Unit :- 1

Electricity & Magnetism

Physical Quantity

Scalar (only Mag.)

$$\boxed{\vec{A}\cdot\vec{B} = |\vec{A}||\vec{B}|\cos\theta}$$

$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$

$\vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$

$$\boxed{\vec{A}\cdot\vec{B} = A_xB_x + A_yB_y + A_zB_z}$$

Vector (Mag. + dirn)

$$\boxed{\vec{A}\times\vec{B} = AB\sin\theta\,\hat{n}}$$

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$$\vec{A}\times\vec{B} = i(A_yB_z - A_zB_y) - j(A_xB_z - B_xA_z) + k(A_xA_y - B_xA_y)$$

GRADIENT

→ The gradient is a vector operation which operates on scalar funcⁿ to produce vector.

→ Symbol ∇ (Nabla) → gradient $\nabla f(x,y,z)$

→ If funcⁿ $f(x,y,z)$

$$\boxed{\nabla = \frac{\delta}{\delta x}\hat{i} + \frac{\delta}{\delta y}\hat{j} + \frac{\delta}{\delta z}\hat{k}}$$

$$\nabla f(x,y,z) = \frac{\delta f(x)}{\delta x}\hat{i} + \frac{\delta f(x)}{\delta y}\hat{j} + \frac{\delta f(x)}{\delta z}\hat{k}$$

Example :- $f(x,y,z) = x + y^2 + z$, Find $\nabla f(x,y,z)$

$$\nabla f(x,y,z) = \frac{\delta}{\delta x}(x+y^2+z)\hat{i} + \frac{\delta}{\delta y}(x+y^2+z)\hat{j} + \frac{\delta}{\delta z}(x+y^2+z)\hat{k}$$

$$= (1)\hat{i} + (2y)\hat{j} + (1)\hat{k}$$

$$\boxed{\nabla f(x,y,z) = 1\cdot\hat{i} + 2y\hat{j} + 1\cdot\hat{k}}$$

Ans

DIVERGENCE

→ Divergence of vector field gives scalar.

→ Symbol = "Div" or "$\nabla.$"

→ Divergence of vector field can be calculated by taking (dot product) of ∇ & vector func^n ($\vec{F}$)

$$\nabla = \frac{\delta}{\delta x}\hat{i} + \frac{\delta}{\delta y}\hat{j} + \frac{\delta}{\delta z}\hat{k}$$

$$\vec{F} = F_x\hat{i} + F_y\hat{j} + F_z\hat{k}$$

$$\boxed{\nabla.\vec{F} = \frac{\delta}{\delta x}F_x + \frac{\delta}{\delta y}F_y + \frac{\delta}{\delta z}F_z}$$

scalar qty.

CURL

→ curl of vector func^n gives vector

→ curl of vector field is obtained by taking a vector product of vector operator (∇) & vector func^c $\vec{F}(x, y, z)$

curl $\vec{F}$ or $\nabla \times \vec{F}(x, y, z)$, $\vec{F}(x,y,z) = F_x\hat{i} + F_y\hat{j} + F_z\hat{k}$

$$\nabla \times \vec{F}(x,y,z) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ F_x & F_y & F_z \end{vmatrix}$$

$$\nabla \times \vec{F}(x,y,z) = \hat{i}\left(\frac{\delta}{\delta y}F_z - \frac{\delta}{\delta z}F_y\right) - j\left(\frac{\delta}{\delta x}F_z + \frac{\delta}{\delta z}F_x\right) + \hat{k}\left(\frac{\delta}{\delta x}F_z + \frac{\delta}{\delta y}F_x\right)$$

11.8 PHYSICAL SIGNIFICANCE OF DIVERGENCE

The divergence of vector field $\vec{E}$ (div $\vec{E}$) is defined as the limiting value of the ratio of the closed surface integral to the volume enclosed by the surface over which integration is carried out, when the volume tends to zero, i.e.,

$$div\,\vec{E} = \underset{v \to 0}{\mathrm{Lt}} \frac{1}{v} \oiint \vec{E} \cdot \vec{ds}$$

where v is the volume enclosed by the source S over which integration is carried out. In vector form div $\vec{E}$ is represented by $\vec{\nabla} \cdot \vec{E}$.

The divergence of a vector field $\vec{E}$ can be expressed as

$$div\,\vec{E} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot (\hat{i}\,E_x + \hat{j}\,E_y + \hat{k}\,E_z) = \vec{\nabla} \cdot \vec{E}$$

where symbol $\vec{\nabla}$ (del) is a vector differential operator and is given by $\vec{\nabla} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$.

The divergence of electric field at any point gives the charge density at that point, which is a scalar quantity.

If a vector function $\vec{E}$ spreads out, *i.e.*, diverges from a point, then it has a positive divergence at that point and point acts as a source of the field $\vec{E}$. Indeed $\vec{\nabla} \cdot \vec{E}$ can be taken to be a measure of the spreading out of the field. On the other hand, if the field converges to a point then $\vec{\nabla} \cdot \vec{E}$ will be negative at that point because the point acts as a sink for the field $\vec{E}$. Finally, if the vector field $\vec{E}$ neither converges nor diverges then $\vec{\nabla} \cdot \vec{E} = 0$, *i.e.*, the flux entering any element of space is same as leaving it and such a vector field is known as **solenoidal vector field.**

11.9 PHYSICAL SIGNIFICANCE OF CURL

The curl of a vector field signifies the whirling nature of the vector field which is sometime also known as rotation. It is directed along the perpendicular to the plane of maximum rotation which can be proved by considering the streamline flow of a liquid in a pipe. There exists a velocity gradient in the liquid and its velocity is maximum at the top and minmum at the bottom. Put a toothed wheel, with its plane along the direction of flow. It will begin to rotate in the direction of flow. Thus, the rotation of the wheel in this position will be maximum and it will be minimum if the toothed wheel is put with its plane perpendicular to the direction of flow. The rotation in any intermediate position will be in between the maximum and the minimum value.

The rotation with maximum value is termed as curl and is a vector quantity. If we denote the flow field in the above example by $\vec{B}$, then rotation may be represented as curl $\vec{B}$. It is represented in accordance with the right handed screw rule, *e.g.* in the above case, the rotation being in clockwise direction, represents the curl into the plane of the paper (Fig. 11.4).

The curl of a vector field $\vec{B}$ can be expressed as curl $\vec{B} = \vec{\nabla} \times \vec{B}$.

Non-zero curl of a vector field implies existance of circulation or velocity or rotation. It suggests a whirling effect or the formation of a vortex. We have seen the vortex in the river or in the tub, when water is allowed to flow out of it through a hole in the bottom (Fig. 11.5). $\vec{\nabla} \times \vec{E} = 0$ signifies that there is no tendency of rotation in an electrostatic field.

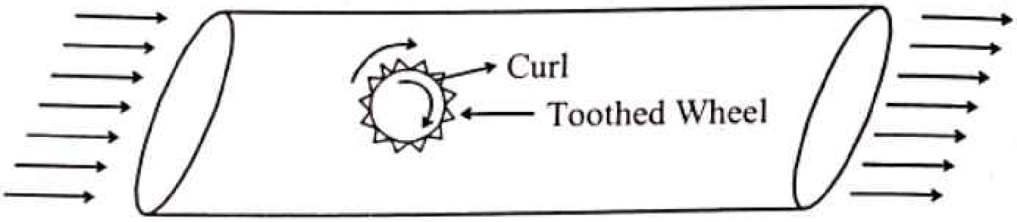

Fig. 11.4 Clockwise rotation of toothed wheel in the direction of flow

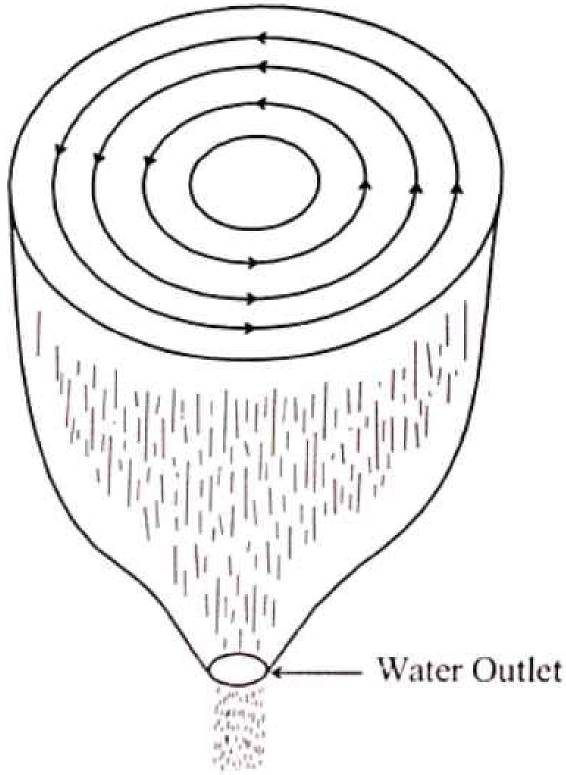

Fig. 11.5 Demonstration of vortex

11.10 GRADIENT OF A SCALAR FIELD AND ITS PHYSICAL SIGNIFICANCE

By itself $\vec{\nabla}$ has no physical significance. It acquires significance only when it operates upon a vector or a scalar function.

Let $U(\vec{r})$ be a scalar field. Then

$$\vec{\nabla}U = \left(\hat{i}\,\frac{\partial}{\partial x} + \hat{j}\,\frac{\partial}{\partial y} + \hat{k}\,\frac{\partial}{\partial z}\right)U = \hat{i}\,\frac{\partial U}{\partial x} + \hat{j}\,\frac{\partial U}{\partial y} + \hat{k}\,\frac{\partial U}{\partial z} \qquad ...(1)$$

$\vec{\nabla}U$ is termed as *gradient of U* and is abbreviated as grad U.

Let $\vec{r}$ be the position vector of a point, whose coordinates are (x, y, z), then

$$\vec{r} = \hat{i}x + \hat{j}y + \hat{k}z$$

also,
$$d\vec{r} = \hat{i}dx + \hat{j}\,dy + \hat{k}\,dz$$

Taking the scalar product $\vec{\nabla}U$ of $d\vec{r}$

$$\vec{\nabla}U \cdot d\vec{r} = \left(\vec{i}\,\frac{\partial U}{\partial x} + \vec{j}\,\frac{\partial U}{\partial y} + \vec{k}\,\frac{\partial U}{\partial z}\right) \cdot (\hat{i}dx + \hat{j}\,dy + \hat{k}dz)$$

$$= \frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy + \frac{\partial U}{\partial z}dz = dU$$

Therefore,

$$\vec{\nabla}U \cdot d\vec{r} = dU \qquad ...(2)$$

In other words ***differential of a scalar function is equal to the scalar product of the gradient of the function and the differential of the position vector.***

From eqn. (2)

$$dU = \vec{\nabla}U \cdot d\vec{r} \quad \text{or} \quad dU = |\vec{\nabla}U||d\vec{r}|\cos\theta$$

where θ is the angle between the direction of $\vec{\nabla}U$ and $d\vec{r}$.

Maximum value of dU is (when $\cos\theta = 1$)

$$dU_{\max} = |\vec{\nabla}U||d\vec{r}|$$

or

$$|\vec{\nabla}U| = \frac{dU_{\max}}{dr} \qquad (\because |d\vec{r}| = dr)$$

Therefore, it is clear that gradient of a scalar function U is the maximum rate of change of U with distance and directed along the normal to the surface having same value of U. Thus, $\vec{\nabla}U$ tells us how U varies in the neighbourhood of a point.

11.11 RELATIONSHIP BETWEEN ELECTRIC FIELD AND ELECTRIC POTENTIAL

We have seen that the electric potential is given by

$$U(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q}{r}$$

Now taking gradient of above expression, we have

$$\vec{\nabla}U = \vec{\nabla}\left(\frac{1}{4\pi\varepsilon_0}\frac{q}{r}\right) = \frac{1}{4\pi\varepsilon_0} q\, \vec{\nabla}\left(\frac{1}{r}\right)$$

Since, $\vec{r} = \hat{i}x + \hat{j}y + \hat{k}z$, we have

$$\vec{\nabla}\left(\frac{1}{r}\right) = \vec{\nabla}(x^2+y^2+z^2)^{-\frac{1}{2}} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2+y^2+z^2)^{-\frac{1}{2}}$$

$$= -\frac{1}{2}(x^2+y^2+z^2)^{-3/2}(\hat{i}\,2x + \hat{j}\,2y + \hat{k}\,2z)$$

$$= -r^{-3}\vec{r} = -\frac{1}{r^2}\frac{\vec{r}}{r} = -\frac{\hat{r}}{r^2} \qquad (\because \vec{r} = |\vec{r}|\hat{r})$$

$$\therefore \vec{\nabla}U = \frac{1}{4\pi\varepsilon_0} q\left(\frac{-\hat{r}}{r^2}\right) = -\frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}$$

or

$$\vec{\nabla}U = -\vec{E} \qquad \left(\because \vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}\right)$$

or

$$\vec{E} = -\vec{\nabla}U \qquad ...(3)$$

Thus, electric field at a point is defined as the gradient of the potential at that point.

11.11.1 Curl of Electric Field $\vec{E}$ is Zero

Since $$\vec{E} = -\nabla U$$

Therefore, $$\text{Curl } \vec{E} = \vec{\nabla}\times\vec{E} = \vec{\nabla}\times[-\vec{\nabla}U]$$

i.e., $$\vec{\nabla}\times\vec{E} = -\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial U}{\partial x} & \frac{\partial U}{\partial y} & \frac{\partial U}{\partial z} \end{vmatrix}$$

$$= -\left[\hat{i}\left(\frac{\partial^2 U}{\partial z\partial y} - \frac{\partial^2 U}{\partial y\partial z}\right) + \hat{j}\left(\frac{\partial^2 U}{\partial x\partial z} - \frac{\partial^2 U}{\partial x\partial z}\right) + \hat{k}\left(\frac{\partial^2 U}{\partial x\partial y} - \frac{\partial^2 U}{\partial x\partial y}\right)\right] = 0.$$

That is, curl of electric field is zero.

11.12 DIV. GRAD U ; LAPLACIAN OPERATOR

$$\text{div. grad } U = \vec{\nabla}\cdot\vec{\nabla}U = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left(\hat{i}\frac{\partial U}{\partial x} + \hat{j}\frac{\partial U}{\partial y} + \hat{k}\frac{\partial U}{\partial z}\right)$$

$$= \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)U$$

$$\therefore \quad \vec{\nabla}\cdot\vec{\nabla}U = \vec{\nabla}^2 U$$

where $\vec{\nabla}^2\left(= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)$ is known as Laplacian operator.

11.13 DIFFERENTIAL VECTOR IDENTITIES

Followings are the possible combinations of differential operators and products which can be applied to various products of two vectors (*e.g.*, E and E′) and scalars (*e.g.* U).

(*i*) $\vec{\nabla}\cdot\vec{\nabla}U = \vec{\nabla}^2 U$

(*ii*) $\vec{\nabla}\cdot\vec{\nabla}\times\vec{E} = 0$

(*iii*) $\vec{\nabla}\times\vec{\nabla}U = 0$

(*iv*) $\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{E}) - \vec{\nabla}^2 E$

(*v*) $\vec{\nabla}(UU') = (\nabla U)U' + U\nabla U'$

(*vi*) $\vec{\nabla}(\vec{E}\times\vec{E}') = (\vec{E}\cdot\vec{\nabla})\vec{E}' + \vec{E}\times(\vec{\nabla}\times\vec{E}') + (\vec{E}\cdot\vec{\nabla})\vec{E} + \vec{E}'(\vec{\nabla}\times\vec{E})$

(*vii*) $\vec{\nabla}\cdot(U\vec{E}) = (\vec{\nabla}U)\cdot\vec{E} + U\vec{\nabla}\cdot\vec{E}$

(*viii*) $\vec{\nabla}\cdot(\vec{E}\times\vec{E}') = (\vec{\nabla}\times\vec{E})\cdot\vec{E} - (\vec{\nabla}\times\vec{E}')\cdot\vec{E}$

(*ix*) $\vec{\nabla}\times(U\vec{E}) = (\vec{\nabla}U)\times\vec{E} + U\vec{\nabla}\times\vec{E}'$

(*x*) $\vec{\nabla}\times(\vec{E}\times\vec{E}') = (\vec{\nabla}\cdot\vec{E}') + \vec{E} - (\vec{\nabla}\cdot\vec{E})\vec{E}' + (\vec{E}'\cdot\vec{\nabla})\vec{E} - (\vec{E}\cdot\vec{\nabla})\vec{E}'$

11.14 TYPES OF VECTOR FIELDS

1. Lamellar Vector Field (LVF): ***A vector field is said to be lamellar (Laminar) if it can be expressed as the gradient of a scalar field.*** Electric field is a lamellar field as it is expressed by gradient of potential U

i.e.,
$$\vec{E} = -\vec{\nabla}U$$

The name 'lamellar' suggests that the field can be divided into layers over which the value of the scalar function whose gradient gives the vector field, remains constant.

The line integral of a lamellar vector is independent of the path followed and only depends on the initial and final points of the path, *e.g.*, the line integral of a lamellar field between two points A and B is given by

$$\int_A^B \vec{E}\cdot d\vec{l} = \int_A^B -\vec{\nabla}U\cdot d\vec{l}$$

$$= -\int_A^B dU = U_A - U_B$$

where U is the electric potential and U_A and U_B are values of electric potential at A and B respectively.

From last equation, we may write

$$\oint \vec{E}\cdot d\vec{l} = \int_A^B \vec{E}\cdot d\vec{l} = U_A - U_B = 0$$

i.e., the closed line integral of a lamellar field is zero.

2. Solenoidal Vector Field (SVF): ***A vector field $\vec{B}$ is said to be solenoidal if its divergence is zero.***

i.e.,
$$\vec{\nabla}\cdot\vec{B} = 0 \qquad ...(4)$$

Solenoidal fields neither have source nor sink of flux and the flux lines entering a closed surface must also leave it. Incompressible fluids, steady magnetic fields, current density under steady state are some of the examples of solenoidal vector fields.

3. Irrotational Vector Field (IVF): ***A vector field $\vec{E}$ whose curl is zero is called irrotational vector field.***

i.e.,
$$\vec{\nabla}\times\vec{E} = 0 \qquad ...(5)$$

This equation is satisfied only by electric field $\vec{E}$.

($\because$ $\vec{E} = -\vec{\nabla}U$ then $\vec{\nabla}\times\vec{E} = \vec{\nabla}\times[-\vec{\nabla}U] = -[\vec{\nabla}\times\vec{\nabla}U]$ But ($\vec{\nabla}\times\vec{\nabla} = 0$ $\therefore$ $\vec{\nabla}\times\vec{\nabla}U = 0$ or $\vec{\nabla}\times\vec{E} = 0$)

4. Rotational Vector Field (RVF): ***A vector field whose curl is non-zero is called rotational vector field.*** Magnetic flux density $\vec{B}$ satisfies this condition and is said to be rotational.

i.e.,
$$\vec{\nabla}\times\vec{B} \neq 0 \qquad ...(6)$$

11.15 STOKES THEOREM

It states that the integral over a surface of the normal component of the curl $\vec{E}$ is equal to the line integral of the tangential component of the $\vec{E}$ around the path enclosing area S, *i.e.*,

$$\oint \vec{E}\,.\,d\vec{l} = \iint_S \vec{\nabla}\times\vec{E}\,.\,d\vec{s}$$

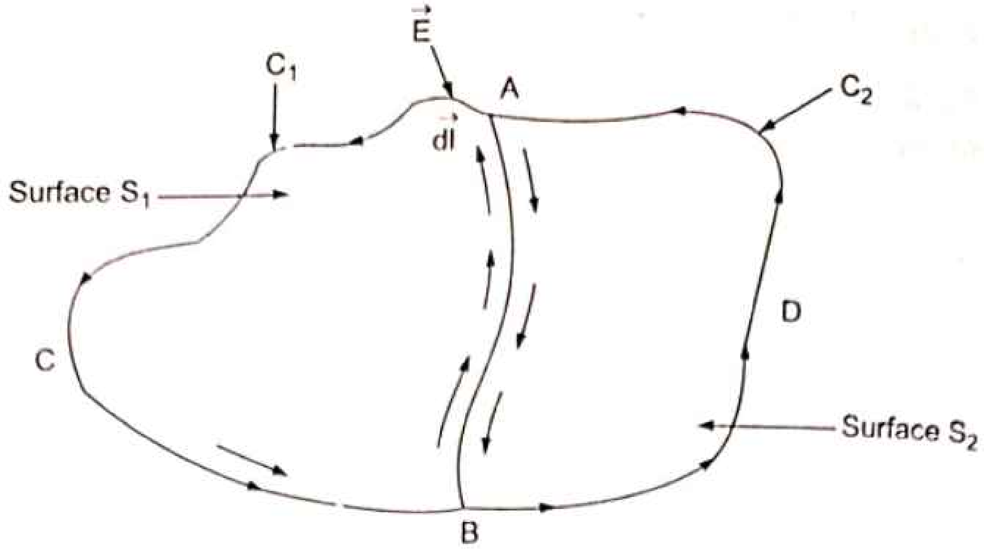

Fig. 11.6 A closed curve c having surface s

Proof: Consider a closed curve c having surface S as shown in Fig. 11.6. Suppose this surface is placed in a vector field, in the present case we have considered an electric field $\vec{E}$. The line integral of $\vec{E}$ along the closed curve c is given by

$$\oint_C \vec{E}\,.\,d\vec{l}$$

For our convenience we have divided the contour c into two small contours c_1 and c_2 having respectively surfaces S_1 and S_2. Therefore, the sum of closed line integral of $\vec{E}$ on both contours c_1 and c_2 gives

$$\oint_{c_1} \vec{E}\,.\,d\vec{l} + \oint_{c_2} \vec{E}\,.\,d\vec{l} = \int_{ACB} \vec{E}\,.\,d\vec{l} + \int_{BA} \vec{E}\,.\,d\vec{l} + \int_{AB} \vec{E}\,.\,d\vec{l} + \int_{BDA} \vec{E}\,.\,d\vec{l}$$

where line integrals are taken in the anticlockwise direction. But the line integrals along AB and BA are equal and opposite, therefore

$$\oint_{c_1} \vec{E}\,.\,d\vec{l} + \oint_{c_2} \vec{E}\,.\,d\vec{l} = \int_{ACB} \vec{E}\,.\,d\vec{l} + \int_{BDA} \vec{E}\,.\,d\vec{l} = \oint_{c} \vec{E}\,.\,d\vec{l}$$

Suppose we divide the area S into a large number of elementary areas, then the above equation will become

$$\oint_{c} \vec{E}\,.\,d\vec{l} = \sum_{i=1}^{i=n} \oint_{c_1} \vec{E}\,.\,d\vec{l}$$

Let ith surface element have area $\vec{ds}_i$, then the normal component of curl $\vec{E}$ is given by

$$\left|\text{curl } \vec{E}\right|_{\text{normal}} = \frac{\oint_{c_i} E\,.\,d\vec{l}}{ds_i}$$

But $$\left|\text{curl } \vec{E}\right|_{\text{normal}} ds_i = \text{curl } \vec{E}\,.\,\vec{ds}_i$$

$\therefore$ $$\text{curl } \vec{E}\,.\,\vec{ds}_i = \oint_{c_i} \vec{E}\,.\,d\vec{l}$$

and
$$\oint_{c_i} \vec{E} \, . \, d\vec{l} \;=\; \sum_{i=1 c_i}^{i=n} \vec{E} \, . \, d\vec{l} \;=\; \sum_{i=1}^{i=n} \text{curl } \vec{E} \, . \, \vec{ds}_i$$

which in limiting case becomes

$$\oint_{c} \vec{E} \, . \, d\vec{l} \;=\; \iint_{s} \text{curl} \vec{E} \, . \, \vec{ds} \;=\; \iint_{s} \vec{\nabla} \times \vec{E} \, . \, \vec{ds}$$

11.17 GAUSS'S LAW

According to this law, the total electric flux through a closed surface is equal to the charge (in Coulomb) enclosed by that surface. If $\vec{D}$ represents the electric flux density through an elementary surface area $\vec{ds}$ drawn about any point, then the flux through this elementary area is

$$d\phi = \vec{D}\cdot\vec{ds}$$

Thus, the total flux through a closed surface is

$$\phi = \int d\phi = \iint_S \vec{D}\cdot\vec{ds}$$

$\therefore$ According to Gauss's law

$$\phi = \oiint \vec{D}\cdot\vec{ds} = q \text{ in SI system, where } q \text{ is charge}$$

and for free space, the above equation can be written as

$$\phi = \oiint \varepsilon_0\vec{E}\cdot\vec{ds} = q$$

11.17.1 Gaussian Surface

An imaginary closed surface of any shape drawn in an electric field for the purpose of solving problems concerning electric flux is called *Gaussian surface*. The shape of the Gaussian surface is chosen on the basis of symmetry of the problem, so that the expression for Gauss's law,

$\phi = \oiint \vec{D}\cdot\vec{ds}$, can be evaluated conveniently.

11.17.2 Proof of Gauss's Law

Suppose a charge q, assumed to be situated at the origin of the co-ordinate axes as shown in Fig. 11.8. Let S be a Gaussian surface around it. Consider an elementary area $\vec{ds}$ at $\vec{r}$. The electric flux through $\vec{ds}$ is given by

$$\phi = \vec{D}\cdot\vec{ds}$$

where $\vec{D}$ is electric flux density at $\vec{r}$.

and
$$\phi = \int d\phi = \oint\!\!\oint \vec{D} \cdot \vec{ds}$$

But the electric displacement vector for free space is

$$\vec{D} = \varepsilon_0 \vec{E} \qquad \text{(in SI)}$$

$$\therefore \qquad \vec{D} = \varepsilon_0 \frac{q}{4\pi\varepsilon_0 r^2}\hat{r} = \frac{q}{4\pi r^2}\hat{r}$$

Hence, we have

$$\phi = \oint\!\!\oint \frac{q}{4\pi r^2}\hat{r} \cdot \vec{ds} = \frac{q}{4\pi}\oint\!\!\oint \frac{\hat{r} \cdot \vec{ds}}{r^2} \qquad \text{...(12)}$$

But
$$\oint\!\!\oint \frac{\hat{r} \cdot \vec{ds}}{r^2} = \int d\Omega = 4\pi$$

i.e., the solid angle subtended by a closed surface on a point inside it is 4π

$$\therefore \qquad \phi = \frac{q}{4\pi} 4\pi = q$$

which is Gauss's Law, *i.e.*, the total electric flux in SI system through a closed surface is equal to the charge (in Coulomb) enclosed by the surface.

From eqn. (12), it is clear that the electric flux ϕ over a surface is equal to

$$\phi = \frac{q}{4\pi}\Omega$$

With $\Omega = \oint\!\!\oint \frac{\hat{r} \cdot \vec{ds}}{r^2}$ as the solid angle subtended by the surface at the location of point charge. If the charge enclosed is zero, the total flux through the surface is zero. It implies that either no lines of electric force are cutting the surface or the number of lines of force entering the surface is equal to the number of lines of force coming out of it.

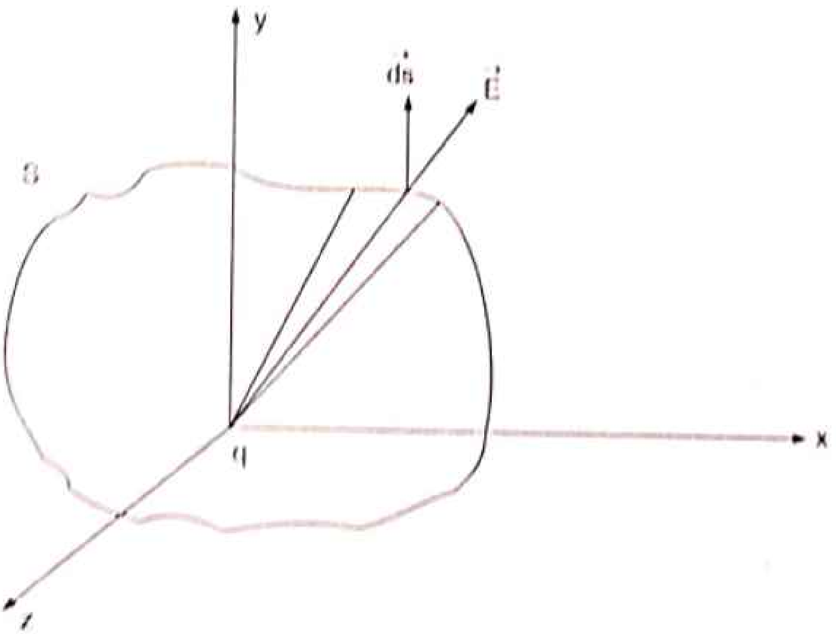

Fig. 11.8 Depiction of Gaussian surface

If the Gaussian surface encloses more than one charge then the Gauss's law can be written as

$$\phi = \oint\!\!\oint \vec{D} \cdot \vec{ds} = \sum_{i=1}^{n} q_i \qquad \text{...(13)}$$

If there is continuous distribution of charges, then we have

$$\phi = \oint\!\!\oint \vec{D} \cdot \vec{ds} = \int dq = \iiint \rho dv \qquad \text{...(14)}$$

where ρ is the volume density of charge.

If the Gaussian surface can be divided into a large number of surfaces $S_1, S_2, S_3, \ldots$ etc., then the expression for Gauss's Law can be written as

$$\phi = \oint\!\!\oint \vec{D} \cdot \vec{ds} = \iint_{S_1} \vec{D}_1 \cdot \vec{ds}_1 + \iint_{S_2} \vec{D}_1 \cdot \vec{ds}_2 + \iint_{S_3} \vec{D}_1 \cdot \vec{ds}_3 + \ldots$$

11.17.3 Differential Form of Gauss's Law

Gauss's law can also be written as

$$\oiint \vec{E}\cdot\vec{ds} = \frac{q}{\varepsilon_0} \qquad (\because\ \vec{D} = \varepsilon_0\vec{E}) \qquad ...(15)$$

From Gauss's divergence theorem (see #11.20), we have

$$\oiint \vec{E}\cdot\vec{ds} = \iiint_v div\ \vec{E}dv \qquad ...(16)$$

Combining eqns. (15) and (16), we have

$$\iiint_v div\ \vec{E}dv = \frac{q}{\varepsilon_0} \qquad ...(17)$$

If q is the total charge contained in the volume distribution having volume density of charge ρ, then we have

$$q = \iiint_v \rho dv \qquad ...(18)$$

From eqns. (17) and (18)

$$\iiint_v div\ \vec{E}dv = \frac{\iiint_v \rho dv}{\varepsilon_0} \qquad ...(19)$$

which gives

$$\text{div } \vec{E} = \frac{q}{\varepsilon_0}$$

But

$$\text{div } \vec{E} = \vec{\nabla}\cdot\vec{E}$$

∴

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0} \qquad ...(20)$$

or

$$\vec{\nabla}\cdot\vec{D} = \rho \qquad ...(21)$$

The above equations have been derived from Gauss's law. $\vec{\nabla}\cdot\vec{E}$ expresses the emergence of electric flux from a point where volume density of charge is ρ. These equations express Gauss's theorem in differential form, because $\vec{\nabla}$ is differential operator.

11.18 LAPLACE'S AND POISSON'S EQUATIONS IN ELECTROSTATICS

We know that the electric field $\vec{E}$ is related to the electric potential U as $\vec{E} = -\vec{\nabla}U$, where U is a scalar function of space coordinates.

Now, according to the differential form of Gauss's law in free space, we have

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0} \text{ in SI system}$$

where ρ is the volume density of charge. Also, $\vec{\nabla}\cdot\vec{E} = -[\vec{\nabla}\cdot\vec{\nabla}U] = -\nabla^2 U$

$$\therefore \qquad -\nabla^2 U = \frac{\rho}{\varepsilon_0}$$

which is the Poisson's equation in electrostatics in SI system.

For free space, $\rho = 0$, therefore, we have

$$\nabla^2 U = 0$$

which is Laplace's equation in electrostatics in SI system. The above equation can also be written as

$$\vec{\nabla} \cdot \vec{E} = \text{div } \vec{E} = 0$$

11.20 GAUSS'S DIVERGENCE THEOREM

It states that the surface integral of the normal component of electric field vector $\vec{E}$ over the bounding surface S of a volume v is equal to the volume integral of the div $\vec{E}$ over the same volume, *i.e.*,

$$\oiint \vec{E} \cdot ds = \iiint_V \operatorname{div} \vec{E} dv$$

Gauss's divergence theorem is applicable to all types of vector fields. Let the bounding surface be divided into a large number of elementary surfaces (see Fig. 11.15), say N, then

$$\oiint \vec{E} \cdot \vec{ds} = \sum_{i=1}^{i=N} \iint_{S_i} \vec{E} \cdot \vec{ds}_i$$

where $\vec{ds}_i$ is the area of the ith surface.

Now dividing and multiplying the right hand side of the above equation by volume v_i, the volume of ith element whose surface area is $\vec{ds}_i$, we have

$$\sum_{i=1}^{i=N} \left(\oiint_{S_i} \frac{\vec{E} \cdot \vec{ds}_i}{v_i} \right) v_i$$

Let us consider the limiting case, *i.e.*, as N approaches infinity, v_i also approaches infinity. Therefore, the above equation becomes

$$\sum_{i=1}^{\infty} \underset{v_i \to \infty}{\mathrm{Lt}} \left(\oint\!\!\oint_{S_i} \frac{\vec{E} \cdot \vec{ds}_i}{v_i} \right) v_i$$

But, we know that

$$\underset{v_i \to \infty}{\mathrm{Lt}} \oint\!\!\oint_{S_i} \frac{\vec{E} \cdot \vec{ds}_i}{v_i} = \mathrm{div}\ \vec{E}$$

Therefore, we have

$$\sum_{i=1}^{\infty} (\mathrm{div}\vec{E})\, v_i$$

or

$$\iiint_v \mathrm{div}\vec{E} dv$$

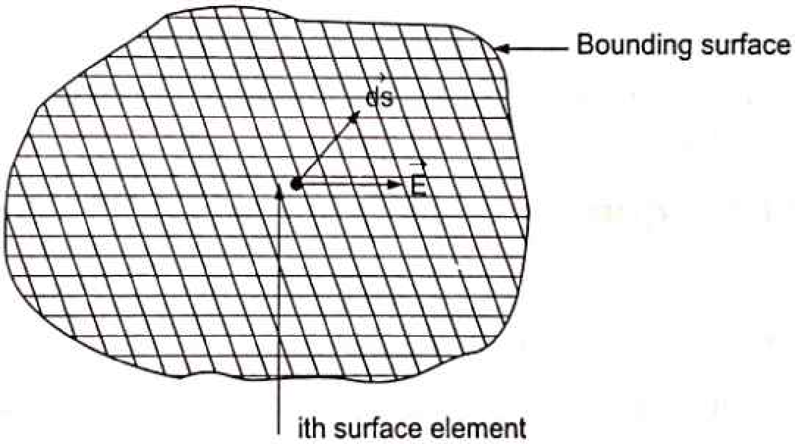

Fig. 11.15 A large surface bounding many elementary surfaces

where we have used the concept that the summation over all the volume elements reduce to volume integrals.

$$\therefore \qquad \sum_{i=1}^{i=N} \left(\iint_{S_i} \vec{E} \cdot \vec{ds}_i \right) = \iiint_v \mathrm{div}\ \vec{E} dv$$

or

$$\iint \vec{E} \cdot \vec{ds} = \iiint_v \mathrm{div}\ \vec{E} dv$$

which is Gauss's divergence theorem.

11.40 MAGNETIC FIELD STRENGTH

We know that if a stationary charge q_0 is placed in a uniform electric field $\vec{E}$, it experiences an electrostatic force $\vec{F}_e$ which is given by

$$\vec{F}_e = q_0 \vec{E} \qquad ...(56)$$

Also if a test charge q_0 is moving with a velocity $\vec{v}$ parallel to a current I at a distance d, experiences a force perpendicular to its own velocity and is given by

$$\vec{F}_m = q_0 v \frac{\mu_0}{4\pi} \frac{2I}{d} \hat{n} \text{ (in SI)} \qquad ...(57)$$

where $\hat{n}$ is the unit vector perpendicular to $\vec{v}$. This force $\vec{F}_m$ is called magnetic force

and
$$B = \frac{\mu_0}{4\pi} \frac{2I}{d} \qquad ...(58)$$

is termed as magnetic field strength. It is a vector quantity. $\vec{B}$ is the magnetic field, produced by the electric current I, whose direction depends upon the direction of current and is given by the right hand rule, *i.e.*, if one grasps the wire carrying electric current with right hand in such a way that the thumb points in the direction of the current and fingers circle the wire, then the direction of the magnetic field is same as the direction of the fingers (Fig. 11.33). In general, the force experienced by a test charge q_0 moving with velocity $\vec{v}$ in a magnetic field $\vec{B}$ is

$$\vec{F}_m = q_0 \vec{v} \times \vec{B} \text{ (in SI)} \qquad ...(59)$$

From eqns. (56) and (59) it is evident that a test charge moving with a velocity $\vec{v}$ in the electric field $\vec{E}$ and magnetic field $\vec{B}$ experiences an electromagnetic force

$$\vec{F}_{em} = q_0 \vec{E} + q_0 \vec{v} \times \vec{B} \qquad ...(60)$$

Here $\vec{F}_{em}$ is also termed as **Lorentz force** and accordingly eqn. (60) may be called as **Lorentz force equation.**

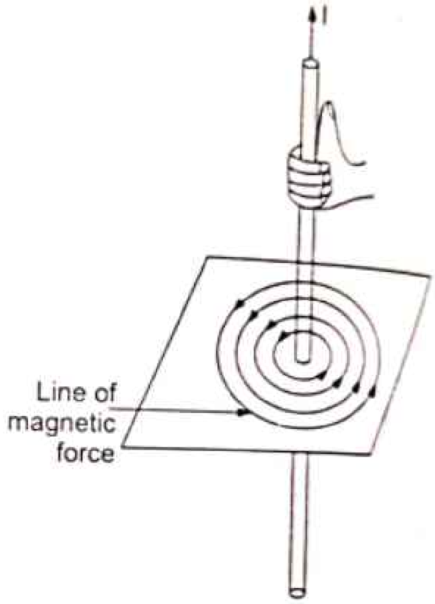

Fig. 11.33 The right hand rule

11.40.1 Ampere's Law

We already know that moving charges produce a magnetic field, which in turn influence any other charge moving through it and it does not affect a stationary charge. Also it is clear that the magnetic effect is maximum when both the charges are moving parallel to each other and is minimum when the charges are moving perpendicular to each other.

It is well known that a current I produces a magnetic field $\vec{B}$ in a plane perpendicular to the direction of flow and its magnitude at a distance d is

$$\vec{B} = \frac{\mu_0}{4\pi}\frac{2I}{d} \text{ (in SI)} \qquad ...(61)$$

Now consider a circular path of radius R in a plane perpendicular to the wire carrying I. Then the magnetic field on every point of the circular path will be tangential to the path (see Fig. 11.34) and is

$$\vec{B} = \frac{\mu_0}{4\pi}\frac{2I}{R}\hat{dl} \text{ (in SI)} \qquad ...(62)$$

where $\hat{dl}$ is the unit vector along the path. The closed line integral of $\vec{B}$ on the circular path is

$$\oint \vec{B}\cdot\vec{dl} = \oint |\vec{B}|\hat{dl}\cdot\vec{dl} = \oint |\vec{B}|(\overline{dl})\hat{dl}\cdot\hat{dl}$$

$$= \oint B\cdot dl \qquad (\because \hat{dl}\cdot\hat{dl} = 1)$$

As B is constant everywhere on the circular path, we have

$$\oint B\cdot dl = B\oint dl = B\times 2\pi R$$

Also, because $d = R$, we have

$$B = \frac{\mu_0}{4\pi}\frac{2I}{R}$$

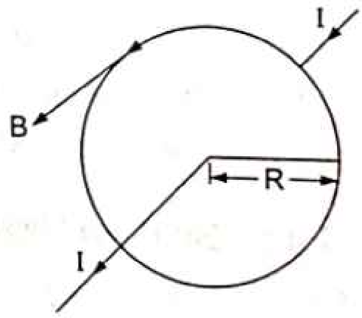

Fig 11.34 Magnetic field on a circular path perpendicular to the wire carrying current I

Thus, the closed line integral of $\vec{B}$ on the circular path of radius R around the current carrying conductor is

$$\oint \vec{B}\cdot\vec{dl} = \frac{\mu_0}{4\pi}\frac{2I}{R}\times 2\pi R$$

$$= \mu_0 I \qquad ...(63)$$

The above equation is known as **Ampere's law** or **Ampere's circuital law.** It is similar to the Gauss's law in electrostatics. Also, if the path goes round the current carrying conductor N times, then

$$\oint \vec{B}\cdot\vec{dl} = N\mu_0 I \qquad ...(64)$$

If $I = 0$; *i.e.*, the current enclosed by the circular path is zero then $\oint \vec{B}\cdot\vec{dl} = 0$.

11.41 BIOT-SAVART'S LAW OR AMPERE'S RULE

Ampere's law can be used to compute B only in the cases where the current distribution is symmetrical and the integral can easily be evaluated. But it is difficult to apply for non-symmetrical charge distribution. Thus there is another law which can easily be used for such cases and is known as Biot-Savart's law. This law is a summarization of the experimental studies regarding the force between current carrying conductors, carried out by Ampere. Ampere concluded that:

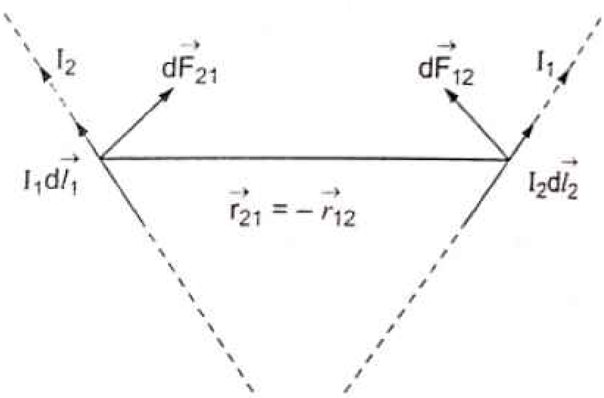

Fig. 11.36 Force between two current carrying conductors

The force on a current element $I_1 d\vec{l}_1$ (a current carrying conductor of length dl_1, having a current I_1) due to another current element $I_2 d\vec{l}_2$ (Fig. 11.36) separated by distance $|\vec{r}_{12}| = |\vec{r}_{21}|$ is given by

$$d\vec{F}_{21} = \frac{\mu_0}{4\pi}\frac{I_1 I_2}{r_{21}^3}\left[d\vec{l}_1 \times \left(d\vec{l}_2 \times \vec{r}_{21}\right)\right] \qquad ...(70)$$

Also from eqn. (69), the force on the current element $I_1 d\vec{l}_1$ due to the magnetic field $d\vec{B}$ is given by

$$d\vec{F}_{21} = I_1 d\vec{l}_1 \times d\vec{B} \qquad ...(71)$$

On comparing (70) and (71), we have

$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{I_2 d\vec{l}_2 \times \vec{r}_{21}}{r_{21}^3} \qquad ...(72)$$

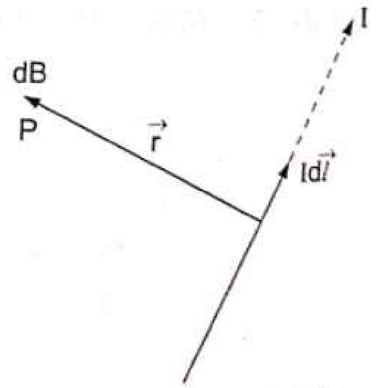

Fig. 11.37 Magnetic field due to a current element $Id\vec{l}$ located at distance $\vec{r}$.

Suppose we omit the suffixes we can say that the magnetic field due to a current element $Id\vec{l}$ at a distance $\vec{r}$ from it (Fig. 11.37) is given by

$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{Id\vec{l}\times\vec{r}}{r^3}$$

which is known as **Biot-Savart's Law** and it states that the magnetic field dB due to a current element is

(i) directly proportional to the current I through it,
(ii) directly proportional to the length dl of the conductor,
(iii) is inversely proportional to the distance $\vec{r}$ of the observation point from current element, and
(iv) is directed along the $d\vec{l}\times\vec{r}$.

11.42 MAGNETIC FLUX

The number of lines of magnetic force crossing normally an area is known as magnetic flux. It is represented by the symbol ϕ_m. If $\vec{B}$ is the magnetic field, the flux $d\phi_m$ through the area element is given by

$$d\phi_m = \vec{B}\cdot d\vec{s} \qquad ...(73)$$

Thus, the flux through any surface S is given by

$$\phi_m = \int d\phi_m$$

$$= \iint_S \vec{B}\cdot d\vec{s} \qquad ...(74)$$

In SI system, B is measured in tesla (T) and ds in metre square (m^2) and hence the unit of flux in SI system is Tm^2. Tm^2 also called weber (wb); *i.e.*, $wb = Tm^2$

From eqn. (73), if $\vec{B}$ is pendendicular to $d\vec{s}$, we have

$$d\phi_m = Bds$$

$$\therefore \qquad B = \frac{d\phi_m}{ds}$$

Thus, B is the magnetic flux per unit area and is called magnetic flux density.

11.43 FARADAY'S LAWS OF ELECTROMAGNETISM

Faraday, through his measurements, observed that:

(i) Whenever the magnetic flux linked with a circuit changes, induced *emf* is set up in the circuit and the induced current may flow through it, lasting so long as the change in flux continues.
(ii) The magnitude of the induced *emf* is proportional to the rate of change of magnetic flux linked with the circuit.

The above statements are called Faraday's laws of electro-magnetism.

Let $d\phi_m$ be the change in magnetic flux in time interval dt. The rate of change of magnetic flux is proportional to the induced emf e and according to Lenz's law it opposes the cause that produces it. Thus, we have

$$e \propto -\frac{d\phi_m}{dt} \qquad (\because \text{ of Lenz's law}) \qquad ...(75)$$

or
$$e = k\frac{d\phi_m}{dt} \qquad ...(76)$$

where k is constant of proportionality, whose value depends upon the units in which various quantities are measured.

In SI units, k is one and eqn. (76) reduces to

$$e = -\frac{d\phi_m}{dt} \qquad ...(77)$$

11.43.1 Differential Form of Faraday's Law

Let us consider a loop of wire which encloses surface area S and is placed in a non-uniform magnetic field $\vec{B}$ (Fig. 11.38). The flux linked with the loop is

$$\phi_m = \iint_S \vec{B}\cdot d\vec{s}$$

Differentiating with respect to time, we have

$$\frac{d\phi_m}{dt} = \iint_S \frac{\partial \vec{B}}{\partial t}\cdot d\vec{s}$$

Here we have written partial differential of $\vec{B}$ as $\vec{B}$ may be a function of both space and time co-coordinates.

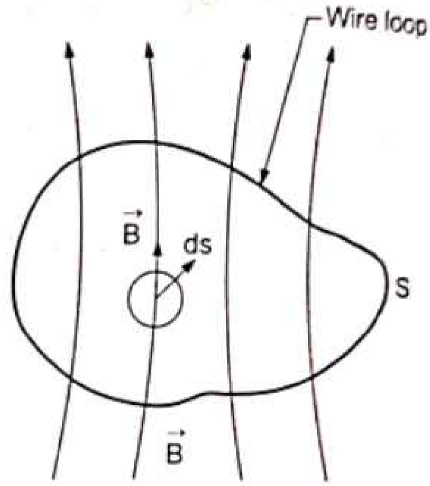

Fig. 11.38 A wire loop enclosing a surface area placed in a non-uniform magnetic field

According to Faraday's laws $\quad e = -\dfrac{d\phi_m}{dt}$

or
$$e = -\iint_S \frac{\partial \vec{B}}{\partial t}\cdot d\vec{s} \qquad ...(78)$$

Also emf is defined as the work done in taking unit charge completely around the closed circuit, therefore we have

$$e = \oint \vec{E}\cdot d\vec{l} \qquad ...(79)$$

where $\vec{E}$ is the electric field.

Form eqns. (78) and (79), we have

$$\oint \vec{E}\cdot d\vec{l} = -\iint_S \frac{\partial \vec{B}}{\partial t}\cdot d\vec{s} \qquad ...(80)$$

According to Stoke's theorem

$$\oint \vec{E}\cdot d\vec{l} = \iint_S (\vec{\nabla}\times\vec{E})\cdot d\vec{s} \qquad ...(81)$$

Therefore, on comparing eqns. (80) and (81), we have

$$\iint_S (\vec{\nabla}\times\vec{E})\cdot d\vec{s} = -\iint_S \frac{\partial \vec{B}}{\partial t}\cdot d\vec{s}$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad ...(82)$$

⇒

which is a differential form of Faraday's laws of electromagnetic induction and is one of Maxwell's equation.

11.44 DIFFERENTIAL FORM OF AMPERE'S LAW

Now let us consider a region of space in which currents are flowing as shown in Fig. 11.39

A steady current distribution can be described by the current density vector $\vec{j} = \vec{j}(\vec{r})$, which may vary from point to point but is time independent. Let S be closed curve in the region. By definition the total current I, through the surface area enclosed by curve S is given by

$$I = \iint_S \vec{j}\cdot d\vec{s} \quad ...(83)$$

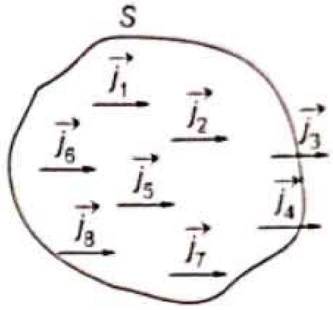

Fig. 11.39 A region of space with various currents flowing through it

From Ampere's law

$$\oint_c \vec{B}\cdot d\vec{l} = \mu_0 I \text{ (in SI)} \quad ...(84)$$

$$\therefore \quad \oint_c \vec{B}\cdot d\vec{l} = \mu_0 \iint_S \vec{j}\cdot d\vec{s} \quad ...(85)$$

Also according to Stokes theorem, the closed line integral of $\vec{B}$ is related to the surface integral as

$$\oint_c \vec{B}\cdot d\vec{l} = \iint_S (\vec{\nabla}\times\vec{B})\cdot d\vec{s} \quad ...(86)$$

Therefore, on comparing eqns. (85) and (86), we have

$$\iint_S (\vec{\nabla}\times\vec{B})\cdot d\vec{s} = \mu_0 \iint_S \vec{j}\cdot d\vec{s}$$

$$\Rightarrow \quad \vec{\nabla}\times\vec{B} = \mu_0\vec{j} \quad ...(87)$$

Equation (87) is another form of Ampere's law and is known as Maxwell's equation of magneto statics.

11.44.1 Modified Ampere's Law (Modifying Equation for Curl of Magnetic Fields to Satisfy Continuity Equation)

Since divergence of a curl is zero, therefore, taking the divergence on both sides of eqn. (87), we have

$$\vec{\nabla}\cdot\vec{\nabla}\times\vec{B} = \vec{\nabla}\cdot\left(\mu_0\vec{j}\right)$$

or

$$0 = \mu_0\left(\vec{\nabla}\cdot\vec{j}\right)$$

or

$$\vec{\nabla}\cdot\vec{j} = 0 \quad ...(88)$$

which is true only for steady currents and for varying currents

$$\vec{\nabla}\cdot\vec{j} = -\frac{\partial \rho}{\partial t} \quad ...(89)$$

$\therefore$ Ampere's law as given in eqn. (87) cannot be valid for varying current and should be modified. Maxwell modified Ampere's law, making it valid for varying current as well, by introducing the concept of displacement current as described below:

We know that

$$\vec{\nabla} \cdot \vec{E} = -\frac{\rho}{\varepsilon_0} \text{ (in SI)}$$

or

$$\rho = \varepsilon_0 \vec{\nabla} \cdot \vec{E}$$

$\therefore$

$$\frac{\partial \rho}{\partial t} = \varepsilon_0 \vec{\nabla} \cdot \frac{\partial \vec{E}}{\partial t}$$

Now add $\vec{\nabla} \cdot \vec{j}$ to both sides, we have

$$\vec{\nabla} \cdot \vec{j} + \frac{\partial \rho}{\partial t} = \vec{\nabla} \cdot \vec{j} + \varepsilon_0 \vec{\nabla} \cdot \frac{\partial \vec{E}}{\partial t}$$

$$= \vec{\nabla}\left(\vec{j} + \varepsilon_0 \frac{\partial \vec{E}}{\partial t}\right) \quad \text{...(90)}$$

But for varying currents

$$\vec{\nabla} \cdot \vec{j} + \frac{\partial \rho}{\partial t} = 0 \quad \text{(from continuity equation)}$$

$\therefore$ We have

$$\vec{\nabla} \cdot \left(\vec{j} + \varepsilon_0 \frac{\partial \vec{E}}{\partial t}\right) = 0 \quad \text{...(91)}$$

Thus for varying currents

$$\vec{\nabla} \cdot \vec{j} \neq 0$$

But

$$\vec{\nabla} \cdot \left(\vec{j} + \varepsilon_0 \frac{\partial \vec{E}}{\partial t}\right) = 0$$

Maxwell, therefore proposed that $\vec{j}$ in Ampere's Law should be replaced by

$$\vec{j} + \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

And hence Ampere's law (eqn. 87) becomes

$$\vec{\nabla} \times \vec{B} = \mu_0 \left(\vec{j} + \varepsilon_0 \cdot \frac{\partial \vec{E}}{\partial t}\right) \quad \text{...(92)}$$

The term $\varepsilon_0 \frac{\partial \vec{E}}{\partial t}$ in above relation is known as displacement current density in vacuum. The name was given by Maxwell. It will be appropriate to mention here that Ampere's law as modified in eqn. (92) is consistent with continuity equation since divergence of both sides in eqn. (92) is zero. If we represent the displacement vector by $\vec{D}$, then in vacuum,

$$\vec{D} = \varepsilon_0 \vec{E} \text{ (in SI)}$$

11.45 SCALAR AND VECTOR POTENTIAL

In case of electric field $\vec{E}$, curl $\vec{E} = 0$ and it is possible to write $\vec{E} = -\vec{\nabla}U$, where U is a scalar function called electric potential.

However, in case of the magnetic field.

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{j}$$

Only in special cases, when $\vec{j} = 0$, $\vec{\nabla} \times \vec{B} = 0$. In all other cases $\vec{\nabla} \times \vec{B} \neq 0$.

Hence $\vec{B}$ cannot be expressed as gradient of a scalar function.

Another relation that defines $\vec{B}$ is

$$\vec{\nabla} \cdot \vec{B} = 0$$

This enables us to write

$$\vec{B} = \vec{\nabla} \times \vec{A} \qquad \text{...(94)}$$

$\therefore$ $$\text{div curl } \vec{A} = 0 \text{ or } \vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) = 0$$

Hence $$\vec{\nabla} \cdot \vec{B} = 0$$

The vector quantity $\vec{A}$ in eqn. (94) is called vector potential.