Linear Time-Invariant Filters: Frequency Preservation

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In this note I will explore and show that Linear Time-Invariant (LTI) filters preserve the frequency content of the signal being filtered.

Background and Setting

Although, I won't show it here, all LTI filters can be represented as convolutional operators, but first let's define what an LTI operator/filter is. O yeah, one more thing, we'll need to define the *shift* operator $\mathcal{S}$ to make the presentation concise.

Let $\mathcal{L}^\infty(\mathbb{R})$ denote the (linear) space of essentially bounded functions on $\mathbb{R}$ (i.e. all Lebesgue measurable functions which are unbounded on a set of measure zero), then a shift operator $\mathcal{S}$ is defined as,

$$\mathcal{S}(f;T)(x) := f(x-T) \qquad \forall\, T \in \mathbb{R},\ f \in \mathcal{L}^\infty(\mathbb{R})$$

Next, a linear time-invariant operator $\mathcal{T}$ on $\mathcal{L}^\infty(\mathbb{R})$,

$$\mathcal{T} : \mathcal{L}^\infty(\mathbb{R}) \to \mathcal{L}^\infty(\mathbb{R})$$

is a linear operator subject to the following conditions,

1) $\mathcal{T}(f+g)(x) = \mathcal{T}(f+g)(x) + \mathcal{T}(f+g)(x)$ for all $f, g \in \mathcal{L}^\infty(\mathbb{R})$ and all $x \in \mathbb{R}$
2) $\mathcal{S}(f;T) \circ \mathcal{T} = \mathcal{T} \circ \mathcal{S}(f;T)$ for all $f \in \mathcal{L}^\infty$ and all $T \in \mathbb{R}$

where the second condition is a succinct way of saying that we can interchangeably filter and shift or shift and filter [technical speak: the operators commute].

Theorem: Let $\mathcal{T}$ be a linear time-invariant operator on $\mathcal{L}^\infty(\mathbb{R})$, then there exists a Lebesgue integrable function $h : \mathbb{R} \to \mathbb{R}$ such that $h \in \mathcal{L}^1(\mathbb{R})$ and,

$$\mathcal{T}(f)(t) = \int_{-\infty}^{\infty} h(\tau) f(t-\tau)\ d\tau := (h * f)(t)$$

Proof: To show that an LTI operator is a convolution-type operator is a lengthy and technical proof, which hopefully I'll get around to typing up in the future.

Frequency Preservation

Now the fun part, let's take i) a complex sinusoid $s(t;\omega) := \mathrm{e}^{i\omega t} = \cos(\omega t) + i\sin(\omega t)$, which is clearly a bounded function for all $t, \omega \in \mathbb{R}$ and ii) an LTI filter $\mathcal{T}$ defined via convolutional kernel $h : \mathbb{R} \to \mathbb{R}$, then,

$$\begin{aligned}
\mathcal{T}(s(\cdot;\omega))(t) &= \int_{-\infty}^{\infty} h(\tau)\mathrm{e}^{i\omega(t-\tau)}\ d\tau = \mathrm{e}^{i\omega t} \underbrace{\int_{-\infty}^{\infty} h(\tau)\mathrm{e}^{-i\omega\tau}\ d\tau}_{z\in\mathbb{C}} \\
&= z\mathrm{e}^{i\omega t} = A\mathrm{e}^{i\phi}\mathrm{e}^{i\omega t} \\
&= \boxed{A\mathrm{e}^{i(\omega t+\phi)}} \\
&= A\mathrm{e}^{i\phi} s(t;\omega)
\end{aligned}$$

Summary

Hence, in the language of linear operator theory, the complex exponentials are eigenfunctions of LTI operators. Without getting too deep into the technical details of basis functions on $\mathcal{L}^2(\mathbb{R})$, but assuming that our function (signal) of interest is expressible as a countable linear combination of complex sinusoids or that it is Fourier transformable, then we have that the frequency content of an LTI filtered signal is preserved.