond p_t)\right) \wedge \Box \bigwedge_{t\in T} \left(p_t \to \Box \bigvee_{t<s} p_s\right).\end{aligned}$$

Clearly, there is a Kripke model based on T where φ is true in r; namely, any one where each p_t holds only in t. Hence, φ is consistent with GLP_1, whereby it is satisfiable in X. Fix a valuation over X and a point $x_r \in X$ such that $x_r \models \varphi$. Thus, x_r satisfies p_r and $\bigwedge_{s,t\in T;\, s\neq r} \neg p_s$ and x_r is a limit point of points satisfying each of p_t, for $t \neq r$. Moreover, by each of the conjuncts above:

1. there is a punctured neighborhood of x_r where each point satisfies p_t for some $t \in T$;

2. there is a punctured neighborhood of x_r where no point satisfies p_r;

3. for each pair of distinct $s, t \in T$, there is a punctured neighborhood of x_r of points satisfying at most one of p_s and p_t;

4. for each pair of distinct $s, t \in T$ with $s < t$, there is a punctured neighborhood of x_r where all points satisfying p_s are limits of points satisfying p_t;

5. for each pair of distinct $s, t \in T$ with $s \not< t$, there is a punctured neighborhood of x_r where all points satisfying p_s are not limits of points satisfying p_t; and

6. there is a punctured neighborhood of x_r where whenever a point x satisfies p_t, then there is a punctured neighborhood of x where each points satisfies one of p_s, with $t < s$.

Let S be the intersection of all those finitely many open neighborhoods of x_r. Clearly, $x \models p_t$ and $t \neq s$ together imply $x \not\models p_s$. We define $f\colon S \to T$ by

$$f(x) = t \text{ if, and only if, } x \models p_t.$$

We claim f is a d-map. Let A_t be an open subset of T of the form

$$\{s \in T\colon t \leq s\},$$

so that

$$f^{-1}(A_t) = \{x \in S\colon x \models p_s \wedge t \leq s\}.$$

This clearly equals S if $t = r$. Otherwise, for each $x \in S$ with $x \models p_s$ and $t \leq s$, there is an open neighborhood U of x where each point satisfies p_u for some $u > s$. But $t < u$, whence $U \subset f^{-1}(A_t)$. This implies that $f^{-1}(A_t)$ is open, and so f is continuous.

Conversely, suppose $U \subset S$ is open, $x \in U$ is such that $x \models p_t$, and $t < s$. Then x is a limit of points satisfying p_s, so that

$$\{y \in S\colon y \models p_s\} \cap U \neq \varnothing,$$

whence $s \in f(U)$. Hence, f is open. Finally if $t \in T$, then $f^{-1}(t)$ is discrete, for t is the image of points satisfying p_t and no point in S can satisfy $p_s \wedge \Diamond p_s$ for any s. Therefore, f is a d-map. □

Hence, the need to check whether a given space $\mathfrak{X}$ satisfies a formula is replaced by the definition of a suitable mapping between $\mathfrak{X}$ and some other space which is known to do so. Moreover, by Lemma 15.2.13.2, this result can be relativized to ambiances (condition (15.1) is preserved). In practice, polymodal analogs of Lemma 15.2.14 do not even require us to use full d-maps, but rather a weaker form of embeddings, as realized by Beklemishev [Bek10].

Definition 15.2.15 (J-frame)**.** A finite polymodal Kripke frame

$$\mathfrak{F} = (W, \{<_n\}_{n<\omega})$$

is called a *J-frame* if each relation is transitive and conversely wellfounded and it satisfies the following two conditions:

(I) For all $x, y \in W$ and all $m < n$: $x <_n y$ implies that for all $z \in W$: $x <_m z$ if, and only if $y <_m z$.

(J) For all $x, y, z \in W$ and all $m < n$: if $x <_m y$ and $y <_n z$, then $x <_m z$.

We call a J-frame a *J_n-frame* if all binary relations past the nth one are empty.

Let $(T, <_0, \ldots, <_N)$ be a frame. Denote by E_n the reflexive, symmetric, and transitive closure of $\bigcup_{n \leq k < \omega} <_k$. The equivalence classes under E_n are called *n-planes*. A natural order is defined on the set of $(n+1)$-planes:

$$\alpha \prec \beta \text{ if, and only if, } x <_n y \text{ for some } x \in \alpha, y \in \beta$$

We say that a J-frame is a *J-tree* if for all n, the $(n+1)$-planes contained in each n-plane form a tree under $<_n$ and if whenever $\alpha < \beta$ for two $(n+1)$-planes α, β, we have $x <_n y$ for all $x \in \alpha$ and $y \in \beta$. This means that each J_n-tree can be thought of as a tree each of whose nodes are themselves J_{n-1}-trees. Below, a node $t \in T$ is a *hereditary k-root* if for no $j \geq k$ and no $s \in T$ do we have $s <_j t$. We also write $x \ll_k y$ if $x <_j y$ for some $j \geq k$ and

$$\ll_k(x) = \{x \in T : x \ll_k y\}.$$

Definition 15.2.16 (J-map)**.** Let $(T, \sigma_0, ..., \sigma_n)$ be a J_n-tree and $(X, \tau_0, ..., \tau_n)$ be a space with $n+1$ topologies. We say that a function $f : X \to T$ is a *J_n-map* if

(j_1) $f : (X, \tau_n) \to (T, \sigma_n)$ is a d-map;

(j_2) $f : (X, \tau_k) \to (T, \sigma_k)$ is open for each k;

(j_3) $f^{-1}(\ll_k(x)), f^{-1}(\{x\} \cup \ll_k(x)) \in \tau_k$ for each $k < n$ and each hereditary $(k+1)$-root x;

(j_4) $f^{-1}x$ is a τ_k-discrete subspace for each $k < n$ and each hereditary $(k+1)$-root x.

Lemma 15.2.17 (Beklemishev [Bek10])**.** *If $f : Y \to Z$ is a J_n-map and $g : X \to Y$ is a d-map, then $f \circ g : X \to Z$ is a J_n-map.*

Lemma 15.2.18 (Beklemishev-Gabelaia [BG13])**.** *For each $\mathcal{L}_n$-formula φ, there exists a J_n-tree T such that if $\mathfrak{X}$ is a GLP_n-space and $f : \mathfrak{X} \to T$ is a surjective J_n-map, then $\mathfrak{X} \models \varphi$.*

We call the tree obtained in Lemma 15.2.18 the *canonical tree for φ*.

15.2.4 Arithmetic, II

We will need the definition of hyperlogarithms and hyperexponentials, due to Fernández-Duque and Joosten [FDJ13a]:

Definition 15.2.19.

1. The *hyperlogarithms* $\{\ell^\xi\}_{\xi\in\mathsf{Ord}}$ are the unique family of pointwise maximal initial[1] functions that satisfy:
   (a) $\ell^1 = \ell$, and
   (b) $\ell^{\alpha+\beta} = \ell^\beta \ell^\alpha$.
2. Let the function e be defined by $\xi \mapsto -1 + \omega^\xi$. The *hyperexponentials* $\{e^\zeta\}_{\zeta\in\mathsf{Ord}}$ are the unique pointwise minimal family of normal functions that satisfy
   (a) $e^1 = e$, and
   (b) $e^{\alpha+\beta} = e^\alpha e^\beta$ for all α and β.

One can verify by induction that the sequence $\{\ell^\xi\gamma\colon \xi \in \mathsf{Ord}\}$ is non-increasing for any ordinal γ. If we set e^0 to be the identity function and $e^\xi 0 = 0$ for all ξ, then one can also describe hyperexponentials recursively by condition 15.2.19.2b, together with the following normality clause:

$$\text{for any } \xi \text{ and any limit } \lambda : e^\xi\lambda = \lim_{\eta\to\lambda} e^\xi\eta; \tag{15.3}$$

and the following fixed-point clause:

$$\text{for any } \xi \text{ and any limit } \lambda : e^\lambda(\xi+1) = \lim_{\eta\to\lambda} e^\eta(e^\lambda\xi + 1). \tag{15.4}$$

The hyperexponential family refines the Veblen hierarchy. We mention some more properties of hyperlogarithms and –exponentials.

Lemma 15.2.20 (see Fernández-Duque and Joosten [FDJ13a] and Fernández-Duque [FD14])**.**

1. *If ξ and δ are nonzero, then $\ell^\xi(\gamma+\delta) = \ell^\xi\delta$; if $\gamma < \delta$ as well, then $\ell^\xi(-\gamma+\delta) = \ell^\xi\delta$. Moreover, if $1 < \xi$, then $\ell^\xi(\gamma\delta) = \ell^\xi\delta$;*
2. *If $\xi < \zeta$, then $\ell^\xi e^\zeta = e^{-\xi+\zeta}$ and $\ell^\zeta e^\xi = \ell^{-\xi+\zeta}$. Furthermore, if $\alpha < e^\xi\beta$, then $\ell^\xi\alpha < \beta$.*

[1] That is, sending initial segments of Ord onto initial segments of Ord.

Sketch of 1. That $\ell^\xi(\gamma+\delta) = \ell^\xi\delta$ is proved by induction on ξ using 15.2.19.1b. From this follows that if $\gamma < \delta$, then

$$\ell^\xi(\delta) = \ell^\xi(\gamma + (-\gamma + \delta)) = \ell^\xi(-\gamma + \delta).$$

Finally, it is proved by induction that $\ell(\gamma\delta) = L\gamma + \ell\delta$, so that if $1 < \xi$, then

$$\ell^\xi(\gamma\delta) = \ell^{-1+\xi}\ell(\gamma\delta) = \ell^{-1+\xi}(L\gamma + \ell\delta) = \ell^{-1+\xi}\ell\delta = \ell^\xi\delta,$$

as desired. □

We now give an alternative characterization of topologies $\tau_{\uparrow\lambda}$ and their rank functions:

Lemma 15.2.21. *Let (X, τ) be a scattered space of rank Θ.*

1. *The topologies $\tau_{\uparrow\lambda}$ are computed as follows:*
   - *$\tau_{\uparrow 0}$ is equal to τ*
   - *$\tau_{\uparrow\lambda}$ generated by τ and all sets of the form*
   $$(\alpha, \beta]^\tau_\xi := \{x \in X : \alpha < \ell^\xi\rho_\tau x \leq \beta\},$$
   for some $-1 \leq \alpha < \beta \leq \Theta$ and some $\xi < \lambda$.
2. *If (X, τ) is a scattered space, then $\rho_{\tau_{\uparrow\lambda}} = \ell^\lambda \circ \rho_\tau$. In particular, the rank function of $\mathcal{I}_\lambda$ is ℓ^λ.*

Sets of the form $[\alpha, \beta]^\tau_\xi$, $[\alpha, \beta)^\tau_\xi$, and $(\alpha, \beta)^\tau_\xi$ are defined in the obvious way. In particular, note that $(\alpha, \beta)^\tau_0 = (\alpha, \beta)^\tau$.

Proof. The second claim follows from Lemma 15.2.13.1 and Lemma 15.2.13.3. We use this to prove the first claim by induction. Suppose $\tau_{\uparrow\lambda}$ is generated by τ and all sets of the form

$$(\alpha, \beta]^\tau_\xi := \{x \in X : \alpha < \ell^\xi\rho_\tau x \leq \beta\},$$

for $\xi < \lambda$. By definition, $\tau_{\uparrow(\lambda+1)} = (\tau_{\uparrow\lambda})_{\uparrow 1}$ is generated by $\tau_{\uparrow\lambda}$ and all sets of the form

$$(\alpha, \beta)^{\tau_{\uparrow\lambda}} := \{x \in X : \alpha < \rho_{\tau_{\uparrow\lambda}} x < \beta\},$$

but $\rho_{\tau_{\uparrow\lambda}} = \ell^\lambda \circ \rho_\tau$ by induction hypothesis. So $(\tau_{\uparrow\lambda})_{\uparrow 1}$ is generated by all sets of the form

$$(\alpha, \beta]^\tau_\xi := \{x \in X : \alpha < \ell^\xi\rho_\tau x \leq \beta\},$$

for $\xi < \lambda + 1$. The limit case is immediate. □

The following lemma provides the key relationship between arithmetic and topology for ordinals:

Lemma 15.2.22 (Fernández-Duque [FD14])**.** *Hyperlogarithms $\ell^\xi\colon (\Theta, \mathcal{I}_{\xi+\zeta}) \to (\Theta, \mathcal{I}_\zeta)$ are d-maps.*

We will make use of the following two lemmata from [AFD17]:

Lemma 15.2.23. *Let (X,τ) be a scattered space and λ be an ordinal. Any x in $(X, \tau_{\uparrow\lambda})$ has a λ-neighborhood U such that whenever $x \neq y \in U$, $\ell^\lambda \rho_0 y < \ell^\lambda \rho_0 x$.*

Lemma 15.2.24. *Let $1 < \lambda$ be an additively indecomposable ordinal and $x \in X$ be such that $\rho_\tau x = e^\lambda\Theta > 0$. Then for any $\tau_{\uparrow\lambda}$-neighborhood V of x, there exist*

- *a set $U \in \tau$, and*
- *ordinals $\eta < e^\lambda\Theta$ and $\zeta < \lambda$,*

such that V contains the set $U \cap (\eta, e^\lambda\Theta]^\tau_\zeta$.

For ranks not of the form $e^\lambda\Theta$, we have a more general result, also from [AFD17]:

Lemma 15.2.25. *Let (X,τ) be a scattered space. Suppose $0 < \lambda$, $0 < \ell^\lambda\xi$, and $\rho_\tau x = \xi$. Then for any $\tau_{\uparrow\lambda}$-neighborhood V of x, there exist*

- *a set $U \in \tau$, and*
- *a finite partial function $r : \lambda \to \mathsf{Ord}$ such that letting*

$$B^X_r(x) = \bigcap_{\zeta \in dom(r)} (r(\zeta), \ell^\zeta \xi]^\tau_\zeta,$$

we have $U \cap B^X_r(x) \subset V$.

We conclude this section with a final observation on logarithms:

Lemma 15.2.26. *Suppose that λ is a limit ordinal, ζ is of the form $e^\lambda\zeta_0$, and $\ell^\lambda\xi < \zeta_0$. Let*

$$\eta = \min\{\beta : \ell^\beta \xi < \zeta\}.$$

Then η is a successor ordinal or zero.

Proof. This is proved by induction on ξ. Suppose towards a contradiction that ξ is least such that η is a limit; clearly $\zeta < \xi$. If η is additively decomposable, say

$$\eta = \eta_0 + \omega^\rho > \omega^\rho,$$

then

$$\ell^\eta \xi = \ell^{\omega^\rho} \ell^{\eta_0} \xi.$$

Now, we must have $\ell^{\eta_0}\xi < \xi$, for otherwise

$$\xi = \ell^{\eta_0}\xi \leq \ell^{\omega^\rho}\xi = \ell^{\omega^\rho}\ell^{\eta_0}\xi \leq \xi,$$

contradicting the fact $\ell^{\omega^\rho}\ell^{\eta_0}\xi < \zeta$; thus, $\ell^{\eta_0}\xi < \xi$. But then, $\zeta \leq \ell^{\eta_0}\xi$ and

$$\ell^\lambda \ell^{\eta_0}\xi = \ell^{\eta_0+\lambda}\xi \leq \ell^\lambda \xi < \zeta_0,$$

so by the induction hypothesis applied to $\ell^{\eta_0}\xi$, the least η' such that

$$\ell^{\eta'}\ell^{\eta_0}\xi < \zeta$$

is a successor ordinal. However, this ordinal is ω^ρ, which is a contradiction.

Thus η is additively indecomposable. Fernández-Duque and Joosten [FDJ13a] computed that, letting

$$\eta^* = \arg\min\{\ell^\nu \xi : 0 \leq \nu < \eta\},$$

i.e., letting η^* be the least ordinal ν which minimizes $\ell^\nu\xi$ in $[0, \eta)$, we have

1. if $0 < \eta^*$, then
$$\ell^\eta \xi = \ell^\eta \ell^{\eta^*}\xi;$$
2. if $0 = \eta^*$, then
$$\ell^\eta \xi = \sup_{\beta\in[0,\xi)} (\ell^\eta \beta + 1).$$

If $0 < \eta^*$, then $\ell^{\eta^*}\xi < \xi$ and one reaches a contradiction as above, using the induction hypothesis on $\ell^{\eta^*}\xi$; thus $0 = \eta^*$. Now, note that $\eta \leq \lambda$, since $\ell^\lambda \xi < \zeta_0 \leq \zeta$. In fact, we must have $\eta < \lambda$, by the displayed equation above. Since $e^\lambda \zeta_0$ is additively indecomposable, for all $\gamma < e^\lambda \zeta_0$, we have

$$\gamma < \gamma + e^\eta \gamma < e^\lambda \zeta_0 = \zeta < \xi.$$

It follows that

$$\begin{aligned}\ell^\eta\xi &= \sup_{\beta\in[0,\xi)}\big(\ell^\eta\beta+1\big)\\ &= \sup_{\beta\in[0,\zeta)}\big(\ell^\eta\beta+1\big)\\ &\geq \sup_{\beta\in[0,\zeta)}\big(\ell^\eta(\gamma+e^\eta\gamma)+1\big)\\ &\geq \sup_{\beta\in[0,\xi)}\big(\gamma+1\big)\\ &\geq \zeta,\end{aligned}$$

which is a contradiction. This proves the lemma. □

15.3 $\vec{\vartheta}$-polytopologies

In this section, we state our completeness theorem and prove it modulo the *product lemma*, which will be proved in the next section. Let us begin with some motivation by recalling the constructions from [BG13] and [FD14]. Let $\mathfrak{X} = (X, \tau)$ be a scattered space; by [AFD17], GL is complete with respect to each topology $\tau_{\uparrow\lambda}$, with $0 < \lambda$, provided $\mathfrak{X}$ is tall enough. Thus, one would attempt to prove completeness of GLP with respect to the polytopology

$$\{\tau_{\uparrow\lambda} : 0 < \lambda < \Lambda\}.$$

However, this is not a GLP-space and thus does not validate the axioms of GLP. The idea is then to replace each topology $\tau_{\uparrow\lambda}$ by a rank-preserving extension and prove completeness for that space. It is not known whether this is possible for arbitrary Λ. What we will do here is, given a formula ϕ consistent with GLP, say, with occurrences of modalities $\lambda_0, \ldots, \lambda_n$, and a (tall enough) scattered space $\mathfrak{X} = (X, \tau)$, we produce a sequence of topologies $\tau_0, \ldots, \tau_n$ such that

1. $(X, \tau_0, \ldots, \tau_n)$ satisfies ϕ, and
2. each τ_i is a rank-preserving extension of $\tau_{\uparrow 1+\lambda_i}$.

Definition 15.3.1 (ϑ-maximal topology)**.** Let ϑ be a nonzero ordinal and (X, τ) be a scattered topological space. We say that τ_* is a *ϑ-extension* of τ if

1. $\tau \subset \tau_*$,

2. $\rho_{\tau_*} = \rho_\tau$, and
3. the identity function $id : (X, \tau) \to (X, \tau_*)$ is continuous at all points x such that
$$\ell^\vartheta \rho_\tau(x) = 0.$$

We say that τ_* is an *ϑ-maximal topology* if there are no proper ϑ-extensions of τ_*.

In particular, when $\vartheta = 1$, ϑ-maximality coincides with the notion of ℓ-maximality from [BG13] . If $\vec{\vartheta} = \{\vartheta_i \colon 0 < i < n\}$ is a finite increasing sequence of ordinals, we write
$$\partial\vec{\vartheta} := \{\partial\vartheta_{i+1} \colon 0 < i < n\},$$
where $\partial\vartheta_{i+1} = -\vartheta_i + \vartheta_{i+1}$ for $0 < i$. For such a sequence $\vec{\vartheta}$, we also write $\partial\vartheta_1 = \vartheta_1$.

Definition 15.3.2. Let us call a polytopological space $\mathfrak{X} = (X, \tau_0, \ldots, \tau_n)$ a *$\vec{\vartheta}$-polytopology* over (X, τ) if $\vec{\vartheta} = \{\vartheta_i \colon 0 < i \leq n\}$ is increasing and

1. τ_0 is a ϑ_1-maximal extension of τ,
2. τ_{i+1} is a $\partial\vartheta_{i+2}$-maximal extension of $(\tau_i)_{\restriction\partial\vartheta_{i+1}}$, for $i + 1 < n$, and
3. $\tau_n = (\tau_{n-1})_{\restriction\partial\vartheta_n}$.

We remark the following consequence of the definition.

Lemma 15.3.3. *Let $\mathfrak{X} = (X, \tau_0, \ldots, \tau_n)$ be a $\vec{\vartheta}$-polytopology over (X, τ). Then, for each i,*
$$\rho_{\tau_i} = \rho_{\tau_{\restriction\vartheta_i}} = \ell^{\vartheta_i} \circ \rho_\tau.$$

Let us refer to the polytopologies considered in [BG13] and [FD14] as BG-*polytopologies*. We will not need that notion below, so we do not define them. $\vec{\vartheta}$-polytopologies are weak versions of BG-polytopologies. For example, suppose $\mathfrak{X}$ is a $\{\omega_1\}$-polytopology over the interval topology on an ordinal. Then τ_1 is a rank-preserving extension of $\mathcal{I}_{\omega_1}$ obtained just by adding sets that would be already included in the corresponding rank-preserving extension of $\mathcal{I}_{\omega_1}$ in any corresponding BG-space of length $\geq \omega_1$ over $\mathcal{I}_1$. We now state the completeness theorem we shall prove:

Theorem 15.3.4. *Let $\vec{\vartheta}$ be an increasing sequence of nonzero ordinals. Denote by* GLP $\restriction \vec{\vartheta}$ *the fragment of* GLP *whose only modalities appear in $\vec{\vartheta}$.*

1. *(Soundness) All theorems of* GLP $\restriction \vec{\vartheta}$ *hold in every $\vec{\vartheta}$-space.*

2. *(Completeness) Let $\Lambda \geq \vartheta_n$ be a limit ordinal and (X, τ) be any scattered space of height $\geq e^{\Lambda}1$. Suppose φ only contains modalities in $\vec{\vartheta}$ and is consistent with $\mathsf{GLP} \restriction \vec{\vartheta}$. Then, there is an open subset U of X such that φ is satisfied on a $\vec{\vartheta}$-polytopology over $(U, \tau_{\restriction 1})$.*

The result also holds also for successor ordinals by replacing $e^{\Lambda}1$ with $e^{1+\Lambda}\omega$. In fact, this general version is what we will prove; the smaller bound in the statement of the theorem follows from the fact that, for limit Λ,

$$\lim_{\lambda \to \Lambda} e^{\lambda}\omega = \lim_{\lambda \to \Lambda} e^{\lambda+1}1 = \lim_{\lambda \to \Lambda} e^{\lambda}1 \leq \lim_{\lambda \to \Lambda} e^{\Lambda}1 = e^{\Lambda}1$$

These bounds are sharp (this follows from Lemma 15.2.14).

Notice that in Theorem 15.3.4.2, we satisfy the consistent formula on a $\vec{\vartheta}$-polytopology over a subspace $(X, \tau_{\restriction 1})$. We cannot in general replace this with $(X, \tau_{\restriction 0})$—consider an ordinal with the initial-segment topology. It is still useful to consider polytopologies of this sort. We will call $\vec{\vartheta}$-polytopologies over the initial-segment topology *improper*.

In the remainder of this chapter, we prove Theorem 15.3.4. Soundness follows from Lemma 15.3.6 below, which in turn follows from Lemma 15.3.5. The proofs are the same as in the case $\vartheta = 1$ from [BG13]. They can also be found in [Agu15].

Lemma 15.3.5. *(X, τ) is a ϑ-maximal space if, and only if, for all $x \in X$ whose rank ρ is such that $\ell^{\vartheta}\rho > 0$ and all $V \in \tau$ with $V \subset [0, \rho)_0^{\tau}$, one of the following holds:*

1. *$V \cup \{x\} \in \tau$, or*
2. *$\rho_{\tau}(U \cap V) < \rho$ for some τ-neighborhood U of x.*

Lemma 15.3.6. *Suppose (X, τ) is ϑ-maximal and $\lambda \geq \vartheta$. Then $\{dA \colon A \subset X\} \subset \tau_{\restriction \lambda}$.*

It follows from Lemma 15.3.6 that all $\vec{\vartheta}$-polytopologies are GLP_n-spaces, which implies soundness.

Lemma 15.3.7 (pullback)**.** *Suppose $\mathfrak{Y} = (Y, \mathcal{S}_0, \ldots, \mathcal{S}_n)$ is a (possibly improper) $\vec{\vartheta}$-polytopology over (Y, σ) and $f \colon (X, \tau) \to (Y, \sigma)$ is a d-map. Then, there exists a $\vec{\vartheta}$-polytopology $\mathfrak{X} = (X, \mathcal{T}_0, \ldots, \mathcal{T}_n)$ over (X, τ) such that*

$$f \colon (X, \mathcal{T}_i) \to (Y, \mathcal{S}_i) \tag{15.5}$$

is a d-map for each $i \leq n$.

Proof. This is essentially the same proof as for the case $\vartheta = 1$ (see [BG13, Lemma 8.5]). The key point is the following claim:

Claim 15.3.8. *Suppose $\mathcal{T}$ and $\mathcal{S}$ are topologies such that $f\colon (X,\mathcal{T}) \to (Y,\mathcal{S})$ is a d-map and $\mathcal{S}'$ is a ϑ-maximal extension of $\mathcal{S}$, then the topology generated by $\mathcal{T}$ and the family*

$$f^{-1}\mathcal{S}' := \{f^{-1}S\colon S \in \mathcal{S}'\}$$

is a ϑ-extension of $\mathcal{T}$. Moreover, $f\colon (X,\mathcal{T}') \to (Y,\mathcal{S}')$ is a d-map for any ϑ-extension $\mathcal{T}'$ of this topology.

To see this suffices, suppose the claim holds. Then, letting $\mathcal{T}_0$ be any ϑ_1-maximal extension of the topology given by $\mathcal{T}$ and $f^{-1}\mathcal{S}_0$, we obtain that (15.5) holds for $i = 0$. Inductively, suppose (15.5) holds for some i. By Lemma 15.2.13.3,

$$f\colon (X,\mathcal{T}_{i\restriction\partial\vartheta_{i+1}}) \to (Y,\mathcal{S}_{i\restriction\partial\vartheta_{i+1}})$$

is a d-map. If $i+1 = n$, then we are done; otherwise, by definition, $\mathcal{S}_{i+1}$ is a $\partial\vartheta_{i+2}$-maximal extension of $\mathcal{S}_{i\restriction\partial\vartheta_{i+1}}$, whereby the claim yields that (15.5) holds for $i+1$ if we set $\mathcal{T}_{i+1}$ to be some $\partial\vartheta_{i+2}$-maximal extension of the topology given by $\mathcal{T}_{i\restriction\partial\vartheta_{i+1}}$ and $f^{-1}\mathcal{S}_{i+1}$. Hence, it suffices to prove the claim.

Proof of the claim. Let $\mathcal{R}$ be the topology given by $\mathcal{T}$ and $f^{-1}\mathcal{S}'$. Using the fact that

$$f\colon (X,\mathcal{T}) \to (Y,\mathcal{S}) \text{ is a d-map,} \tag{15.6}$$

it is not hard to see that $f\colon (X,\mathcal{R}) \to (Y,\mathcal{S}')$ is also a d-map. By definition, $\mathcal{T} \subset \mathcal{R}$, whence $\mathcal{R}$ is a rank-preserving extension of $\mathcal{T}$. Let $x \in X$ be such that $\ell^{\vartheta}\rho_{\mathcal{T}}x = 0$. We need to show that $id\colon (X,\mathcal{T}) \to (X,\mathcal{R})$ is continuous at x. This follows from the fact that $\mathcal{S}'$ is a ϑ-extension of $\mathcal{S}$: for any $\mathcal{R}$-neighborhood of x of the form $f^{-1}V$, we have that V is a $\mathcal{S}'$-neighborhood of $f(x)$ and $\ell^{\vartheta}\rho_{\mathcal{S}'}f(x) = 0$, so that $f(x) \in U$ for some $U \in \mathcal{S}$ with $U \subset V$. Therefore, $x \in f^{-1}U \subset f^{-1}V$ and $f^{-1}U \in \mathcal{T}$, by (15.6).

Now let $\mathcal{T}'$ be any ϑ-extension of $\mathcal{R}$. Clearly, $f\colon (X,\mathcal{T}') \to (Y,\mathcal{S}')$ is continuous and pointwise discrete. Suppose towards a contradiction that $x \in X$ and $W \in \mathcal{T}'$ witness a failure of f being open. Let

$$\rho := \rho_{\mathcal{T}}x = \rho_{\mathcal{T}'}x = \rho_{\mathcal{S}'}f(x).$$

Note that we must have $0 < \ell^{\vartheta}\rho$. Without loss of generality, we may assume ρ is the least possible rank of a counterexample and W contains no other

point of rank $\geq \rho$, so that $W = W_0 \cup \{x\}$, for some $W_0 \in \mathcal{T}$ with $W_0 \subset [0,\rho)_0^{\mathcal{T}}$. We will arrive at a contradiction using Lemma 15.3.5: since f is rank-preserving, we have that $\ell^{\vartheta}\rho > 0$, $f(W_0) \in \mathcal{S}'$, and $f(W_0) \subset [0,\rho)_0^{\mathcal{S}'}$. Hence, by Lemma 15.3.5, one of the following holds:

1. $f(W_0) \cup \{f(x)\} \in \mathcal{S}'$, or
2. $\rho_{\mathcal{S}'}(U \cap f(W_0)) < \rho$ for some $\mathcal{S}'$-neighborhood U of $f(x)$.

It must be the second one that holds, for $f(W_0) \cup \{f(x)\} = f(W)$ is not $\mathcal{S}'$-open by hypothesis. Observe that $f^{-1}U \cap W$ is a $\mathcal{T}'$-neighborhood of x and thus contains points with rank of every ordinal up to, and including, ρ. Because W contains only one point of rank ρ and f is rank-preserving,

$$(f^{-1}U \cap W) \setminus f^{-1}(x) = f^{-1}U \cap W_0.$$

It follows that the set

$$f^{-1}U \cap W_0$$

contains points with rank of every ordinal up to, but not including, ρ. However, this is impossible by 2 above, because ρ is a limit ordinal. This finishes the proof of the claim and the lemma. □

Lemma 15.3.7 is still true in the degenerate case $\sigma = \mathcal{I}_0$. In this case, notice that $\mathcal{I}_0$ is already ϑ-maximal for every ϑ, for there is only one point of each rank. The proof of Theorem 15.3.9 below is postponed to the next section.

Theorem 15.3.9 (Product Lemma)**.** *Assume ς is a nonzero additively indecomposable ordinal, $([1,\kappa], \mathcal{T}_0, \ldots, \mathcal{T}_n)$ is a $\vec{\vartheta}$-polytopology over $\mathcal{I}_\varsigma$, and $([1,\lambda], \mathcal{S}_0, \ldots, \mathcal{S}_n)$ is a $\vec{\vartheta}$-polytopology over $\mathcal{I}_0$. Suppose moreover that*

$$\kappa < e^{\varsigma+\vartheta_n}\omega$$

and that

$$\lambda < e^{\vartheta_n}\omega.$$

Fix $\{\kappa_0, \ldots, \kappa_m\}$ a finite subset of $[1,\kappa]$. Let

$$\xi = \text{ the unique ordinal equal to } \ell^{\varsigma}[1,\kappa_m].$$

Let $\Theta = e^{\varsigma}(\xi + (-1+\lambda)) < e^{\varsigma+\vartheta_n}\omega$ and define $X_\uparrow = [1,\Theta] \cap [\xi,\infty)_\varsigma$ and $X_\downarrow = [1,\Theta] \cap [0,\xi)_\varsigma$. Then, there exist:

1. *A $\vec{\vartheta}$-polytopology $([1,\Theta], \mathcal{R}_0, \ldots, \mathcal{R}_n)$ over $\mathcal{I}_\varsigma$.*

2. *Functions* $\pi_0 \colon [1,\Theta] \to [1,\kappa]$ *and* $\pi_1 \colon [1,\Theta] \to [1,\lambda]$ *such that:*

   - $\pi_0 \colon (X_\downarrow, \mathcal{R}_i) \to ([1,\kappa], \mathcal{T}_i)$ *is a surjective d-map for each* i*;*
   - $\pi_1 \colon (X_\uparrow, \mathcal{R}_i) \to ([1,\lambda], \mathcal{S}_i)$ *is a surjective d-map for each* i*;*
   - $X_\uparrow \subset d_{\mathcal{R}_0}\pi_0^{-1}(\kappa_i)$ *for any* $i < m$*;*
   - $\pi_1 = 1 + (-\xi + \ell^\varsigma)$*;*
   - *the polytopology* $(\mathcal{R}_0, \ldots, \mathcal{R}_n)$*, when restricted to* $X_\uparrow$*, is the one obtained from Lemma 15.3.7 by pulling back via* π_1*;*
   - $\pi_1^{-1}(\{\lambda\}) = \{\Theta\}$.

Theorem 15.3.9 is the main new ingredient of our proof. With it, we can adapt the usual proofs to obtain completeness. First, we need an embedding lemma:

Lemma 15.3.10. *Let* $(T, <_0, <_1, ..., <_n)$ *be a finite* J_n*-tree with root* r *and* $\vec{\vartheta}$ *be an increasing* n*-sequence of nonzero ordinals. Then, for any* $\varsigma > 0$*, there exist*

- *a* $\vec{\vartheta}$*-polytopology* $([1,\Theta], \mathcal{T}_0, \ldots, \mathcal{T}_n)$ *over* $([1,\Theta], \mathcal{I}_\varsigma)$ *such that* $\Theta < e^{\varsigma + \vartheta_n}\omega$*; and*
- *a surjective* J_n*-map* $f : ([1,\Theta], \mathcal{T}_0, \ldots, \mathcal{T}_n) \to T$ *such that* $f^{-1}(r) = \{\Theta\}$.

Proof. The proof is by induction on n. The base case follows from [AFD17, Theorem 6.11], so we assume that the result holds for all $m < n$ and proceed by a subsidiary double induction on

1. ς, which we decompose as $\varsigma_0 + \omega^\rho$, and
2. $hgt_0(T)$, the height of $<_0$,

in that order. Let $hgt_i(T)$ be the height of $<_i$. We need to consider various cases:

Case I: $\omega^\rho < \varsigma$. By the induction hypothesis (for ς) applied to ω^ρ, there are:

- a $\vec{\vartheta}$-polytopology $([1,\Theta_0], \mathcal{S}_0, \ldots, \mathcal{S}_n)$ over $([1,\Theta_0], \mathcal{I}_{\omega^\rho})$ such that $\Theta_0 < e^{\omega^\rho + \vartheta_n}\omega$; and
- a surjective J_n-map $g : ([1,\Theta_0], \mathcal{S}_0, \ldots, \mathcal{S}_n) \to T$ such that $g^{-1}(r) = \{\Theta_0\}$.

By Lemma 15.2.22,

$$\ell^{\varsigma_0}\colon ([1, e^{\varsigma_0}\Theta_0], \mathcal{I}_\varsigma) \to ([1, \Theta_0], \mathcal{I}_{\omega^\rho})$$

is a d-map, whence by Lemma 15.3.7, there is a $\vec{\vartheta}$-polytopology

$$\mathfrak{X} = ([1, e^{\varsigma_0}\Theta_0], \mathcal{T}_0, \ldots, \mathcal{T}_n)$$

over $([1, e^{\varsigma_0}\Theta_0], \mathcal{I}_\varsigma)$ such that

$$\ell^{\varsigma_0}\colon ([1, e^{\varsigma_0}\Theta_0], \mathcal{T}_i) \to ([1, \Theta_0], \mathcal{S}_i)$$

is a d-map for each $i \leq n$. It follows that $f := g \circ \ell^{\varsigma_0}$ is a surjective J_n-map from $\mathfrak{X}$ onto T. Finally,

$$\Theta := e^{\varsigma_0}\Theta_0 < e^{\varsigma_0 + \omega^\rho + \vartheta_n}\omega = e^{\varsigma + \vartheta_n}\omega.$$

Case II: ς is additively indecomposable and $0 = hgt_0(T)$, so that $<_0 = \varnothing$. Let

$$\vec{\vartheta}^* = -\vartheta_1 + \vec{\vartheta} \restriction [2, n] = (-\vartheta_1 + \vartheta_2, -\vartheta_1 + \vartheta_3, \ldots, -\vartheta_1 + \vartheta_n).$$

By induction hypothesis (applied to n), there are:

- a $\vec{\vartheta}^*$-polytopology $([1, \Theta_0], \mathcal{S}_1, \ldots, \mathcal{S}_n)$ over $([1, \Theta_0], \mathcal{I}_{\vartheta_1})$ such that
$$\Theta_0 < e^{\vartheta_1 + (-\vartheta_1 + \vartheta_n)}\omega = e^{\vartheta_n}\omega;$$
- a surjective J_{n-1}-map $g : ([1, \Theta_0], \mathcal{S}_1, \ldots, \mathcal{S}_n) \to (T, <_1, \ldots, <_n)$ such that $g^{-1}(r) = \{\Theta_0\}$.

Note that each $\mathcal{S}_i$ is a rank-preserving extension of $\mathcal{I}_{\vartheta_i}$. Thus, a simple computation shows that $([1, \Theta_0], \mathcal{I}_0, \mathcal{S}_1, \ldots, \mathcal{S}_n)$ is a(n improper) $\vec{\vartheta}$-polytopology. Clearly,

$$\ell^{\varsigma}\colon ([1, e^{\varsigma}\Theta_0], \mathcal{I}_\varsigma) \to ([1, \Theta_0], \mathcal{I}_0)$$

is a d-map. By Lemma 15.3.7, there exists a $\vec{\vartheta}$-polytopology $\mathfrak{X}$ of the form $([1, e^{\varsigma}\Theta_0], \mathcal{T}_0, \ldots, \mathcal{T}_n)$ over $([1, e^{\varsigma}\Theta_0], \mathcal{I}_\varsigma)$ such that

$$\ell^{\varsigma}\colon ([1, e^{\varsigma}\Theta_0], \mathcal{T}_i) \to ([1, \Theta_0], \mathcal{S}_i)$$

is a d-map for each $i \leq n$. Let $\Theta := e^{\varsigma}\Theta_0 < e^{\varsigma + \vartheta_n}\omega$. Let $f := g \circ \ell^{\varsigma}$. We have that

$$f\colon ([1, \Theta], \mathcal{T}_1, \ldots \mathcal{T}_n) \to (T, <_1, \ldots, <_n) \text{ is a } \mathrm{J}_{n-1}\text{-map.} \tag{15.7}$$

We claim (15.7) holds for the full space $\mathfrak{X}$ and the structure $(T, <_0, \ldots, <_n)$, i.e., f is already a J_n-map: condition (j_1) is given by definition; (j_2) is satisfied trivially since the topology induced by $<_0$ is discrete; (j_3) and (j_4) hold for $k \neq 0$ because of (15.7) and for $k = 0$ because r is the sole hereditary 1-root of T.

Case III: ς is additively indecomposable and $0 < hgt_0(T)$. Let $r_1, \ldots, r_m$ be all $<_0$-successors of r that are hereditary 1-roots and (following earlier notation) let $\ll_0(r_i)$ denote the generated subtrees. Also let $\ll_1(r)$ denote the subtree consisting of all nodes that are $<_0$-incomparable with r (i.e., the $<_0$-roots). By induction hypothesis (applied to $hgt_0(T)$), there exist $\vec{\vartheta}$-polytopologies

$$\mathfrak{X}_i = ([1, \kappa_i], \mathcal{T}_0^i, \ldots, \mathcal{T}_n^i)$$

over $([1, \kappa_i], \mathcal{I}_\varsigma)$ and surjective J_n-maps

$$f_i \colon \mathfrak{X}_i \to (\ll_0(r_i), <_0, \ldots, <_n)$$

for $0 < i$. Let

$$\kappa = \kappa_1 + \ldots + \kappa_m$$

and

$$\mathfrak{X} = ([1, \kappa], \mathcal{T}_0, \ldots \mathcal{T}_n)$$

be the topological sum. We also denote by

$$f^* : \mathfrak{X} \to \left(\bigcup_{0<i\leq n} \ll_0(r_i), <_0, \ldots, <_n \right)$$

the sum of the functions f_i. We may define an improper $\vec{\vartheta}$-polytopology

$$\mathfrak{Y}' = ([1, \lambda], \mathcal{I}_0, \mathcal{S}_1, \ldots, \mathcal{S}_n)$$

over $\mathcal{I}_0$ as in Case II in such a way that there is a J_{n-1}-map

$$g \colon ([1, \lambda], \mathcal{S}_1, \ldots, \mathcal{S}_n) \to (\ll_1(r), <_1, \ldots, <_n)$$

such that, letting $f_0 = \ell^\varsigma \circ g$ and

$$\mathfrak{Y} = [1, e^\varsigma \lambda]$$

then

$$f_0 \colon \mathfrak{Y} \to (\ll_1(r), <_0, ..., <_n)$$

is a J_n-map. In fact, if we instead define

$$f_0(\zeta) = g\Big(1 + \big(- \xi + \ell^\varsigma(\zeta)\big)\Big)$$

then

$$f_0 : \Big([1, e^\varsigma(\xi + (-1 + \lambda))] \cap [\xi, \infty)_\varsigma, \hat{\mathcal{S}}_0, \ldots, \hat{\mathcal{S}}_n \Big) \to (\ll_1(r), <_0, \ldots, <_n)$$

is also a J_n-map, where $\hat{\mathcal{S}}_0, \ldots, \hat{\mathcal{S}}_n$ is the polytopology obtained from $\mathcal{I}_0, \mathcal{S}_1, \ldots, \mathcal{S}_n$ via Lemma 15.3.7. Let

$$\Theta = e^{\varsigma}(\xi + (-1 + \lambda))$$

and write $X_\uparrow = [1, \Theta] \cap [\xi, \infty)_\varsigma$ and $X_\downarrow = [1, \Theta] \cap [0, \xi)_\varsigma$. By the Product Lemma, there are:

1. A $\vec{\vartheta}$-polytopology $([1, \Theta], \mathcal{R}_0, \ldots, \mathcal{R}_n)$ over $\mathcal{I}_\varsigma$.
2. Functions $\pi_0 \colon [1, \Theta] \to [1, \kappa]$ and $\pi_1 \colon [1, \Theta] \to [1, \lambda]$ such that:
   - $\pi_0 \colon (X_\downarrow, \mathcal{R}_i) \to ([1, \kappa], \mathcal{T}_i)$ is a surjective d-map for each i;
   - $\pi_1 \colon (X_\uparrow, \mathcal{R}_i) \to ([1, \lambda], \mathcal{S}_i)$ is a surjective d-map for each $i > 0$;
   - $X_\uparrow \subset d_{\mathcal{R}_0} \pi_0^{-1}(\kappa_i)$ for any $i < m$;
   - $\pi_1 = 1 + (-\xi + \ell^\varsigma)$;
   - the polytopology $(\mathcal{R}_0, \ldots, \mathcal{R}_n)$, when restricted to $X_\uparrow$, is the one obtained from Lemma 15.3.7 by pulling back via π_1;
   - $\pi_1^{-1}(\{\lambda\}) = \{\Theta\}$.

Thus, when restricted to $X_\uparrow$, the topologies $\mathcal{R}_i$ and $\hat{\mathcal{S}}_i$ coincide, as do the mappings π_1 and f_0, and so

$$f_0 : (X_\uparrow, \mathcal{R}_0, \ldots, \mathcal{R}_n) \to (\ll(r), <_0, \ldots, <_n)$$

is a J_n-map. We define a function

$$f : ([1, \Theta], \mathcal{R}_0, \ldots, \mathcal{R}_n) \to (T, <_0, \ldots, <_n)$$

given by:

$$f(x) = \begin{cases} f^*(\pi_0(x)) & \text{if } x \in X_\downarrow \\ f_0 & \text{if } x \in X_\uparrow \end{cases}$$

Since $X_\uparrow$ and $X_\downarrow$ are $\mathcal{I}_{\varsigma+1}$-clopen and $1 \leq \partial\vartheta_1$, it follows that $X_\uparrow$ and $X_\downarrow$ are $\mathcal{R}_i$-clopen for all $0 < i$. The facts that f_0 and f^* are J_n-maps and that the projection π_0 is a d-map yield condition (j_1), as well as conditions (j_2)–(j_4) for $0 < i$. We verify the remaining ones:

(j_2) Let U be $\mathcal{R}_0$-open. If $U \subset X_\downarrow$, then $f(U)$ is $<_0$-open, as $\pi_0 \circ f^*$ is a J_n-map. If $U \cap X_\uparrow \neq \varnothing$, then we claim $f(U)$ is $<_0$-open in T. Indeed, since $X_\uparrow \subset d_{\mathcal{R}_0} \pi_0^{-1}(\kappa_i)$ for any $0 < i \leq m$, then there are ordinals $\xi_0, \ldots, \xi_m \in U$ such that $\pi_0(\xi_i) = \kappa_i$ for each i. But then $\pi_0(U \cap X_\downarrow)$ contains a neighborhood U_i of each κ_i and by choice of f^*, $f^*(U_i) = \ll_0(r_i)$.

(j_3) Any hereditary 1-root x is either r or in some T_i. In the former case, $f^{-1}(\ll_0(r))$ and $f^{-1}(\{r\} \cup \ll_0(r)) \in \tau_0$ equal $[1, \Theta)$ and $[1, \Theta]$, respectively. In the latter case, the result follows from the continuity of π_0 and the fact that f^* is a J_n-map.

(j_4) Again, $f^{-1}(\{r\}) = \{\Theta\}$ and for any hereditary 1-root $x \neq r$, $f^{-1}(x) = \pi_0^{-1} f^{*-1}(x)$ is discrete because $f^{*-1}(x)$ is discrete and π_0 is pointwise discrete.

Therefore, f is a indeed a surjective J_n-map.

Since we have considered all cases, the lemma follows. □

We can now finish the proof of Theorem 15.3.4. Let $\Lambda \geq \vartheta_n$ and (X, τ) be any scattered space of height $\geq e^{1+\Lambda}\omega$. Suppose φ only contains modalities in $\vec{\vartheta}$ and is consistent with $\mathsf{GLP} \restriction \vec{\vartheta}$. We need to show that φ is satisfied on a $\vec{\vartheta}$-polytopology over $(X, \tau_{\restriction 1})$.

We use Lemmata 15.2.18 and 15.3.10 to find

1. the canonical tree T for φ with root r,
2. a $\vec{\vartheta}$-polytopology $([1, \Theta], \mathcal{T}_0, \ldots, \mathcal{T}_n)$ over $([1, \Theta], \mathcal{I}_1)$ such that $\Theta < e^{1+\vartheta_n}\omega$, and
3. a surjective J_n-map $f : ([1, \Theta], \mathcal{T}_0, \ldots, \mathcal{T}_n) \to T$ such that $f^{-1}(r) = \{\Theta\}$.

By Lemma 15.2.18,

$$([1, \Theta], \mathcal{T}_0, \ldots, \mathcal{T}_n) \not\models \varphi.$$

Now, by assumption, (X, τ) is a scattered space such that

$$e^{1+\vartheta_n}\omega \leq e^{1+\Lambda}\omega \leq hgt(X, \tau).$$

Choose $k < \omega$ such that

$$\Theta < e^{1+\vartheta_n} k$$

and choose a point $x \in X$ of rank $e^{1+\vartheta_n} k$. Let U be a neighborhood of x. Without loss of generality, (U, τ) is a scattered space of height $e^{1+\vartheta_n} k + 1$. By Lemma 15.3.7, there is a $\vec{\vartheta}$-polytopolgy

$$\mathfrak{U} = (U, \mathcal{S}_0, \ldots, \mathcal{S}_n)$$

over $(U, \tau_{\restriction 1})$ such that

$$\rho_\tau : (U, \mathcal{S}_i) \to ([1, e^{1+\vartheta_n} k], \mathcal{T}_i)$$

is a d-map for each $i \leq n$. By Lemmata 15.2.18 and 15.2.17, φ is then satisfiable in $\mathfrak{U}$. This completes the proof of Theorem 15.3.4.

15.4 Proof of the Product Lemma

For convenience, we restate the lemma:

Theorem 15.4.1 (Product Lemma)**.** *Assume ς is a nonzero additively indecomposable ordinal, $([1,\kappa],\mathcal{T}_0,\dots,\mathcal{T}_n)$ is a $\vec{\vartheta}$-polytopology over $\mathcal{I}_\varsigma$, and $([1,\lambda],\mathcal{S}_0,\dots,\mathcal{S}_n)$ is a $\vec{\vartheta}$-polytopology over $\mathcal{I}_0$. Suppose moreover that*

$$\kappa < e^{\varsigma+\vartheta_n}\omega$$

and that

$$\lambda < e^{\vartheta_n}\omega.$$

Fix $\{\kappa_0,\dots,\kappa_m\}$ a finite subset of $[1,\kappa]$. Let

$$\xi = \text{ the unique ordinal equal to } \ell^\varsigma[1,\kappa_m].$$

Let $\Theta = e^\varsigma(\xi+(-1+\lambda)) < e^{\varsigma+\vartheta_n}\omega$ and define $X_\uparrow = [1,\Theta]\cap[\xi,\infty)_\varsigma$ and $X_\downarrow = [1,\Theta]\cap[0,\xi)_\varsigma$. Then, there exist:

1. *A $\vec{\vartheta}$-polytopology $([1,\Theta],\mathcal{R}_0,\dots,\mathcal{R}_n)$ over $\mathcal{I}_\varsigma$.*
2. *Functions $\pi_0\colon[1,\Theta]\to[1,\kappa]$ and $\pi_1\colon[1,\Theta]\to[1,\lambda]$ such that:*
   - *$\pi_0\colon(X_\downarrow,\mathcal{R}_i)\to([1,\kappa],\mathcal{T}_i)$ is a surjective d-map for each i;*
   - *$\pi_1\colon(X_\uparrow,\mathcal{R}_i)\to([1,\lambda],\mathcal{S}_i)$ is a surjective d-map for each i;*
   - *$X_\uparrow\subset d_{\mathcal{R}_0}\pi_0^{-1}(\kappa_i)$ for any $i<m$;*
   - *$\pi_1 = 1+(-\xi+\ell^\varsigma)$;*
   - *the polytopology $(\mathcal{R}_0,\dots,\mathcal{R}_n)$, when restricted to $X_\uparrow$, is the one obtained from Lemma 15.3.7 by pulling back via π_1;*
   - *$\pi_1^{-1}(\{\lambda\}) = \{\Theta\}$.*

A slight change in notation will make the proof easier: instead of starting with an ordinal κ and a finite subset $\{\kappa_1,\dots,\kappa_m\}$ of $[1,\kappa]$, we will start with a set of ordinals $\{\kappa_1,\dots,\kappa_m\}$ and define κ as their sum.

So let ς be an additively indecomposable ordinal (possibly equal to 1). Let $\lambda,\kappa_1,\dots,\kappa_m$ be nonzero ordinals such that $\kappa_1\le\dots\le\kappa_m$ and write

$$\kappa = \kappa_1+\ldots+\kappa_m.$$

We will often speak of ordinals $\mod m$ and of "$\kappa_{\iota \mod m}$." In particular,

$$\kappa_0 = \kappa_{m \mod m}$$

is the greatest among all of κ_1, $\kappa_2, \ldots, \kappa_m$. By convention, we assume that $\omega \equiv 0 \mod m$.

Fix $\vec{\vartheta}$-polytopologies $([1,\kappa], \mathcal{T}_0, \ldots, \mathcal{T}_n)$ and $([1,\lambda], \mathcal{S}_0, \ldots, \mathcal{S}_n)$ over $\mathcal{I}_\varsigma$ and $\mathcal{I}_0$, respectively and

$$\xi = \text{ the unique ordinal equal to } \ell^\varsigma[1,\kappa_m].$$

Note that ξ exists and is a successor ordinal. Write ζ for the predecessor of ξ. We have $\kappa_m < e^\varsigma\xi$, for otherwise $e^\varsigma\xi$ belongs to the interval $[1,\kappa_m]$ and so $\xi \in \ell^\varsigma[1,\kappa_m]$, contrary to its definition. Now, condition (15.4) states that

$$e^\varsigma(\zeta+1) = \lim_{\alpha\to\varsigma} e^\alpha(e^\varsigma\zeta+1).$$

Since $e^\varsigma\zeta \leq \kappa_m < e^\varsigma\xi$, it follows that

$$e^\alpha(e^\varsigma\zeta+1) \leq \kappa_m$$

for some greatest ordinal α, which we will denote by ν, i.e.,

$$e^\nu(e^\varsigma\zeta+1) \leq \kappa_m < e^{\nu+1}(e^\varsigma\zeta+1).$$

The proof of the Product Lemma is distributed among a series of lemmata and definitions throughout this section.

Definition 15.4.2. We define the *characteristic sequence for* ς, $\mathtt{cs}(\varsigma)$ as follows:

- If $\varsigma = 1$, then $\mathtt{cs}(\varsigma)$ is the sequence $\{\varsigma_\iota\}_{\iota<\omega}$ with constant value 0.
- If ς is a limit ordinal, then $\mathtt{cs}(\varsigma)$ is the sequence $\{\varsigma_\iota\}_{\iota<\varsigma}$ given by $\varsigma_\iota = \max\{\iota, \nu\cdot\iota\}$.

Remark 15.4.3. The case $\varsigma = 1$ needs to be considered separately. It corresponds to the notion of d-products from [BG13]. In our context, the Product Lemma will be sufficient, but it is also possible to ignore the case $\varsigma = 1$ of the Product Lemma and instead employ the d-product construction directly (cf. [Agu15]).

Clearly, if ς is multiplicatively indecomposable, then the sequence $\mathtt{cs}(\varsigma)$ is cofinal in ς. We shall assume for notational simplicity that ς is multiplicatively indecomposable, as opposed to simply additively indecomposable, hereafter. If it were not, however, multiplicatively indecomposable, then it must still be a limit of points in the sequence $\mathtt{cs}(\varsigma)$. If ς were not multiplicatively indecomposable, then the following proof goes through if one replaces the sequence $\mathtt{cs}(\varsigma)$ with the appropriate initial segment throughout.

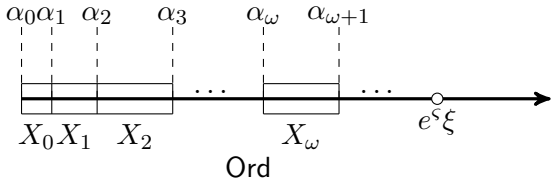
15.4.1 The partition

For the rest of the section, we shall assume that ν is nonzero; this will avoid various awkward case distinctions. If ν were in fact 0, or even finite, then the computations carried out below would in fact be simpler, and similar to the situation in [AFD17].

Definition 15.4.4 $(\Theta, X_\uparrow, X_\downarrow)$**.** For each $\iota < \varsigma$, we take κ_ι to mean the unique κ_i such that $\iota \equiv i \mod m$. In particular, $\kappa_0 = \kappa_m \geq \kappa_i$ for any i. Set:

- $\alpha_0 = 1$;
- $\alpha_{\iota+1} = \beta_\iota + 1$;
- $\alpha_\iota = e^{\varsigma_\iota}(e^\varsigma \zeta + 1)$, at limit stages; and
- $\beta_\iota = 1 + e^{\varsigma_\iota}(\kappa_0 + \kappa_\iota)$

Let $X_\iota = [\alpha_\iota, \beta_\iota]$, $Y_\iota = X_\iota \cap [0, \kappa_0]_{\varsigma_\iota}$, and $Z_\iota = X_\iota \setminus Y_\iota$. We will soon prove (Lemma 15.4.7) that the family $\{X_\iota \colon \iota < \varsigma\}$ partitions $e^\varsigma \xi$:



Definition 15.4.5. As in the statement of the theorem, we define:

$$\begin{aligned}\Theta = &\text{ least } \alpha \text{ such that } 1 + (-\xi + \ell^\varsigma \alpha) = \lambda \\ &= e^\varsigma(\xi + (-1 + \lambda)) \\ &= e^\varsigma(\ell^\varsigma[1, \kappa] + (-1 + \lambda)).\end{aligned}$$

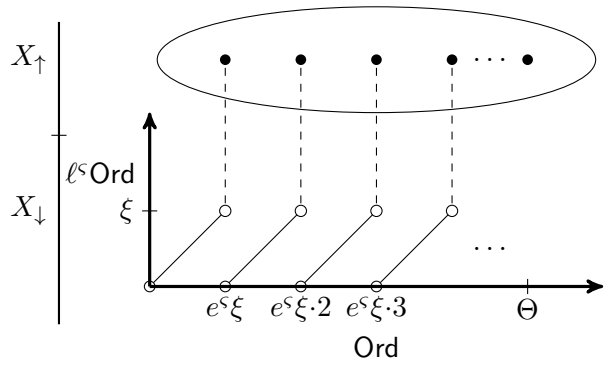
Observe that if $\lambda < e^{\vartheta_n}\omega$ and $\kappa < e^{\varsigma + \vartheta_n}\omega$, then

$$e^\varsigma(\ell^\varsigma[1, \kappa] + (-1 + \lambda)) < e^{\varsigma + \vartheta_n}\omega,$$

as desired. We also set:

1. $X_\downarrow = [1, \Theta] \cap [0, \xi)_\varsigma$;
2. $X_\uparrow = [1, \Theta] \cap [\xi, \infty)_\varsigma$.

See the following picture:



Lemma 15.4.6. *The sets X_ι, Y_ι, and Z_ι are $\mathcal{I}_\varsigma$-clopen.*

Proof. The sets X_ι are clearly already $\mathcal{I}_1$-clopen. That the sets Y_ι are $\mathcal{I}_\varsigma$-clopen follows from the fact that $\varsigma_\iota < \varsigma$; consequently, so too are the sets Z_ι. □

Lemma 15.4.7. *Suppose ς is a limit ordinal. Then, the sets $\{X_\gamma : \gamma < \varsigma\}$ in Definition 15.4.4 form a partition of $e^\varsigma\xi$.*

Proof. From [AFD17, Lemma 2.8] follows that

$$\sup_{\iota\in\varsigma\cap\mathsf{Lim}} \alpha_\iota = e^\varsigma\xi,$$

so it suffices to show that

$$\lim_{\iota\to\gamma} \alpha_\gamma = e^{\nu\cdot\gamma}(e^\varsigma\zeta + 1) \text{ for each limit } \gamma < \varsigma. \tag{15.8}$$

Write $\gamma = \gamma^* + \omega^\rho$, where ρ is nonzero. Recall that the functions e^ι are normal. First, we have:

$$\begin{aligned}
\lim_{\iota\to\gamma} \alpha_\iota = \lim_{\iota\to\gamma} \beta_\iota &= \lim_{\iota\to\gamma} e^{\nu\cdot\iota}(\kappa_0 + \kappa_\iota) \\
&\leq \lim_{\iota\to\gamma} e^{\nu\cdot\iota}(\kappa_0 \cdot 2) \\
&\leq \lim_{\iota\to\gamma} e^{\nu\cdot\iota} e(\kappa_0 + 1) \\
&= \lim_{\iota\to\gamma} e^{\nu\cdot\iota}(\kappa_0 + 1) \\
&\leq \lim_{\iota\to\gamma} e^{\nu\cdot\iota}(\kappa_0 + \kappa_\iota),
\end{aligned}$$

so $\lim_{\iota\to\gamma}\alpha_\iota = \lim_{\iota\to\gamma} e^{\nu\cdot\iota}(\kappa_0+1)$. Using the decomposition of γ:

$$\begin{aligned}\lim_{\iota\to\gamma}\alpha_\iota &= \lim_{\iota\to\gamma} e^{\nu\cdot\iota}(\kappa_0+1)\\ &= \lim_{\iota\to\omega^\rho} e^{\nu\cdot(\gamma^*+\iota)}(\kappa_0+1)\\ &= \lim_{\iota\to\omega^\rho} e^{\nu\cdot\gamma^*+\nu\cdot\iota}(\kappa_0+1)\\ &= e^{\nu\cdot\gamma^*}\left(\lim_{\iota\to\omega^\rho} e^{\nu\cdot\iota}(\kappa_0+1)\right). \end{aligned} \tag{15.9}$$

By choice of ν, we have

$$e^\nu(e^\varsigma\zeta+1) \le \kappa_0 < e^{\nu+1}(e^\varsigma\zeta+1),$$

and thus

$$e^\nu(e^\varsigma\zeta+1)+1 \le \kappa_0+1 \le e^{\nu+1}(e^\varsigma\zeta+1),$$

so normality implies that

$$\begin{aligned}\lim_{\iota\to\omega^\rho} e^{\nu\cdot\iota}\Big(e^\nu(e^\varsigma\zeta+1)+1\Big) &\le \lim_{\iota\to\omega^\rho} e^{\nu\cdot\iota}(\kappa_0+1)\\ &\le \lim_{\iota\to\omega^\rho} e^{\nu\cdot\iota}e^{\nu+1}(e^\varsigma\zeta+1)\\ &= \lim_{\iota\to\omega^\rho} e^{\nu\cdot\iota}(e^\varsigma\zeta+1)\\ &\le \lim_{\iota\to\omega^\rho} e^{\nu\cdot\iota}\Big(e^\nu(e^\varsigma\zeta+1)+1\Big).\end{aligned}$$

This implies that

$$\lim_{\iota\to\omega^\rho} e^{\nu\cdot\iota}(\kappa_0+1) = \lim_{\iota\to\omega^\rho} e^{\nu\cdot\iota}(e^\varsigma\zeta+1). \tag{15.10}$$

Now, recall that if ν^* is a limit, then, for every ordinal γ, we have

$$e^{\nu^*}(\gamma+1) = \lim_{\iota\to\nu^*} e^\iota(e^{\nu^*}\gamma+1).$$

Using this (for $\nu^* = \nu\cdot\omega^\rho$ and $\gamma = e^\varsigma\zeta$) and the additive indecomposability of ς, we obtain:

$$\begin{aligned}\lim_{\iota\to\omega^\rho} e^{\nu\cdot\iota}(e^\varsigma\zeta+1) &= \lim_{\iota\to\nu\cdot\omega^\rho} e^\iota(e^\varsigma\zeta+1)\\ &= \lim_{\iota\to\nu\cdot\omega^\rho} e^\iota(e^{\nu\cdot\omega^\rho+\varsigma}\zeta+1)\\ &= \lim_{\iota\to\nu\cdot\omega^\rho} e^\iota(e^{\nu\cdot\omega^\rho}e^\varsigma\zeta+1)\\ &= e^{\nu\cdot\omega^\rho}(e^\varsigma\zeta+1).\end{aligned}$$

Putting this together with equations (15.9) and (15.10),

$$\begin{aligned} e^{\nu\cdot\gamma}(e^{\varsigma}\zeta+1) = e^{\nu\cdot\gamma^*}e^{\nu\cdot\omega^\rho}(e^{\varsigma}\zeta+1) &= e^{\nu\cdot\gamma^*}\left(\lim_{\iota\to\omega^\rho} e^{\nu\cdot\iota}(e^{\varsigma}\zeta+1)\right) \\ &= e^{\nu\cdot\gamma^*}\left(\lim_{\iota\to\omega^\rho} e^{\nu\cdot\iota}(\kappa_0+1)\right) \\ &= \lim_{\iota\to\gamma}\alpha_\iota, \end{aligned}$$

which proves equation (15.8). This finishes the proof of the lemma. □

15.4.2 Projections

Definition 15.4.8 (Projections)**.** We define the functions π_0 and π_1:

(↓) $\pi_0\colon [0,e^{\varsigma}\xi)\to[1,\kappa]$ is defined by:

$$\pi_0(\alpha)=\begin{cases} 1+\ell^{\varsigma_\iota}\alpha & \text{if } \ell^{\varsigma_\iota}\alpha\le\kappa_0 \\ \kappa_1+\cdots+\kappa_{i-1}+1+(-(1+\kappa_0)+\ell^{\varsigma_\iota}\alpha) & \text{otherwise;}\end{cases}$$

where ι and i are the unique ordinals such that $\alpha\in X_\iota$, $\iota\equiv i \mod m$, and $i<m$.

(↓) The function π_0 is extended to all of $X_\downarrow$: given $\alpha\in X_\downarrow$, let η be least such that $\ell^\eta\alpha<e^{\varsigma}\xi$. Then, $\pi_0(\alpha)=\pi_0(\ell^\eta\alpha)$.

(↑) $\pi_1\colon X_\uparrow\to[1,\lambda]$ is defined by

$$\pi_1\alpha = 1+(-\xi+\ell^{\varsigma}\alpha).$$

Clearly,

$$\pi_1 : (X_\uparrow,\mathcal{I}_\varsigma)\to([1,\lambda],\mathcal{I}_0)$$

is a surjective d-map. It is not immediately clear whether π_0 is a d-map; this we verify below.

Observe that, since $X_\downarrow=[1,\Theta]_0\cap[0,\xi)_\varsigma$ by definition, the ordinal η in the second clause above must be strictly smaller than ς.

Let us look at the definition of π_0 a bit more closely. Consider a typical element α^* of $X_\downarrow$ and generate the sequence

$$\{\ell^\nu\alpha^* : \nu\in Ord\}.$$

Let η be least such that $\alpha:=\ell^\eta\alpha^*<e^{\varsigma}\xi$. Then α belongs to some cell

$$X_\iota=[\alpha_\iota,\beta_\iota].$$

By definition, β_ι is of the form $e^{\varsigma_\iota}(\kappa_0 + \kappa_i)$, where $\iota \equiv i \mod m$, so

$$\ell^{\varsigma_\iota}\alpha \in [0, \kappa_0 + \kappa_i].$$

Recall that we assumed for simplicity that $\nu \neq 0$ (see p. 419). We distinguished two cases: if $\ell^{\varsigma_\iota}\alpha \in [0, \kappa_0]$, then we had set

$$\pi_0\alpha^* = \pi_0\alpha = 1 + \ell^{\varsigma_\iota}\alpha \in [1, \kappa_0].$$

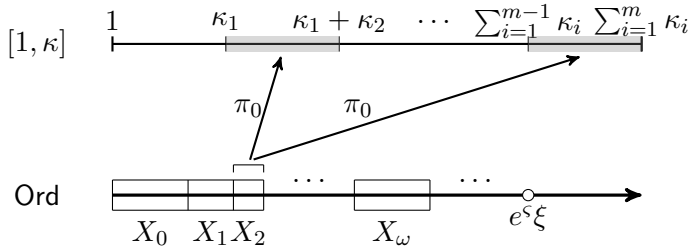
Otherwise, $\ell^{\varsigma_\iota}\alpha \in [\kappa_0 + 1, \kappa_0 + \kappa_i]$, so that

$$(-1 + \kappa_0) + \ell^{\varsigma_\iota}\alpha \in [0, \kappa_i].$$

Its definition then places $\pi_0\alpha^* = \pi_0\alpha$ within the interval

$$[\kappa_1 + \cdots + \kappa_{i-1} + 1, \kappa_1 + \cdots + \kappa_{i-1} + \kappa_i].$$

Here is the picture:



Lemma 15.4.9. *The projection function π_0 has the following properties:*

1. *If $\iota \equiv i \mod m$, then $\pi_0 \colon (X_\iota, \mathcal{I}_\varsigma) \to ([1, \kappa_0] \sqcup [1, \kappa_i], \mathcal{I}_\varsigma)$ is a surjective d-map. If, in addition, $\varsigma = 1$, then it is a homeomorphism.*
2. *$\pi_0 \colon (X_\downarrow, \mathcal{I}_\varsigma) \to ([1, \kappa], \mathcal{I}_\varsigma)$ is a surjective d-map.*
3. *$\pi_0(\beta_\iota) = \kappa_i$, where $\iota \equiv i \mod m$.*

Proof. By Lemma 15.4.6, the sets Y_ι and Z_ι are $\mathcal{I}_\varsigma$-clopen in X_ι. Consider generalized cells of the form

$$\begin{aligned} X_\iota^\eta &= \{x \in X_\downarrow : \ell^{\eta+1}x < e^{\varsigma}\xi \text{ and } e^{\varsigma}\xi \geq \ell^\eta x \text{ and } \ell^{\eta+1}x \in X_\iota\} \\ &= \{x \in X_\downarrow : \ell^{\eta+1}x \leq e^{\varsigma}\xi \text{ and } e^{\varsigma}\xi > \ell^\eta x \text{ and } \ell^{\eta+1}x \in X_\iota\} \\ &= X_\downarrow \cap [0, e^{\varsigma}\xi]_{\eta+1} \cap (e^{\varsigma}\xi, \infty)_\eta \cap [\alpha_\iota, \beta_\iota]_{\eta+1}. \end{aligned}$$

(The second equality follows from the definition of $X_\downarrow$.) Observe that $[\alpha_\iota, \beta_\iota]_{\eta+1}$ is an $\mathcal{I}_\varsigma$-clopen interval if $\eta < \varsigma$, even when α_ι is a limit ordinal; this is because α_ι is always an isolated point in $\mathcal{I}_\varsigma$ by its definition. Similarly, the sets

$$Y_\iota^\eta = X_\iota^\eta \cap [0, \kappa_0]_{\eta+\varsigma_\iota}, Z_\iota^\eta = X_\iota^\eta \cap (\kappa_0, \infty)_{\eta+\varsigma_\iota}$$

are $\mathcal{I}_\varsigma$-clopen. The definition of π_0 is the same within each Y_ι^η and within each Z_ι^η, and in each of those sets, π_0 is defined as a combination of additions, substractions, and logarithms and is thus a d-map (recall that ς is additively indecomposable). Additionally, by Lemma 15.2.26, if $x \in X_\downarrow$, then the least η such that $\ell^\eta x < e^\varsigma \xi$ is a successor ordinal. It follows that the collection of all Y_ι^η and Z_ι^η forms a clopen partition of $X_\downarrow$. Since π_0 is a d-map on each element of the partition, it is a d-map on all of $X_\downarrow$.

Moreover, if $\varsigma = 1$, then $\mathtt{cs}(\varsigma)$ is the constant sequence with value 0 and so X_ι is homeomorphic to $[1, \kappa_0] \sqcup [1, \kappa_i]$. This gives item 1. Item 2 is obtained by a similar argument, as each X_ι is $\mathcal{I}_\varsigma$-clopen. Item 3 follows readily from the definition. □

Lemma 15.4.10. *$\pi_0^{-1}\alpha$ is $\mathcal{I}_\varsigma$-dense in $X_\uparrow$ for any $\alpha \in [1, \kappa]$.*

Proof. We can even provide witnesses for the density. Let i be least such that

$$\kappa_1 + \cdots + \kappa_{i-1} < \alpha \leq \kappa_1 + \cdots + \kappa_i$$

and let $\alpha_0 := -(\kappa_1 + \ldots + \kappa_{i-1} + 1) + \alpha$.

If $\varsigma = 1$, then the result is clear, as any $\mathcal{I}_1$-neighborhood of any $\beta \in X_\uparrow$ contains an interval of the form $[\delta, \beta]_0$ and, by construction, β is a limit of endpoints of cells X_ι. In particular, the interval $[\delta, \beta]$ contains some cell X_ι with $\iota \equiv i \mod m$ and $\alpha \in \pi_0(X_\iota) = [1, \kappa_0] \sqcup [1, \kappa_i]$ by Lemma 15.4.9.1, from which the result follows.

So suppose $\varsigma \neq 1$. Let $\beta \in X_\uparrow$, so that β has $\mathcal{I}_\varsigma$-rank $\rho \geq \xi$ and U be an $\mathcal{I}_\varsigma$-neighborhood of β. We distinguish two cases:

Case I: $\beta = e^\varsigma \xi$; thus $\rho = \xi$. We can apply Lemma 15.2.24 (over the interval topology) to obtain a $\mathcal{I}_\varsigma$-neighborhood base of $e^\varsigma \xi$ consisting of sets of the form

$$(\eta, e^\varsigma \rho]_\gamma$$

for $\eta < e^\varsigma \rho$ and $\gamma < \varsigma$.

Hence, we may assume U is a neighborhood of $e^\varsigma \xi$ of the form $(\eta, e^\varsigma \rho]_\gamma$. We need to find some ordinal $\chi \in U \cap X_\downarrow$ such that $\pi_0 \chi = \alpha$. Let μ be some successor ordinal $\equiv i \mod m$ large enough so that

1. $\gamma < \varsigma_\mu < \varsigma$, and

2. $\eta < e^{-\gamma+\varsigma_\mu}(1+\kappa_0+\alpha_0)$.

This is certainly possible, as it follows from Lemma 15.4.7 that:

$$\lim_{\iota\to\varsigma} e^{\varsigma_\iota}(1+\kappa_0+\alpha_0) = \lim_{\iota\to\varsigma}\alpha_\iota = e^{\varsigma}\xi.$$

Claim 15.4.11. *Let* $\chi := e^{\varsigma_\mu}(1+\kappa_0+\alpha_0)$. *Then* $\chi \in U \cap X_\mu$.

Proof. Clearly, $\chi < e^{\varsigma}\xi$. By choice of μ, we have

$$\eta < e^{-\gamma+\varsigma_\mu}(1+\kappa_0+\alpha_0) = \ell^\gamma(e^{\varsigma_\mu}(1+\kappa_0+\alpha_0)), \tag{15.11}$$

and since $\varsigma_\mu < \varsigma$, we have

$$\ell^\gamma e^{\varsigma_\mu}(1+\kappa_0+\alpha_0) < e^{\varsigma}\xi. \tag{15.12}$$

From (15.11) and (15.12) follows that $e^{\varsigma_\mu}(1+\kappa_0+\alpha_0) \in U$. It remains to prove that $\chi \in X_\mu$. Denote by μ^* the immediate predecessor of μ. Notice that

$$e^{\varsigma_{\mu^*}}(\kappa_0+\kappa_{i-1}) < e^{\varsigma_\mu}(1+\kappa_0+\alpha_0) \leq e^{\varsigma_\mu}(\kappa_0+\kappa_i), \tag{15.13}$$

so that $e^{\varsigma_\mu}(1+\kappa_0+\alpha_0) \in X_\mu$. This proves the claim. □

We show that $\pi_0(e^{\varsigma_\mu}(1+\kappa_0+\alpha_0)) = \alpha$. Since $\mu \equiv i \mod m$ and

$$\kappa_0 < 1+\kappa_0+\alpha_0 = \ell^{\varsigma_\mu}e^{\varsigma_\mu}(1+\kappa_0+\alpha_0),$$

we have:

$$\begin{aligned}\pi_0(e^{\varsigma_\mu}(1+\kappa_0+\alpha_0)) &= \kappa_1+\ldots+\kappa_{i-1}+1+(-(1+\kappa_0)+\ell^{\varsigma_\mu}e^{\varsigma_\mu}(1+\kappa_0+\alpha_0))\\ &= \kappa_1+\ldots+\kappa_{i-1}+1+\alpha_0\\ &= \kappa_1+\ldots+\kappa_{i-1}+1+(-(\kappa_1+\ldots+\kappa_{i-1}+1)+\alpha)\\ &= \alpha.\end{aligned}$$

Since U was arbitrary, this finishes the proof in this case.

Case II: $e^\varsigma\xi < \beta$. It is enough to consider the case $\ell^\varsigma\beta = \xi$, as any $\mathcal{I}_\varsigma$-neighborhood of any point of higher rank contains a point of rank ξ. As observed by Fernández-Duque and Joosten [FDJ13a], there is a least $\eta^* < \varsigma$ such that

$$\ell^{\eta^*}\beta = e^\varsigma\xi.$$

Since ς is additively indecomposable, η^* must be a successor ordinal, say $\eta + 1$. Thus,

$$\ell^{\eta+1}\beta = e^\varsigma\xi < \ell^\eta\beta,$$

and so $\ell^\eta\beta$ must be of the form $\beta' + e^\varsigma\xi$. The Hyperexponential Normal Form theorem [AFD17, Proposition 2.13] states that every ordinal γ can be uniquely written in the form $e^{\gamma_0}\gamma_1$, where γ_1 is either additively decomposable or 1. If $\gamma_1 = 1$, then let us call the expression $e^{\gamma_0}1$ the *normal form expansion* of γ. Inductively, the normal form expansion of

$$\gamma = e^{\gamma_0}(\gamma_1 + \gamma_2)$$

is defined to be

$$\mathtt{nf}(\gamma) = e^{\gamma_0}(\gamma_1 + \mathtt{nf}(\gamma_2)).$$

Observe that $\eta < \varsigma$, for otherwise we would have

$$\xi < e^\varsigma\xi \leq \ell^\varsigma\beta,$$

contradicting the choice of β. Thus, if one writes out the normal form expansion of β, one obtains an expression of the form

$$e^{\beta_0}\Big(\beta_1 + e^{\beta_1}(\dots(\beta' + \mathtt{nf}(e^\varsigma\xi))\dots)\Big),$$

where $e^\varsigma\xi < \beta' + e^\varsigma\xi$ and all the exponents to the left of $\beta' + \mathtt{nf}(e^\varsigma\xi)$ add up to η. Consider the sequence $\{\beta(\iota) : \iota < e^\varsigma\xi\}$, where $\beta(\iota)$ is the ordinal one obtains if one substitutes ι for the rightmost occurrence of $\mathtt{nf}(e^\varsigma\xi)$ in the normal form expansion of β (the occurrence indicated in the equation displayed above). If U is a $\mathcal{I}_\varsigma$-neighborhood of β, then, by Lemma 15.2.25, U contains a set of the form

$$B_r(\beta) = \bigcap_{i\in\mathrm{dom}(r)} (r(i), \ell^i\beta]_i,$$

where $r : \varsigma \to \beta + 1$ is a finite partial function. It follows that every such set U, if nonempty, contains cofinally many ordinals of the form $\beta(\iota)$. Since

$e^\varsigma\xi < \beta' + e^\varsigma\xi$ and $e^\varsigma\xi$ is additively indecomposable, we must have $e^\varsigma\xi < \beta'$, so it follows that for each $\beta(\iota)$,

$$\pi_0\beta(\iota) = \pi_0\ell(\beta' + \iota) = \pi_0\ell\iota.$$

An argument as in Case I shows that there is some $\iota < e^\varsigma\xi$ such that $\beta(\iota) \in U$ and $\pi_0\beta(\iota) = \alpha$. □

15.4.3 The polytopology

It remains to define a $\vec{\vartheta}$-polytopology $(\mathcal{R}_0, \ldots, \mathcal{R}_n)$ on $([1,\Theta], \mathcal{I}_\varsigma)$ such that the projection mappings

$$\pi_0 : (X_\downarrow, \mathcal{R}_i) \to ([1,\kappa], \mathcal{T}_i)$$

and

$$\pi_1 : (X_\uparrow, \mathcal{R}_i) \to ([1,\lambda], \mathcal{S}_i)$$

remain d-maps. Recall that n denotes the length of $\vec{\vartheta}$.

To begin, we observe that since

$$\pi_0 : (X_\downarrow, \mathcal{I}_\varsigma) \to ([1,\kappa], \mathcal{I}_\varsigma)$$

is a d-map by Lemma 15.4.9.2, we may apply Lemma 15.3.7 to obtain a $\vec{\vartheta}$-polytopology $(X_\downarrow, \hat{\mathcal{T}}_0, \ldots, \hat{\mathcal{T}}_n)$ over $\mathcal{I}_\varsigma$ such that

$$\pi_0 : (X_\downarrow, \hat{\mathcal{T}}_i) \to ([1,\kappa], \mathcal{T}_i)$$

is a d-map for each $0 \leq i \leq n$.

This $\vec{\vartheta}$-polytopology $(\hat{\mathcal{T}}_0, \ldots, \hat{\mathcal{T}}_n)$ is not, however, a topology on $[1,\Theta]$, so we need to extend it. For each i, let $\hat{\mathcal{R}}_i$ be the smallest topology on $[1,\Theta]$ extending $\mathcal{I}_{\varsigma+\vartheta_i}$ and containing all sets in $\hat{\mathcal{T}}_i$. Since $X_\uparrow$ is $\mathcal{I}_{\varsigma+1}$-clopen, $\hat{\mathcal{R}}_i$ is simply equal to $\mathcal{I}_{\varsigma+\vartheta_i}$ when restricted to $X_\uparrow$, for $0 < i$. We are closer to our goal, but not done yet, since the space

$$([1,\Theta], \hat{\mathcal{R}}_0, \ldots, \hat{\mathcal{R}}_n)$$

might not be a $\vec{\vartheta}$-polytopology, as e.g., $\hat{\mathcal{R}}_0$ might not be ϑ_1-maximal around points in $X_\uparrow$.

Note that $-\xi + \ell^\varsigma$ is the rank function of $(X_\uparrow, \mathcal{I}_\varsigma)$ (viewed as a subspace of $([1,\Theta], \mathcal{I}_\varsigma)$) and so

$$-\xi + \ell^\varsigma : (X_\uparrow, \mathcal{I}_\varsigma) \to ([0, -1+\lambda], \mathcal{I}_0)$$

is a d-map. Since $[0, -1+\lambda]$ is homeomorphic to $[1, \lambda]$, it follows that

$$1 + (-\xi + \ell^{\varsigma}) : (X_{\uparrow}, \mathcal{I}_{\varsigma}) \to ([1, \lambda], \mathcal{I}_0)$$

is also a d-map. But this is precisely equal to π_1. Having only one point of each rank, the space $([1, \lambda], \mathcal{I}_0)$ has no proper rank-preserving extensions, and in particular is ϑ_1-maximal. By the claim within the proof of Lemma 15.3.7, if $(X_{\uparrow}, \mathcal{R})$ is any ϑ_1-extension of $(X_{\uparrow}, \mathcal{I}_{\varsigma})$, then

$$\pi_1 : (X_{\uparrow}, \mathcal{R}) \to ([1, \lambda], \mathcal{I}_0)$$

remains a d-map. Let $([1, \Theta], \mathcal{R}_0)$ be a ϑ_1-maximal extension of $([1, \Theta], \hat{\mathcal{R}}_0)$. Then, $\mathcal{R}_0$ only adds neighborhoods around points of rank some ρ such that $0 < \ell^{\vartheta_1}\rho$ and, moreover, only neighborhoods around points in $X_{\uparrow}$, since $(X_{\downarrow}, \hat{\mathcal{T}}_0)$ was already ϑ_1-maximal. Given a point $x \in X_{\uparrow}$, and recalling that ξ, the minimum $\mathcal{I}_{\varsigma}$-rank of points in $X_{\uparrow}$, is a successor ordinal, we see that

$$\begin{aligned} 0 < \ell^{\vartheta_1}\rho_{([1,\Theta],\mathcal{R}_0)}x \text{ if, and only if, } & 0 < \ell^{\vartheta_1}\ell^{\varsigma}x \\ \text{if, and only if, } & 0 < \ell^{\vartheta_1}\rho_{(X_{\uparrow},\mathcal{I}_{\varsigma})}x. \end{aligned}$$

Thus, the space $(X_{\uparrow}, \mathcal{R}_0)$ is a ϑ_1-extension of $(X_{\uparrow}, \mathcal{I}_{\varsigma})$. It follows that

$$\pi_1 : (X_{\uparrow}, \mathcal{R}_0) \to ([1, \lambda], \mathcal{I}_0)$$

remains a d-map. We may now apply Lemma 15.3.7 to obtain a $\vec{\vartheta}$-polytopology

$$(X_{\uparrow}, \mathcal{R}_0, \hat{\mathcal{S}}_1, \ldots, \hat{\mathcal{S}}_n)$$

over $(X_{\uparrow}, \mathcal{R}_0)$ such that

$$\pi_1 : (X_{\uparrow}, \hat{\mathcal{S}}_i) \to ([1, \lambda], \mathcal{S}_i)$$

is a d-map for each $1 \leq i \leq n$. For each $1 \leq i \leq n$, we let $\mathcal{R}_i$ be the disjoint union

$$(X_{\downarrow}, \hat{\mathcal{T}}_i) \sqcup (X_{\uparrow}, \hat{\mathcal{S}}_i).$$

The sets $X_{\uparrow}$ and $X_{\downarrow}$ are $\mathcal{R}_{0\uparrow\vartheta_1}$-clopen and so it follows that

$$([1, \Theta], \mathcal{R}_0, \mathcal{R}_1, \ldots, \mathcal{R}_n)$$

is a $\vec{\vartheta}$-polytopology. Moreover, we have seen that

$$\pi_0 : (X_{\downarrow}, \hat{\mathcal{T}}_i) \to ([1, \kappa], \mathcal{T}_i)$$

is a d-map for each $0 \leq i \leq n$ and that

$$\pi_1 : (X_\uparrow, \mathcal{R}_i) \to ([1, \lambda], \mathcal{S}_i)$$

is also a d-map for each $0 \leq i \leq n$. The other conditions in the statement of the Product Lemma are easy to check from the construction, so its proof is complete.

Bibliography

[Aan74] S. Aanderaa. Inductive Definitions and Their Closure Ordinals. In J. E. Fenstad and P. G. Hinman, editors, *Generalized Recursion Theory*, pages 207–220. 1974.

[ABa] J. P. Aguilera and M. Baaz. Unsound inferences make proofs shorter. *J. Symbolic Logic*. To appear.

[ABb] J. P. Aguilera and D. W. Blue. Provably-Δ_1 games. Forthcoming.

[Aba85] M. Abashidze. Ordinal completeness of the Gödel-Löb modal system (in Russian). *Intensional logics and the logical structure of theories*, pages 49–73, 1985.

[Acz75] P. Aczel. Games, Quantifiers, and Inductive Definitions. In Stig Kanger, editor, *Proceedings of the Third Scandinavian Logic Symposium*, pages 1–14. North-Holland, 1975.

[AFD17] J. P. Aguilera and D. Fernández-Duque. Strong completeness of provability logic for ordinal spaces. *J. Symbolic Logic*, 2017.

[Agua] J. P. Aguilera. Determinate logic and the axiom of choice. Forthcoming.

[Agub] J. P. Aguilera. Determined admissible sets. Forthcoming.

[Aguc] J. P. Aguilera. F_σ games and reflection in $L(\mathbb{R})$. Forthcoming.

[Agud] J. P. Aguilera. Long Borel Games. Forthcoming.

[Ague] J. P. Aguilera. The order of reflection. Forthcoming.

[Aguf] J. P. Aguilera. Shortening clopen games. Forthcoming.

[Agug] J. P. Aguilera. σ-Projective Determinacy. Forthcoming.

[Aguh] J. P. Aguilera. A topological completeness theorem for transfinite provability logic. Forthcoming.

[Agu15] J. P. Aguilera. Uncountable topological interpretations of provability logic. *B. Sc. Thesis*, 2015.

[AM] J. P. Aguilera and S. Müller. The consistency strength of long projective determinacy. Forthcoming.

[AMS] J. P. Aguilera, S. Müller, and P. Schlicht. Long games and σ-projective sets. Forthcoming.

[AR74] P. Aczel and W. Richter. Inductive Definitions and Reflecting Properties of Admissible Ordinals. In J. E. Fenstad and P. G. Hinman, editors, *Generalized Recursion Theory*, pages 301–381. 1974.

[AS76] F. G. Abramson and G. E. Sacks. Uncountable Gandy Ordinals. *J. London Math. Soc.*, 14(2):387–392, 1976.

[Bag] J. Bagaria. Derived topologies on ordinals and stationary reflection. *Trans. Amer. Math. Soc.* To appear.

[Bag15] J. Bagaria. Topologies on Ordinals and the Completeness of Polymodal Provability Logics. *In preparation*, 2015.

[Bar75] J. Barwise. *Admissible Sets and Structures*. Perspectives in Mathematical Logic. Springer-Verlag, Berlin, 1975.

[BBI10] L. D. Beklemishev, G. Bezhanishvili, and T. Icard. On topological models of GLP. *Ways of Proof Theory. Essays in honor of Wolfram Pohlers*, pages 135–155, 2010.

[Bek04] L. D. Beklemishev. Provability algebras and proof-theoretic ordinals, I. *Ann. Pure Appl. Logic*, 128:103–123, 2004.

[Bek10] L. D. Beklemishev. Kripke semantics for provability logic GLP. *Ann. Pure Appl. Logic*, 161:756–774, 2010.

[Bek11] L. D. Beklemishev. Ordinal completeness of bimodal provability logic GLB. *Proceedings of the 8th International Tbilisi conference on Logic, Language, and Computation, TbiLLC'09*, pages 1–15, 2011.

[BG13] L. D. Beklemishev and D. Gabelaia. Topological completeness of the provability logic GLP. *Ann. Pure Appl. Logic*, 164(12):1201–1223, 2013.

[BG14] L. D. Beklemishev and D. Gabelaia. Topological interpretations of provability logic. *Leo Esakia on Duality in Modal and Intuitionistic Logics*, pages 257–290, 2014.

[BGM71] K. J. Barwise, R.O. Gandy, and Y. N. Moschovakis. The Next Admissible Set. *J. Symbolic Logic*, 36:108–120, 1971.

[BH11] M. Baaz and S. Hetzl. On the non-confluence of cut-elimination. *J. Symbolic Logic*, 76:313–340, 2011.

[BHW12] M. Baaz, S. Hetzl, and D. Weller. On the complexity of proof deskolemization. *J. Symbolic Logic*, 77:669–686, 2012.

[BL94] M. Baaz and A. Leitsch. On skolemization and proof complexity. *Fund. Inform.*, 20:353–379, 1994.

[BL99] M. Baaz and A. Leitsch. Cut normal forms and proof complexity. *Ann. Pure Appl. Logic*, 97:127–177, 1999.

[Bla75] A. Blass. Equivalence of Two Strong Forms of Determinacy. *Proc. Amer. Math. Soc.*, 52:373–376, 1975.

[Bla90] A. Blass. Infinitary combinatorics and modal logic. *J. Symbolic Logic*, 55(2):761–778, 1990.

[BM10] G. Bezhanishvili and P. J. Morandi. Scattered and hereditarily irresolvable spaces in modal logic. *Arch. Math. Logic*, 49:343–365, 2010.

[BMM03] G. Bezhanishvili, R. Mines, and P.J. Morandi. Scattered, Hausdorff-reducible, and hereditarily irresolvable spaces. *Topol. Appl.*, 132(3):291–306, 2003.

[BMS15] J. Bagaria, M. Magidor, and H. Sakai. Reflection and Indescribability in the Constructible Universe. *Israel J. Math.*, 208:1–11, 2015.

[Bri16] H. Brickhill. A Generalisation of Closed Unbounded and Stationary Sets. *In preparation*, 2016.

[Cen74] D. Cenzer. Ordinal Recursion and Inductive Definitions. In J. E. Fenstad and P. G. Hinman, editors, *Generalized Recursion Theory*, pages 221–264. 1974.

[CFFDJLM15] A. Cordón-Franco, D. Fernández-Duque, J. J. Joosten, and Félix Lara-Martín. Predicativity through transfinite reflection. 2015.

[Esa81] L. Esakia. Diagonal constructions, Löb's formula and Cantor's scattered space (in Russian). *Studies in logic and semantics*, 132(3):128–143, 1981.

[FD14] D. Fernández-Duque. The polytopologies of transfinite provability logic. *Arch. Math. Logic*, 53(3-4):385–431, 2014.

[FDJ11] D. Fernández-Duque and J. J. Joosten. Models of transfinite provability logic. *J. Symbolic Logic*, 78(2):543–561, 2011.

[FDJ13a] D. Fernández-Duque and J. J. Joosten. Hyperations, Veblen progressions, and transfinite iteration of ordinal functions. *Ann. Pure Appl. Logic*, 164(7-8):785–801, 2013.

[FDJ13b] D. Fernández-Duque and J. J. Joosten. The omega-rule interpretation of transfinite provability logic. 2013.

[Fef62] S. Feferman. Transfinite recursive progressions of axiomatic theories. *J. Symbolic Logic*, pages 259–316, 1962.

[FNS10] G. Fuchs, I. Neeman, and R. Schindler. A Criterion for Coarse Iterability. *Arch. Math. Logic*, 49, 2010.

[Fri71] H. M. Friedman. Higher set theory and mathematical practice. *Ann. Math. Logic*, 2(3):325 – 357, 1971.

[GH79] R. Gostanian and K. Hrbacek. A New Proof That $\pi^1_1 < \sigma^1_1$. *Zeitsch. f. math. Logik und Grundlagen d. Math.*, 25:407–408, 1979.

[GM53] D. Gale and Stewart F. M. Infinite games with perfect information. In H. W. Kuhn and A. W. Tucker, editors, *Contributions to the Theory of Games*, volume 2, pages 245–266. Princeton University Press, 1953.

[Gos79] R. Gostanian. The Next Admissible Ordinal. *Ann. Math. Logic*, 17:171–203, 1979.

[Gos80] R. Gostanian. Constructible Models of Subsystems of ZF. *J. Symbolic Logic*, 34:237–250, 1980.

[Gö31] K. Gödel. Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme I. *Monatsh. f. Math. Phys.*, 38:173–198, 1931.

[Har78] L. Harrington. Analytic Determinacy and $0^{\#}$. *Journal of Symbolic Logic*, 43:685–693, 1978.

[HK] L. A. Harrington and A. S. Kechris. On Monotone vs. Non-monotone Induction. Handwritten notes dated September 1975. 9 pp.

[HK76] L. A. Harrington and A. S. Kechris. On Monotone vs. Non-monotone Induction. *Bull. Amer. Math. Soc.*, 82:888–890, 1976.

[HK81] L. A. Harrington and A. S. Kechris. On the determinacy of games on ordinals. *Annals of Mathematical Logic*, 20:109–154, 1981.

[Ica11] T. F. Icard. A topological study of the closed fragment of GLP. *J. Log. Comp.*, 21(4):683–696, 2011.

[Jec03a] T. Jech. *Set Theory*. 2003.

[Jec03b] T. J. Jech. *Set Theory*. Springer Monographs in Mathematics. Springer, 2003.

[Jen72] R. B. Jensen. The fine structure of the constructible hierarchy. *Ann. Math. Logic*, 4:229–308, 1972.

[JS13] R. B. Jensen and J. R. Steel. K without the measurable. *J. Symb. Log.*, 78(3):708–734, 2013.

[Kan08] A. Kanamori. *The Higher Infinite: Large Cardinals in Set Theory from Their Beginnings*. Springer Monographs in Mathematics. Springer, 2008.

[Kec16] A. S. Kechris. On Spector Classes. In A. S. Kechris, B. Löwe, and J. R. Steel, editors, *Ordinal definability and recursion theory, The Cabal Seminar, Volume III*. Cambridge University Press, 2016.

[KKMW08] A. S. Kechris, E. M. Kleinberg, Y. N. Moschovakis, and W. H. Woodin. The Axiom of Determinacy, Strong Partition Properties and Nonsingular Measures. In A. S. Kechris, B. Löwe, and J. R. Steel, editors, *Games, Scales and Suslin Cardinals, The Cabal Seminar, Volume I*. Cambridge University Press, 2008.

[KS85] A. Kechris and R. Solovay. On the Relative Consistency Strength of Determinacy Hypotheses. *Trans. Amer. Math. Soc.*, 290(1), 1985.

[KW83] A. S. Kechris and W. H. Woodin. Equivalence of Partition Properties and Determinacy. *Proc. Natl. Acad. Sci. USA*, 80:1783–1786, 1983.

[KW10] P. Koellner and W. H. Woodin. Large Cardinals from Determinacy. In M. Foreman and A. Kanamori, editors, *Handbook of Set Theory*. Springer, 2010.

[Lar04] P. B. Larson. *The Stationary Tower: Notes on a Course by W. Hugh Woodin*, volume 32 of *University Lecture Series*. AMS, 2004.

[Lar17] P. B. Larson. *Extensions of the Axiom of Determinacy*. 10 2017. Preliminary version available at http://www.users.miamioh.edu/ larsonpb/adplusbook_public.pdf.

[LS67] A. Lévy and R. M. Solovay. Measurable Cardinals and the Continuum Hypothesis. *Israel J. Math.*, 5:234–248, 1967.

[MÏ8] S. Müller. The Axiom of Determinacy Implies Dependent Choices in Mice. *Submitted*, 2018. Preprint available at `https://muellersandra.github.io/publications/`.

[Mar] D. A. Martin. Untitled Book on Determinacy. Unpublished.

[Mar70] D. A. Martin. Measurable cardinals and analytic games. *Fundamenta Mathematicae*, 66:287–291, 1970.

[Mar75] D. A. Martin. Borel Determinacy. *Ann. Math.*, 102(2):363–371, 1975.

[Mar08a] D. A. Martin. The largest countable this, that, and the other. In A. S. Kechris, B. Löwe, and J. R. Steel, editors, *Games, Scales and Suslin Cardinals, The Cabal Seminar, Volume I.* Cambridge University Press, 2008.

[Mar08b] D. A. Martin. The Real Game Quantifier Propagates Scales. In A. S. Kechris, B. Löwe, and J. R. Steel, editors, *Games, Scales and Suslin Cardinals, The Cabal Seminar, Volume I.* Cambridge University Press, 2008.

[Mat15] A. R. D. Mathias. Provident Sets and Rudimentary Set Forcing. *Fundamenta Mathematicae*, 230:99–148, 2015.

[Mos71] Y. N. Moschovakis. Uniformization in a playful universe. *Bull. Amer. Math. Soc.*, 77:731–736, 1971.

[Mos72] Y. N. Moschovakis. The Game Quantifier. *Proc. Amer. Math. Soc.*, 31:245–250, 1972.

[Mos74] Y. N. Moschovakis. Structural Characterizations of Classes of Relations. In J. E. Fenstad and P. G. Hinman, editors, *Generalized Recursion Theory*, pages 53–79. 1974.

[Mos08] Y. N. Moschovakis. Inductive Scales on Inductive Sets. In A. S. Kechris, B. Löwe, and J. R. Steel., editors, *Games, Scales, and Suslin Cardinals. The Cabal Seminar, Volume I*, pages 94–101. The Association of Symbolic Logic, 2008.

[Mos09] Y. N. Moschovakis. *Descriptive set theory, second edition*, volume 155 of *Mathematical Surveys and Monographs*. AMS, 2009.

[MS89] D. A. Martin and J. R. Steel. A proof of projective determinacy. *J. Amer. Math. Soc.*, 2:71–125, 1989.

[MS94] W. J. Mitchell and J. R. Steel. *Fine structure and iteration trees*. Lecture notes in logic. Springer-Verlag, Berlin, New York, 1994.

[MS04] W. J. Mitchell and R. Schindler. A universal extender model without large cardinals in V. *Journal of Symbolic Logic*, 69(2):371–386, 2004.

[MS08] D. A. Martin and J. R. Steel. The Extent of Scales in $L(\mathbb{R})$. In A. S. Kechris, B. Löwe, and J. R. Steel., editors, *Games, Scales, and Suslin Cardinals. The Cabal Seminar, Volume I*, pages 110–120. The Association of Symbolic Logic, 2008.

[MS11] A. Montalbán and R. A. Shore. The Limits of Determinacy in Second-Order Arithmetic. *Proc. London Math. Soc.*, 104:223–252, 2011.

[MS17] S. Müller and G. Sargsyan. $\mathrm{HOD}^{M_n(x,g)}$ is a core model. 2017. In preparation.

[MSW16] S. Müller, R. Schindler, and W. H. Woodin. Mice with Finitely many Woodin Cardinals from Optimal Determinacy Hypotheses. 2016. Submitted.

[MT08] M. O. MedSalem and K. Tanaka. Weak Determinacy and Iterations of Inductive Definitions. In C. T. Chong, Q. Fend, T. A. Slaman, W. H. Woodin, and Y. Yang, editors, *Computational Prospects of Infinity*. National University of Singapore, 2008.

[Myc64] J. Mycielski. On the axiom of determinateness. *Fund. Math.*, 53(2):205–224, 1964.

[MZ06] G. Moser and R. Zach. The Epsilon Calculus and Herbrand Complexity. *Studia Logica*, 82(1):133–155, 2006.

[Nee95] I. Neeman. Optimal Proofs of Determinacy. *Bulletin of Symbolic Logic*, 1(3):327–339, 09 1995.

[Nee02] I. Neeman. Optimal Proofs of Determinacy II. *Journal of Mathematical Logic*, 2(2):227–258, 11 2002.

[Nee04] I. Neeman. *The Determinacy of Long Games*. De Gruyter series in logic and its applications. Walter de Gruyter, 2004.

[Nee05] I. Neeman. *An introduction to proofs of determinacy of long games*, pages 43—-86. Lecture Notes in Logic. Cambridge University Press, 2005.

[Nee07] I. Neeman. Games of length ω_1. *J. Math. Log*, 07(01):83–124, 2007.

[Ore82] V. P. Orevkov. Lower Bounds for Increasing Complexity of Derivations after Cut Elimination (in Russian). *J. Soviet Math.*, pages 2337–2350, 1982.

[Rat05a] M. Rathjen. An Ordinal Analysis of Parameter-Free Π^1_2-Comprehension. *Arch. Math. Logic*, 44:263–362, 2005.

[Rat05b] M. Rathjen. An Ordinal Analysis of Stability. *Arch. Math. Logic*, 44:1–62, 2005.

[Rud97] M. Rudominer. Mouse sets. *Ann. Pure Appl. Logic*, 87(1):1–100, 1997.

[Sar09] G. Sargsyan. *A tale of hybrid mice*. PhD thesis, University of California, Berkeley, 2009.

[Sar13] G. Sargsyan. On the prewellorderings associated with directed systems of mice. *J. Symb. Log.*, 78(3):735–763, 2013.

[Sar14] G. Sargsyan. Nontame mouse from the failure of square at a singular strong limit cardinal. *J. Math. Log.*, 14, 06 2014.

[Sch06] R. Schindler. Core models in the presence of Woodin cardinals. *J. Symb. Log.*, 71(4):1145–1154, 2006.

[Seg71] K. Segerberg. An essay in classical modal logic. *Filosofiska Föreningen och Filosofiska Institutionen vid Uppsala Universitet*, 1971.

[Sim78] S. G. Simpson. Short Course on Admissible Recursion Theory. In J. E. Fenstad, R. O. Gandy, and G. E. Sacks, editors, *Generalized Recursion Theory, II*, pages 355–390. 1978.

[SS99] E. Schimmerling and J. R. Steel. The Maximality of the Core Model. *Trans. Amer. Math. Soc.*, 351(8), 1999.

[SS09] R. Schindler and J. R. Steel. The self-iterability of $L[E]$. *J. Symb. Log.*, 74(3):751–779, 2009.

[SS14] R. Schindler and J. R. Steel. *The core model induction*. 2014. Book draft.

[SS15] G. Sargsyan and J. Steel. The mouse set conjecture for sets of reals. *J. Symbolic Logic*, 80(2):671–683, 2015.

[SSZ02] R. Schindler, J. R. Steel, and M. Zeman. Deconstructing inner model theory. *Journal of Symbolic Logic*, 67:721–736, 2002.

[ST] G. Sargsyan and N. D. Trang. *The Largest Suslin Axiom*. Preprint available at `http://sites.math.rutgers.edu/~gs481/lsa.pdf`.

[ST16] F. Schlutzenberg and N. D. Trang. Scales in hybrid mice over $\mathbb{R}$. *Forthcoming*, 2016.

[Sta79] R. Statman. Lower Bounds on Herbrand's Theorem. *Proc. Amer. Math. Soc.*, 75:104–107, 1979.

[Stea] J. R. Steel. An optimal consistency strength lower bound for $\mathsf{AD}_{\mathbb{R}}$. Unpublished.

[Steb] J. R. Steel. The Derived Model Theorem. Unpublished.

[Ste77] J. R. Steel. *Determinateness and Subsystems of Analysis*. PhD thesis, University of California at Berkeley, 1977.

[Ste93] J. R. Steel. Inner models with many Woodin cardinals. *Ann. Pure Appl. Logic*, 65:185–209, 1993.

[Ste95] J. R. Steel. Projectively well-ordered inner models. *Ann. Pure Appl. Logic*, 74, 1995.

[Ste96] J. R. Steel. *The core model iterability problem*, volume 8 of *Lecture Notes in Logic*. Springer-Verlag, Berlin, New York, 1996.

[Ste08a] J. R. Steel. Closure Properties of Pointclasses. In A. S. Kechris, B. Löwe, and J. R. Steel, editors, *Wadge Degrees and Projective Ordinals, The Cabal Seminar, Volume II*. Cambridge University Press, 2008.

[Ste08b] J. R. Steel. Scales in $K(\mathbb{R})$. In A. S. Kechris, B. Löwe, and J. R. Steel, editors, *Games, Scales and Suslin Cardinals, The Cabal Seminar, Volume I*. Cambridge University Press, 2008.

[Ste08c] J. R. Steel. Scales in $L(\mathbb{R})$. In A. S. Kechris, B. Löwe, and J. R. Steel, editors, *Games, Scales and Suslin Cardinals,*

The Cabal Seminar, Volume I. Cambridge University Press, 2008.

[Ste10] J. R. Steel. An Outline of Inner Model Theory. In M. Foreman and A. Kanamori, editors, *Handbook of Set Theory*. Springer, 2010.

[SW16] J. R. Steel and W. H. Woodin. HOD as a core model. In A. S. Kechris, B. Löwe, and J. R. Steel, editors, *Ordinal definability and recursion theory, The Cabal Seminar, Volume III*. Cambridge University Press, 2016.

[SZ10] R. Schindler and M. Zeman. Fine Structure. In M. Foreman and A. Kanamori, editors, *Handbook of Set Theory*. Springer, 2010.

[Tak80] G. Takeuti. *Proof Theory*. 1980.

[Tan90] K. Tanaka. Weak Axioms of Determinacy and Subsystems of Analysis I: Δ^0_2 Games. *Zeitsch. f. math. Logik und Grundlagen d. Math*, 36:481–491, 1990.

[Tra13] N. D. Trang. *Generalized Solovay Measures, the HOD analysis, and the Core Model Induction*. 2013. PhD Thesis. UC Berkeley.

[Tra14] N. D. Trang. Determinacy in $L(\mathbb{R}, \mu)$. *J. Math. Log.*, 14(1), 2014.

[Tra15] N. D. Trang. Structure theory of $L(\mathbb{R}, \mu)$ and its applications. *J. Symb. Log.*, 80(1):29–55, 2015.

[Uhl16] S. Uhlenbrock. *Pure and Hybrid Mice with Finitely Many Woodin Cardinals from Levels of Determinacy*. PhD thesis, WWU Münster, 2016.

[vBB06] J. van Benthem and G. Bezhanishvili. *Modal logics of space*. Institute for Logic, Language and Computation, 2006.

[Wel11] P. D. Welch. Determinacy in Strong Cardinal Models. *J. Symbolic Logic*, 76:719–728, 2011.

[Wil12] T. M. Wilson. *Contributions to Descriptive Inner Model Theory*. PhD thesis, UC Berkeley, 2012.

[Woo88] W. H. Woodin. Supercompact cardinals, sets of reals, and weakly homogeneous trees. *Proc. Natl. Acad. Sci. USA*, 1988.

[Woo02] W. H. Woodin. Beyond Σ^2_1 absoluteness. *Proceedings of the International Congress of Mathematicians, Vol. I*, pages 515–524, 2002.

[Woo10] W.H. Woodin. *The Axiom of Determinacy, Forcing Axioms, and the Nonstationary Ideal*. De Gruyter series in logic and its applications. De Gruyter, 2010.

[Woo16] W. H. Woodin. Fine Structure at the Finite Levels of Supercompactness. 2016. Circulated manuscript. 710 pages.

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