

WAVE FUNCTION →

The wave function is represented by the mathematical symbol ψ or ϕ . The wave function ψ as such has no physical significance but the square of its magnitude, i.e. $|\psi|^2$ for a particular position and time represents the probability of finding the particle at that time.

Since $|\psi|^2$ is proportional to the probability P of finding a particle, the integral of $|\psi|^2$ over some space must give a finite value i.e.

$$\int_a^b |\psi|^2 dv = \text{a finite value} = N \text{ say.}$$

If $\int_{-\infty}^{\infty} |\psi|^2 dv = 0$, the particle cannot exist

If the particle at any position is present in a volume element dv then

$$\int_{-\infty}^{\infty} |\psi|^2 dv = 1 \quad \text{or} \quad \int_{-\infty}^{\infty} P dv = 1$$

If ψ^* is the complex conjugate of ψ then

$$\int |\psi|^2 dv = \int \psi \psi^* dv = N$$

Here, $\psi/\sqrt{N}$ is known as normalised wave function and the coefficient of ψ that normalises it, i.e., $1/\sqrt{N}$ is known as normalisation constant.

Schrodinger Wave Equation :→

(7)

(a) Time-dependent Schrodinger Eqⁿ →

The wave function ψ of a freely moving particle along x -direction may be given as,

$$\psi = A e^{-i(\omega t - kx)} \quad \text{--- (1)}$$

Since $\omega = 2\pi\nu$ & $k = 2\pi/\lambda$

$$\psi = A e^{-2\pi i(\nu t - x/\lambda)} \quad \text{--- (2)}$$

as $E = h\nu = 2\pi\hbar\nu$ & $\lambda = \frac{h}{p} = \frac{2\pi\hbar}{p}$

p = momentum of the particle

$$\psi = A e^{(-i/\hbar)[Et - Px]} \quad \text{--- (3)}$$

Differentiating eqⁿ (3) twice with respect

to x , $\frac{\partial^2 \psi}{\partial x^2} = -\frac{p^2}{\hbar^2} \psi \quad \text{--- (4)}$

& differentiating eqⁿ (3) once with respect

to t ,

$$\frac{\partial \psi}{\partial t} = -\frac{i}{\hbar} E \psi \quad \text{--- (5)}$$

Now for a particle total energy is the sum of kinetic energy and potential energy.

$$E = T + V = \frac{p^2}{2m} + V$$

$$E\psi = \frac{p^2}{2m} \psi + V\psi \quad \text{--- (6)}$$

using (4) & (5)

(8)

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + V\psi \quad \text{--- (7)}$$

It is known as the time-dependent Schrodinger's eqⁿ.

Here V is the function of x & t

In three dimensions,

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} \right) + V\psi$$

$$i\hbar \frac{\partial \psi}{\partial t} = \left[-\frac{\hbar^2}{2m} \nabla^2 + V \right] \psi \quad \text{--- (8)}$$

(b) Time Independent Schrodinger eqⁿ.

Now $\psi = A e^{-\frac{i}{\hbar}(Et - Px)}$

$$= A e^{-\frac{iE}{\hbar}t} e^{\frac{iP}{\hbar}x} = A e^{-\frac{iE}{\hbar}t} \phi(x)$$

--- (9)

Here,

$e^{-\frac{iE}{\hbar}t}$ & $e^{\frac{iP}{\hbar}x}$ are time dependent

and position dependent functions.

Differentiating eqⁿ (9) with respect to t

$$\frac{\partial \psi}{\partial t} = -\frac{iE}{\hbar} A e^{-\frac{iE}{\hbar}t} \phi(x)$$

--- (10)

& with respect to x (twice)

$$\frac{\partial^2 \psi}{\partial x^2} = A e^{-\frac{iE}{\hbar}t} \frac{\partial^2 \phi}{\partial x^2} \quad \text{--- (11)}$$

Substituting (10) & (11) in (7), we get (9)

$$E \phi e^{-i(E/\hbar)t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \phi}{\partial x^2} e^{-i(E/\hbar)t} + V[\phi e^{-i(E/\hbar)t}]$$

$$E \phi = -\frac{\hbar^2}{2m} \frac{\partial^2 \phi}{\partial x^2} + V \phi \quad \text{--- (12)}$$

In three dimensions,

$$-\frac{\hbar^2}{2m} \Delta^2 \phi + V \phi = E \phi$$

This is the time-independent Schrodinger's ~~Eqn~~ Eqn.

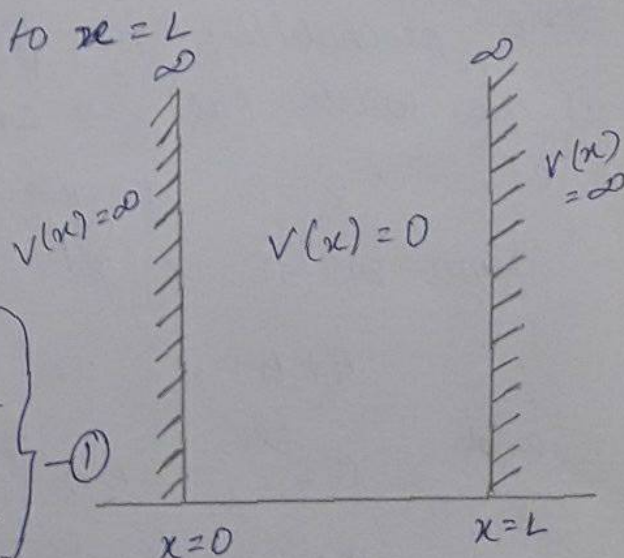
Particle in one dimensional box →

Let a particle be restricted to move in one dimension within a certain region from $x=0$ to $x=L$

The Potential function $V(x)$ can be defined as follows,

$$V(x) = \infty \quad \text{for } x \leq 0 \text{ \& } x \geq L$$

$$= 0 \quad \text{for } 0 < x < L$$



From Schrodinger eqn

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{2m}{\hbar^2} E \psi = 0 \quad \text{--- (2)}$$

(10)

Since ψ is a function of x only, the partial derivatives can be replaced by total derivatives, i.e.

$$\frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2} E \psi = 0$$

$$\text{or } \frac{d^2\psi}{dx^2} + \alpha^2 \psi = 0 \quad \text{--- (3)}$$

$$\text{where } \alpha = \sqrt{\frac{2mE}{\hbar^2}} \quad \text{--- (4)}$$

The general solution of eqn (3) can be given as,

$$\psi = A e^{i\alpha x} + B e^{-i\alpha x} \quad \text{--- (5)}$$

The probability of finding the particle at the walls (at $x=0$ and $x=L$) is zero

i.e. $\psi = 0$ at $x=0$ & $x=L$

from eqn (5) we get

$$A + B = 0 \quad \text{or } B = -A \quad \text{--- (6)}$$

and

$$A e^{i\alpha L} + B e^{-i\alpha L} = 0$$

$$A e^{i\alpha L} - A e^{-i\alpha L} = 0 \quad \{\text{using (6)}\}$$

$$2iA \sin \alpha L = 0$$

Since

$A \neq 0$, therefore $\sin \alpha L = 0$

$$\alpha L = n\pi \quad \text{--- (7)}$$

$$(n=0, 1, 2, 3, \dots)$$

Now,

$$\psi = A e^{i\alpha x} + B e^{-i\alpha x}$$

$$= A e^{i\alpha x} - A e^{-i\alpha x}$$

$$= 2iA \sin \alpha x$$

$$\psi = A' \sin \alpha x$$

$$\psi_n = A' \sin \frac{n\pi}{L} x \quad \text{--- (8)}$$

Since the probability of finding the particle within the box ($0 < x < L$) is unity, we have

$$\int_0^L \psi_n(x) \psi_n^*(x) dx = 1$$

$$\int_0^L \psi_n(x) \psi_n^*(x) dx = 1$$

$$\int_0^L A'^2 \sin^2 \left(\frac{n\pi}{L} x \right) dx = 1$$

$$A'^2 \frac{L}{2} = 1$$

$$A' = \sqrt{\frac{2}{L}} \quad \text{--- (9)}$$

Therefore from eqn (8) & (9)

$$\boxed{\psi_n(x) = \sqrt{\frac{2}{L}} \sin \left(\frac{n\pi}{L} \right) x} \quad \text{--- (10)}$$

Wave function for the particle in a one dimensional box.

Eigen Value $\rightarrow$

From eqn (4) & (7)

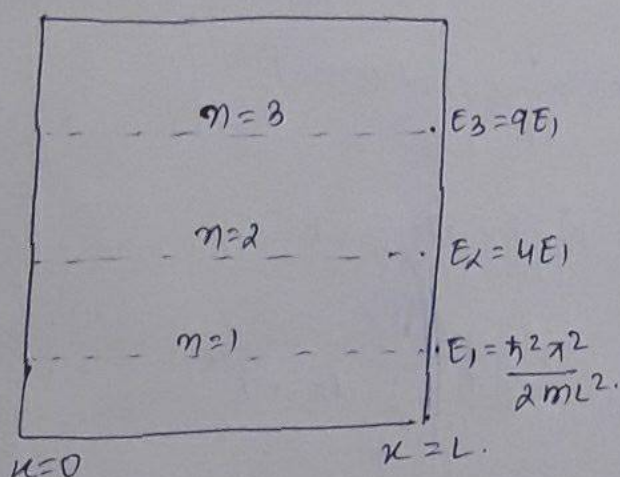
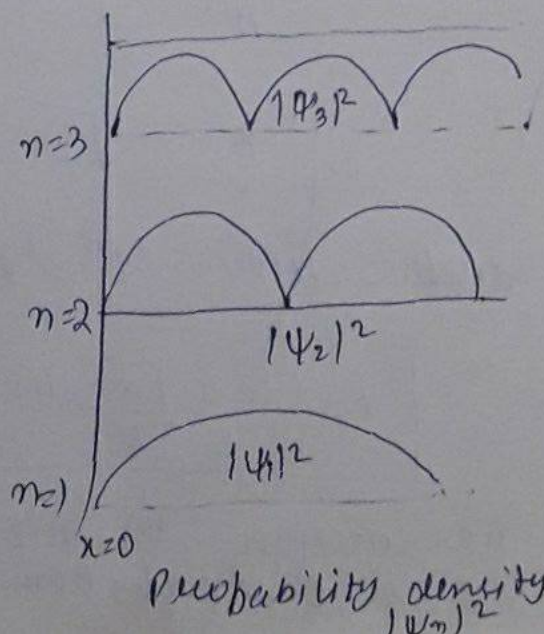
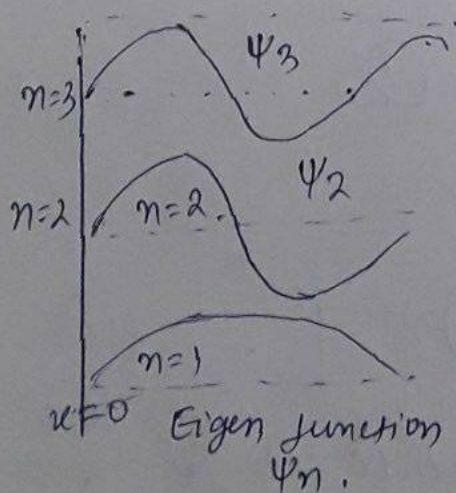
$$\alpha L = n\pi$$

$$\alpha = \frac{n\pi}{L}$$

$$\sqrt{\frac{2mE}{\hbar^2}} = \frac{n\pi}{L} \quad [\text{using (4)}]$$

$$E_n = n^2 \frac{\pi^2 \hbar^2}{2mL^2} \quad \text{--- (11)}$$

$$n = 1, 2, 3, \dots$$

Wave function $\rightarrow$ 

Particle in Three dimensional Box

$$\text{Wave function } \psi(\vec{r}) = \sqrt{\frac{8}{abc}} \sin\left(\frac{n_x \pi}{a} x\right) \sin\left(\frac{n_y \pi}{b} y\right) \sin\left(\frac{n_z \pi}{c} z\right)$$

Energy eigen value.

$$E = \frac{\hbar^2}{2m} (k_x^2 + k_y^2 + k_z^2)$$

$$E = \frac{\hbar^2}{2m} \left[\frac{n_x^2 \pi^2}{a^2} + \frac{n_y^2 \pi^2}{b^2} + \frac{n_z^2 \pi^2}{c^2} \right]$$

$$E = \frac{\hbar^2 \pi^2}{2m} \left[\frac{n_x^2}{a^2} + \frac{n_y^2}{b^2} + \frac{n_z^2}{c^2} \right]$$

Unit - 3

Coherence

The atom or molecule of any substance in general remains in ground state i.e. the state having lowest energy. If it absorbs some energy by any interaction then it goes to excited state, where it remains for an interval of the order of 10^{-8} sec, and then it returns to lower energy state by emitting energy. The emission is totally random and called spontaneous emission.

Therefore the phase difference between light waves obtained from two different sources never remains constant and even, the phase difference between the wave keep on changing with respect to time. The sources, which emit such type of light, are called incoherent sources.

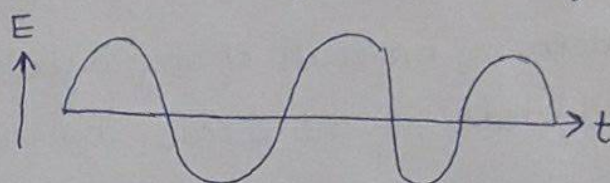
whereas the source of light from which phase difference ~~from~~ between the waves from them reaching at any point in space does not change with time rather it remains constant, are called coherent sources. The property of the wave is known as coherence.

So coherence is an important property of light which refers to the connection between the phase of light waves and time ~~to the~~ at a point in space. Coherence effects are mainly divided into two categories: Temporal and spatial.

The temporal coherence is related directly to the finite bandwidth of the sources, whereas the spatial coherence is related to the finite size of the source.

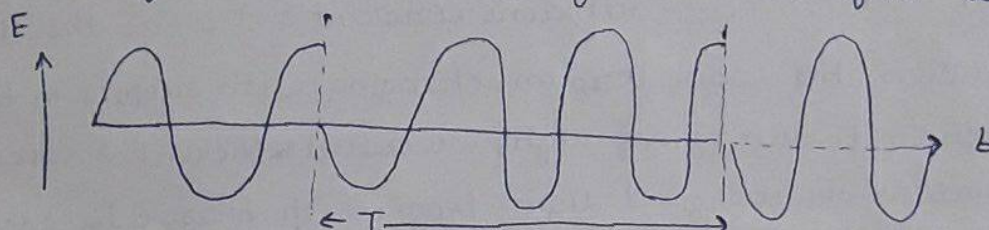
Temporal Coherence - In temporal coherent wave there is definite relationship between the phase of wave at a given instant of time.

Perfectly coherent light wave would have a constant amplitude of vibration at any time, while its phase would vary linearly with time. However no actual source of light produces an



Perfectly coherent light wave

ideal sinusoidal field for all values of time. When an excited atom returns to the initial state, it emits pulse of short duration of 10^{-10} sec. Thus the field remains sinusoidal for time intervals of the order of 10^{-10} second



Incoherent light wave

The ~~average~~ average time interval for which definite phase relationship exists (i.e. the field remains sinusoidal) is known as 'Coherence time' or temporal coherence of the light beam. Coherence time is denoted by T . The distance L for which the field is sinusoidal is given by $L = TC$ — (1)

L = coherence length

Coherence light in terms of frequency $\rightarrow$ for monochromatic light the frequency range is zero i.e. $\Delta \nu = 0$. But the frequency range of waves emitted by an actual light source is greater than zero. The reason of this is that every spectral line has a finite wavelength spread i.e. for sodium D_2 line ($\lambda = 5896 \text{ \AA}$) $\Delta \lambda = 0.06 \text{ \AA}$)

The measure of the finite purity of spectral lines Q is represented by

$$Q = \frac{\lambda}{\Delta\lambda} \quad \text{--- (2)}$$

The time of coherence is of the order of the reciprocal of frequency range i.e. $\tau = \frac{1}{\Delta\nu}$ --- (3)

from eq (1) & (3) coherence length $= L = \tau c = \frac{c}{\Delta\nu}$ --- (4)

coherence length is terms of wavelength.

for light waves $\nu = \frac{c}{\lambda}$ --- (5)

Differentiating, we get $\Delta\nu = -\frac{c}{\lambda^2} \Delta\lambda$

$$|\Delta\nu| = c \frac{\Delta\lambda}{\lambda^2} \quad \text{--- (6)}$$

coherence time $\tau = \frac{1}{\Delta\nu} = \frac{\lambda^2}{c \Delta\lambda}$ --- (7)

from eq (1) $\tau = \frac{1}{\Delta\nu} = \frac{\lambda^2}{c \Delta\lambda}$ --- 8

from eq (2) & (8) coherence length $L = \frac{\lambda}{\Delta\lambda} \times \lambda$
 $= Q \lambda = \text{Purity factor} \times \text{wavelength}$

Here Q is called purity factor which is equal to the wavelength of the light emitted by the source per unit wavelength interval. The value of Q defines purity of the light emitted

$$\text{Purity factor } Q = \frac{\lambda}{\Delta\lambda} \quad \text{--- (9)}$$

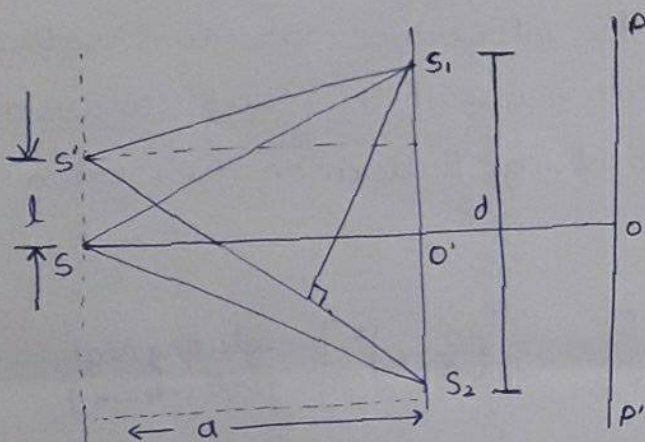
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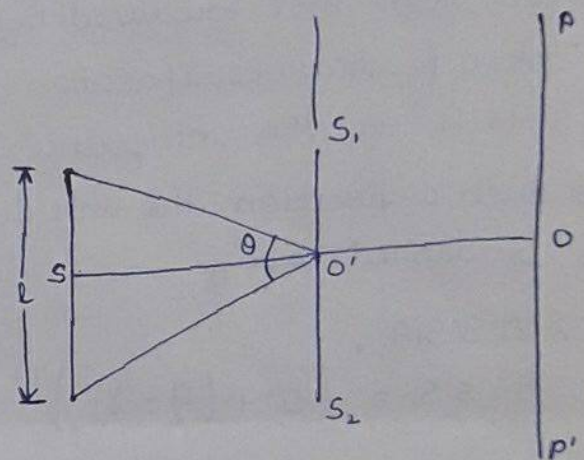
Spatial Coherence and Size of the Source

It is the ~~phase~~ phase relationship between radiation field at different points in a direction perpendicular to direction of propagation in space. The spatial coherence means the phase difference for any two fixed point in a plane, normal to the wave propagation, does not vary with time. It is known as spatial coherence.

The coherence property of light is associated with the finite dimension / size of the source.



(a) Young's double slit experiment with two independent point source



(b) The Same experiment with an extended source

Thus, an ideal point source exhibits spatial coherence, as the wave produced by it are likely to have the same phase at point in space, which are equidistant from the source. On the other hand, an extended source is bound to exhibit lesser spatial coherence.

Let S represents a point source placed in front of screen with two holes S_1 and S_2 and the interference pattern between the waves emitting from S_1 and S_2 as observed on the screen pp' .

The point O on the screen is such that

$$S_1O = S_2O \text{ and } SS_1 = SS_2$$

We assume S to be nearly monochromatic so that it produce interference fringes of good contrast on the screen ~~pp~~ pp' .

Now we consider another similar source S' at a distance l from S and assume that the waves from S and S' have no phase relationship. In such a case the interference pattern observed on the screen PP' will be superposition of the interference patterns formed due to S and S' .

If separation l is slowly increased from zero, the contrast of the fringes on the screen PP' becomes poorer because of the fact that the interference pattern produced by S' is slightly shifted from that produced by S .

For a particular separation (l), the interference maximum produced by S falls on the interference minima produced by S' and vice versa. For such a position the interference fringe pattern on the screen PP' is washed away.

In $\Delta S'S_1A$,

$$S'S_1 = \left[a^2 + \left(\frac{d}{2} - l \right)^2 \right]^{1/2} \approx a \left[1 + \frac{1}{2a^2} \left(\frac{d}{2} - l \right)^2 \right] \quad \text{(neglecting high order term)}$$

$$= a + \frac{1}{2a} \left(\frac{d}{2} - l \right)^2 \quad \text{--- (1)}$$

In $\Delta S'S_2A$

$$S'S_2 = \left[a^2 + \left(\frac{d}{2} + l \right)^2 \right]^{1/2} \approx a \left[1 + \frac{1}{2a^2} \left(\frac{d}{2} + l \right)^2 \right]$$

$$= a + \frac{1}{2a} \left(\frac{d}{2} + l \right)^2 \quad \text{--- (2)}$$

where we have assumed $a \gg l, d$

for disappearance of fringe $S'S_2 - S'S_1 = \lambda/2$

$$\left\{ a + \frac{1}{2a} \left(\frac{d}{2} + l \right)^2 \right\} - \left\{ a + \frac{1}{2a} \left(\frac{d}{2} - l \right)^2 \right\} = \lambda/2$$

$$\frac{1}{2a} \left\{ \left(\frac{d}{2} \right)^2 + l^2 + dl - \left(\frac{d}{2} \right)^2 - l^2 + dl \right\} = \lambda/2$$

$$\frac{dl}{a} = \frac{\lambda}{2} \Rightarrow \boxed{l = \frac{\lambda a}{2d}} \quad \text{--- (3)}$$

If we have an extended incoherent source whose linear dimension is equal to $\frac{\lambda a}{d}$ then for every point on the source there is

point at a distance $\frac{\lambda a}{2d}$ which produces fringes which are shifted by half a fringe width. Therefore the interference pattern will not be seen eqⁿ ③ show that the smaller is the size of the source the better is the spatial coherence. Spatial coherence of the wave decreases with increasing size of the light source.

Thus for an extended incoherent source, the interference fringes of good contrast will be observed only when $l \ll \lambda a/d$

Now if θ is the angle subtended by the source at the slits then $\theta \approx \frac{l}{a}$ [∵ angle = $\frac{\text{arc}}{\text{radius}}$]

Therefore, above condition for obtaining fringes of good contrast takes the form

$$d \ll \frac{\lambda a}{l} \Rightarrow d \ll \frac{\lambda}{\theta} \Rightarrow d \ll l_w$$

The distance $l_w = \frac{\lambda}{\theta}$ is referred to as lateral (spatial) coherence width. It (l_w) depends inversely on θ .

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Visibility as a measure of coherence —

The interference of light produces (maximum intensity) and dark (~~max~~ minimum intensity) fringes. The difference among bright and dark fringes is a measure of the quality of interference and its visibility. It is a dimensionless quantity which measures contrast or the quality of interference pattern.

The visibility V of interference fringes is defined as the ratio of difference between maximum intensity and minimum intensity of the sum of these intensities i.e.

$$V = \frac{I_{\max} - I_{\min}}{I_{\max} + I_{\min}}$$

—————①—————

This ratio is a measure of the "degree of coherence" of the light waves that produce the interference

Obviously if Coherence is poor, visibility will also be poor and so contrast will be poor. If two wave trains of light of equal intensities overlap in their entire lengths it results in zero minimum intensity ($I_{\min} = 0$) and visibility $V = 1$. Such waves are said to be in perfect coherence with each other and result in maximum and minima of the highest degree of contrast.

If wave trains do not overlap at all, then $I_{\max} = I_{\min}$ and $V = 0$. Hence, no fringes are seen and the waves do not have any coherence.

Let us consider two waves of equal intensities I_0 superimpose but these wave have 'c' part of their intensity as coherent part and $(1-c)$ part as incoherent. On superposition, the coherent part shall interfere and so the amplitude of the wave are added up. Intensity is proportional to the square of amplitude so the intensity at maximum due to coherent part shall be $4I_0c$. while at minimum it shall be zero.

The incoherent parts do not interfere and intensities are added everywhere. Therefore, intensity both at maximum and minimum will be $2(1-c)I_0$.

On superposition the maximum and minima have intensities

$$I_{\max} = 4I_0c + 2(1-c)I_0 \quad \text{--- (2)}$$

$$I_{\min} = 0 + 2(1-c)I_0 \quad \text{--- (3)}$$

from eqn (1) $V = \frac{4I_0c + 2(1-c)I_0 - 2(1-c)I_0}{4I_0c + 2(1-c)I_0 + 2(1-c)I_0}$

$$= \frac{4I_0c}{4I_0c + 4I_0 - 4I_0c} = c \Rightarrow V = c \quad \text{--- (4)}$$

which shows that the visibility (degree of contrast) of the fringes produced by interference of two waves of equal intensities is equal to the degree of coherence of between the waves. It can be shown that if wave have different ~~int~~ intensities

say I_1 & I_2 the visibility is given by $V = \frac{2\sqrt{I_1 I_2}}{I_1 + I_2} c \quad \text{--- (5)}$

Value $V = 0.8$ is good whereas $V = 0.2$ is poor