The RMS Transform

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With technical details aside but explored below, can the root-mean-square of a function decrease with increasing parameter value? Short answer - sometimes.

Setting

Consider the space of positive and piecewise continuous functions defined on the *non-negative* real line,

$$\mathcal{PC}_+[0,\infty]$$

and the space of positive and piecewise continuous functions defined on the *positive* real line,

$$\mathcal{PC}_+(0,\infty)$$

then the root-mean-square (RMS) transform,

$$\mathcal{R} : \mathcal{PC}_+[0,\infty] \to \mathcal{PC}_+(0,\infty)$$

defined as,

$$\mathcal{R}(f)(t) := \sqrt{\frac{1}{t}\int_0^t f(\tau)\ d\tau}$$

is well-defined in the Riemann or Lebesgue sense with the standard topology on $\mathbb{R}$ and associated Borel σ-algebra.

Question (technical)

Given any $0 < t_1 < t_2 < \infty$, then what are the necessary and sufficient conditions on $f \in \mathcal{PC}_+[0,\infty]$ such that $\mathcal{R}(f)(t_1) > \mathcal{R}(f)(t_2)$?

Solution

Let $f \in \mathcal{PC}_+[0,\infty]$ and $0 < t_1 < t_2 < \infty$, then

$$\begin{aligned}
\mathcal{R}(f)(t_1) > \mathcal{R}(f)(t_2) &\iff \mathcal{R}(f)(t_1)^2 > \mathcal{R}(f)(t_2)^2 \\
&\iff \frac{1}{t_1}\int_0^{t_1} f(\tau)\ d\tau > \frac{1}{t_2}\int_0^{t_2} f(\tau)\ d\tau \\
&\iff t_2\int_0^{t_1} f(\tau)\ d\tau > t_1\int_0^{t_2} f(\tau)\ d\tau \\
&\iff t_2\int_0^{t_1} f(\tau)\ d\tau > t_1\left(\int_0^{t_1} f(\tau)\ d\tau + \int_{t_1}^{t_2} f(\tau)\ d\tau\right) \\
&\iff (t_2 - t_1)\int_0^{t_1} f(\tau)\ d\tau > t_1\int_{t_1}^{t_2} f(\tau)\ d\tau \\
&\iff \boxed{\frac{1}{t_1}\int_0^{t_1} f(\tau)\ d\tau > \frac{1}{(t_2 - t_1)}\int_{t_1}^{t_2} f(\tau)\ d\tau}
\end{aligned}$$

Example

Guided by the above condition, let $f \in \mathcal{PC}_+[0,\infty]$ be defined as follows,

$$f(t) = \begin{cases} 4 & 0 \le t \le 1 \\ 1 & t > 1 \end{cases}$$

and let $t_1 = 1$ and $t_2 = 2$, then

$$\mathcal{R}(f)(1) = \sqrt{\frac{1}{1}\int_0^1 4 \ d\tau} = 2$$

and

$$\mathcal{R}(f)(2) = \sqrt{\frac{1}{2}\left(\int_0^1 4 \ d\tau + \int_1^2 1 \ d\tau\right)} = \sqrt{\frac{5}{2}}$$

$$< 2 = \mathcal{R}(f)(1)$$