Questions

Q1.

(i) A sequence of numbers is defined by

$$u_1 = 6, \qquad u_2 = 27$$

$$u_{n+2} = 6u_{n+1} - 9u_n \qquad n \geq 1$$

Prove by induction that, for $n \in \mathbb{Z}^+$

$$u_n = 3^n(n + 1)$$

(6)

(ii) Prove by induction that, for $n \in \mathbb{Z}^+$

$$f(n) = 3^{3n-2} + 2^{3n+1} \text{ is divisible by } 19$$

(6)

(Total for question = 12 marks)

Q2.

Prove by induction that, for $n \in \mathbb{Z}^+$,

$$f(n) = 8^n - 2^n$$

is divisible by 6

(6)

(Total 6 marks)

Q3.

Prove by induction that, for $n \in \mathbb{Z}^+$,

$f(n) = 2^{2n-1} + 3^{2n-1}$ is divisible by 5.

(6)

(Total 6 marks)

Q4.

Prove by induction, that for $n \in \mathbb{Z}^+$,

(a) $\begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}^n = \begin{pmatrix} 3^n & 0 \\ 3(3^n - 1) & 1 \end{pmatrix}$,

(6)

(b) $f(n) = 7^{2n-1} + 5$ is divisible by 12.

(6)
(Total 12 marks)

Q5.

A sequence of numbers u_1 , u_2 , u_3 , u_4 , ... is defined by

$$u_{n+1} = 4u_n + 2, \; u_1 = 2$$

Prove by induction that, for $n \in$ ℤ$^+$

$$u_n = \frac{2}{3}(4^n - 1)$$

(5)

(Total 5 marks)

Q6.

Prove by induction that, for

$n \in \mathbb{Z}^+$, $\sum_{r=1}^{n} (2r-1)^2 = \frac{1}{3}n(2n-1)(2n+1)$.

(5)

(Total 5 marks)

Q7.

Given that

$$f(n) = 3^{4n} + 2^{4n+2},$$

(a) show that, for $k \in \mathbb{Z}^+$, $f(k + 1) - f(k)$ is divisible by 15,

(4)

(b) prove that, for $n \in \mathbb{Z}^+$, $f(n)$ is divisible by 5,

(3)

(c) show that it is not true that, for all positive integers n, $f(n)$ is divisible by 15.

(1)

(Total 8 marks)

Q8.
Prove by induction that, for n ∈ ℤ$^+$,

$$\sum_{r=1}^{n} \frac{1}{r(r+1)} = \frac{n}{n+1}$$

(5)

(Total 5 marks)

Q9.
A series of positive integers u_1, u_2, u_3, ... is defined by

$$u_1 = 6 \text{ and } u_{n+1} = 6u_n - 5, \text{ for } n \geqslant 1.$$

Prove by induction that $u_n = 5 \times 6^{n-1} + 1$, for $n \geqslant 1$.

(5)

(Total 5 marks)

Q10.

Prove by induction that, for $n \in \mathbb{Z}^+$,

(a) f(n) = $5^n + 8n + 3$ is divisible by 4,

(7)

(b) $\begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix}^n = \begin{pmatrix} 2n+1 & -2n \\ 2n & 1-2n \end{pmatrix}$

(7)
(Total 14 marks)

Q11.
A sequence of numbers is defined by

$$u_1 = 2,$$
$$u_{n+1} = 5u_n - 4, \qquad n \geqslant 1.$$

Prove by induction that, for $n \in$ ℤ$^+$, $u_n = 5^{n-1} + 1$.

(4)

(Total 4 marks)

Q12.

$f(n) = 2^n + 6^n$

(a) Show that $f(k + 1) = 6f(k) - 4(2^k)$.

(3)

(b) Hence, or otherwise, prove by induction that, for $n \in \mathbb{Z}^+$, $f(n)$ is divisible by 8.

(4)

(Total 7 marks)

Q13.

(i) Prove by induction that, for $n \in \mathbb{Z}^+$.

$$\begin{pmatrix} 1 & 0 \\ -1 & 5 \end{pmatrix}^n = \begin{pmatrix} 1 & 0 \\ -\frac{1}{4}(5^n - 1) & 5^n \end{pmatrix}$$

(6)

(ii) Prove by induction that, for $n \in \mathbb{Z}^+$.

$$\sum_{r=1}^{n}(2r-1)^2 = \frac{1}{3}n\left(4n^2-1\right)$$

(6)

(Total for question = 12 marks)

Q14.

Prove by mathematical induction that, for $n \in \mathbb{N}$

$$\sum_{r=1}^{n} \frac{1}{(2r-1)(2r+1)} = \frac{n}{2n+1}$$

(Total for question = 6 marks)

Q15.

Prove by induction that, for $n \in \mathbb{Z}^+$

$$f(n) = 2^{n+2} + 3^{2n+1}$$

is divisible by 7

(Total for question = 6 marks)

Q16.

(i) Prove by induction that, for all positive integers *n*,

$$\sum_{r=1}^{n} \frac{1}{r(r+1)} = \frac{n}{n+1}$$

(5)

(ii) Prove by induction that, for all positive integers n,

$$f(n) = 3^{2n+4} - 2^{2n}$$

is divisible by 5

(5)

(Total for question = 10 marks)

Q17.

(a) Prove by induction that, for all positive integers *n*,

$$\sum_{r=1}^{n} r(r+1)(2r+1) = \frac{1}{2}n(n+1)^2(n+2)$$

(6)

(b) Hence, show that, for all positive integers *n*,

$$\sum_{r=n}^{2n} r(r+1)(2r+1) = \frac{1}{2}n(n+1)(an+b)(cn+d)$$

where *a*, *b*, *c* and *d* are integers to be determined.

(3)

(Total for question = 9 marks)

Q18.

Prove by induction that for all positive integers *n*

$$\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$$

(6)

(Total for question = 6 marks)

Q19.

(i) Prove by induction that for $n \in \mathbb{Z}^+$

$$\begin{pmatrix} 5 & -8 \\ 2 & -3 \end{pmatrix}^n = \begin{pmatrix} 4n+1 & -8n \\ 2n & 1-4n \end{pmatrix}$$

(6)

(ii) Prove by induction that for $n \in \mathbb{Z}^+$

$$f(n) = 4^{n+1} + 5^{2n-1}$$

is divisible by 21

(6)

(Total for question = 12 marks)

Q20.

(a) Prove by induction

$$\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$$

(5)

(b) Using the result in part (a), show that

$$\sum_{r=1}^{n} (r^3 - 2) = \frac{1}{4}n(n^3 + 2n^2 + n - 8)$$

(3)

(c) Calculate the exact value of $\sum_{r=20}^{50} (r^3 - 2)$

(3)

(Total 11 marks)

Q21.

(a) Prove by induction, that for $n \in \mathbb{Z}^+$,

$$\sum_{r=1}^{n} r(2r-1) = \frac{1}{6}n(n+1)(4n-1)$$

(6)

(b) Hence, show that

$$\sum_{r=n+1}^{3n} r(2r-1) = \frac{1}{3}n(an^2+bn+c)$$

where *a*, *b* and *c* are integers to be found.

(4)

(Total 10 marks)

Q22.
(a) Prove by induction that, for any positive integer *n*,

$$\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$$

(5)

(b) Using the formulae for $\sum_{r=1}^{n} r$ and $\sum_{r=1}^{n} r^3$, show that

$$\sum_{r=1}^{n}(r^3 + 3r + 2) = \frac{1}{4}n(n+2)(n^2+7)$$

(5)

(c) Hence evaluate $\sum_{r=15}^{25}(r^3 + 3r + 2)$

(2)

(Total 12 marks)

Q23.

(a) Prove by induction that

$$\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$$

(6)

Using the standard results for $\sum_{r=1}^{n} r$ and $\sum_{r=1}^{n} r^2$,

(b) show that

$$\sum_{r=1}^{n}(r+2)(r+3)=\frac{1}{3}n(n^2+an+b),$$

where *a* and *b* are integers to be found.

(5)

(c) Hence show that

$$\sum_{r=n+1}^{2n}(r+2)(r+3)=\frac{1}{3}n(7n^2+27n+26)$$

(3)
(Total 14 marks)

Q24.

(a) Prove by induction that for all positive integers *n*,

$$\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$$

(6)

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(b) Use the standard results for $\sum_{r=1}^{n} r^3$ and $\sum_{r=1}^{n} r$ to show that for all positive integers *n*,

$$\sum_{r=1}^{n} r(r+6)(r-6) = \frac{1}{4}n(n+1)(n-8)(n+9)$$

(4)

(c) Hence find the value of *n* that satisfies

$$\sum_{r=1}^{n} r(r+6)(r-6) = 17\sum_{r=1}^{n} r^2$$

(5)

(Total for question = 15 marks)