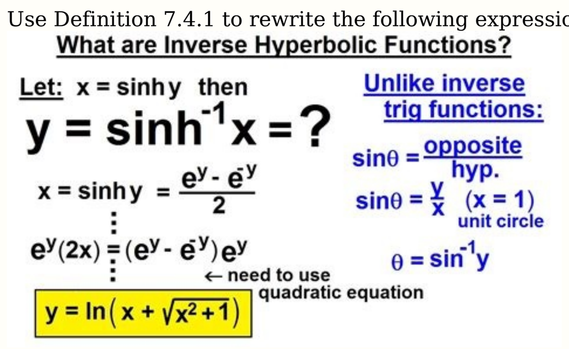
Inverse hyperbolic functions integrals

Math.Info » Pre-Calculus/Calculus » List of Integrals of Inverse Hyperbolic Functions Derivatives and Integrals Involving Inverse Hyperbolic Functions Loading... Found a content error? Tell us The hyperbolic functions are functions that have many applications to mathematics, physics, and engineering. Among many other applications, they are used to describe the formation of satellite rings around planets, to describe the shape of a rope hanging from two points, and have application to the theory of special relativity. This section defines the hyperbolic functions and describes many of their properties, especially their usefulness to calculus. ††margin: (cosθ,sinθ)A=θ2x2+y2=1-11-11xy(coshθ,sinhθ)A=θ2x2-y2=1-22-22xy Figure 7.4.1: Using trigonometric functions to define points on a circle and hyperbolic functions to define points on a hyperbola. Λ These functions are sometimes referred to as the “hyperbolic trigonometric functions” as there are many connections between them and the standard trigonometric functions. Figure 7.4.1 demonstrates one such connection. Just as cosine and sine are used to define points on the circle defined by x2+y2=1, the functions hyperbolic cosine and hyperbolic sine are used to define points on the hyperbola x2-y2=1.
We begin with their definitions. (a) coshx=ex+e-x2 (b) sinhx=ex-e-x2 (c) tanhx=sinhxcoshx (d) sechx=1coshx (e) cschx=1sinhx (f) cothx=coshxsinhx The hyperbolic functions are graphed in Figure 7.4.2. In the graphs of coshx and sinhx, graphs of ex/2 and e-x/2 are included with dashed lines. As x gets “large,” coshx and sinhx each act like ex/2; when x is a large negative number, coshx acts like e-x/2 whereas sinhx acts like -e-x/2. ††margin: Pronunciation Note: “cosh” rhymes with “gosh,” “sinh” rhymes with “pinch,” and “tanh” rhymes with “ranch,” Λ Notice the domains of tanhx and sechx are (-∞,∞), whereas both cothx and cschx have vertical asymptotes at x=0. Also note the ranges of these functions, especially tanhx: as x→∞, both sinhx and coshx approach ex/2, hence tanhx approaches 1. f(x)=coshx-3-2-1123-10-5510xyf(x)=sinhx-3-2-1123-10-5510xyf(x)=tanhxf(x)=cothx-3-2-1123-22xyf(x)=sechxf(x)=cschx-3-2-1123-3-2-1123xy Figure 7.4.2: Graphs of the hyperbolic functions. Watch the video: Hyperbolic Functions — The Basics from The following example explores some of the properties of these functions that bear remarkable resemblance to the properties of their trigonometric counterparts.

$$\int \sinh^{-1}\left(\frac{x}{a}\right) dx = x\sinh^{-1}\left(\frac{x}{a}\right) - \sqrt{x^2+a^2} + c$$
$$\int \cosh^{-1}\left(\frac{x}{a}\right) dx = x\cosh^{-1}\left(\frac{x}{a}\right) - \sqrt{x^2-a^2} + c$$
$$\int \tanh^{-1}\left(\frac{x}{a}\right) dx = x\tanh^{-1}\left(\frac{x}{a}\right) + \frac{a}{2}\ln|a^2-x^2| + c$$
$$\int \operatorname{csch}^{-1}\left(\frac{x}{a}\right) dx = x\operatorname{csch}^{-1}\left(\frac{x}{a}\right) + a\ln\left(\frac{x+\sqrt{x^2+a^2}}{a}\right) + c$$
$$\int \operatorname{sech}^{-1}\left(\frac{x}{a}\right) dx = x\operatorname{sech}^{-1}\left(\frac{x}{a}\right) + a\ln\left(\frac{x+\sqrt{a^2-x^2}}{a}\right) + c$$
$$\int \coth^{-1}\left(\frac{x}{a}\right) dx = x\coth^{-1}\left(\frac{x}{a}\right) + \frac{a}{2}\ln|x^2-a^2| + c$$

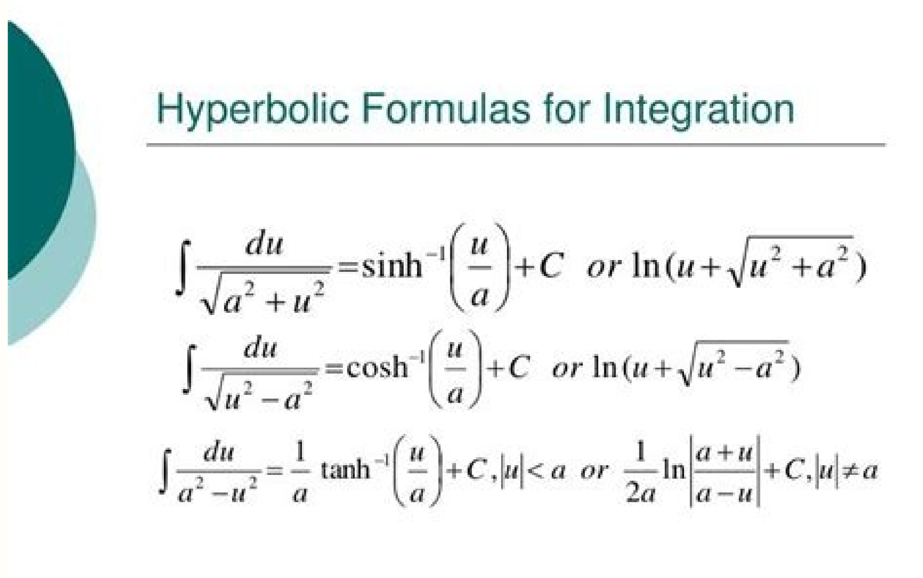
Use Definition 7.4.1 to rewrite the following expressions. (a) (b) (c) (d) (e) (f) Solution (a) cosh2x-sinh2x=(ex+e-x2)2-(ex-e-x2)2=e2x+2exe-x+e-2x4-e2x-2exe-x+e-2x4=44=1.



So cosh2x-sinh2x=1. (b) tanh2x+sech2x=sinh2xcosh2x+1cosh2x=sinh2x+1cosh2x   Now use identity from #1.=cosh2xcosh2x=1. So tanh2x+sech2x=1. (c) 2coshxsinhx=2(ex+e-x2)(ex-e-x2)=2·e2x-e-2x4=e2x-e-2x2=sinh(2x). Thus 2coshxsinhx=sinh(2x). (d) ddx(coshx)=ddx(ex+e-x2)=ex-e-x2=sinhx. So ddx(coshx)=sinhx. (e) ddx(sinhx)=ddx(ex-e-x2)=ex+e-x2=coshx. So ddx(sinhx)=coshx. (f) ddx(tanhx)=ddx(sinhxcoshx)=coshxcoshx-sinhxsinhxcosh2x=1cosh2x=sech2x. So ddx(tanhx)=sech2x. The following Key Idea summarizes many of the important identities relating to hyperbolic functions. Each can be verified by referring back to Definition 7.4.1. (a) (b) (c) (d) (e) (f) (g) Derivatives (a) (b) (c) (d) ddx(sechx)=-sechxtanhx (e) ddx(cschx)=-cschxcothx (f) Integrals (a) (b) (c) ∫tanhxdx=ln(coshx)+C (d) ∫cothxdx=ln|sinhx|+C We practice using Key Idea 7.4.1. Evaluate the following derivatives and integrals.
(a) ddx(cosh2x) (b) ∫sech2(7t-3)dt (c) ∫0ln2coshxdx Solution (a) Using the Chain Rule directly, we have ddx(cosh2x)=2sinh2x.

$$\frac{d}{dx}(\sinh ax) = a\cosh ax$$
$$\frac{d}{dx}(\cosh ax) = a\sinh ax$$
$$\frac{d}{dx}(\tanh ax) = a\operatorname{sech}^2 ax$$
$$\frac{d}{dx}(\operatorname{csch} ax) = -a\operatorname{csch} ax\coth ax$$
$$\frac{d}{dx}(\operatorname{sech} ax) = -a\operatorname{sech} ax\tanh ax$$
$$\frac{d}{dx}(\coth ax) = -a\operatorname{csch}^2 ax$$

Just to demonstrate that it works, let’s also use the Basic Identity found in Key Idea 7.4.1: cosh2x=cosh2x+sinh2x. ddx(cosh2x)=ddx(cosh2x+sinh2x) =2coshxsinhx+2sinhxcoshx =4coshxsinhx. Using another Basic Identity, we can see that 4coshxsinhx=2sinh2x.



We get the same answer either way. (b) We employ substitution, with u=7t-3 and du=7dt. Applying Key Idea 7.4.1 we have: ∫sech2(7t-3)dt=17tanh(7t-3)+C. (c) ∫0ln2coshxdx=sinhx|0ln2=sinh(ln2)-sinh0=sinh(ln2). We can simplify this last expression as sinhx is based on exponentials: sinh(ln2)=eln2-e-ln22=2-1/22=34. Just as the inverse trigonometric functions are useful in certain integrations, the inverse hyperbolic functions are useful with others. Figure 7.4.3 shows the restrictions on the domains to make each function one-to-one and the resulting domains and ranges of their inverse functions. Their graphs are shown in Figure 7.4.4. Because the hyperbolic functions are defined in terms of exponential functions, their inverses can be expressed in terms of logarithms as shown in Key Idea 7.4.2. It is often more convenient to refer to sinh-1x than to ln(x+x2+1), especially when one is working on theory and does not need to compute actual values. On the other hand, when computations are needed, technology is often helpful but many hand-held calculators lack a convenient sinh-1x button. (Often it can be accessed under a menu system, but not conveniently.) In such a situation, the logarithmic representation is useful. The reader is not encouraged to memorize these, but rather know they exist and know how to use them when needed.

$$\int \sinh x\, dx = \cosh x + c$$
$$\int \cosh x\, dx = \sinh x + c$$
$$\int \tanh x\, dx = \ln(\cosh x) + c$$
$$\int \coth x\, dx = \ln|\sinh x| + c$$
$$\int \operatorname{sech} x\, dx = \tan^{-1}|\sinh x| + c$$
$$\int \operatorname{csch} x\, dx = \ln\left|\tanh\frac{x}{2}\right| + c$$

Function Domain Range Function Domain Range coshx [0,∞) [1,∞) cosh-1x [1,∞) [0,∞) sinhx (-∞,∞) (-∞,∞) sinh-1x (-∞,∞) (-∞,∞) tanhx (-∞,∞) (-1,1) tanh-1x (-1,1) (-∞,∞) sechx [0,∞) (0,1] sech-1x (0,1] [0,∞) cschx (-∞,0)∪(0,∞) (-∞,0)∪(0,∞) csch-1x (-∞,0)∪(0,∞) (-∞,0)∪(0,∞) cothx (-∞,0)∪(0,∞) (-∞,-1)∪(1,∞) coth-1x (-∞,-1)∪(1,∞) (-∞,0)∪(0,∞) Figure 7.4.3: Domains and ranges of the hyperbolic and inverse hyperbolic functions. y=cosh-1xy=coshx510510xy y=sinhxy=sinh-1x-1010-10-5510xy y=coth-1xy=tanh-1x-22-22xy y=sech-1xy=csch-1x-3-2-1123-3-2-1123xy Figure 7.4.4: Graphs of the hyperbolic functions and their inverses. Now let’s consider the inverses of the hyperbolic functions. We begin with the function f(x)=sinhx. Since f′(x)=coshx>0 for all real x, f is increasing and must be one-to-one. y =ex-e-x2 2y =ex-e-x   (now multiply by ex) 2yex =e2x-1   (a quadratic form) (ex)2-2yex-1 =0   (use the quadratic formula) ex =2y±4y2+42 ex =y±y2+1   (use the fact that ex>0) ex =y+y2+1 x =ln(y+y2+1) Finally, interchange the variable to find that In a similar manner we find that the inverses of the other hyperbolic functions are given by: (a) cosh-1x=ln(x+x2-1);   x≥1 (b) tanh-1x=12ln(1+x1-x); |x|<1 (c) sech-1x=ln(1+1-x2x);   01 (f) csch-1x=ln(1x+1+x2|x|);   x≠0 The following Key Ideas give the derivatives and integrals relating to the inverse hyperbolic functions. In Key Idea 7.4.4, both the inverse hyperbolic and logarithmic function representations of the antiderivative are given, based on Key Idea 7.4.2. Again, these latter functions are often more useful than the former. (a) ddx(cosh-1x)=1x2-1;   x>1 (b) ddx(sinh-1x)=1x2+1 x≠0 (c) ddx(tanh-1x)=11-x2; |x|<1 (d) ddx(sech-1x)=-1x1-x2;   01 1. ∫1x2-a2dx =cosh-1(xa)+C; 00 =ln(x+x2+a2)+C 3. ∫1a2-x2dx ={1atanh-1(xa)+C|x|<|a|1acoth-1(xa)+C|a|<|x| =12aln|a+xa-x|+C 4. ∫1xa2-x2dx =-1asech-1(xa)+C; 00 =1aln|xa+a2+x2|+C We practice using the derivative and integral formulas in the following example.
Evaluate the following. (a) (b) (c) Solution (a) Applying Key Idea 7.4.3 with the Chain Rule gives: ddx[cosh-1(3x-25)]=1(3x-25)2-1·35. (b) Multiplying the numerator and denominator by (-1) gives a second integral can be solved with a direct application of item #3 from Key Idea 7.4.4, with a=1. Thus ∫1x2-1dx =-∫11-x2dx ={-tanh-1(x)+Cx2<1-coth-1(x)+C1Four Key Ideas were presented, each including quite a bit of information. Do not view this section as containing a source of information to be memorized, but rather as a reference for future problem solving. Key Idea 7.4.4 contains perhaps the most useful information. Know the integration forms it helps evaluate and understand how to use the inverse hyperbolic answer and the logarithmic answer. The next section takes a brief break from demonstrating new integration techniques. It instead demonstrates a technique of evaluating limits that return indeterminate forms. This technique will be useful in Section 8.6, where limits will arise in the evaluation of certain definite integrals. 1. In Key Idea 7.4.1, the equation ∫tanhxdx=ln(coshx)+C is given. Why is “ln|coshx|” not used — i.e., why are absolute values not necessary? 2. The hyperbolic functions are used to define points on the right hand portion of the hyperbola x2-y2=1, as shown in Figure 7.4.1. How can we use the hyperbolic functions to define points on the left hand portion of the hyperbola? 3.
Suppose sinht=5/12. Find the values of the other five hyperbolic functions at t. 4. Suppose tanht=-3/5. Find the values of the other five hyperbolic functions at t. In Exercises 5–12., verify the given identity using Definition 7.4.1, as done in Example 7.4.1. 5. 6. 7. 8.
9. ddx[sechx]=-sechxtanhx 10. 11.
∫tanhxdx=ln(coshx)+C 12. ∫cothxdx=ln|sinhx|+C In Exercises 13–24., find the derivative of the given function. 13. 14. 15.
16. 17. 18. 19. 20. 21. 22. 23. 24.
In Exercises 25–30., find the equation of the line tangent to the function at the given x-value. 25. 26. 27. 28. 29. 30. In Exercises 31–38., evaluate the given indefinite integral. 31. 32. 33. 34. 35. 36. 37. 38. ∫sechxdx   (Hint: multiply by coshxcoshx; set u=sinhx.) In Exercises 39–40., evaluate the given definite integral. 39. 40. 41. In the bottom graph of Figure 7.4.1 (the hyperbola), it is stated that the shaded area is θ/2. Verify this claim by setting up and evaluating an appropriate integral (and note that θ is just a positive number, not an angle).
Hint: Integrate with respect to y, and consult the table of Integration Rules in the Appendix if necessary. 7.3 Exponential and Logarithmic Functions7.5 L’Hôpital’s Rule Generated on Sun Nov 21 19:48:25 2021 by LaTeXML For example, inverse hyperbolic sine can be written as???\text{arcsinh}??? or as???\sinh^{-1}???Some people argue that the ???\text{arcsinh}??? form should be used instead of ???\sinh^{-1}??? because ???\sinh^{-1}??? can be misinterpreted as ???1/\sinh???.
Whichever form you prefer, you see both, so you should be able to recognize both and understand that they mean the same thing.The general rules for the six inverse hyperbolic functions are???\int{\text{arcsinh}{(ax)}}\ dx=x\text{arcsinh}{(ax)}-\frac{\sqrt{a^2x^2+1}}{a}+C??????\int{\text{arccosh}{(ax)}}\ dx=x\text{arccosh}{(ax)}-\frac{\sqrt{ax+1}\sqrt{ax-1}}{a}+C??????\int{\text{arctanh}{(ax)}}\ dx=x\text{arctanh}{(ax)}+\frac{\ln{(1-a^2x^2)}}{2a}+C??????\int{\text{arccoth}{(ax)}}\ dx=x\text{arccoth}{(ax)}+\frac{\ln{(a^2x^2-1)}}{2a}+C??????\int{\text{arcsech}{(ax)}}\ dx=x\text{arcsech}{(ax)}-\frac{2}{a}\arctan{\sqrt{\frac{1-ax}{1+ax}}}+C??????\int{\text{arccsch}{(ax)}}\ dx=x\text{arccsch}{(ax)}+\frac{1}{a}\text{arccoth}{\sqrt{\frac{1}{a^2x^2}+1}}+C???We also have a few other standard inverse hyperbolic integrals that are based on the standard inverse hyperbolic derivatives. In the following formulas, ???u??? represents a function.???\int{\frac{1}{\sqrt{a^2+u^2}}}\ du=\text{arcsinh}{\left(\frac{u}{a}\right)}+C??? where ???a>0??????\int{\frac{1}{\sqrt{u^2-a^2}}}\ du=\text{arccosh}{\left(\frac{u}{a}\right)}+C??? where ???u>a>0??????\int{\frac{1}{a^2-u^2}}\ du=\frac{1}{a}\text{arctanh}{\left(\frac{u}{a}\right)}+C??? if ???u^2a^2??????\int{\frac{1}{u\sqrt{a^2-u^2}}}\ du=-\frac{1}{a}\text{arcsech}{\left(\frac{u}{a}\right)}+C??? where ???0