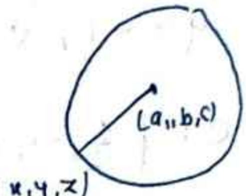
* **Sphere** – Locus of a moving point s.t its distance from a fixed point always remains constant, this fixed point is known as centre of sphere & fixed distance is known as radius of sphere.
(Atleast one Que of 10M)

- $(x-a)^2+(y-b)^2+(z-c)^2=r^2$

  Centre (a,b,c) & Radius $= r$



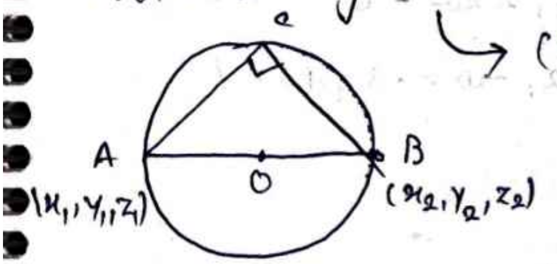
- General Eq^n: $x^2+y^2+z^2+2ux+2vy+2wz+d=0$

$$\text{Centre} = (-u,-v,-w) \equiv \left(-\frac{\text{coeff of } x}{2}, -\frac{\text{coeff of } y}{2}, -\frac{\text{coeff of } z}{2}\right)$$

$$r = \sqrt{u^2+v^2+w^2-d}$$

- Diametric form

$$\rightarrow (x-x_1)(x-x_2)+(y-y_1)(y-y_2)+(z-z_1)(z-z_2)=0$$



Q $3x^2+3y^2+3z^2-2x+3y-4z-6=0$, find radius & centre of sphere?

Ans $\div 3$
$S: x^2+y^2+z^2-\frac{2}{3}x+y-\frac{4}{3}z-2=0$

[Always make coeff of x^2, y^2 & z^2 unity i.e 1]

$$\text{Centre} = \left[-\left(\frac{-2}{3 \cdot 2}\right), -\left(\frac{1}{2}\right), -\left(\frac{-4}{3 \cdot 2}\right)\right]$$

$$= \left[\frac{1}{3}, -\frac{1}{2}, \frac{2}{3}\right] \quad ; \quad 2u = -\frac{2}{3}, \; 2v = 1, \; 2w = -\frac{4}{3}$$

$$u = -\frac{1}{3}, \; v = \frac{1}{2}, \; w = -\frac{2}{3}$$

$$d = -2$$

$$\text{Radius} = \sqrt{u^2+v^2+w^2-d}$$

$$= \sqrt{\frac{1}{9}+\frac{4}{9}+\frac{1}{4}+2}$$

$r =$

Q $x^2+y^2+z^2-2x+4y-2=0$, find centre & radius?

Ans $2u=-2,\ 2v=4,\ 2w=0$ & $d=-2$

$u=-1,\ v=2,\ w=0$ & $d=-2$

$C:\left(\frac{1}{2},-1,0\right)$ & $r=\sqrt{1+4+2}$ $r=\sqrt{7}$

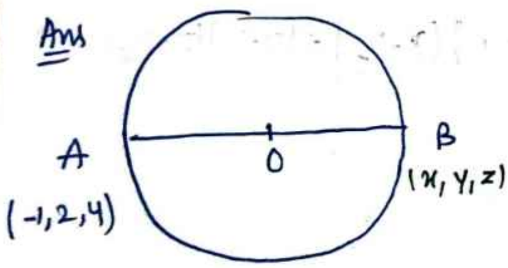
~~Q $x^2-y^2+z^2+2x-3y+4=0$~~ find Centre & Radius

Ans
~~$2u=2,\ 2v \neq$~~

Q Solved Ex-7 $x^2+y^2+z^2-2x+4y-6z-7=0$

One end of Diameter $(-1,2,4)$, other end of Diameter is?

Ans



$2u=-2,\ 2v=4,\ 2w=-6$

$u=-1,\ v=2,\ -w=-3,\ d=-7$

$C.\ (1,-2,3)$

$\frac{x-1}{2}=1 \Rightarrow x=3$

$\frac{y+2}{2}=-2 \Rightarrow y=-6$

$\frac{z+4}{2}=3 \Rightarrow z=2$

$(x,y,z)=(3,-6,2)$

Q** $(1,0,0)$, $(0,1,0)$ & $(0,0,1)$, find eq^n of sphere ~~with min radius~~ which passes through & has it radius as small as possible?

Imp

(Q6 of Book, Pg 7)

Ans Let Eq^n of Sphere $= x^2+y^2+z^2+2ux+2vy+2wz+d=0$ —(1)

At $(1,0,0)$: $1+2u+d=0 \therefore u = \frac{-(d+1)}{2}$

At $(0,1,0)$: $1+2v+d=0 \therefore v = \frac{-(d+1)}{2}$ —(2)

At $(0,0,1)$: $1+2w+d=0$, $w = \frac{-(d+1)}{2}$

$$r^2 = u^2+v^2+w^2-d$$

$$f(d) = r^2 = \frac{3}{4}(d+1)^2 - d$$

$$f'(d) = \frac{3}{4} \times 2(d+1) - 1 = 0$$

$$\Rightarrow \frac{3}{2}(d+1) = 1 \Rightarrow d+1 = \frac{2}{3}$$

$$\left[d = -\frac{1}{3}\right]$$

$$f''(d) = \frac{3}{2} > 0 \Big|_{\text{At } d = -\frac{1}{3}}$$

$\therefore$ → Min^m Occurs at d : Now Replacing by $-1/3$ in eq^n (2)

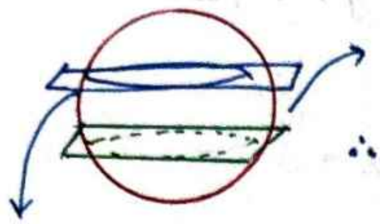
$u = v = w = -\frac{1}{2}\left(-\frac{1}{3}+1\right)$

$\therefore u = v = w = d = -1/3$

(1) $\therefore$ Eq^n of Sphere

$$\left[x^2+y^2+z^2-\frac{2}{3}x-\frac{2}{3}y-\frac{2}{3}z-\frac{1}{3}=0\right]$$

* Intersection of sphere & a plane

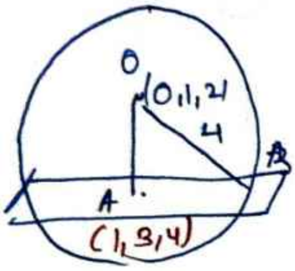


Circle: $S=0, U=0$

$\therefore$ Eq^n of sphere passing through circle: $S+\lambda U=0$

Great circle → largest Radius.

Q5 find centre & radius of circle $x^2+y^2+z^2-2y-4z=11$, $x+2y+2z=15$



centre = $(0,1,2)$

$r = \sqrt{0^2+1^2+2^2+11} = 4$

Eq^n of line OA

$$\frac{x-0}{1} = \frac{y-1}{2} = \frac{z-2}{2} = k$$

$A = (k, 2k+1, 2k+2)$

lies on Plane

$9k = 9$

$k + 4k + 2 + 4k + 4 = 15 \rightarrow \boxed{k=1}$

Centre of Circle → $A:(1,3,4)$

$OA = \sqrt{(1-0)^2+(3-1)^2+(4-2)^2} = \sqrt{1+4+4} = 3$

$r_c = \sqrt{4^2-3^2} = \sqrt{16-9}$

$r_c = \sqrt{7}$

Radius of circle

U.S.Q (9) find the eq^n of sphere through the circle $x^2+y^2+z^2+2x+3y+6=0$

$; x-2y+4z-9=0$ & passes through the centre of

$x^2+y^2+z^2-2x+4y-6z+5=0$

Ans Eq^n of sphere passing through circle - $S+\lambda U=0$

$$x^2+y^2+z^2+2x+3y+6+\lambda(x-2y+4z-9)=0 \quad -(1)$$

• Centre of $x^2+y^2+z^2\boxed{-2x+4y-6z}+5=0 \rightarrow (1,-2,3)$

* ~~(2) so~~ $(1,-2,3)$ satisfies eq^n (1)

$$1+4+9+2-6+6+\lambda(1+4+12-9)=0$$

$$16+\lambda 8=0 \Rightarrow \lambda=-2$$

(1)

$$x^2+y^2+z^2+2x+3y+6-2(x-2y+4z-9)=0$$

$$\left[S: x^2+y^2+z^2+7y-8z+24=0\right]$$

U.S (17) find the eq^n of sphere having the following circle as a great circle $x^2+y^2+z^2=16,\ 2x-3y+6z=7$

Ans $S_1: x^2+y^2+z^2-16+\lambda(2x-3y+6z-7)=0$ i.e $S+\lambda U$

$$x^2+y^2+z^2+2\lambda x-3\lambda y+6\lambda z-16-7\lambda=0 \quad -(1)$$

$$C:\left(-\lambda, \frac{3}{2}\lambda, -3\lambda\right)$$

lie on plane $2x-3y+6z=7$

$$2(-\lambda)-3\cdot\frac{3}{2}\lambda+6\cdot(-3\lambda)=7$$

$-4\lambda-9\lambda-36\lambda=14$

~~$2\lambda=14$~~ $49\lambda=14$

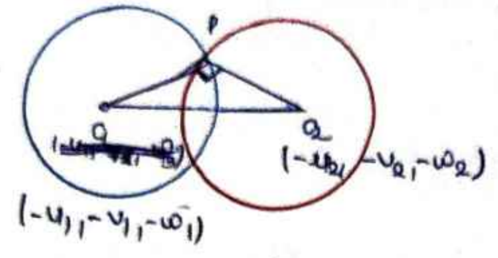
~~$\lambda=14/23$~~ $\lambda=\frac{14}{49}$

$$\left[\lambda=\frac{2}{7}\right]$$

(1)

$$\left[S: x^2+y^2+z^2-\frac{4}{7}x+\frac{6}{7}y-\frac{12}{7}z+2=0\right]$$

* Orthogonality of two sphere



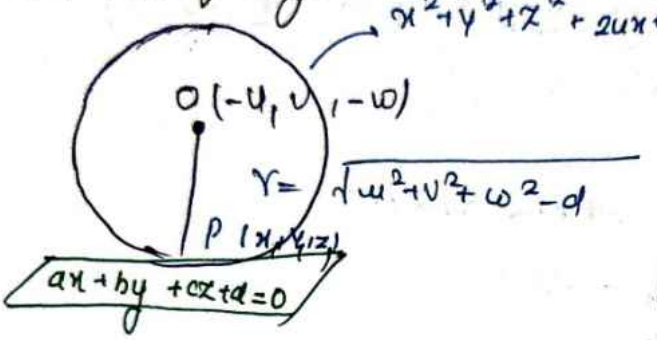
$S_1 : x^2+y^2+z^2+2u_1x+2v_1y+2w_1z+d_1=0$

$S_2 : x^2+y^2+z^2+2u_2x+2v_2y+2w_2z+d_2=0$

for orthogonality

$$2u_1u_2+2v_1v_2+2w_1w_2=d_1+d_2$$

* Condition of Tangent



$x^2+y^2+z^2+2ux+2vy+2wz+d=0$

$OP = r$

$$\left|\frac{-au-bv-cw+d}{\sqrt{a^2+b^2+c^2}}\right| = \sqrt{u^2+v^2+w^2-d}$$

* Eqⁿ of Tangent plane

$$xx_1+yy_1+zz_1+u(x+x_1)+(y+y_1)v+w(z+z_1)+d=0$$

* Q Prove that every sphere through the circle
$x^2+y^2-2ax+r^2=0,\ z=0$

Cuts orthogonally, $x^2+z^2=r^2,\ y=0$
to the sphere passing through

Eqⁿ of sphere passing through circle; $S+\lambda u=0$

$S_1: x^2+y^2+z^2-2ax+r^2=0,\ z=0$, $S_2: x^2+y^2+z^2=r^2,\ y=0$

$\therefore$ ~~$x^2+y^2+z^2$~~ $S_1: (x^2+y^2+z^2-2ax+r^2)+\lambda z=0$

$S_2: (x^2+y^2+z^2-r^2)+\mu y=0$

Cuts orthogonally

$2u_1u_2 + 2v_1v_2 + 2w_1w_2 = 0$

$$2\left[(0)(0) + (0)\left(\frac{u}{2}\right) + 2\left(\frac{\lambda}{2}\right)(0)\right] = -r^2 + r^2$$

$\boxed{0 = 0} \rightarrow \therefore$ Both are orthogonal.

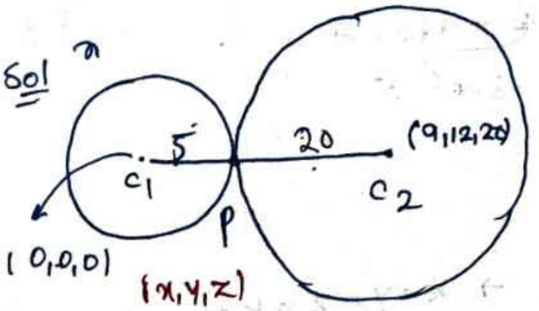
5

Q10 Show that the spheres $x^2+y^2+z^2=25$

$x^2+y^2+z^2-18x-24y-40z+225=0$

touch externally & find the co-ordinates of their common point

Sol



$r_1 = 5$, $c_1 = (0,0,0)$

$r_2 = 20$, $c_2 = (9,12,20)$

* If touching externally then

$C_1P + C_2P = C_1C_2$

$r_1 + r_2 = C_1C_2$

$5 + 20 = 25 \rightarrow$ Touches externally ~~Checked~~

$C_1C_2 = \sqrt{9^2+12^2+20^2}$

$C_1C_2 = 25$

$$x = \frac{5 \times 9 + 20 \times 0}{5+20} = \frac{45}{25}$$

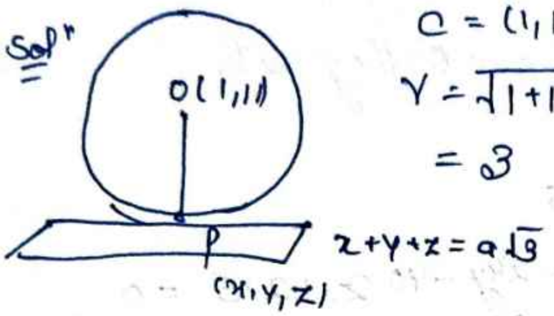
$$y = \frac{5 \times 12 + 20 \times 0}{5+20} = \frac{60}{25}$$

$$z = \frac{5 \times 20 + 20 \times 0}{5+20} = 4$$

$\Rightarrow P(x,y,z) = \left(\frac{45}{25}, \frac{60}{25}, \frac{100}{25}\right)$

(11) Find the value of 'a' for which the plane $x+y+z = a\sqrt{3}$ touches the sphere $x^2+y^2+z^2-2x-2y-2z=6$, also find the point of contact.

Sol^n



$C = (1,1,1)$

$r = \sqrt{1+1+1+6}$

$= 3$

$\Rightarrow OP = r$

$$\left|\frac{1+1+1-a\sqrt{3}}{\sqrt{3}}\right| = 3$$

$3 - a\sqrt{3} = \pm 3$

$3 = 3 + a\sqrt{3}$

$3 = -3 + a\sqrt{3}$

$3 - a\sqrt{3} = \pm 3$

$a = \sqrt{3} \mp 3$

eq^n of line OP

$$\frac{x-1}{1} = \frac{y-1}{1} = \frac{z-1}{1} = k \rightarrow x = y = z = k+1$$

$P = (k+1,\ k+1,\ k+1)$

lie on plane $x+y$

$$(k+1) + (k+1) + (k+1) = (\sqrt{3} \mp 3)\sqrt{3}$$

$$3k + \cancel{3} = \cancel{3} \mp 3\sqrt{3}$$

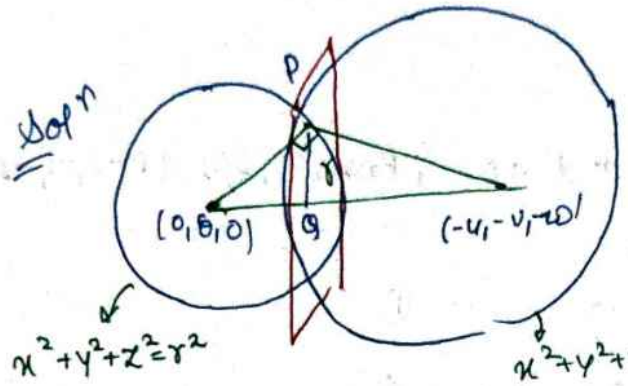
$$k = \mp\sqrt{3}$$

$$P(\mp\sqrt{3}+1,\ \mp\sqrt{3}+1,\ \mp\sqrt{3}+1)$$

S.E.7 Two spheres of radii r_1 & r_2 cuts orthogonally through the radius of common circle $= \dfrac{r_1 r_2}{\sqrt{r_1^2 + r^2}}$

→ USQ3, 9

Solⁿ



$u^2+v^2+w^2 = d + r_2^2$

$u^2+v^2+w^2 = r_1^2 + r_2^2$

↑

$r_2 = \sqrt{u^2+v^2+w^2-d}$

Cuts orthogonally

$$2(0)+2(0)+2(0) = -r_1^2 + d$$

$$\boxed{d = r_1^2}$$

Intersection plane : $S_1 - S_2 = 0$

$$2ux + 2vy + 2wz + d + r_1^2 = 0$$

$$ux + vy + wz + r_1^2 = 0$$

C_1Q is perpendicular from $C_1(0,0,0)$ to plane $ux+vy+wz+r_1^2=0$

$$C_1Q = \left| \frac{0 + r_1^2}{\sqrt{u^2+v^2+w^2}} \right|$$

$$C_1Q = \left| \frac{r_1^2}{\sqrt{r_1^2+r_2^2}} \right|$$

In ΔC_1PQ

$$r^2 = r_1^2 - C_1Q^2$$

$$r^2 = r_1^2 - \frac{r_1^4}{r_1^2+r_2^2} \rightarrow r^2 = \frac{r_1^4 + r_1^2 r_2^2 - r_1^4}{r_1^2+r_2^2}$$

↑

$$r = \frac{\pm r_1 r_2}{\sqrt{r_1^2+r_2^2}}$$

I(C)

Q13

Sol^n $x^2+y^2+z^2+2ux+2vy+2wz+d=0$; Passes $(0,3,0)$ & $(-2,-1,-4)$

At $(0,3,0)$ → $0+9+0+0+6v+0+d=0$ — ①

At $(-2,-1,-4)$ → $4+1+16-4u-2v-8w+d=0$ — ②

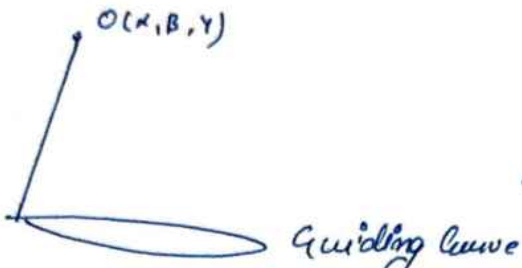
∘ Cuts Orthogonally : $x^2+y^2+z^2+x-3z$

$$x^2+y^2+z^2+\frac{x}{2}+\frac{\sqrt{3}y}{2}+\frac{4}{2}=0$$

$$\therefore\ 2u\cdot\frac{1}{2}+2v\cdot(0)+2w\cdot-\frac{3}{2}=d-2 \quad —③$$

$$2u\cdot\frac{1}{4}+2v\cdot\left(\frac{3}{4}\right)+2w\cdot 0=d+2 \quad — 4$$

Cone



- A surface generated by a straight line which passes through a fixed point & intersects a given curve is called a cone.

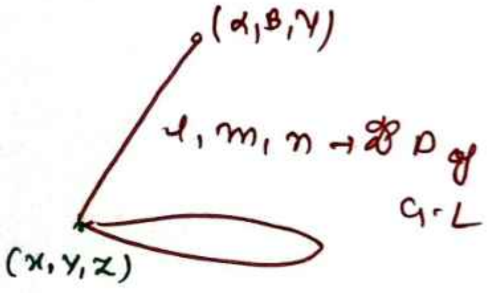
- fixed point is known as Vertex of Cone.
- Intersecting Surface → Guiding Curve
- Variable straight line → generating line of Cone.

→ The eqn of cone with vertex at origin is homogeneous eqn of second degree in x, y & z.

Q 2 find the eqn of the cone whose vertex is (α, β, γ) & Base is guiding curving $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 ; z = 0$ — ①

Soln Eqn of line passing through (α, β, γ)

$$\frac{x - \alpha}{l} = \frac{y - \beta}{m} = \frac{z - \gamma}{n}$$



- Put $z = 0$

$$\frac{x - \alpha}{l} = \frac{y - \beta}{m} = \frac{-\gamma}{n}, z = 0$$

$$x = \alpha - \frac{l\gamma}{n}, \ y = \beta - \frac{m\gamma}{n}, \ z = 0 \quad \text{— ②}$$

① ↳

$$\frac{\left(\alpha - \frac{l\gamma}{n}\right)^2}{a^2} + \frac{\left(\beta - \frac{l\gamma}{n}\right)^2}{b^2} = 1$$

$$\frac{1}{a^2}(\alpha n - l\gamma)^2 + \frac{1}{b^2}(\beta n - m\gamma)^2 = n^2$$

Eliminating eqn of l, m, & n from ②

$$\frac{1}{a^2}\left[\alpha[z-\gamma) - (x-\alpha)\gamma\right]^2 + \frac{1}{b^2}\left[\beta(z-\gamma) - (y-\beta)\gamma\right]^2 = [z-\gamma]^2$$

$$\frac{1}{a^2}\left[\alpha z - \cancel{\alpha\gamma} - x\alpha + \cancel{\alpha\gamma}\right]^2 + \frac{1}{b^2}\left[\beta z - \cancel{\beta\gamma} - y\gamma + \cancel{\beta\gamma}\right]^2 = [z-\gamma]^2$$

$$\frac{1}{a^2}\left[\alpha z - x\alpha\right]^2 + \frac{1}{b^2}\left[\beta z - y\gamma\right]^2 = [z-\gamma]^2$$

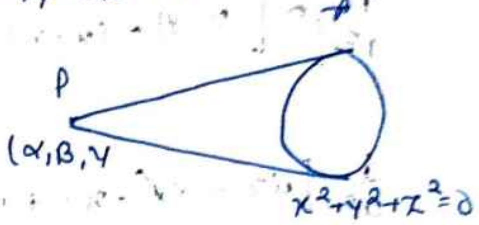
Q If the straight line $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$ intersects the curve $ax^2 + by^2 = 1$; $z = 0$, then Prove that

$$a(\alpha n - \gamma l)^2 + b(\beta n - \gamma m)^2 = n^2$$

Enveloping Cone

- The locus of Tangent lines drawn from a given point to a given surface is called an enveloping cone or Tangent cone.

- Eqⁿ of enveloping cone of the sphere : $x^2+y^2+z^2=a^2$ with vertex at (α,β,γ)



- Eqⁿ of PA: passing through (α,β,γ)

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} = r$$

$$x = \alpha + lr,\ y = \beta + mr,\ z = nr + \gamma$$

SPHERE

$$(lr+\alpha)^2 + (\beta + mr)^2 + (\gamma + nr)^2 = a^2$$

$$\rightarrow r^2(l^2+m^2+n^2) + 2r(l\alpha + m\beta + n\gamma) + (\alpha^2+\beta^2+\gamma^2-a^2) = 0$$

$\therefore$ for equal root $B^2 = 4AC$

$$\cancel{4}\cancel{r^2}(l\alpha + m\beta + n\gamma)^2 = \cancel{4}\cancel{r^2}(l^2+m^2+n^2)(\alpha^2+\beta^2+\gamma^2-a^2)$$

$$\left[(x-\alpha)\alpha + (y-\beta)\beta + (z-\gamma)\gamma\right]^2 = \left[(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2\right]\left[\alpha^2+\beta^2+\gamma^2-a^2\right]$$

Q Show that eqn to the cone whose vertex is Origin & Base the curve $z=c$, $f(x,y)=0$. is $f\left(\frac{xc}{z}, \frac{yc}{z}\right)=0$

Soln Eqn of line

$$\frac{x-0}{l} = \frac{y-0}{m} = \frac{z-0}{n}$$

Put $z=c$

$$x = \frac{lc}{n}, \; y = \frac{mc}{n} \; ; \; z=c$$

$$f(x,y)=0$$

$$\downarrow$$

$$f\left(\frac{lc}{n}, \frac{mc}{n}\right)=0$$

$$\downarrow$$

$$f\left(\frac{xc}{z}, \frac{yc}{z}\right)=0$$; Hence proved.

Q find eqn of the cone whose vertex is the point (α, β, γ) & Guiding Curve $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, $z=0$

S-B III rd

Ans Eqn of line passing throug (α,β,γ)

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$$

Put $z=0$

$$x = \alpha - \frac{l\gamma}{n}, \; y = \beta - \frac{m\gamma}{n} \; ; \; z=0$$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

$$\frac{1}{a^2}\left(\alpha - \frac{l\gamma}{n}\right)^2 + \frac{1}{b^2}\left(\beta - \frac{m\gamma}{n}\right)^2 = 1$$

$$\frac{1}{a^2}(\alpha n - l\gamma)^2 + \frac{1}{b}(\beta n - m\gamma)^2 = n^2$$

Using Eq^n.

$$\frac{1}{a^2}\left(\alpha(z-v) - (x-\alpha)v\right]^2 + \frac{1}{b^2}\left(\beta(z-v) - (y-\beta)v\right]^2 = (z-v)^2$$

$$\frac{1}{a^2}\left[\alpha z - xv\right]^2 + \frac{1}{b^2}\left[\beta z - yv\right]$$

IIA, 6

Q find eq^n of cone whose vertex is (α, β, γ) & Guiding Curve is parabola $y^2 = 4ax$; $z = 0$

Sol^n Eq^n of line passing through (α, β, γ)

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-v}{n}$$

Put $z = 0$

$$x = \alpha - \frac{lv}{n}, \quad y = \beta - \frac{mv}{n}, \quad z = 0$$

$$y^2 = 4ax$$

$$(\beta n - mv)^2 = 4a(\alpha n - lv)n$$

$$\left[\beta(z-v) - (y-\beta)m\right]^2 = 4a\left[\alpha(z-v) - (x-\alpha)v\right](z-v)$$

$$\left[\beta z - vy\right]^2 = 4a(z-v)\left[\alpha z - xv\right]$$

Q find the eq^n of cone through origin $(0,0,0)$ & which passes intersection of curves $x^2+y^2+z^2+2ax+b=0$, $lx+my+nz=p$

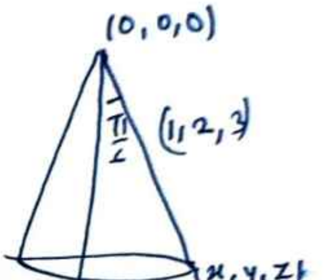
Sol^n

$$x^2+y^2+z^2+2ax(1)+b(1)^2 \qquad \frac{lx+my+nz}{p}=1$$

$$x^2+y^2+z^2+2ax\left(\frac{lx+my+nz}{p}\right)+b\left(\frac{lx+my+nz}{p}\right)^2$$

$$p^2(x^2+y^2+z^2)+2axp(lx+my+nz)+b(lx+my+nz)^2$$

Q $(0,0,0)$, Axis $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$, $\theta=\frac{\pi}{6}$

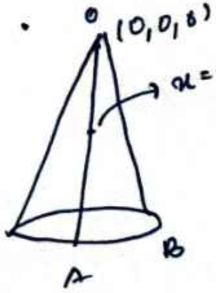


U.S III rq

R.C. Δ $V(0,0,0)$, $\frac{x}{1}=\frac{y}{1}=\frac{z}{1}$

& $2x=3y=-5z$ is one of the generators

$$\frac{x}{15}=\frac{y}{10}=\frac{z}{-6}$$



DR of OA = 1, 1, 1

DR of OB = x, y, z

Angle b/w OA & OB

$$\cos\theta=\frac{x+y+z}{\sqrt{1^2+1^2+1^2}\sqrt{x^2+y^2+z^2}}=\frac{x+y+z}{\sqrt{3}\cdot\sqrt{x^2+y^2+z^2}} \quad -(1)$$

DR of OC : 15, 10, -6

$$\cos\theta=\frac{1.15+1.10+1.-6}{\sqrt{3}\cdot\sqrt{10^2+15^2+6^2}}=\frac{\cancel{19}}{\sqrt{3}\,\cancel{\sqrt{19^2}}}=\frac{1}{\sqrt{3}}$$

eq^n (1)

$$\frac{1}{\cancel{\sqrt{3}}}=\frac{x+y+z}{\cancel{\sqrt{3}}\cdot\sqrt{x^2+y^2+z^2}} \Rightarrow x^2+y^2+z^2=(x+y+z)^2$$

$$[xy+yz+zx=0]$$

S.E I, II, IV, U.S.Q, 1, 3, 5