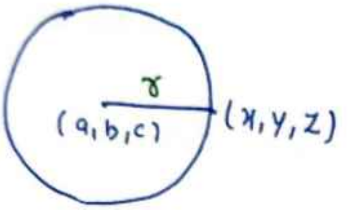
* Sphere

• **Definition:** The locus of a point in space which remains at a constant distance from a fixed point is called a sphere.

• Eq^n of sphere whose centre is (a,b,c) & radius is r



$$S : (x-a)^2+(y-b)^2+(z-c)^2 = r^2$$

* NOTE · Standard eq^n of a sphere : If the centre of sphere is origin then eq^n is : $x^2+y^2+z^2=r^2$

Ex - Eq^n of sphere whose centre is $(1,1,2)$ & radius is 4

Ans $a=1,\ b=1,\ c=2$ & $r=4$

$$\therefore S : (x-1)^2+(y-1)^2+(z-2)^2 = 4^2$$

• General Eq^n of sphere : $x^2+y^2+z^2+2ux+2vy+2wz \pm d=0$

• It represent the eq^n of sphere whose centre $=(-u,-v,-w)$

i.e $\left(-\dfrac{\text{coeff of } x}{2},\ -\dfrac{\text{coeff of } y}{2},\ -\dfrac{\text{coeff. of } z}{2}\right)$.

and

$$\text{radius} = \sqrt{u^2+v^2+w^2-d}$$

Q find the centre & radius of the sphere $x^2+y^2+z^2-2x+4y-6z=11$

Ans Here $u=-1,\ v=2,\ w=-3$ & $d=-11$

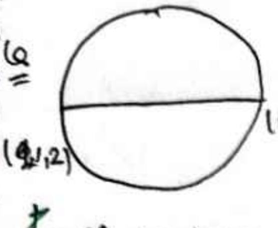
$\therefore$ Centre $=(1,-2,3)$

& $r=\sqrt{1^2+2^2+3^2-(-11)} = \sqrt{25}$

$r=5$

• Diameter form : The eq^n of sphere the ends of whose diameter are (x_1, y_1, z_1) & (x_2, y_2, z_2) is ↓

$$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)+(z-z_1)(z-z_2)=0$$



Ans Then the eq^n of sphere is

$$(x-1)(x)+(y-1)^2+(z-2)^2=0$$

* eq^n of sphere through four Non-coplanar points :

• Let (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) & (x_4, y_4, z_4) be four points & the eq^n of sphere through these points be

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0 \quad -(1)$$

→ All these points satisfy eq^n(1), then we get u, v, w & d after solving four eq^n.

Ex : find the eq^n to the sphere through the points $(0,0,0)$, $(0,1,-1)$, $(-1,2,0)$ & $(1,2,3)$

Ans Let the eq^n of sphere be : $x^2+y^2+z^2+2ux+2vy+2w+d=0$ $-(1)$ –

• If (1) passes through $(0,0,0)$ then $d=0$

At $(0,1,-1) \Rightarrow 1+1+2v-2w=0$

$$[v-w=-1] \quad -(2)$$

At $(-1,2,0) \Rightarrow 1+4-2u+4v=0 \Rightarrow [2u-4v=5] \quad -(3)$

At $(1,2,3) \Rightarrow 1+4+9+2u+4v+6w=0$

$$[2u+4v+6w=-14] \quad -(4)$$

$$\begin{array}{r} 2u+4v=5 \\ 2u-4v+6w=-14 \\ \hline 4u+10w \\ 8v+6 \end{array}$$

• taking -1 common in (3) :- $-2u+4v+5=0$

$$\begin{array}{r} 2u+4v+6w+14=0 \\ + \\ \hline 8v+6w+19=0 \\ 6v-6w+6=0 \\ \hline 14v=-25 \end{array}$$

• Multiply eq^n 2 by 6 ⇒ → $\left[v=-25/14\right]$

- Put Value of v in eq^n (2)

$$\frac{-25}{14} + (-w) = -1 \Rightarrow -w = -1 + \frac{25}{14}$$

$$\left[w = -\frac{11}{14} \right]$$

- Put value of v in eq^n (3)

$$2u - 2\left(-\frac{25}{14}\right) = 5$$

$$2u + \frac{\cancel{3}50}{7} = 5 \Rightarrow 2u = \frac{35-50}{7}$$

Now Put value of u, v, w in eq^n 1 $\quad u = -\frac{15}{14}$

* Eq^n: $7(x^2+y^2+z^2) - 15x - 25y - 11z = 0$

Q find eq^n of sphere passing through $(0,0,0), (a,0,0), (0,b,0)$ & $(0,0,c)$

Ans Let the eq^n of sphere be : $x^2+y^2+z^2+2ux+2vy+2wz = 0$ --(1)

- If (1) passes through $(0,0,0)$, we get $d=0$

So eq^n of sphere is $x^2+y^2+z^2+2ux+2vy+2wz = 0$ -- (2)

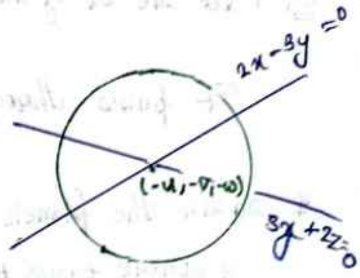
* If (2) passes through $(a,0,0), (0,b,0), (0,0,c)$, then we get

$$\begin{aligned} a^2 + 2ua = 0 &\Rightarrow u = -a/2 \\ b^2 + 2vb = 0 &\Rightarrow v = -b/2 \\ c^2 + 2wc = 0 &\Rightarrow w = -c/2 \end{aligned}$$

- Eq^n of Sphere : S: $(x^2+y^2+z^2) - ax - by - cz = 0$

Q. find the eq^n of sphere having its centre on the line $2x-3y=0=5y+2z$ & passing through the points $(0,-2,-4)$ & $(2,-1,-1)$.

Sol^n - Let the eq^n of sphere be : $x^2+y^2+z^2+2ux+2vy+2wz+d=0$ —(1)



. If (1) passes through $(0,-2,-4)$ & $(2,-1,-1)$

we get $0^2+(-2)^2+(-4)^2+2u(0)+2v(-2)+2w(-4)+d=0$

$$-4v-8w+d+20=0 \quad -(2)$$

& $2^2+(-1)^2+(-1)^2+2u(2)+2v(-1)+2w(-1)+d=0$

$$4u-2v-2w+d+6=0 \quad -(3)$$

• Now given that $(-u,-v,-w)$ lie on the line ∴

$$-2u+3v=0 \quad -(4) \longrightarrow u=\frac{3v}{2}$$

$\cancel{5}$

$$-5v-2w=0 \quad -(5) \longrightarrow w=-\frac{5v}{2}$$

* (3)-(2), we get : $4u+2v+6w-14=0$

$$\boxed{2u+v+3w=7} \quad -(6)$$

$$3v+v-\frac{3\times5v}{2}=7 \Rightarrow 6v+2v-15v=14$$

$$\therefore \boxed{v=-2} \Rightarrow \begin{cases} u=-3 \\ w=5 \end{cases}$$

• Putting values of u,v & w in eq^n (2) ⇒ $-4(-2)-8(5)+20+d=0 \Rightarrow d=12$

∴ eq^n of sphere : $x^2+y^2+z^2+2(-3)x+2(-2)y+2(5)z+12=0$

$$x^2+y^2+z^2-6x-4y+10z+12=0$$

Q. A plane passes through a fixed point (a,b,c) & cut the axes in A, B, C.

Ex.4 of Book: Show that the locus of centre of sphere OABC is $\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=2$

Solⁿ: Let the eqⁿ of the plane be - $\frac{x}{\alpha}+\frac{y}{\beta}+\frac{z}{\gamma}=1$ —(I)

It passes through (a,b,c) $\therefore$ $\frac{a}{\alpha}+\frac{b}{\beta}+\frac{c}{\gamma}=1$ — (II)

* Again the plane(I) passes through $(\alpha,0,0)$, $(0,\beta,0)$, $(0,0,\gamma)$

& sphere passes through O, A, B & C $\therefore$

$$S: x^2+y^2+z^2-\alpha x-\beta y-\gamma z=0$$

Centre: $\left(\frac{\alpha}{2},\frac{\beta}{2},\frac{\gamma}{2}\right)$ & if the centre is (x,y,z)

this implies $\Rightarrow \alpha=2x, \beta=2y, \gamma=2z$

* Substituting value of α, β, γ in (II) we get

$$\frac{a}{2x}+\frac{b}{2y}+\frac{c}{2z}=1 \Rightarrow \frac{a}{x}+\frac{b}{y}+\frac{c}{z}=2$$

1.6 of Book

Q The eqⁿ of sphere through the origin & the points where the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ meets the co-ordinate axes. OR Circum-scribing the tetrahedron whose faces are $x=y=z=0$ & $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$

Ans Let the Reqʳ eqⁿ of Sphere Be : $x^2+y^2+z^2+2ux+2vy+2wz+d=0$

→ Since it passes through $(0,0,0)$ $\therefore$ $d=0$

∘ Again this sphere passes through the points $A(a,0,0)$, $B(0,b,0)$, $C(0,0,c)$ $\therefore$

$$a^2+2au=0 \Rightarrow 2u=-a$$
$$b^2+2bv=0 \Rightarrow 2v=-b$$
$$c^2+2cw=0 \Rightarrow 2w=-c$$

$\therefore$ Eqⁿ of Reqʳ Sphere: $x^2+y^2+z^2-ax-by-cz=0$

Q* find eqn of sphere which passes through (1,0,0),(0,1,0) & (0,0,1) & has its radius as small as possible?

Ans. Let Eqn of sphere : $x^2+y^2+z^2+2ux+2vy+2wz+d=0$ — (1)

- As sphere passes through (1,0,0), (0,1,0) & (0,0,1) ∴

$$\left.\begin{aligned}1+2u+d&=0\\1+2v+d&=0\\1+2w+d&=0\end{aligned}\right\} \Rightarrow u=v=w=\frac{-(d+1)}{2} \quad -(2)$$

- $r^2 = u^2+v^2+w^2-d$ — (3)

- Put values of u, v, & w in 3 ∴ ~~$f(x) = \frac{3}{4}(d+1)^2-d=0$~~ $= r^2$

$$r^2 = \frac{3}{4}(d+1)^2 - d = f(d)$$

- Now $f'(d) = \frac{3}{4}\times 2(d+1)-1$

for Maxm or minm Value : $f'(d)=0$ ∴ $\frac{3}{2}(d+1)-1=0$

$$\left[d=-\frac{1}{3}\right] \to$$ at this value Max or min will occur.

$$f''(d) = \frac{3}{2}$$

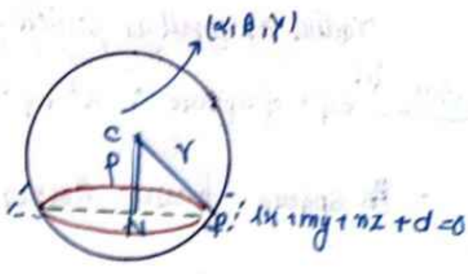
~~if at d~~ $f''(d)=\frac{3}{2} > 0$ ∴ minm occurs at $d=-1/3$

- Replace $d=-1/3$ in eqn (2) ∴ $u=v=w= -\frac{1}{2}(-1/3+1) = -1/3$

Eqn of Sphere : $\left[x^2+y^2+z^2-\frac{2}{3}x-\frac{2}{3}y-\frac{2}{3}z=0\right]$

* Plane Section of Sphere

- The section of sphere by a plane is a circle.



* In Δ CNP : $NP^2 = CP^2 - CN^2$

$$= r^2 - p^2$$

$$\therefore NP = \sqrt{r^2 - p^2}$$

where $CP = r$ = radius of sphere

NP = radius of sphere.

$CN = p$ = perpendicular length from centre on plane.

- If plane sectioning sphere is $lx + my + nz + d = 0$

Then eq^n of line CN is: $\dfrac{x-\alpha}{l} = \dfrac{y-\beta}{m} = \dfrac{z-\gamma}{n} = q$

- Equation of circle

- The combined eq^n of sphere & plane : $x^2+y^2+z^2+2ux+2vy+2wz+d=0$
&
$lx+my+nz=p$

is an equat^n of circle.

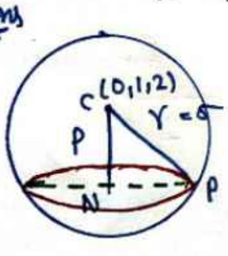
NOTE : 1) Centre of circle is the foot of perpendicular from centre of sphere to the plane

2) Radius of circle = $\sqrt{r^2-p^2}$

Ex find the co-ordinate of centre & radius of circle :

$$x^2+y^2+z^2-4z-2y-20=0, \; x+2y+2z-21=0$$

Ans



centre of sphere : $(0,1,2)$

Radius of sphere, $r = \sqrt{0^2+1^2+2^2-(-20)} = 5$

$$NP = R_c = \sqrt{5^2 - p^2}$$

* Perpendicular distance of a point (x_1,y_1,z_1) from a plane $lx+my+nz+d=0$ is

$$p = \left| \frac{lx_1 + my_1 + nz_1 + d}{\sqrt{l^2+m^2+n^2}} \right|$$

• Here $p = \left| \dfrac{0+2(1)+2(2)-21}{\sqrt{1^2+2^2+2^2}} \right| = 5 \therefore R_c = \sqrt{5^2-5^2} = 0$

• Eq^n of line CN: $\dfrac{x-0}{1} = \dfrac{y-1}{2} = \dfrac{z-2}{2} = q$

$$x = q,\ y = 2q+1,\ z = 2q+2$$

• As (x,y,z) passes throug $x+2y+2z-21=0$

$$\therefore\ q + 2(2q+1) + 2(2q+2) - 21 = 0$$

$$9q - 15 = 0 \Rightarrow q = \frac{15}{9} \Rightarrow (x,y,z) = \left(\frac{15}{9}, \frac{10}{3}+1, \frac{10}{3}+2\right)$$

$$\therefore \text{Centre of Circle} = \left(\frac{5}{3}, \frac{13}{3}, \frac{16}{3}\right)$$

* Sphere passing through a given circle

• Let the Eq^n of circle ~~& plane~~ be

$$S \equiv x^2+y^2+z^2+2ux+2vy+2wz+d=0$$
$$P \equiv lx+my+nz+p=0$$

• Then eq^n of sphere passing through given circle is

$$S+\lambda P=0$$

Ex The eq^n of sphere which passes through (α,β,γ) & the Circle $x^2+y^2+z^2=a^2;\ z=0$

Ans The eq^n of sphere is $x^2+y^2+z^2-a^2+\lambda z=0$ ie $S+\lambda P=0$ - (1)

eq (1) passes through (α,β,γ) $\therefore$ $\alpha^2+\beta^2+\gamma^2-a^2+\lambda\gamma=0$

$$\lambda = \frac{-(\alpha^2+\beta^2+\gamma^2-a^2)}{\gamma}$$

eq^n(1)

$$x^2+y^2+z^2-a^2 - \frac{(\alpha^2+\beta^2+\gamma^2-a^2)}{\gamma}\cdot z = 0$$

$S:$ ~~$x^2+y^2+z^2=(\alpha^2+\beta^2+\gamma^2)$~~ $S: \left[\gamma(x^2+y^2+z^2-a^2) = z(\alpha^2+\beta^2+\gamma^2-a^2)\right]$

Eqⁿ of Sphere through intersection of two sphere

- Let the eqⁿ of the given spheres be:

$$S_1 : x^2+y^2+z^2+2u_1x+2v_1y+2w_1z+d_1=0$$

$$S_2 : x^2+y^2+z^2+2u_2x+2v_2y+2w_2z+d_2=0$$

→ Then the eqⁿ of sphere passing through their intersectⁿ is

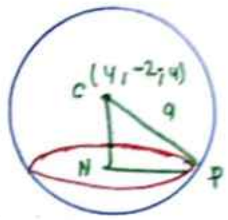
$$\left[S_1+\lambda S_2=0\right]$$

Q find the Radius & centre of the circle $x^2+y^2+z^2-8x+4y+8z-45=0$
$x-2y+2z=3$

<u>Ans.</u> $S : x^2+y^2+z^2-8x+4y+8z-45=0$

Centre : $(4,-2,-4)$ & $r = \sqrt{4^2+2^2+4^2+45} = 9$

Radius of sphere = 9



- p = length of $\perp^r$ from C to plane $x-2y+2z=3$

$$p = \left|\frac{4-2(-2)+2(-4)-3}{\sqrt{1+2^2+2^2}}\right| = 1$$

radius of circle = $\sqrt{r^2-p^2} = \sqrt{9^2-1^2}$

$= \sqrt{80}$

- Eqⁿ of line CN

$$\frac{x-4}{1} = \frac{y+2}{-2} = \frac{z+4}{2} = q \Rightarrow x = q+4,\ y = -2q-2,\ z = 2q-4$$

(x,y,z) lies on $x-2y+2z=3$ ∴ $q+4-2(-(q+1)2)+2(2q-4)=3$

$q = 3/9 = 1/3$

$$\therefore (x,y,z) = \left(\frac{1}{3}+4,\ -2\left(\frac{1}{3}+1\right),\ \frac{2}{3}-4\right) = \left(\frac{13}{3}, \frac{-8}{3}, \frac{-10}{3}\right)$$

i.e. Centre of Circle

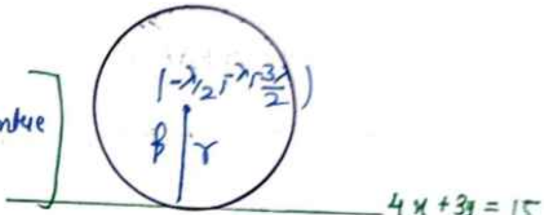
Q The eq^n of sphere which passes through the circle $x^2+y^2+z^2=5$, $x+2y+3z=3$ & touch the plane $4x+3y=15$.

Ans The eq^n of Req^r Sphere: $x^2+y^2+z^2-5+\lambda\,(x+2y+3z-3)=0$

$$\therefore\ x^2+y^2+z^2+\lambda x+2\lambda y+3\lambda z-(3\lambda+5)=0 \quad \text{--- (1)}$$

Centre of sphere : $\left(-\frac{\lambda}{2},-\lambda,-\frac{3\lambda}{2}\right)$

[radius of sphere = perpendicular distance from centre to plane]



cond^n of Tangency

$$\therefore \sqrt{\frac{\lambda^2}{4}+\frac{4\lambda^2}{4}+\frac{9\lambda^2}{4}+\frac{4(3\lambda+5)}{4}} = \left|\frac{4(-\lambda/2)+3(-\lambda)-15}{\sqrt{4^2+3^2}}\right|$$

$$\frac{\sqrt{14\lambda^2+12\lambda+20}}{2} = \frac{5\lambda+15}{5}$$

Squaring

$$\frac{14\lambda^2+12\lambda+20}{4} = (\lambda+3)^2$$

$$\frac{7\lambda^2}{2}+3\lambda+5 = \lambda^2+6\lambda+9$$

$$\frac{5\lambda^2}{2}-3\lambda-4=0 \Rightarrow 5\lambda^2-6\lambda-8=0$$

$$5\lambda-10\lambda+4\lambda-8=0$$

$$(5\lambda+4)(\lambda-2)=0 \Rightarrow \left[\lambda=-\frac{4}{5},2\right]$$

Putting values of λ in (1) we get two spheres

$$x^2+y^2+z^2+2x+4y+3z-11=0$$

&

$$5x^2+5y^2+5z^2-4x-8y-12z-13=0$$

Q Prove that the following circles lie on the same sphere & find its equation ?.

$x^2+y^2+z^2-2x+3y+4z-5=0$; $5y+6z+1=0$ & $x^2+y^2+z^2-3x-4y+5z-6=0$; $x+2y-7z=0$

Ans: ~~The eqn of sphere~~ $S_1+\lambda P_1=0$

$$x^2+y^2+z^2-2x+3y+4z-5+\lambda(5y+6z+1)=0$$

$$x^2+y^2+z^2-2x+(3+5\lambda)y+(4+6\lambda)z-5+\lambda=0 \quad -(1)$$

$\rightarrow S_2+\lambda P_2=0$

$$x^2+y^2+z^2-3x-4y+5z-6+\mu(x+2y-7z)=0$$

$$x^2+y^2+z^2+(\mu-3)x+(2\mu-4)y+(5-7\mu)z-6=0 \quad -(2)$$

- If $S_1=S_2 \Rightarrow$ eqn (1) & eqn (2) must be identical ;- on comparing coeff

$$-2=\mu-3 \Rightarrow [\mu=1]$$

$$3+5\lambda=2\mu-4 \rightarrow 3+5\lambda=-2 \Rightarrow [\lambda=-1]$$

eqn (1)

Eqn of Reqr sphere :- $x^2+y^2+z^2-2x-2y-2z-6=0$

1.5 Intersection of a Sphere & line

- Let $x^2+y^2+z^2+2ux+2vy+2wz+d=0 \quad -(1)$

and $\dfrac{x-\alpha}{l}=\dfrac{y-\beta}{m}=\dfrac{z-\gamma}{n}=r(\text{say}) \quad -(2)$

Be the eqn of sphere & line.

- The co-ordinate of any point on the line are $(lr+\alpha, mr+\beta, nr+\gamma)$

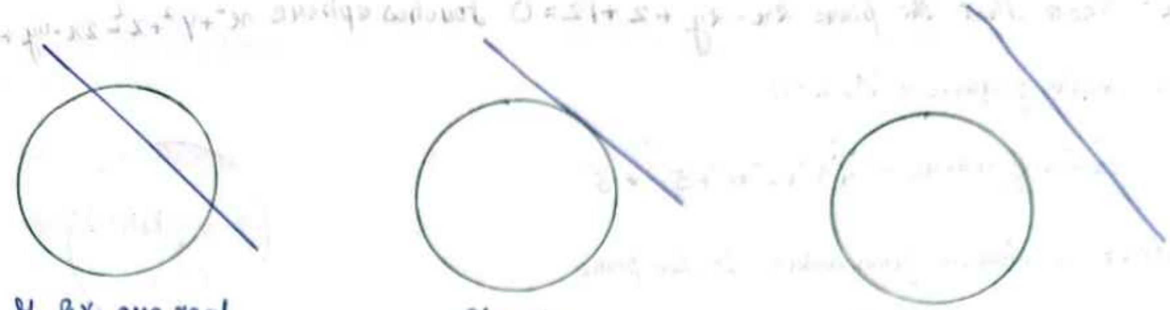
If the line intersect the sphere, then this point satisfy the eqn of sphere

So, $r^2(l^2+m^2+n^2)+2r\left[l(\alpha+u)+m(\beta+v)+n(\gamma+w)\right]$

$$+(\alpha^2+\beta^2+\gamma^2+2u\alpha+2v\beta+2w\gamma+d)=0$$

- This is quadr eqn in r. so we get two roots as r_1 & r_2 :- Substitute r_1 & r_2 in point.

- These two points can be real, coincident or imaginary according to value of r i.e real, equal or imaginary.



r_1 & r_2 are real & distinct　　　$r_1 = r_2$　　　r_1 & r_2 are imaginary

* Tangent Line & Tangent Plane of Sphere.

- A line is said to be tangent line to the sphere if it intersect a sphere in two coincident points & locus of all tangent line at a point on a sphere is called a Tangent plane at that point.

• eqⁿ of Tangent Plane

- The eqⁿ of Tangent plane at the point (x, y, z) to the sphere $x^2+y^2+z^2+2ux+2vy+2wz+d=0$ is:

$$xx_1 + yy_1 + zz_1 + u(x+x_1) + v(y+y_1) + w(z+z_1) + d = 0$$

OR $T=0$

$x^2 \to xx_1$
$y^2 \to yy_1$
$z^2 \to zz_1$
$2x \to x+x_1$
$2y \to y+y_1$
$2z \to z+z_1$

Ex The eqⁿ of tangent plane at $(1,1,1)$ to $x^2+y^2+z^2+4x-7=0$

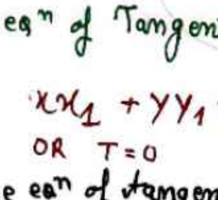
Ans The eqⁿ of Tangent Plane : $xx_1 + yy_1 + zz_1 + 2(x+x_1) - 7 = 0$

$(x_1, y_1, z_1) = (1,1,1)$ ∴ $x + y + z + 2(x+1) = 7$

Tangent Plane : $x+y+z+2x = 5$

• Condⁿ for Tangent Plane

- The condⁿ for a plane $lx+my+nz=p$ to touch the sphere $x^2+y^2+z^2+2ux+2vy+2wz+d=0$



→ Perpendicular distance from centre to Plane = Radius of Sphere.

$$\frac{-ul-vm-wn-p}{\sqrt{l^2+m^2+n^2}} = \sqrt{u^2+v^2+w^2-d}$$

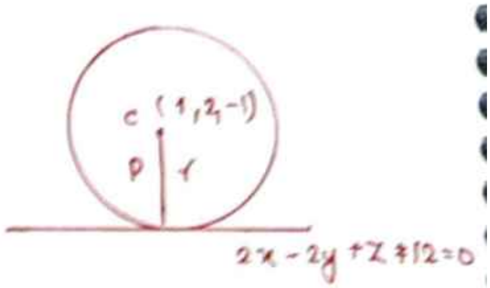
$$\left\{ (ul+vm+wn+p)^2 = (l^2+m^2+n^2)(u^2+v^2+w^2-d) \right\}$$

• Ex - Show that the plane $2x - 2y + z + 12 = 0$ touches sphere $x^2+y^2+z^2-2x-4y+2z-3=0$

Ans Centre of sphere $= (1, 2, -1)$

&

radius of sphere $= \sqrt{1^2+2^2+1^2+3} = 3$



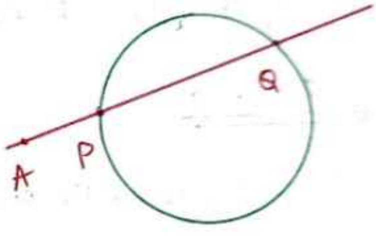
Now, $p = \perp^r$ distance from centre to the plane

$$p = \frac{2(1) - 2(2) + 1(-1) + 12}{\sqrt{2^2+2^2+1^2}} = \frac{9}{\sqrt{9}}$$

$$p = 3$$

* Here $r = p$ $\therefore$ The plane touches the sphere.

Power of a point



• Let A be any point (x_1, y_1, z_1) &

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0$$

→ Then power of point A = AP. AQ

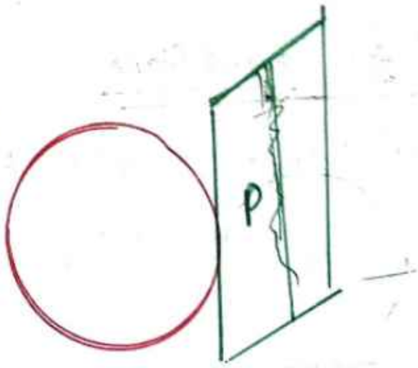
* The power of point $A = x_1^2+y_1^2+z_1^2+2ux_1+2vy_1+2wz_1+d$

(Put value of (x_1, y_1, z_1) in eqn of Sphere) • NOTE: The power of point is constant.

Q The power of point $A(0,1,1)$ w.r.t $x^2+y^2+z^2+2x+4z+5=0$

Ans The power of Point $A(0,1,1)$ is $0+1+1+0+4+5 = 11$

Plane of Contact



• The Plane in which all the tangent lines through Point P lie is called plane of contact of sphere.

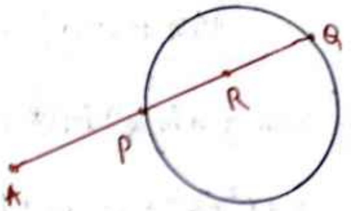
• The plane of contact of sphere

$x^2+y^2+z^2+2ux+2vy+2wz+d=0$ wrt $P(x_1,y_1,z_1)$ is

P.O.C: $xx_1 + yy_1 + zz_1 + u(x+x_1) + v(y+y_1) + w(z+z_1) + d = 0$

• Pole and Polar Plane

• If a variable line through a fixed point A meet the sphere at the ~~sphere~~ points P, Q & R is the point lying on this line such that : $AR = \frac{2AP \cdot AQ}{AP + AQ}$



→ Then locus of R is called the Polar Plane of A w.r.t the sphere & A is called the pole of this polar plane.

• The Polar plane of (x_1, y_1, z_1) w.r.t. sphere $x^2+y^2+z^2+2ux+2vy+2wz+d=0$ is –

$$xx_1 + yy_1 + zz_1 + u(x+x_1) + v(y+y_1) + w(z+z_1) + d = 0$$ [Here (x_1, y_1, z_1) doesn't lie on sphere]

Example - The pole of the plane $lx+my+nz=p$ w.r.t sphere $x^2+y^2+z^2=a^2$

Ans The polar of (x_1, y_1, z_1) w.r.t $x^2+y^2+z^2=a^2$ is

$$xx_1 + yy_1 + zz_1 = a^2 \quad \text{— (1)}$$

Given that $lx+my+nz=p$ is polar — (2)

from (1) & (2) $\Rightarrow \frac{l}{x_1} = \frac{m}{y_1} = \frac{n}{z_1} = \frac{p}{a^2} \Rightarrow x_1 = \frac{a^2 l}{p},\ y_1 = \frac{a^2 m}{p},\ z_1 = \frac{a^2 n}{p}$

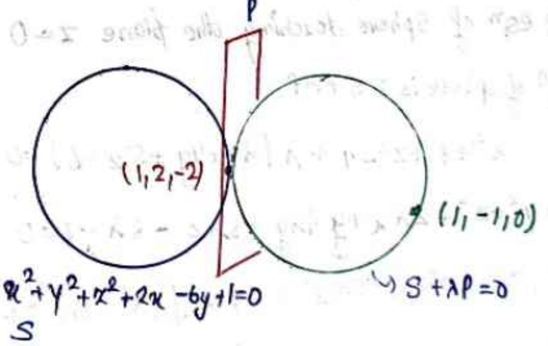
which is pole.

Q find the eqn of sphere which touches the sphere $x^2+y^2+z^2+2x-6y+1=0$ at $(1,2,-2)$ & Passes through the point $(1,-1,0)$

Ans The eqn of Tangent Plane at $(1,2,-2)$ to the sphere $x^2+y^2+z^2+2x-6y+1=0$ is

$$x + 2y - 2z + (x+1) - 3(y+2) + 1 = 0$$

$$\therefore 2x - y - 2z - 4 = 0$$



* Now, eqn of sphere passing through S & P is $S + \lambda P = 0$

$$\therefore x^2+y^2+z^2+2x-6y+1+\lambda(2x-y-2z-4) = 0 \quad \text{— (*)}$$

• This sphere also passes through $(1,-1,0)$ ∴

$$1+1^2+1^2+0+2(1)-6(-1)+\lambda(2(1)-(-1)+0-4)=0$$

$$11+(-\lambda)=0 \Rightarrow \lambda=11$$

∴ Putting value of λ in eqⁿ in (*), we get Req^r Eqⁿ i.e

$$x^2+y^2+z^2+2x-6y+1+22x-11y-22z-44=0$$

$$S: x^2+y^2+z^2+24x-17y-22z-43=0$$

Q If any tangent plane to sphere $x^2+y^2+z^2=r^2$ makes intercepts a, b & c on the co-ordinate axis, then prove that $\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}=\frac{1}{r^2}$

Ans Let the tangent plane at the point (x_1,y_1,z_1) be $xx_1+yy_1+zz_1=r^2$ —(1)

• Given that plane passes through $(a,0,0)$; $(0,b,0)$ & $(0,0,c)$ ∴

$$ax_1=r^2 \Rightarrow x_1=r^2/a$$
$$by_1=r^2 \Rightarrow y_1=r^2/b \quad -(2)$$
$$cz_1=r^2 \Rightarrow z_1=r^2/c$$

• But the point lie on the sphere ∴ $x_1^2+y_1^2+z_1^2=r^2$

So $\frac{r^4}{a^2}+\frac{r^4}{b^2}+\frac{r^4}{c^2}=r^2$

Hence ⇒ $\left[\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}=\frac{1}{r^2}\right]$

Q The eqⁿ of sphere touching the plane $z=0$ & passing through the circle $x^2+y^2+z^2=1$, $2x+4y+5z=6$

Ans Eqⁿ of sphere is : $S+\lambda P$

$$x^2+y^2+z^2-1+\lambda(2x+4y+5z-6)=0$$

$\rightarrow x^2+y^2+z^2+2\lambda x+4\lambda y+5\lambda z-6\lambda-1=0$ —(1) * Eqⁿ 1 touch the plane $z=0$

Centre $=(-\lambda,-2\lambda,-5\lambda/2)$ & $r=\sqrt{4\lambda^2+4\lambda^2+\frac{25\lambda^2}{4}+6\lambda+1}$

* Perpendicular Distance, $p=\left|-5\lambda/2/\sqrt{1^2}\right| = 5\lambda/2$ * Condⁿ of Tangency ⇒ $p=r$ ∴ $p^2=r^2$

$$\frac{25\lambda^2}{4}=\frac{45\lambda^2+4+24\lambda}{4} \Rightarrow 20\lambda^2+24\lambda+4=0$$

$\therefore\ 5\lambda^2+6\lambda+1=0$; $\lambda = \dfrac{-6 \pm \sqrt{36-4(5)(1)}}{2(5)}$ $\therefore\ \lambda = \dfrac{-6\pm 4}{10}$

$$\left[\lambda = -1, -\frac{1}{5}\right]$$

- Putting values of λ in eq^n ①, we get two spheres.

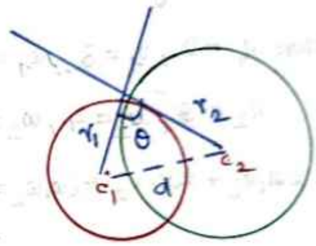
$$x^2+y^2+z^2-2x-4y-6z+5=0$$

&

$$5(x^2+y^2+z^2)-2x-4y-6z+1=0$$

L^6 = Angle of intersection of Two spheres.

- The angle of intersection of two spheres is angle b/w the tangent to the two spheres at their point of intersection



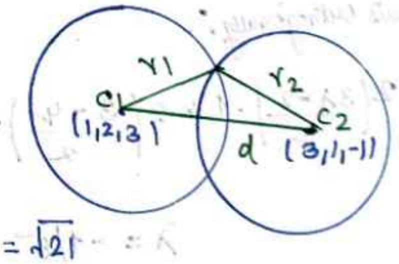
- Let C_1 & C_2 be two centre of spheres & r_1 & r_2 be their respective radii

- Then angle of spheres intersec^n of spheres at P = Angle b/w these radii

$$\cos\theta = \frac{r_1^2+r_2^2-d^2}{2r_1r_2}, \quad \theta = \cos^{-1}\left(\frac{r_1^2+r_2^2-d^2}{2r_1r_2}\right)$$

Q The angle of intersection of spheres : $x^2+y^2+z^2-2x-4y-6z+10=0$
$x^2+y^2+z^2-6x-2y+2z+2=0$

Ans $C_1 = (1,2,3)$ & $C_2 = (3,1,-1)$



$r_1 = \sqrt{1^2+2^2+3^2-10} = 2$

$r_2 = \sqrt{3^2+1^2+1^2-2} = 3$

$C_1C_2 = d = \sqrt{(3-1)^2+(1-2)^2+(-1-3)^2} = \sqrt{21}$

$$\theta = \cos^{-1}\left(\frac{2^2+3^2-(\sqrt{21})^2}{2\cdot 2\cdot 3}\right) \Rightarrow \theta = \cos^{-1}\left(\frac{-8}{12}\right) \Rightarrow \left\{\theta = \cos^{-1}\frac{2}{3}\right\}$$

&

$\cos^{-1}(-\theta) = \cos^{-1}\theta$

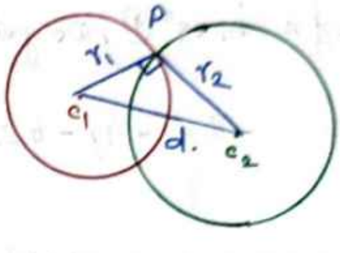
* Orthogonal Spheres

- If the angle of intersection of two spheres is 90°, then they are called Orthogonal Spheres.

* Condⁿ of Orthogonality of Two spheres.

- if $\angle C_1 P C_2 = 90°$

  then $(C_1C_2)^2 = r_1^2 + r_2^2$



Let $x^2+y^2+z^2+2u_1x+2v_1y+2w_1z+d_1=0$

& $x^2+y^2+z^2+2u_2x+2v_2y+2w_2z+d_2=0$

be two spheres, Then condⁿ for orthogonality is → $\left[2u_1u_2+2v_1v_2+2w_1w_2 = d_1+d_2\right]$

Q Show that the two sphere $x^2+y^2+z^2+6y+2z+8=0$ & $x^2+y^2+z^2+6x+8y+4z+20=0$ are Orthogonal.

Ans Here $u_1 = 0,\ v_1 = 3,\ w_1 = 1$ & $d_1 = 8$

$u_2 = 3,\ u_2 = 4,\ w_2 = 2,\ d_2 = 20$

So $2u_1u_2 + 2v_1v_2 + 2w_1w_2 = 0 + 2\cdot3\cdot4 + 2\cdot1\cdot2$

$= 28$

& $d_1 + d_2 = 20 + 8 = 28$] equal ∴ They are orthogonal.

Q find the eqⁿ of sphere that passes through the circle $x^2+y^2+z^2-2x+3y-4z+6=0$, $3x-4y+5z-15=0$ & cuts the sphere $x^2+y^2+z^2+2x+4y-6z+11=0$ Orthogonally.

Ans The eqⁿ of sphere passing through given circle is

$$x^2+y^2+z^2-2x+3y-4z+6+\lambda(3x-4y+5z-15)=0$$

- It cuts Orthogonally ∴

$$2\left(\frac{3\lambda-2}{2}\right)\cdot1 + 2\left(\frac{3-4\lambda}{2}\right)(2) + 2\left(\frac{5\lambda-4}{2}\right)(-3) = (6-15\lambda)+11$$

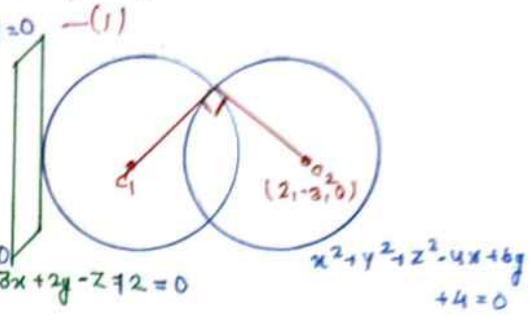
$$\lambda = -1/5$$

- Then we get sphere : $x^2+y^2+z^2-2x+3y-4z+6-1/5(3x-4y+5z-15) =$

↓ $\left[5x^2+5y^2+5z^2-18x+19y-25z+45=0\right]$

Q find the eq^n of Sphere which touches the plane $3x+2y-z+2=0$ at the point $(1,-2,1)$ & Also cut the sphere $x^2+y^2+z^2-4x+6y+4=0$ orthogonally.

Ans Let the eq^n of Sphere be : $x^2+y^2+z^2+2ux+2vy+2wz+d=0$ —(1)



$\therefore$ Tangent plane at $(1,-2,1)$ is

$x-2y+z+u(x+1)+v(y-2)+w(z+1)+d=0$

$\rightarrow (u+1)x+(v-2)y+(w+1)z+(u-2v+w+d)=0$

But given plane is $3x+2y-z+2=0$

$\therefore \dfrac{u+1}{3} = \dfrac{v-2}{2} = \dfrac{w+1}{-1} = \dfrac{u-2v+w+d}{2} = k$

So, $u=3k+1$, $v=2k+2$, $w=-k-1$ & $u-2v+w+d=2$ —*

Putting value of u,v,w in *, we get $\boxed{d=4k+6}$

• Given that eq^n (1) cuts orthogonally to given sphere. $[x^2+y^2+z^2-4x+6y+4=0]$

$\therefore 2u_1u_2+2v_1v_2+2w_1w_2 = d_1+d_2$

$u_1 = 3k-1$, $u_2=-2$

$v_1 = 2k+2$, $v_2=3$

$w_1 = -k-1$, $w_2=0$

$d_1 = 4k+6$, $d_2=4$

$\therefore -4(3k-1)+6(2k+2)+0 = 4+4k+6$

$\cancel{12k}-\cancel{12k}+4+12=10+4k$

$\cancel{2k=6}$ $\quad 6=4k \Rightarrow \boxed{k=3/2} \longrightarrow d=12$

$u_1 = 7/2$

$v_1 = 5$

$w_1 = -5/2$

Then Eq^n of Req^r sphere: $\{x^2+y^2+z^2+7x+10y-5z+12=0\}$

* Radical Plane of two sphere

- The radical plane of two spheres is the locus of a point such that its power is same w.r.t both plane

- Eqn : let $S_1 = x^2+y^2+z^2+2u_1x+2v_1y+2w_1z+d_1=0$

  & $S_2 = x^2+y^2+z^2+2u_2x+2v_2y+2w_2z+d_2=0$

- let (x,y,z) be any point then By Defn : $S_1 = S_2$

  $\therefore\ 2(u_1-u_2)x+2(v_1-v_2)+2(w_1-w_2)+d_1-d_2=0$

* System of Coaxial spheres.

- let $S_1=0$ & $S_2=0$ be two spheres then : System of Co-axial Spheres is $\boxed{S_1+\lambda S_2=0}$

* Limiting Point of Coaxial System

- The limiting point of system of coaxial spheres is sphere of zero radius

  let $S_1+\lambda S_2=0$ Be system of coaxial spheres, find its centre & radius r_1

- Put $r_1=0$ & we get λ then put value of λ in centre & we get limiting point.

Q find the limiting points of system of coaxial spheres $x^2+y^2+z^2+3x-3y+6=0$ and $x^2+y^2+z^2-6y+6z+6=0$

Ans The system of coaxial spheres: $x^2+y^2+z^2+3x-3y+6+\lambda(x^2+y^2+z^2-6y-6z+6)=0$

$(1+\lambda)x^2+(1+\lambda)y^2+(1+\lambda)z^2+3x-3(1+2\lambda)y-6\lambda z+6(1+\lambda)=0$

$$x^2+y^2+z^2+\frac{3}{1+\lambda}x-\frac{3(1+2\lambda)}{(1+\lambda)}y-\frac{6\lambda}{(1+\lambda)}z+6=0$$

Centre $=\left(\frac{-3}{2(1+\lambda)}, -\frac{3}{2}\left[\frac{1+2\lambda}{1+\lambda}\right], -\frac{6\lambda}{2(1+\lambda)}\right)$ & $r=\sqrt{\frac{9}{4(1+\lambda)^2}+\frac{9}{4}\frac{(1+2\lambda)^2}{(1+\lambda)^2}+\frac{9\lambda^2}{(1+\lambda)^2}-6}$

for limiting points $r=0$ $\therefore$ $9+9(1+2\lambda)^2+36\lambda^2-24(1+\lambda)^2=0$

$48\lambda^2-12\lambda-6=0 \Rightarrow 8\lambda^2-2\lambda-1=0 \Rightarrow (2\lambda-1)(4\lambda+1)=0$

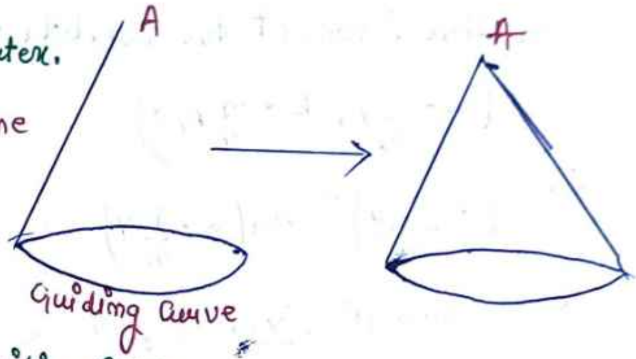
$\lambda=\frac{1}{2}, \frac{-1}{4}$ → Put in centre $\therefore$ limiting Points are $(-1,2,1)$ & $(-2,1,-1)$.

Cone

L-57 Cone

- A surface generated by straight line which passes through a fixed point and intersect a given curve is called a cone.

(i) Vertex: The fixed point is called the vertex.

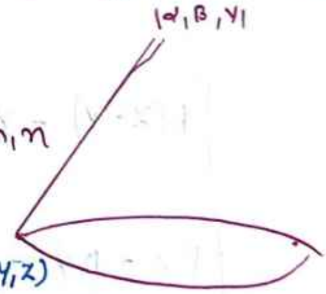


(ii) Generator: Any line on the surface of the cone is called the generating line.

(iii) Guiding Curve: The given curve with which generator intersect is called the guiding curve.

- Eq^n of Cone whose Vertex & Guiding Curve are given.
- The eq^n of cone whose Vertex is (α, β, γ) & Guiding Curve is $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 ; z = 0$
- Process: The eq^n of guiding curve is $f(x,y) = ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 ; z = 0$

Vertex (α, β, γ)



* Let Generating line be : $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} = k$ (1) l, m, n

- This generating line & Guiding Curve intersect at

$$\left(\alpha - \frac{l}{n}\gamma, \beta - \frac{m}{n}\gamma, 0\right)$$

- This point satisfy guiding Curve $f(x,y) = 0$

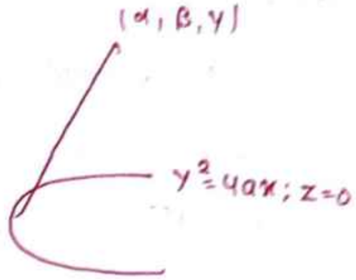
i.e $f\left(\alpha - \frac{l}{n}\gamma, \beta - \frac{m}{n}\gamma, 0\right) = 0$ (2)

* Eliminate l, m & n B/w (1) & (2), then we get required eq^n of Cone.

• Ex - The eq^n of cone whose vertex is (α, β, γ) & Guiding Curve is $y^2 = 4ax$; $z = 0$.

Ans The eq^n of Generating line

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{\overset{=0}{z}-\gamma}{n} \quad -(1)$$



The line intersect the Parabola at :

$$P\left(\alpha - \frac{l}{n}\gamma,\ \beta - \frac{m}{n}\gamma,\ 0\right)$$

$$\therefore \left(\beta - \frac{m}{n}\gamma\right)^2 = 4a\left(\alpha - \frac{l}{n}\gamma\right)$$

Now, from (1), $\frac{y-\beta}{m} = \frac{z-\gamma}{n} \Rightarrow \frac{m}{n} = \frac{y-\beta}{z-\gamma}$

$$\frac{x-\alpha}{l} = \frac{z-\gamma}{n} \Rightarrow \frac{l}{n} = \frac{x-\alpha}{z-\gamma}$$

$$\therefore \left\{\beta - \left(\frac{y-\beta}{z-\gamma}\right)\gamma\right\}^2 = 4a\left\{\alpha - \left(\frac{x-\alpha}{z-\gamma}\right)\gamma\right\}$$

$$\left[\beta(z-\gamma) - \gamma(y-\beta)\right]^2 = 4a(z-\gamma)\left\{\alpha z - \cancel{\alpha\gamma} - x\gamma + \cancel{\alpha\gamma}\right\}$$

$$\left[\beta z - \cancel{\beta\gamma} - y\gamma + \cancel{\beta\gamma}\right]^2 = 4a(z-\gamma)(\alpha z - x\gamma)$$

$$\left[\beta z - \gamma y\right]^2 = 4a(z-\gamma)(\alpha z - x\gamma) \rightarrow \text{Req}^r \text{ eq}^n \text{ of Cone.}$$

*** Eq^n of Cone whose vertex is at origin**

• The eq^n of a cone with vertex at the origin is homogeneous eq^n of second degree in x, y, z.

i.e $-\left\{ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0\right\}$

Q Eqⁿ of cone through Origin & which passes through the intersection of curves $x^2+y^2+z^2+2ax+b=0$, $lx+my+nz=p$.

Solⁿ: The curve is $x^2+y^2+z^2+2ax+b=0$

$$lx+my+nz=p$$

We know that eqⁿ of cone through Origin is always homogeneous

So

$$x^2+y^2+z^2+2ax\left[\frac{lx+my+nz}{p}\right]+b\left[\frac{lx+my+nz}{p}\right]^2=0$$

Homogenizing the curve using $\frac{lx+my+nz}{p}=1$

$\therefore$ Cone: $p^2(x^2+y^2+z^2)+2pxa(lx+my+nz)+b(lx+my+nz)^2=0$

* Enveloping Cone

- The locus of tangent line drawn from a given point to a surface is called an enveloping cone.

→ Eqⁿ of enveloping cone of sphere $x^2+y^2+z^2=a^2$ with vertex at the point (α,β,γ) is $\boxed{SS_1=T^2}$ * → for any curve

Where $S = x^2+y^2+z^2-a^2$

$S_1=\alpha^2+\beta^2+\gamma^2-a^2$

$T=\alpha x+\beta y+\gamma z-a^2$

Q Enveloping Cone of Sphere $x^2+y^2+z^2-2y-2=0$ with its vertex at $(1,1,1)$ [$+2x$]

Solⁿ given $S=x^2+y^2+z^2-2y-2=0$, $S_1=1$, ~~$T = 2x+$~~

$$T = xx_1+yy_1+zz_1+(x+x_1)-(y+y_1)-2$$

$$T=2x+z-2$$

Enveloping cone $SS_1=T^2$

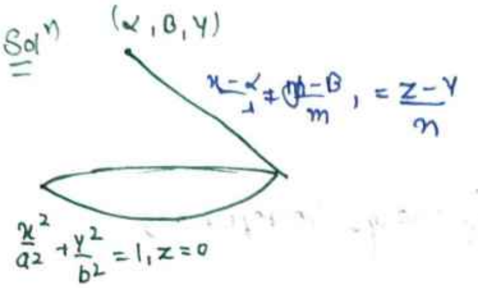
$\therefore (x^2+y^2+z^2+2x-2y-2)(1)=(2x+z-2)^2$

$\Rightarrow 3x^2-y^2-10x+2y-4z+4xz+6=0$

Q The eqn of cone whose vertex is (α, β, γ) & Base of Guiding curve

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \; z = 0$$

Sol



line : $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$

$z = 0$ ∴

$$x = \alpha - \frac{l\gamma}{n}, \quad y = \beta - \frac{m\gamma}{n}$$

Point $\left(\alpha - \frac{l\gamma}{n}, \beta - \frac{m\gamma}{n}\right)$

satisfy $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$\cancel{\frac{1}{a^2}\left[\frac{\alpha - l\gamma}{n}\right]^2 + \frac{1}{b^2}\left[\frac{\beta - m\gamma}{n}\right]^2} \quad \frac{1}{a^2}\left\{\alpha - \frac{l\gamma}{n}\right\}^2 + \frac{1}{b^2}\left\{\beta - \frac{m\gamma}{n}\right\}^2$$

* Eliminating l, m, n

$$\frac{x-\alpha}{z-\gamma} = \frac{l}{n}, \quad \frac{y-\beta}{z-\gamma} = \frac{m}{n}$$

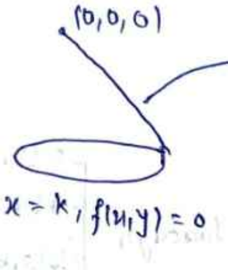
$$\therefore \frac{1}{a^2}\left\{\alpha - \left(\frac{x-\alpha}{z-\gamma}\right)\gamma\right\}^2 + \frac{1}{b^2}\left\{\beta - \left(\frac{y-\beta}{z-\gamma}\right)\gamma\right\}^2 = 0$$

$$\frac{1}{a^2}\left\{\alpha z - \cancel{\alpha\gamma} - x\gamma + \cancel{\alpha\gamma}\right\}^2 + \frac{1}{b^2}\left\{\beta z - \cancel{\beta\gamma} - y\gamma + \cancel{\beta\gamma}\right\}^2 = 0$$

$$\frac{1}{a^2}(\alpha z - \gamma x)^2 + \frac{1}{a^2}(\beta z - \gamma y)^2 = 0$$

Q Prove that the eq^n of cone whose vertex is Origin & base of the curve $z = k$, $f(x,y) = 0$ is $f\left(\frac{xk}{z}, \frac{yk}{z}\right) = 0$

Sol^n:



$$\frac{x-0}{l} = \frac{y-0}{m} = \frac{z-0}{n} \quad \text{--- (1)}$$

$$\frac{x}{l} = \frac{y}{m} = \frac{k}{n}$$

$$P(x,y,z) = \left(\frac{kl}{n}, \frac{km}{n}, k\right)$$

Satisfies $f(x,y) = 0$

$$\therefore f\left(\frac{kl}{n}, \frac{km}{n}\right) = 0 \quad \text{--- (2)}$$

from eq^n (1)

$$\frac{x}{z} = \frac{l}{n} \;\&\; \frac{y}{z} = \frac{m}{n}$$

$\therefore$ (2) becomes $f\left(\frac{xk}{z}, \frac{yk}{z}\right) = 0 \rightarrow$ Eq^n of Cone

Q The eq^n of cone whose generating line passes through Origin & satisfying the relation $3l^2 + 5m^2 - 2n^2 = 0$

Sol^n: Eq^n of line $\rightarrow \frac{x}{l} = \frac{y}{m} = \frac{z}{n} = k$ (as line passes through origin).

$$\therefore \cancel{x = z}\quad l = \frac{x}{k},\; m = \frac{y}{k},\; n = \frac{z}{k}$$

Now

$$\frac{3x^2}{k^2} + \frac{5y^2}{k^2} - \frac{2z^2}{k^2} = 0$$

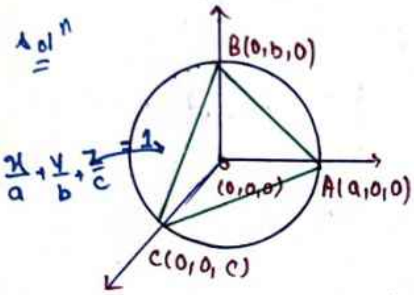
cone: $3x^2 + 5y^2 - 2z^2 = 0$

Q The plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$, meets the co-ordinate Axes in A, B & C respectively.

Prove that the eq^n to the cone generated by their lines drawn from O to Meet the Circle ABC is

$$yz\left(\frac{b}{c}+\frac{c}{b}\right)+zx\left(\frac{c}{a}+\frac{a}{c}\right)+xy\left(\frac{a}{b}+\frac{b}{a}\right)=0$$

Sol^n



Eq^n of sphere of (OABC) which passes through O(0,0,0), A(a,0,0), B(0,b,0), C(0,0,c)

$$x^2+y^2+z^2-ax-by-cz=0$$

Homogenizing this using $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$

$\therefore$

$$x^2+y^2+z^2-ax\left(\frac{x}{a}+\frac{y}{b}+\frac{z}{c}\right)-by\left(\frac{x}{a}+\frac{y}{b}+\frac{z}{c}\right)-cz\left(\frac{x}{a}+\frac{y}{b}+\frac{z}{c}\right)=0$$

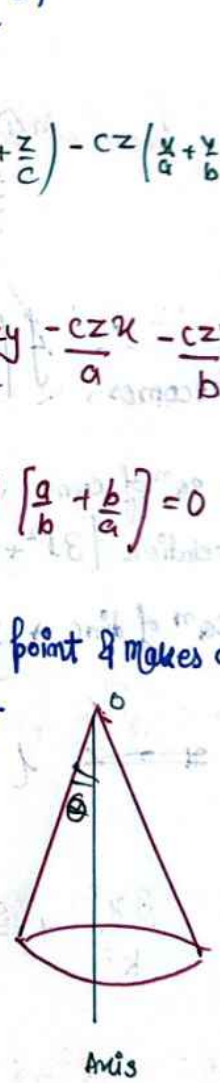
$$\cancel{x^2}+\cancel{y^2}+\cancel{z^2}-\cancel{x^2}-\frac{axy}{b}-\frac{azx}{c}-\frac{bxy}{a}-\cancel{y^2}-\frac{bzy}{c}-\frac{czx}{a}-\frac{czy}{b}-\cancel{z^2}=0$$

$$yz\left(\frac{b}{c}+\frac{c}{b}\right)+zx\left[\frac{c}{a}+\frac{a}{c}\right]+xy\left[\frac{a}{b}+\frac{b}{a}\right]=0$$

L-8

Right circular Cone

- A surface generated by a line which passes through a fixed point & makes an constant Angle α with a fixed line through the fixed point is called the Right Circular Cone.



- Eq^n of right Circular Cone whose Vertex is (α, β, γ), Semivertical angle θ & the Axis is the line

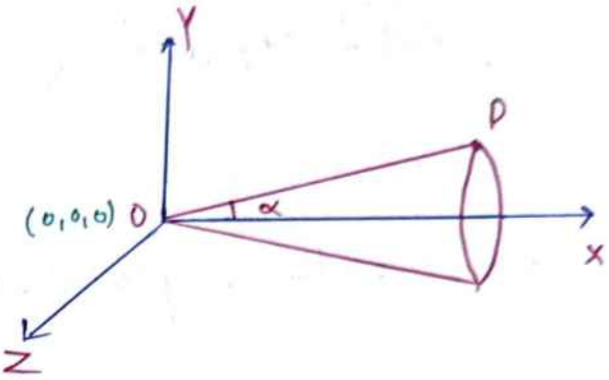
$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$$

$$\cos\theta=\frac{l(x-\alpha)+m(y-\beta)+n(z-\gamma)}{\sqrt{l^2+m^2+n^2}\sqrt{(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2}}\rightarrow \text{Req}^{r}\ \text{Eq}^n$$

Q The eq^n of right circular cone whose vertex is $(0,0,0)$, axis is X-Axis & Semivertical angle is α.

Sol^n The eq^n of X Axis is

$$\frac{x}{1} = \frac{y}{0} = \frac{z}{0}$$



The eq^n of Right circular cone is

$$\cos\alpha = \frac{1.x+0+0}{\sqrt{x^2+y^2+z^2}}$$

$$x^2+y^2+z^2 = \frac{x^2}{\cos^2\alpha} \Rightarrow y^2+z^2 = x^2(\sec^2\alpha - 1)$$

$$\left[y^2+z^2 = x^2\tan^2\alpha\right]$$

NOTE:1, The eq^n of right circular cone whose vertex is $(0,0,0)$, axis is y axis & semi vertical Angle is $\alpha \Rightarrow \left[x^2+z^2 = y^2\tan^2\alpha\right]$

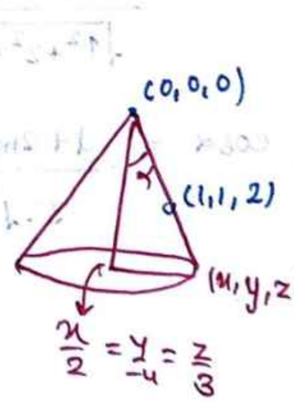
Note-2, The eq^n of right circular cone whose vertex is $(0,0,0)$, axis is z axis & Semivertical angle is $\alpha \Rightarrow \left[x^2+y^2 = z^2\tan^2\alpha\right]$

Q find the eq^n of right circular cone with vertex at the origin, axis the line $\frac{x}{2} = \frac{y}{-4} = \frac{z}{3}$ & which passes through the point $(1,1,2)$.

Sol^n Let α be the semi-vertical angle of the cone,

The Direction Ratio's of generator are $1, 1, 2$

The Direction Ratio's of axis of cone are $2, -4, 3$



$$\cos\alpha = \frac{2x-4y+3z}{\sqrt{2^2+4^2+3^2}\cdot\sqrt{x^2+y^2+z^2}} \quad -(1)$$

At $(1,1,2)$

$$\cos\alpha = \frac{2-4+6}{\sqrt{29}\cdot\sqrt{6}} = \frac{4}{\sqrt{29}\sqrt{6}} - (2) -$$

from (2) putting value of $\cos\alpha$ in (1)

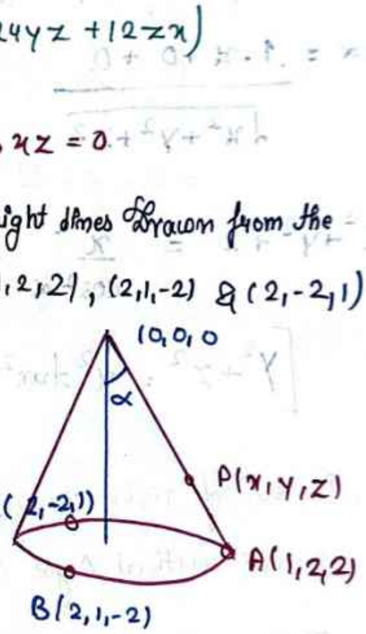
$$\frac{4}{\cancel{\sqrt{29}}\sqrt{6}} = \frac{2x-4y+3z}{\cancel{\sqrt{29}}\sqrt{x^2+y^2+z^2}}$$ SQUARING

$$16(x^2+y^2+z^2) = 6(2x-4y+3z)^2$$

$$8(x^2+y^2+z^2) = 3\left(4x^2+16y^2+9z^2-16xy-24yz+12zx\right)$$

$\therefore$ CONE: $4x^2+40y^2+19z^2-48xy-72yz+36xz=0$

Q find the eqⁿ of right Circular cone generated by straight lines drawn from the origin to cut the circle through the 3 points $(1,2,2)$, $(2,1,-2)$ & $(2,-2,1)$.



Solⁿ Let the vertex of cone be origin $(0,0,0)$

$\therefore$ if α is semivertical angle of the cone and l,m,n be the Direction Ratios of the ~~line~~ Axis of Cone, Then.

Now

$$\cos\alpha = \frac{lx+my+nz}{\sqrt{l^2+m^2+n^2}\sqrt{x^2+y^2+z^2}}$$

for, $A(1,2,2)$

$$\cos\alpha = \frac{l\cdot 1+m\cdot 2+n\cdot 2}{\sqrt{1^2+2^2+2^2}\sqrt{l^2+m^2+n^2}}$$

$$\cos\alpha = \frac{l+2m+2n}{3\cdot\sqrt{l^2+m^2+n^2}} \quad (1)$$

for, $B(2,1,-2)$

$$\cos\alpha = \frac{2l+1\cdot m-2\cdot n}{\sqrt{2^2+1^2+2^2}\sqrt{l^2+m^2+n^2}}$$

$$\cos\alpha = \frac{2l+m-2n}{3\cdot\sqrt{l^2+m^2+n^2}} \quad (2)$$

for, $C(2,-2,1)$

$$\cos\alpha = \frac{l\cdot 2+m\cdot(-2)+n\cdot 1}{\sqrt{2^2+2^2+1^2}\sqrt{l^2+m^2+n^2}}$$

$$\cos\alpha = \frac{2l-2m+n}{3\sqrt{l^2+m^2+n^2}} \quad (3)$$

• By equating eqⁿ 1 & 2

$$\frac{l+2m+2n}{\cancel{3\sqrt{l^2+m^2+n^2}}} = \frac{2l+m-2n}{\cancel{3\sqrt{l^2+m^2+n^2}}} \Rightarrow [-l+m+4n=0] \text{ —(4)}$$

• By equating eqⁿ 2 & 3

$$\frac{2l+m-2n}{\cancel{3\sqrt{l^2+m^2+n^2}}} = \frac{2l-2m+n}{\cancel{3\sqrt{l^2+m^2+n^2}}} \Rightarrow m-n=0 \Rightarrow [m=n] \text{ —(5)}$$

• By eqⁿ 4 & 5

$$-l+5n=0 \Rightarrow [l=5n] \rightarrow \frac{l}{5}=n=m$$

$$\frac{l}{5}=\frac{n}{1}=\frac{m}{1} \rightarrow \text{~~Ans~~}$$

OR $[l=5m \;\&\; m=n]$

Now from 1

$$\cos\alpha = \frac{l+2m+2n}{3\sqrt{l^2+m^2+n^2}}$$

$$\cos\alpha = \frac{5m+2m+2n}{3\cdot\sqrt{m^2(25+1+1)}} \Rightarrow \cos\alpha = \frac{\cancel{m}(9)}{3\cdot\cancel{m}\sqrt{27}} = \frac{\cancel{9}}{\cancel{3}\cdot 3\cdot\sqrt{3}}$$

$\rightarrow \boxed{\cos\alpha = \frac{1}{\sqrt{3}}}$; if $P(x,y,z)$ be any point on cone then

$$\cos\alpha = \frac{lx+my+nz}{\sqrt{l^2+m^2+n^2}\sqrt{x^2+y^2+z^2}}$$

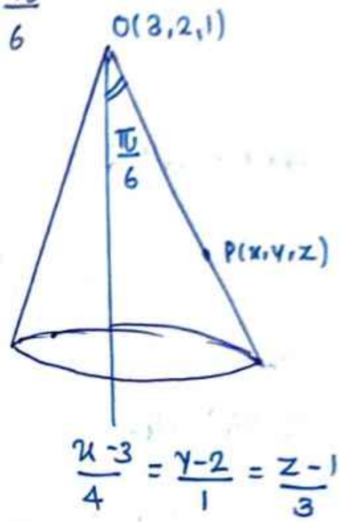
$$\frac{1}{\sqrt{3}} = \frac{5x+y+z}{\sqrt{27}\sqrt{x^2+y^2+z^2}}$$

$$9(x^2+y^2+z^2)$$
$$(5x+y+z)^2$$

$$\left[\text{CONE: } 8x^2-4y^2-4z^2+5xy+yz+5zx=0\right]$$

Q find the eqⁿ of cone whose vertex is (3, 2, 1), axis is the line :-

$$\frac{x-3}{4} = \frac{y-2}{1} = \frac{z-1}{3}$$ & Semi vertical angle is $\frac{\pi}{6}$



Solⁿ Take any point P (x, y, z) on the cone

∴ D.R's ~~theory~~ of the generator through P are

$x-3,\ y-2,\ z-1$

Also the D.R's of the Axis of the cone ~~is~~ are 4, 1, 3

∴

$$\cos\frac{\pi}{6} = \frac{4(x-3)+1(y-2)+3(z-1)}{\sqrt{16+1+9}\cdot\sqrt{(x-3)^2+(y-2)^2+(z-1)^2}}$$

$$\frac{\sqrt{3}}{2} = \frac{4x+y+3z-17}{\sqrt{26}\cdot\sqrt{x^2+y^2+z^2-6x-4y-2x+14}}$$

* Squaring & simplyfying ↓

[CONE: $7x^2+37y^2+21z^2-16xy-48zx+88x-98y+126z-32=0$]

L-9

Condition for the General eqⁿ of second Degree to Represent a Cone.

• Let the general eqⁿ of Second Degree be : $ax^2+by^2+cz^2+2fyz+2gzx+2hxy$
$+2ux+2vy+2wz+d=0$

It represent a cone if $\begin{vmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & d \end{vmatrix} = 0$

Q Prove that the eq^n $ax^2+by^2+cz^2+2ux+2vy+2wz+d=0$ represent a cone if $\frac{u^2}{a}+\frac{v^2}{b}+\frac{w^2}{c}=d$

Sol^n Using the cond^n
$$\begin{vmatrix} a & 0 & 0 & u \\ 0 & b & 0 & v \\ 0 & 0 & c & w \\ u & v & w & d \end{vmatrix} = 0$$

$$\therefore\ a\begin{pmatrix} b & 0 & v \\ 0 & c & w \\ v & w & d \end{pmatrix} - u\begin{pmatrix} 0 & b & 0 \\ 0 & 0 & c \\ u & v & w \end{pmatrix}$$

$$\cancel{a[b(cd-w^2)]-u[}\quad a\{b(cd-w^2)+v(-cv)\}+u(b(-cu))=0$$

$$abcd - aw^2b - av^2c - u^2bc = 0$$

$$abcd = w^2ab + v^2ac + u^2bc$$

$$\left[\frac{u^2}{a}+\frac{v^2}{b}+\frac{w^2}{c}=d\right]$$

Tangent Plane of a Cone

- The eq^n of the tangent plane to the cone $ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$ at the point (α,β,γ) is-

$$x(a\alpha+h\beta+g\gamma)+y(h\alpha+b\beta+f\gamma)+z(g\alpha+f\beta+c\gamma)=0$$

Cond^n for Tangency

The cond^n that the plane $lx+my+nz=0$ may touch the cone $ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$ is

$$\begin{vmatrix} a & h & g & l \\ h & b & f & m \\ g & f & c & n \\ l & m & n & 0 \end{vmatrix} = 0$$

- Expanding the Determinant

$$Al^2 + Bm^2 + Cn^2 + 2Fmn + 2Gnl + 2Hlm = 0$$

$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$, Jiski value chahiye uska cofactor nikalo.

$A = bc - f^2$, $B = ac - g^2$, $C = ab - h^2$

$F = hg - af$, $G = hf - gb$, $H = fg - ch$

Reciprocal Cone

The locus of the line through vertex Normal to tangent planes of a given cone is called reciprocal cone.

* Let the eq^n of cone be - $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$

Then reciprocal cone is, $Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gzx + 2Hxy = 0$

$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$ → use this for values of A, B, C, F, G, H.

Cond^n for three mutually perpendicular generator

- Let the eq^n of cone be :- $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$

Then, the cond^n for three mutually perpendicular generator is -

$$\boxed{a + b + c = 0}$$ → Sum of coeff of x^2, y^2, z^2 equals 0

Q The reciprocal cone of $ax^2 + by^2 + cz^2 + 2hxy + 2fyz + 2gzx = 0$ may have 3 mutually perpendicular generator if $bc + ca + ab = f^2 + g^2 + h$

Ans Reciprocal Cone: $Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gzx + 2Hxy = 0$

for 3 mutually $\perp^r$ gen^r, $A + B + C = 0$; $A = bc - f^2$

$\therefore bc + ac + ab - (f^2 + g^2 + h^2) = 0$ $\quad B = ac - g^2$

$\rightarrow ab + bc + ac = f^2 + g^2 + h^2$ $\quad C = ab - h^2$

Q Show that the cone $ax^2+by^2+cz^2=0$ & $\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}=0$ are Reciprocal to Each other.

Sol^n Let $ax^2+by^2+cz^2=0$ Be eq^n of cone, Then Reciprocal Cone is $Ax^2+By^2+Cz^2=0$

$$\begin{array}{c} A \\ B \\ C \end{array} \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix}, \quad A=bc, \quad B=ac, \quad C=ab$$

$\therefore$ Reciprocal Cone: $bcx^2+acy^2+abz^2$

$\div abc$

RC: $\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}=0$

Q If $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$ represent ~~of~~ one of three mutually $\perp^r$ gen^r of the cone $5yz-8xz-3xy=0$, Then eq^n of other two gen^r is_

Sol^n Let $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$ be the eq^n of remaining gen^r of cone $5yz-8xz-3xy=0$

This gen^r will be $\perp^r$ to $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$ & lie on the cone $\therefore$

$l+2m+3n=0 \quad -(1)$

$5mn-8ln-3lm=0 \quad -(2)$

Eliminating l in (2) from (1)

$5mn+8(2m+3n)n+3(2m+3n)m=0$

$5mn+(8n+3m)(2m+3n)=0$

$6m^2+30mn+24n^2=0$

$m^2+5mn+4n^2=0 \rightarrow m^2+mn+4mn+4n^2=0$

$m(m+4n)+n(m+4n)=0 \quad \therefore (m+n)(m+4n)=0$

$\therefore m=-n$ or $m=-4n$

$\downarrow \qquad\qquad \downarrow$

$l=-n \qquad l=5n$

$\therefore$ Other 2 gen^r are, $\frac{x}{1}=\frac{y}{1}=\frac{z}{-1}$ & $\frac{x}{5}=\frac{y}{-4}=\frac{z}{1}$

Q If a right circular cone has three mutually perpendicular generators, then show that the semi-vertical angle is $\tan^{-1}\sqrt{2}$?

Sol^n Let us consider the eq^n of the right circular cone with vertex at the Origin, Axis - Z Axis & Semi-Vertical Angle θ

It's eq^n is given by $x^2+y^2-z^2\tan^2\theta=0$ — (1)

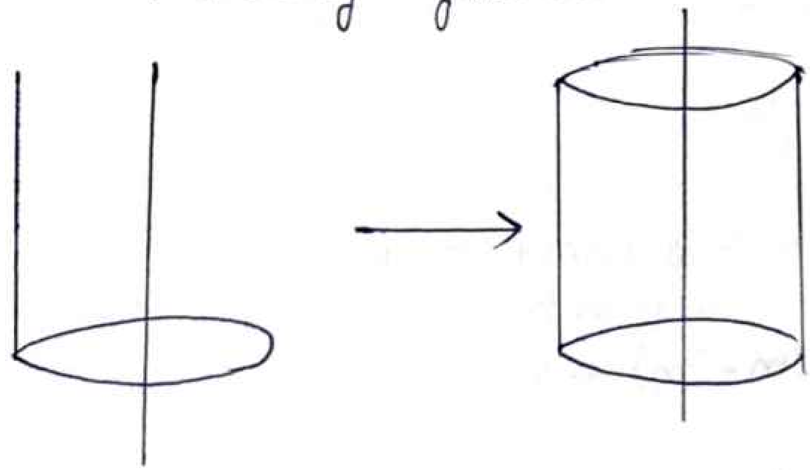
If the cone (1) has 3 mutually $\perp^r$ gen^r, then sum of coeff of x^2, y^2 & z^2 in eq^n (1) must be zero.

$\therefore\ 1+1-\tan^2\theta=0$

$$2=\tan^2\theta \rightarrow \tan\theta=\sqrt{2} \Rightarrow \boxed{\theta=\tan^{-1}\sqrt{2}}$$

* Cylinder

- The cylinder is a surface generated by a variable straight line which is parallel to a fixed line & intersecting a given curve.



1) Generator - The line which generates the surface of the cylinder.

2) Axis - The fixed straight line is called the Axis

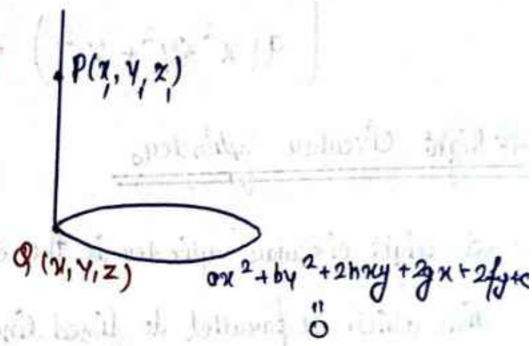
3) Guiding Curve - The curve with which the generator's intersect.

Equation of Cylinder.

- The eq^n of Cylinder whose generators are parallel to the line $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$

& Base is the conic $- ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0;\ z = 0$

→ The eq^n of generator : $\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$



This meets the base at $z = 0$; so

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{-z_1}{n}$$

$$x = x_1 - \frac{z_1 l}{n},\quad y = y_1 - \frac{z_1 m}{n},\quad z = 0$$

$$\therefore Q(x, y, z) = Q\left(x_1 - \frac{z_1 l}{n},\ y_1 - \frac{z_1 m}{n},\ 0\right)$$

This point satisfy the eq^n of Base . i.e

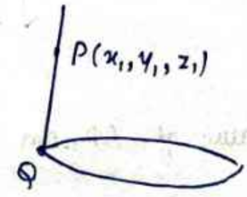
$$a\left(x_1 - z_1\frac{l}{n}\right)^2 + b\left(y - z_1\frac{m}{n}\right)^2 + 2h\left(x_1 - z_1\frac{l}{n}\right)\left(y_1 - z_1\frac{M}{n}\right)$$

$$+ 2g\left(x_1 - z_1\frac{l}{n}\right) + 2f\left(y_1 - z_1\frac{m}{n}\right) + c = 0$$

→ Solve this eq^n to get req^r eq^n of cylinder.

Q The eq^n of cylinder whose gen^r is ∥ to the line $\frac{x}{1} = \frac{y}{-2} = \frac{z}{3}$ & Guiding curve is $x^2 + 2y^2 = 1;\ z = 0$

Sol^n The eq^n of gen^r : $\frac{x - x_1}{1} = \frac{y - y_1}{-2} = \frac{z - z_1}{3}$ —(1)



for intersecting point Q, put $z = 0$ in (1), we get

$$Q\left(x_1 - \frac{z_1}{3},\ y_1 + \frac{2z_1}{3},\ 0\right)$$

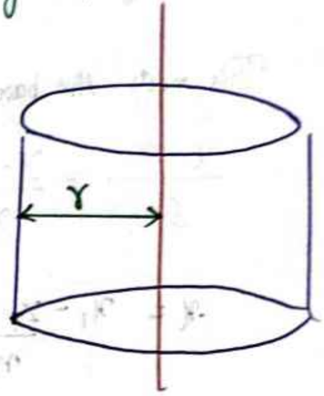
- The point Q satisfy the ellipse, so $\left(x_1 - \frac{z_1}{3}\right)^2 + 2\left(y_1 + \frac{2z_1}{3}\right)^2 = 1$

$$(3x_1 - z_1)^2 + 2(3y_1 + 2z_1)^2 = 9$$

Then, locus is $(3x - z)^2 + 2(3y + 2z)^2 = 9$

$$\left[\,9(x^2 + 2y^2 + z^2) - 6xz + 24yz = 9\,\right] \rightarrow \text{Eq}^n \text{ of Cylinder}$$

Right circular Cylinders



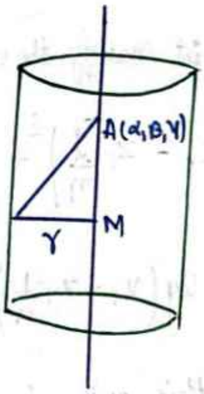
- A right circular cylinder is the surface generated by variable line which is parallel to fixed line & is at a constant Distance.

- Eq^n of Right Circular ~~Cone~~ Cylinder

- The eq^n of Right Circular ~~Cone~~ Cylinder whose axis is the line $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$ and radius is r.

~~Here~~ Then

$$AP^2 = AM^2 + PM^2 \quad -(1)$$



Here

$$AP = \sqrt{(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2}$$

$$PM = r$$

$$AM = \frac{l(x-\alpha) + m(y-\beta) + n(z-\gamma)}{\sqrt{l^2+m^2+n^2}}$$

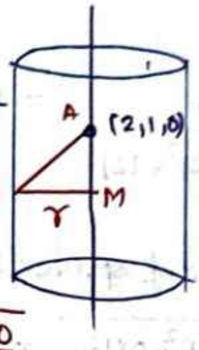
Put value of AP, PM & AM in eq^n (1);

→ We get eq^n of Req^r Right Circular Cylinder.

Q The eq^n of right circular ~~oh~~ cylinder whose axis is $\frac{(x-2)}{2} = \frac{(y-1)}{1} = \frac{z}{3}$ and which passes through (0,0,1)

Sol^n $AP^2 = PM^2 + AM^2$

$(2-0)^2 + (1-0)^2 + (0-1)^2 = r^2 + \left[\frac{2(2-0)+1(1-0)+3(0-1)}{\sqrt{2^2+1^2+3^2}}\right]^2$



$r^2 = 6 - \frac{4}{14} \Rightarrow r^2 = \frac{40}{7} \Rightarrow r = \sqrt{\frac{40}{7}}$

$\therefore$ Now for $P(x,y,z)$; $AP^2 = PM^2 + AN^2$

$(2-x)^2 + (1-y)^2 + (0-z)^2 = \frac{40}{7} + \left[\frac{2(x-2)+1(y-1)+3(z-0)}{\sqrt{14}}\right]^2$

$\downarrow$

$(x-2)^2 + (y-1)^2 + z^2 - \left[\frac{2(x-2)+(y-1)+3z}{\sqrt{14}}\right]^2 = \frac{40}{7}$

$14\left[(x-2)^2 + (y-1)^2 + z^2\right] - \left[4(x-2)^2 + (y-1)^2 + 9z^2 + 4(x-2)(y-1) + 12z(x-2) + 6z(y-1)\right] = 80$

$\rightarrow 10(x-2)^2 + 13(y-1)^2 + 5z^2 - 4xy + 4x + 8y - 8 - 12zx + 24x - 6zy + 6z = 80$

$10(x^2 - 4x + 4) + 13(y^2 - 2y + 1) + 5z^2 - 4xy - 12zx - 6zy + 30z + 4x + 8y - 8 = 80$

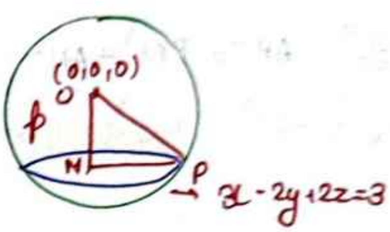
$10x^2 + x(4-40) + 40 + 13y^2 + y(8-26) + 30z + 13 - 8 - 6zy - 4xy + 5z^2 - 12zx = 80$

$\rightarrow 10x^2 + 13y^2 + 5z^2 + 36x - 18y + 30z + 45 - 12zx - 6zy - 4xy = 80$

Req^r eq^n of Right Circular cone : $\left[10x^2 + 13y^2 + 5z^2 - 6yz - 12zx - 4xy - 36x - 18y + 30z - 35 = 0\right]$

Q* The eqⁿ of Right circular Cylinder Whose guiding curve is Circle

$$x^2+y^2+z^2=9,\ x-2y+2z=8.$$



Solⁿ $p = \dfrac{3}{\sqrt{1+(-2)^2+(2)^2}} = 1$

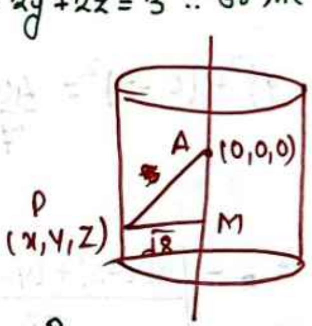
& OP = radius of sphere = 3

$\therefore\ MP = \sqrt{OP^2-OM^2} = \sqrt{9-1}$

$= \sqrt{8}$

→ Here axis is perpendicular to plane $x-2y+2z=3$ ∴ So the eqⁿ of Axis is

$$\frac{x}{1}=\frac{y}{-2}=\frac{z}{2}$$



$(x-0)^2+(y-0)^2+(z-0)^2$

$\parallel$

$$(\sqrt{8})^2+\left[\frac{1(x-0)-2(y-0)+2(z-0)}{\sqrt{1+(-2)^2+(2)^2}}\right]^2 \Rightarrow x^2+y^2+z^2=8+\left[\frac{x-2y+2z}{3}\right]^2$$

$$x^2+y^2+z^2=8+\frac{1}{9}\left\{x^2+4y^2+4z^2-4xy-8yz+4xz\right\}$$

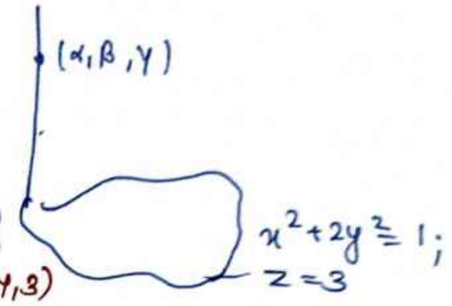
$$9(x^2+y^2+z^2)=8\cdot 9+x^2+4y^2+4z^2-4xy-8yz+4xz$$

∴ Reqᵈ eqⁿ of Right Circular cylinder is

$$\left[8x^2+5y^2+5z^2+4xy+8yz-4xz-72=0\right]$$

Q The eq^n of cylinder whose gen^r are parallel to $\frac{x}{1} = \frac{y}{-2} = \frac{z}{3}$ & guiding curve is $x^2 + 2y^2 = 1;\ z = 3$.

Sol^n Eq^n of Axis : $\frac{x}{1} = \frac{y}{-2} = \frac{z}{3}$



$\therefore$ Eq^n of Generator : $\frac{(x-\alpha)}{1} = \frac{(y-\beta)}{-2} = \left(\frac{z-\gamma}{3}\right)$

This intersect guiding curve at Q ; $z = 3$

~~So Q(x, y, z) = (α − ((3−γ)/3), β − ((3+γ)/1)~~

So $x = \alpha + \frac{3-\gamma}{3} = \alpha + 1 - \frac{\gamma}{3}$; $y = \beta + \frac{2\gamma}{3} - \frac{2 \cdot 3}{3}$

$= \beta - 2 + \frac{2\gamma}{3}$

$\therefore Q(x, y, z) \equiv Q\left(\alpha + 1 - \frac{\gamma}{3}, \beta - 2 + \frac{2\gamma}{3}, 3\right)$

This point lies on ellipse $\therefore \left(\alpha + 1 - \frac{\gamma}{3}\right)^2 + 2\left(\beta - 2 + \frac{2\gamma}{3}\right)^2 = 1$

$$(3\alpha + 3 - \gamma)^2 + 2(3\beta - 6 + 2\gamma)^2 = 9$$

The locus is $(3x + 3 - z)^2 + 2(3y + 2z - 6)^2 = 9$

Req^r Eq^n of cylinder.