Particular Integral

$$f(D)\, y = Q(x). \quad \text{--- (1)}$$

We see that

If $y = \frac{1}{f(D)} Q(x)$.

then

$$f(D) \left\{ \frac{1}{f(D)} Q(x) \right\} = Q(x).$$

so $y = \frac{1}{f(D)} Q(x)$ is a Particular Solution of equation (1).

case I Let $f(D) = D$. then

$$P.I. = y = \frac{1}{f(D)} Q(x) = \frac{1}{D} Q(x) = \int Q(x)\, dx.$$

case II Let $f(D) = D - \alpha$. then

$$PI = y = \frac{1}{D-\alpha} Q(x)$$

$\Rightarrow$ $(D-\alpha)\, y = Q(x)$

$\frac{dy}{dx} - \alpha y = Q(x)$ which is Linear DE.

hence solution is

$$y e^{-\alpha x} = \int e^{-\alpha x} Q(x)\, dx$$

$$y = e^{\alpha x} \int e^{-\alpha x} Q(x)\, dx = PI$$

So if $\boxed{\frac{1}{D-\alpha} Q(x) = e^{\alpha x} \int Q(x)\, e^{-\alpha x}\, dx = P.I.}$

Example:- Solve $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = e^{4x}$.

$\Rightarrow \quad (D^2 - 5D + 6)y = e^{4x}$.

A.E $\quad m^2 - 5m + 6 = 0$.

$(m-2)(m-3) = 0. \quad \Rightarrow m = 2, 3.$

Real Non repeated,

$CF = c_1 e^{2x} + c_2 e^{3x}$.

Now we find P.I.

$$P.I. = \frac{1}{f(D)} Q(x) = \frac{1}{(D^2-5D+6)} e^{4x}.$$

$$= \frac{1}{(D-2)(D-3)} e^{4x}$$

$$= \left[\frac{1}{D-3} - \frac{1}{D-2}\right] e^{4x}$$

$$= \frac{1}{D-3} e^{4x} - \frac{1}{D-2} e^{4x}$$

using above method we have

$$= e^{3x}\int e^{-3x} e^{4x} dx - e^{2x}\int e^{-2x} e^{4x} dx.$$

$$= e^{3x} e^{x} - e^{2x} \frac{e^{2x}}{2}$$

$$= e^{4x} - \frac{e^{4x}}{2} = \boxed{\frac{1}{2} e^{4x} = PI}$$

Hence general solution is

$y = CF + PI$

$$PI = \frac{1}{(D^2+a^2)} \sec ax.$$

$$= \frac{1}{(D+ia)(D-ia)} \sec ax$$

$$= \frac{1}{2ia}\left[\frac{1}{D-ia} - \frac{1}{D+ia}\right] \sec ax.$$

$$= \frac{1}{2ia}\left[\frac{1}{D-ia}\sec ax - \frac{1}{D+ia}\sec ax\right]$$

$$= \frac{1}{2ia}\left\{ e^{iax}\int e^{-iax}\sec ax\,dx - e^{-iax}\int e^{iax}\sec ax\,dx \right\}.$$

$$= \frac{1}{2ia}\left\{ e^{iax}\int \sec ax(\cos ax - i\sin ax)\,dx - e^{-iax}\int \sec ax(\cos ax + i\sin ax)\,dx \right\}$$

$$= \frac{1}{2ia}\left\{ e^{iax}\left(\int dx - i\int \tan x\,dx\right) - e^{-iax}\left(\int dx + i\int \tan x\,dx\right)\right\}.$$

$$= \frac{1}{2ia}\left\{ e^{iax}\left(x + \frac{i}{a}\log \cos xa\right) - e^{-iax}\left(x + \frac{i}{a}\log \cos ax\right)\right\}.$$

$$= \frac{1}{2ia}\left\{ x(e^{iax} - e^{-ix}) + \frac{i}{a}\log \cos ax\,(e^{iax} + e^{-iax})\right\}.$$

$$= \frac{1}{2ia}\left\{ 2xi\sin ax + \frac{2i}{a}\log \cos ax \cdot \cos ax\right\}.$$

$$\boxed{PI = \frac{x}{a}\sin ax + \frac{1}{a^2}\cos ax \log \cos ax}$$

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Special Cases to find P.I.

① $Q(x) = e^{ax}$ a is constant.

i.e $PI = \frac{1}{f(D)} e^{ax}$.

Case I if $f(a) \neq 0$ then.

$$PI = \frac{1}{f(D)} e^{ax} = \frac{e^{ax}}{f(a)}. \qquad f(a) \neq 0.$$

Case II if $f(a) = 0$.

$\Rightarrow f(D) = (D-a)^r \phi(D)$ where $\phi(a) \neq 0$.

in this condition.

$$PI = \frac{1}{f(D)} e^{ax} = \frac{1}{(D-a)^r \phi(D)} e^{ax} = \frac{x^r}{\lfloor r} \cdot \frac{e^{ax}}{\phi(a)}$$

Example:- $3\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - y = e^{x/2} + 2e^{3x}$.

Sol. operator form:-

$$(3D^2 + 2D - 1)y = e^{x/2} + 2e^{3x}.$$

A.E $\Rightarrow$

$$3m^2 + 2m - 1 = 0$$

$$(3m-1)(m+1) = 0 \qquad m = 1/3, -1$$

$$\boxed{CF = c_1 e^{1/3 x} + c_2 e^{-x}.}$$

$$PI = \frac{1}{3D^2+2D-1}\left[e^{x/2} + 2e^{3x}\right]$$

$$= \frac{1}{(3D-1)(D+1)} e^{x/2} + \frac{2}{(3D-1)(D+1)} e^{3x}$$

$$= \frac{e^{x/2}}{\left(3\left(\frac{1}{2}\right)-1\right)\left(\frac{1}{2}+1\right)} + \frac{2}{(3\times3-1)\ (3+1)}\ e^{3x}$$

$$\boxed{PI = \frac{4}{3} e^{x/2} + \frac{1}{16} e^{3x}}$$

$$\boxed{\frac{1}{f(D)} e^{ax} = \frac{e^{ax}}{f(a)} \quad \therefore f(a) \neq 0}$$

Hence General Solution is given by.

$$\boxed{y = c_1 e^{x/3} + c_2 e^{-x} + \frac{4}{3} e^{x/2} + \frac{1}{16} e^{3x}.}$$

<u>Example 2:</u> <u>Solve</u> $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = e^x.$

<u>Sol.</u> Given

$$(D^2 - 3D + 2)y = e^x.$$

A.E

$$m^2 - 3m + 2 = 0$$

$$(m-1)(m-2) = 0 \qquad m = 1, 2.$$

$$CF = c_1 e^x + c_2 e^{2x}.$$

$$PI = \frac{1}{(D-1)(D-2)} e^x = \frac{x}{L1} \frac{e^x}{(+1-2)}$$

$$= -xe^x$$

$$\boxed{PI = -xe^x}$$

$$\boxed{\text{Since } \frac{1}{(D-\alpha)^r \phi(D)} e^{\alpha x} = \frac{x^r}{Lr} \frac{e^{\alpha x}}{\phi(\alpha)} \quad \text{provided } \phi(\alpha) \neq 0}$$

<u>Sol.</u> $y = c_1 e^x + c_2 e^{2x} - xe^x$

Important Questions:

① Solve $(D^2+6D+9)y = 2e^{-3x}$. $y=(c_1+c_2x)e^{-3x}+x^2e^{-3x}$.

② $(D^4-m^4)y = \cosh mx$.

Sol. $y = c_1e^{mx}+c_2e^{-mx}+c_3\cos mx+c_4\sin mx+\frac{x}{4m^3}\sinh mx$

Hint

$$PI = \frac{1}{D^4-m^4}\cosh mx$$

$$= \frac{1}{(D-m)(D+m)(D^2+m^2)}\cosh mx$$

$$= \frac{1}{2}\frac{1}{(D-m)(D+m)(D^2+m^2)}e^{mx}+\frac{1}{2}\frac{1}{(D-m)(D+m)(D^2+m^2)}e^{-mx}.$$

$$= \frac{1}{2}\frac{x}{\lfloor 1}\frac{e^{mx}}{(m+m)(m^2+m^2)}+\frac{1}{2}\frac{x}{\lfloor 1}\frac{e^{-mx}}{(-m-m)(m^2+m^2)}$$

$$= \frac{xe^{mx}}{8m^3}-\frac{xe^{-mx}}{8m^3} = \frac{x}{4m^3}\left(\frac{e^{mx}-e^{-mx}}{2}\right)$$

$$= \boxed{\frac{x}{4m^3}\sinh mx. = PI}$$

③ $\frac{d^2y}{dx^2}-y = \cosh y$. $y = c_1e^{x}+c_2e^{-x}+\frac{1}{2}x\sinh x$

Q4 $\frac{d^5y}{dx^5}-m^2\frac{d^3y}{dx^3} = e^{ax}$. $y = c_1+c_2x+c_3x^2+c_4e^{mx}+c_5e^{-mx}+\frac{e^{ax}}{a^3(a^2-m^2)}$

Q6 $\{(D^2+1)(D-1)^2\}y = e^{x}$. $y = (c_1+c_2x)e^{x}+(c_3+c_4x)\cos x+(c_5+c_6x)\sin x+\frac{1}{8}x^2e^{x}$.

Q7 $(D^3+1)y = (e^{x}+1)^2$

$$y = c_1e^{-x}+e^{x/2}\left(c_2\cos\frac{\sqrt{3}}{2}x+c_3\sin\frac{\sqrt{3}}{2}x\right)+\frac{1}{9}e^{2x}+e^{x}+1$$