Beamforming Weights

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Setup:

Suppose we have a data collection instrument with N sensors/channels. Each sensor records either a sample or finite-duration time-series (stochastic processs) with some additive (stationary) noise. The noise in each channel follows the same (finite second moment) distribution and is independent from other channels. Next, we carry out a series of experiments where we collect $k \leq N$ usable samples/time-series and want to sum these amplitudes together to form a single output - beamform. In the case of the time-series we'll be considering a filtered time gate (or single sample) from each process.

Question:

Given the above setup, the question we're considering here is whether to scale the output by some function of k or not.

Solution:

Let S_i be a sample from the i-th channel. In the case of time-series we'll be considering $S_i(t)$ for some fixed t, where,

$$S_i = S_i(t) = \int_{-\infty}^{\infty} h(t-\tau) T_i(\tau - \phi_i(t))\, d\tau$$

where T_i is the time-series from the i-th channel, then the following collection,

$$S_i \overset{i.i.d}{\sim} \mathcal{D}, \quad i \in \{1, ..., N\}$$

of *independently and identically distributed (i.i.d)* random variables satisfies,

$$\mathbb{E}[S_i] = \mu_i$$

$$\mathsf{Var}(S_i) = \sigma^2$$

Next, define,

$$X_i := S_i - \mu_i \implies \mathbb{E}[X_i] = 0$$

$$\hat{S}_k := \sum_{i=1}^{k} S_i = \sum_{i=1}^{k} X_i + \sum_{i=1}^{k} \mu_i$$

$$\hat{X}_k := \sum_{i=1}^{k} X_i$$

then for any $c \in \mathbb{R}$

$$\mathbb{E}[c\hat{X}_k] = 0$$

$$\mathbb{E}[c\hat{S}_k] = c \sum_{i=1}^{k} \mu_i$$

and by independence

$$\mathsf{Var}(c\hat{X}_k) = c^2\mathsf{Var}\left(\sum_{i=1}^{k} X_i\right) = c^2 k\sigma^2$$

$$\mathsf{Var}(c\hat{S}_k) = c^2\sum_{i=1}^{k}\mathsf{Var}(S_i) = c^2\sum_{i=1}^{k}\mathsf{Var}(X_i) = c^2 k\sigma^2$$

The (amplitude) *signal-to-random noise ratio* (SRNR) is defined as,

$$\mathsf{SRNR}(c\hat{S}_k) := \frac{\mathbb{E}\big[c\hat{S}_k\big]}{\sqrt{\mathsf{Var}\big(c\hat{S}_k\big)}} = \frac{c\sum_{i=1}^{k}\mu_i}{\sqrt{c^2 k\sigma^2}}$$

$$= \frac{1}{\sigma\sqrt{k}}\sum_{i=1}^{k}\mu_i$$

and if $\mu_i \approx \mu$ for all i,

$$\mathsf{SRNR}(c\hat{S}_k) := \sqrt{k}\frac{\mu}{\sigma}$$

Therefore, if we're considering a *delay (filter)-and-sum* beamformer with variable number of input signals, then scaling the output $\hat{S}_k$ of the beamformer **does not change** the *signal-to-random noise ratio.*

With the RMS value of the signal $\hat{S}_k$ given as,

$$\mathsf{RMS}(\hat{S}_k) = \sqrt{\mathbb{E}[(\hat{S}_k)^2]} = \sqrt{\mathsf{Var}(\hat{S}_k) + \mathbb{E}[\hat{S}_k]^2} = \sqrt{k\sigma^2 + \left(\sum_{i=1}^{k}\mu_i\right)^2}$$

and assuming $\mu_i \approx \mu$ for all i,

$$\boxed{\mathsf{RMS}(\hat{S}_k) = \sqrt{k\sigma^2 + k^2\mu^2} = \sqrt{k}\sqrt{\sigma^2 + k\mu^2}}$$

Summary:

From the above equation we observe that whenever the inputs S_i carry low signal-to-random noise ratio (i.e. $\sigma \gg \mu$) we get,

$$\mathsf{RMS}(\hat{S}_k) \approx \sigma\sqrt{k}$$

and we should normalize the beamformer output $\hat{S}_k$ by $\sqrt{k}$ to maintain **constant** RMS value. However, if the signal-to-random noise is high (i.e. $\sigma \ll \mu$) then,

$$\mathsf{RMS}(\hat{S}_k) \approx \mu k$$

and we should normalize the beamformer output $\hat{S}_k$ by k to maintain **constant** RMS value.