What Dix's Conversion Formula is Not and its Misuses

by Adam Gogacz

> The ubiquity of depth estimation from observed/derived RMS velocity field cannot be overstated; it's everywhere. Anywhere there are time-migrations or beamforming or travel-time wavefront curvature fitting methods, such as in, seismic, acoustic, or ground-penetrating radar (GPR), we'll eventually see maps or cross-sections showing features (e.g. horizons, boulders, cables, UXOs) in depth. But how was that depth estimate arrived at? Well, it's always a home-brewed variant of Dix's conversion, but seldom, if ever, are the assumptions conveyed to the final user (e.g. engineer). This note is intended to pop the bonnet and see what's going on in there, and maybe in a follow-up note, I'll explore the range of depth estimation errors associated with these arbitrary assumptions.

Long story short (see [1] for some background), the kinematics of a reflected wave in a horizontally layered (parallel bedding) medium are well approximated[1] by the celebrated **NMO** (normal move-out) equation,

$$t^2(t_0, h) = t_0^2 + \frac{h^2}{[v_{rms}(t_0)]^2}$$

where

- h is the distance between a transmitter and a sensor, both located on the same horizontal plane,
- t_0 is the two-way (down and up) travel time to some layer boundary for co-located transmitter and sensor,
- $v_{rms}(t_0)$ is the root-mean-square of the velocities of each traversed layer, where the velocity in each layer is assumed to be homogenous and isotropic (i.e. constant in all directions and throughout the entire layer),
- $t^2(t_0, h)$ is the two-way travel time to t_0-determined layer boundary, but as recorded by a sensor h distance units away from the source.

Riemann integration to the rescue

By sticking to piecewise continuous functions, we can work entirely with Riemann integration, and there will be no need for Lebesgue to get involved. In the above description the RMS velocity v_{rms} was defined as,

$$v_{rms}(t) := \sqrt{\frac{1}{t}\sum_{i=1}^{n} v_i^2 \Delta t_i} \qquad \text{where } t = \sum_{i=1}^{n} \Delta t_i$$

where Δt_i is the two-way travel time through the i-th layer. Equivalently, we can work with,

$$v_{rms}^2(t) := \frac{1}{t}\sum_{i=1}^{n} v_i^2 \Delta t_i$$

and if the point or instantaneous velocity, v, of the medium under consideration is expressible as a piecewise continuous function, then,

$$v_{rms}^2(t) := \frac{1}{t}\int_0^t v^2(\tau)\ d\tau \qquad t \geq 0$$

[1]The approximation is valid for a small transmitter to sensor distance h.

Multiplying both sides by t leads to,

$$t\ v_{rms}^2(t) = \int_0^t v^2(\tau)\ d\tau$$

and hence for any $0 \leq t_1 < t_2$,

$$\boxed{t_2\ v_{rms}^2(t_2) - t_1\ v_{rms}^2(t_1) = \int_{t_1}^{t_2} v^2(\tau)\ d\tau}$$

Observations of v_{rms} and arbitrariness of upsampling[2] strategies

Now, suppose our experimental design only allows for coarse'ish sampling of v_{rms}, but what we're ultimately seeking is the knowledge of average interval velocities for a much finer sampled grid. looking at the boxed equation above, it's perfectly clear that once we integrate v^2 over some interval, then we cannot reconstruct the function from the integral value, not without some assumptions either about v_{rms} or v.

Assuming v is a step (piecewise constant) function

Suppose we sample v_{rms} at t_0 and t_1 ($t_0 < t_1$), but we would like to know the average value of v on each sub-interval of $t_0 = t_{0_0} < t_{0_1} < \cdots < t_{0_m} = t_1$ (*upsampling* step) and suppose further that on each sub-interval $[t_{0_i}, t_{0_{i+1}}]$ the function v is constant, then on $[t_0, t_1]$ we can express v as,

$$v(t) = \sum_{j=1}^{m} a_j \chi_{[t_{0_{j-1}}, t_{0_j}]}(t) \qquad a_j > 0 \text{ for all } 1 \leq j \leq m$$

where $\chi_{[t_{0_{j-1}}, t_{0_j}]}$ is the characteristic function of the interval $[t_{0_{j-1}}, t_{0_j}]$ (i.e. 1 on the interval and 0 outside of the interval). Hence,

$$\begin{aligned} t_1\ v_{rms}^2(t_1) - t_0\ v_{rms}^2(t_0) &= \int_{t_0}^{t_1} v^2(\tau)\ d\tau \\ &= \int_{t_0}^{t_1} \sum_{j=1}^{m} a_j^2 \chi_{[t_{0_{j-1}}, t_{0_j}]}(\tau)\ d\tau \\ &= \sum_{j=1}^{m} a_j^2 (t_{0_j} - t_{0_{j-1}}) \end{aligned}$$

This is it, and this is the punch line; all we need is for the a_j's to satisfy the above equation. So, if we only know v_{rms} at t_0 and t_1 and we don't make any additional assumptions, then for any partitioning of an interval $[t_0, t_1]$, we can construct an **infinite collection of functions** which will give us the observed/sampled v_{rms} function, and in this case, we have $m - 1$ degrees of freedom.

Assuming v_{rms} is piecewise linear - bad idea

Another frequently and erroneously used assumption is that v_{rms} is piecewise linear; that is, v_{rms} is continuous and made up of linear sections; this cannot happen unless the entire v_{rms} is linear.

[2]Upsampling in this context implies the assignment of values to v or an average v on some sub-interval.

The point here is that v_{rms} is everywhere **differentiable**[3], which implies *no kinks* allowed. For the skeptics, recall from above,

$$t\ v_{rms}^2(t) = \int_0^t v^2(\tau)\ d\tau \qquad \text{differentiate both sides}$$

$$v_{rms}^2(t) + 2t\ v_{rms}(t)\frac{dv_{rms}}{dt}(t) = v^2(t) \qquad \text{by the Fundamental Theorem of Calculus}$$

$$\implies \frac{dv_{rms}}{dt}(t) = \frac{v^2(t) - v_{rms}^2(t)}{2t\ v_{rms}(t)}$$

Hence, the derivative is well-defined as $v_{rms}(t) > 0$ for all $t > 0$, and therefore v_{rms} cannot have any kinks.

Spline interpolating v_{rms}

If we fit splines through the observation points such that the splines are differentiable everywhere, then calculus requirements are satisfied; however, any choice of splines is arbitrary and therefore, the obtained $v(t)$ is equally arbitrary.

Summary and Consequences

At the upsampling and conversion stage, many software implementations make simplifying assumptions such as (erroneous) piecewise linearity of v_{rms} or constraints on the distribution of $\{a_j\}$; however, all these constraints, which yield some sort of 'unique' solution, are in the end completely arbitrary.

Perhaps, the most significant consequence of the above-described arbitrariness is when RMS velocities are used for depth conversion. One can carry out many case studies to understand the impact of an underlying assumption about the structure of v_{rms} or v itself and its impact. From the Bayesian perspective, the depth conversion is always carried out using the prior!

In practice, the client is given values such as depth to horizon or depth to buried infrastructure, and without understanding the methods, takes these values at face value because the operator/contractor said so. But the final nail in the coffin is that, too often, the operators themselves don't understand the software they're using and the built-in assumptions.

References

[1] Hubral & Krey

[3] v_{rms} is differentiable by the Fundamental Theorem of Calculus.