Exam Questions – Chapter 9 - Vectors

Q1.

The plane Π passes through the point A and is perpendicular to the vector $\mathbf{n}$

Given that

$$\overrightarrow{OA} = \begin{pmatrix} 5 \\ -3 \\ -4 \end{pmatrix} \quad \text{and} \quad \mathbf{n} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$$

where O is the origin,

(a) find a Cartesian equation of Π.

$$r \cdot \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} = k$$

$$\begin{pmatrix} 5 \\ -3 \\ -4 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} = 15 + 3 - 8$$

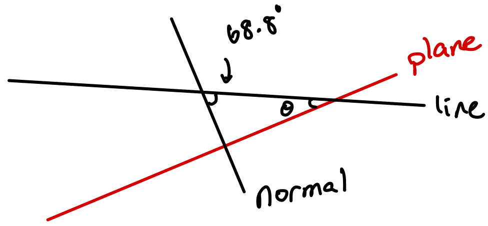
$$= 10$$

$$3x - y + 2z = 10 \;//$$

(2)

With respect to the fixed origin O, the line l is given by the equation

$$\mathbf{r} = \begin{pmatrix} 7 \\ 3 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ -5 \\ 3 \end{pmatrix}$$



The line l intersects the plane Π at the point X.

(b) Show that the acute angle between the plane Π and the line l is 21.2° correct to one decimal place.

angle between normal and line

$$\frac{\begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ -5 \\ 3 \end{pmatrix}}{\sqrt{3^2+(-1)^2+2^2}\sqrt{(-1)^2+(-5)^2+3^2}} = \cos\theta$$

$$\frac{-3+5+6}{\sqrt{14}\sqrt{35}} = \cos\theta$$

$$\cos\theta = 0.3614$$

$$\theta = 68.8^\circ$$

angle between plane and line

$$90^\circ - 68.8 = 21.2^\circ \;//$$

(4)

(c) Find the coordinates of the point X.

$$r = \begin{pmatrix} 7-\lambda \\ 3-5\lambda \\ -2+3\lambda \end{pmatrix}$$

$$\begin{pmatrix} 7-\lambda \\ 3-5\lambda \\ -2+3\lambda \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} = 10$$

$$21 - 3\lambda - 3 + 5\lambda - 4 + 6\lambda = 10$$

$$8\lambda + 14 = 10$$

$$8\lambda = -4$$

$$\lambda = \frac{-1}{2}$$

$$X = \begin{pmatrix} 7+\frac{1}{2} \\ 3+\frac{5}{2} \\ -2-\frac{3}{2} \end{pmatrix}$$

$$X = \begin{pmatrix} 7.5 \\ 5.5 \\ -3.5 \end{pmatrix} \;//$$

(4)

(Total for question = 10 marks)

Q2.

The surface of a horizontal tennis court is modelled as part of a horizontal plane, with the origin on the ground at the centre of the court, and

- **i** and **j** are unit vectors directed across the width and length of the court respectively
- **k** is a unit vector directed vertically upwards
- units are metres

After being hit, a tennis ball, modelled as a particle, moves along the path with equation

$$\mathbf{r} = (-4.1 + 9\lambda - 2.3\lambda^2)\mathbf{i} + (-10.25 + 15\lambda)\mathbf{j} + (0.84 + 0.8\lambda - \lambda^2)\mathbf{k}$$

where λ is a scalar parameter with $\lambda \geq 0$

Assuming that the tennis ball continues on this path until it hits the ground,

(a) find the value of λ at the point where the ball hits the ground.

hits ground, so $k = 0$

$0.84 + 0.8\lambda - \lambda^2 = 0$

$\lambda = \cancel{-\frac{3}{5}}, \frac{7}{5}$

given $\lambda \geq 0$ $\qquad \lambda = \frac{7}{5}$ //

(2)

The direction in which the tennis ball is moving at a general point on its path is given by

$$(9 - 4.6\lambda)\mathbf{i} + 15\mathbf{j} + (0.8 - 2\lambda)\mathbf{k}$$

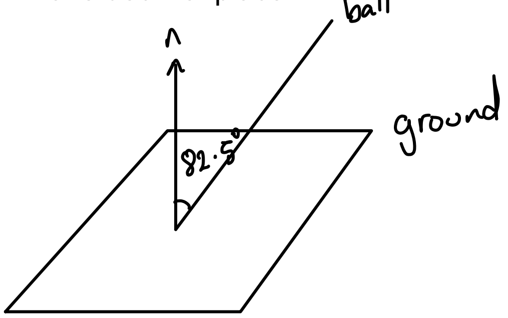
(b) Write down the direction in which the tennis ball is moving as it hits the ground.

$\left(9 - 4.6\left(\frac{7}{5}\right)\right)\underline{i} + 15\underline{j} + \left(0.8 - 2 \times \frac{7}{5}\right)\underline{k}$

$= 2.56\underline{i} + 15\underline{j} - 2\underline{k}$ //

(1)

(c) Hence find the acute angle at which the tennis ball hits the ground, giving your answer in degrees to one decimal place.



Normal to ground $= \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$\left| \dfrac{\begin{pmatrix} 2.56 \\ 15 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}{\sqrt{2.56^2 + 15^2 + (-2)^2}\sqrt{1^2}} \right| = \cos\theta$

$\left| \dfrac{-2}{\sqrt{\frac{147221}{625}}\sqrt{1}} \right| = \cos\theta$

$\theta = 82.5°$

$90 - 82.5$

$= 7.5°$ //

(4)

The net of the tennis court lies in the plane **r**. **j** = 0

(d) Find the position of the tennis ball at the point where it is in the same plane as the net.

tennis ball

$$\begin{pmatrix} -4.1 + 9\lambda - 2.3\lambda^2 \\ -10.25 + 15\lambda \\ 0.84 + 0.8\lambda - \lambda^2 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 0$$

$$-10.25 + 15\lambda = 0$$

$$15\lambda = 10.25$$

$$\lambda = \frac{41}{60}$$

$$\begin{pmatrix} -4.1 + 9\left(\frac{41}{60}\right) - 2.3\lambda^2 \\ -10.25 + 15\lambda \\ 0.84 + 0.8\left(\frac{41}{60}\right) - \left(\frac{41}{60}\right)^2 \end{pmatrix} = \begin{pmatrix} 0.976 \\ 0 \\ 0.9197 \end{pmatrix} //$$

(3)

The maximum height above the court of the top of the net is 0.9 m.

Modelling the top of the net as a horizontal straight line,

(e) state whether the tennis ball will pass over the net according to the model, giving a reason for your answer.

$0.9197 > 0.9$ So vertical height is over the net, so the ball will pass over. //

(1)

With reference to the model,

(f) decide whether the tennis ball will actually pass over the net, giving a reason for your answer.

The ball will have a radius, so may hit the net, the ball is not a particle. //

(2)

(Total for question = 13 marks)

Q3.

A mining company has identified a mineral layer below ground.

The mining company wishes to drill down to reach the mineral layer and models the situation as follows.

With respect to a fixed origin *O*,

- the ground is modelled as a horizontal plane with equation $z = 0$
- the mineral layer is modelled as part of the plane containing the points $A(10, 5, -50)$, $B(15, 30, -45)$ and $C(-5, 20, -60)$, where the units are in metres

(a) Determine an equation for the plane containing *A*, *B* and *C*, giving your answer in the form $\mathbf{r} \cdot \mathbf{n} = d$

$$\overrightarrow{AB} = \begin{pmatrix} 15-10 \\ 30-5 \\ -45+50 \end{pmatrix} = \begin{pmatrix} 5 \\ 25 \\ 5 \end{pmatrix} \qquad \overrightarrow{AC} = \begin{pmatrix} -5-10 \\ 20-5 \\ -60+50 \end{pmatrix} = \begin{pmatrix} -15 \\ 15 \\ -10 \end{pmatrix}$$

Normal to $\begin{pmatrix} 5 \\ 25 \\ 5 \end{pmatrix}$ and $\begin{pmatrix} -15 \\ 15 \\ -10 \end{pmatrix}$

$$\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 5 & 25 & 5 \\ -15 & 15 & -10 \end{vmatrix} = \underline{i}(-250-75) - \underline{j}(-50+75) + \underline{k}(75+375)$$

$$= -325\underline{i} - 25\underline{j} + 450\underline{k}$$

$$= 25(-13\underline{i} - \underline{j} + 18\underline{k})$$

$$\mathbf{r} \cdot \begin{pmatrix} -13 \\ -1 \\ 18 \end{pmatrix} = d$$

$$\begin{pmatrix} 10 \\ 5 \\ -50 \end{pmatrix} \cdot \begin{pmatrix} -13 \\ -1 \\ 18 \end{pmatrix} = d$$

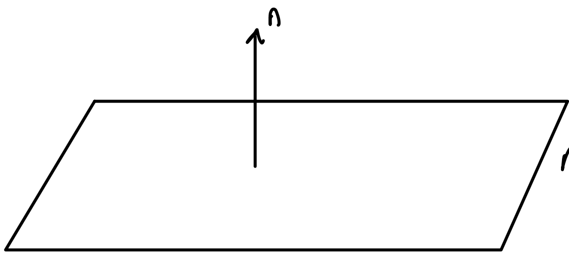
$$-130 - 5 - 900 = d$$

$$d = -1035$$

$$\mathbf{r} \cdot \begin{pmatrix} -13 \\ -1 \\ 18 \end{pmatrix} = -1035 \,//$$

(5)

(b) Determine, according to the model, the acute angle between the ground and the plane containing the mineral layer. Give your answer to the nearest degree.



angle between planes = angle between normals to planes

$$\frac{\begin{pmatrix} -13 \\ -1 \\ 18 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}{\sqrt{13^2+1^2+18^2} \times \sqrt{1^2}} = \cos\theta$$

$$\frac{18}{\sqrt{494}} = \cos\theta$$

$$\theta = 35.917^\circ$$

$$\theta = 36^\circ \,//$$

(3)

The mining company plans to drill vertically downwards from the point (5, 12, 0) on the ground to reach the mineral layer.

(c) Using the model, determine, in metres to 1 decimal place, the distance the mining company will need to drill in order to reach the mineral layer.

downwards from $\begin{pmatrix}5\\12\\0\end{pmatrix} = \begin{pmatrix}5\\12\\\lambda\end{pmatrix}$

$$\begin{pmatrix}5\\12\\\lambda\end{pmatrix}\cdot\begin{pmatrix}-13\\-1\\18\end{pmatrix} = -1035$$

$$-65 - 12 + 18\lambda = -1035$$

$$18\lambda = -958$$

$$\lambda = -53.2 \text{ m}$$

Distance = 53.2m down //

(2)

(d) State a limitation of the assumption that the mineral layer can be modelled as a plane.

The mineral layer will not likely be perfectly flat and not likely a plane. //

(1)

(Total for question = 11 marks)

Q4.

An octopus is able to catch any fish that swim within a distance of 2 m from the octopus's position.

A fish F swims from a point A to a point B.

The octopus is modelled as a fixed particle at the origin O.

Fish F is modelled as a particle moving in a straight line from A to B.

Relative to O, the coordinates of A are $(-3, 1, -7)$ and the coordinates of B are $(9, 4, 11)$, where the unit of distance is metres.

(a) Use the model to determine whether or not the octopus is able to catch fish F.

$$\vec{AB} = \begin{pmatrix} 9+3 \\ 4-1 \\ 11+7 \end{pmatrix} = \begin{pmatrix} 12 \\ 3 \\ 18 \end{pmatrix}$$

$$\text{Fish} = \begin{pmatrix} -3 \\ 1 \\ -7 \end{pmatrix} + \lambda \begin{pmatrix} 12 \\ 3 \\ 18 \end{pmatrix} = \begin{pmatrix} -3+12\lambda \\ 1+3\lambda \\ -7+18\lambda \end{pmatrix}$$

$$\left| \begin{pmatrix} -3+12\lambda \\ 1+3\lambda \\ -7+18\lambda \end{pmatrix} \right| = \sqrt{(-3+12\lambda)^2 + (1+3\lambda)^2 + (-7+18\lambda^2)}$$

$$= \sqrt{9 - 72\lambda + 144\lambda^2 + 1 + 6\lambda + 9\lambda^2 + 49 - 252\lambda + 324\lambda^2}$$

$$= \sqrt{477\lambda^2 - 318\lambda + 59}$$

$$= \sqrt{53(9\lambda^2 - 6\lambda + 1) + 6}$$

$$= \sqrt{53(3\lambda - 1)^2 + 6}$$

Min distance $= \sqrt{6}$ or $2.449 > 2$

So the octopus does not catch the fish //

(7)

(b) Criticise the model in relation to fish F.

The fish is unlikely to travel in an exact straight line //

(1)

(c) Criticise the model in relation to the octopus.

The octopus may move towards an oncoming fish //

(1)

(Total for question = 9 marks)

Q5.

Part of the mains water system for a housing estate consists of water pipes buried beneath the ground surface. The water pipes are modelled as straight line segments. One water pipe, W, is buried beneath a particular road. With respect to a fixed origin O, the road surface is modelled as a plane with equation $3x - 5y - 18z = 7$, and W passes through the points $A(-1, -1, -3)$ and $B(1, 2, -3)$. The units are in metres.

(a) Use the model to calculate the acute angle between W and the road surface.

$W: \quad \overrightarrow{AB} = \begin{pmatrix} 1 & +1 \\ 2 & +1 \\ -3 & +3 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}$

$W = \begin{pmatrix} -1 \\ -1 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}$

$$\frac{\begin{pmatrix} 3 \\ -5 \\ -18 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}}{\sqrt{358}\sqrt{13}} = \cos\theta$$

$$\frac{6 - 15}{\sqrt{358}\sqrt{13}} = \cos\theta$$

$\theta = 97.58°$

$90 - 97.58°$

$= -7.58°$

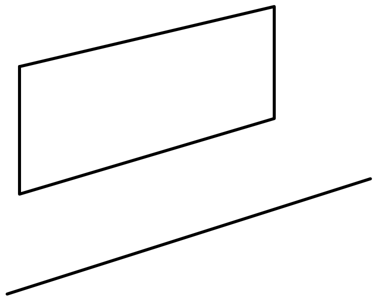
So angle is

$7.58°$ //

(5)

A point $C(-1, -2, 0)$ lies on the road. A section of water pipe needs to be connected to W from C.

(b) Using the model, find, to the nearest cm, the shortest length of pipe needed to connect C to W.



C to W $= \begin{pmatrix} -1 + 2\lambda + 1 \\ -1 + 3\lambda + 2 \\ -3 \end{pmatrix}$

$= \begin{pmatrix} 2\lambda \\ 3\lambda + 1 \\ -3 \end{pmatrix}$

$\begin{pmatrix} 2\lambda \\ 3\lambda + 1 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} = 0$

$4\lambda + 9\lambda + 3 = 0$

$13\lambda = -3$

$\lambda = -\frac{3}{13}$

C to W $= \begin{pmatrix} -\frac{6}{13} \\ \frac{4}{13} \\ -3 \end{pmatrix}$

$\left\| \begin{pmatrix} -\frac{6}{13} \\ \frac{4}{13} \\ -3 \end{pmatrix} \right\| = \sqrt{\left(-\frac{6}{13}\right)^2 + \left(\frac{4}{13}\right)^2 + (-3)^2}$

$= \sqrt{\frac{121}{13}}$

$= 3.05\ \text{m}$ //

(6)

(Total for question = 11 marks)

Q6.

The line l_1 has equation $\mathbf{r} = \begin{pmatrix} -2 \\ 2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$ where λ is a scalar parameter.

The line l_2 is parallel to $\begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$

(a) Show that l_1 and l_2 are perpendicular.

$$\begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} = 3 + 0 - 3$$

$$= 0$$

So perpendicular //

(2)

The plane π contains the line l_1 and is perpendicular to $\begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$

(b) Determine a Cartesian equation of π

Plane goes through $\begin{pmatrix} -2 \\ 2 \\ 0 \end{pmatrix}$ perpendicular to $\begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$

$$\mathbf{r} \cdot \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} = d$$

$$\begin{pmatrix} -2 \\ 2 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$$

$$= -2 + 4$$

$$= 2$$

$$\mathbf{r} \cdot \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} = 2$$

$$x + 2y - 3z = 2 \;//$$

(2)

(c) Verify that the point $A(3, 1, 1)$ lies on π

$$3 + 2(1) - 3(1)$$

$$= 3 + 2 - 3$$

$$= 2$$ So lies on plane //

(1)

Given that

- the point of intersection of π and l_2 has coordinates (2, 3, 2)
- the point $B(p, q, r)$ lies on l_2
- the distance AB is $2\sqrt{5}$
- p, q and r are positive integers

(d) determine the coordinates of B.

$$x + 2y - 3z = 2$$

$$l_2: \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} \qquad A = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$$

$$B: \begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} \qquad |\overrightarrow{AB}| = 2\sqrt{5}$$

$$B: \begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} 2 + \lambda \\ 3 + 2\lambda \\ 2 - 3\lambda \end{pmatrix}$$

$$\overrightarrow{AB} = \begin{pmatrix} 2 + \lambda - 3 \\ 3 + 2\lambda - 1 \\ 2 - 3\lambda - 1 \end{pmatrix} \qquad \left|\begin{pmatrix} -1 + \lambda \\ 2 + 2\lambda \\ 1 - 3\lambda \end{pmatrix}\right| = 2\sqrt{5}$$

$$= \begin{pmatrix} -1 + \lambda \\ 2 + 2\lambda \\ 1 - 3\lambda \end{pmatrix} \qquad \sqrt{(-1+\lambda)^2 + (2+2\lambda)^2 + (1-3\lambda)^2} = 2\sqrt{5}$$

$$1 - 2\lambda + \lambda^2 + 4 + 8\lambda + 4\lambda^2 + 1 - 6\lambda + 9\lambda^2 = 20$$

$$14\lambda^2 + 6 = 20$$

$$14\lambda^2 = 14$$

$$\lambda = \pm 1$$

(6)

$$\lambda = 1 \qquad B: \begin{pmatrix} 2+1 \\ 3+2 \\ 2-3 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \\ -1 \end{pmatrix}$$

$r > 0$ so not $\lambda = 1$

(Total for question = 11 marks)

$$\lambda = -1 \qquad B: \begin{pmatrix} 2-1 \\ 3-2 \\ 2+3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix} //$$

Q7.
The line *l* has equation

$$\frac{x+2}{1}=\frac{y-5}{-1}=\frac{z-4}{-3}$$

The plane *Π* has equation

$$\mathbf{r}.(\mathbf{i}-2\mathbf{j}+\mathbf{k})=-7$$

Determine whether the line *l* intersects *Π* at a single point, or lies in *Π*, or is parallel to *Π* without intersecting it.

$$l_1: \begin{pmatrix}-2\\5\\4\end{pmatrix}+\lambda\begin{pmatrix}1\\-1\\-3\end{pmatrix}$$

$$\begin{pmatrix}-2+\lambda\\5-\lambda\\4-3\lambda\end{pmatrix}\cdot\begin{pmatrix}1\\-2\\1\end{pmatrix}=-7$$

$$(-2+\lambda)-2(5-\lambda)+4-3\lambda=-7$$

$$-2+\lambda-10+2\lambda+4-3\lambda=-7$$

$$-8\neq-7$$

Contradiction so no interestion.

Line is parallel to Π //

(Total for question = 5 marks)

Q8.
All units in this question are in metres.

A lawn is modelled as a plane that contains the points L (–2, –3, –1), M (6, –2, 0) and N (2, 0, 0), relative to a fixed origin O.

(a) Determine a vector equation of the plane that models the lawn, giving your answer in the form $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$

$\overrightarrow{LM} = \begin{pmatrix} 6+2 \\ -2+3 \\ 0+1 \end{pmatrix} = \begin{pmatrix} 8 \\ 1 \\ 1 \end{pmatrix}$ $\qquad$ $\overrightarrow{LN} = \begin{pmatrix} 2+2 \\ 0+3 \\ 0+1 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \\ 1 \end{pmatrix}$

$r = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 3 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 8 \\ 1 \\ 1 \end{pmatrix}$ //

(3)

(b) (i) Show that, according to the model, the lawn is perpendicular to the vector $\begin{pmatrix} 1 \\ 2 \\ -10 \end{pmatrix}$

(ii) Hence determine a Cartesian equation of the plane that models the lawn.

i) $\begin{pmatrix} 4 \\ 3 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ -10 \end{pmatrix}$
$= 4 + 6 - 10$
$= 0$

$\begin{pmatrix} 8 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ -10 \end{pmatrix}$
$= 8 + 2 - 10$
$= 0$

Hence perpendicular to plane //

ii) $r \cdot \begin{pmatrix} 1 \\ 2 \\ -10 \end{pmatrix} = k$ $\qquad$ $\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ -10 \end{pmatrix} = k$

$k = 2$

$x + 2y - 10z = 2$ //

(4)

There are two posts set in the lawn.
There is a washing line between the two posts.
The washing line is modelled as a straight line through points at the top of each post with coordinates P (–10, 8, 2) and Q (6, 4, 3).

(c) Determine a vector equation of the line that models the washing line.

$\overrightarrow{PQ} = \begin{pmatrix} 6+10 \\ 4-8 \\ 3-2 \end{pmatrix} = \begin{pmatrix} 16 \\ -4 \\ 1 \end{pmatrix}$ $\qquad$ $r = \begin{pmatrix} 6 \\ 4 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 16 \\ -4 \\ 1 \end{pmatrix}$ //

(2)

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(d) State a limitation of one of the models.

The lawn may not be flat.

The washing line may not be straight

(1)

The point R (2, 5, 2.75) lies on the washing line.

(e) Determine, according to the model, the shortest distance from the point R to the lawn, giving your answer to the nearest cm.

formula book

$x + 2y - 10z = 2$

$x + 2y - 10z - 2 = 0$

$$\frac{|n_1\alpha + n_2\beta + n_3\gamma + d|}{\sqrt{n_1^2 + n_2^2 + n_3^2}}$$

$$= \frac{|1 \times 2 + 5 \times 2 + 2.75 \times -10 - 2|}{\sqrt{1^2 + (-2)^2 + (-10)^2}}$$

$$= 1.71\text{ m}$$

(2)

Given that the shortest distance from the point R to the lawn is actually 1.5 m,

(f) use your answer to part (e) to evaluate the model, explaining your reasoning.

It is 21cm off therefore not a suitable model.

(1)

(Total for question = 13 marks)

Q9.

The drainage system for a sports field consists of underground pipes.

This situation is modelled with respect to a fixed origin O.

According to the model,

- the surface of the sports field is a plane with equation $z = 0$
- the pipes are straight lines
- one of the pipes, P_1, passes through the points $A(3, 4, -2)$ and $B(-2, -8, -3)$
- a different pipe, P_2, has equation $\dfrac{x-1}{2} = \dfrac{y-3}{4} = \dfrac{z+1}{-2}$
- the units are metres

(a) Determine a vector equation of the line representing the pipe P_1

P_1

$$\begin{pmatrix} 3+2 \\ 4+8 \\ -2+3 \end{pmatrix} = \begin{pmatrix} 5 \\ 12 \\ 1 \end{pmatrix} \qquad r = \begin{pmatrix} 3 \\ 4 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 5 \\ 12 \\ 1 \end{pmatrix} //$$

(2)

(b) Determine the coordinates of the point at which the pipe P_1 meets the surface of the playing field, according to the model.

$$-2 + \lambda = 0$$
$$\lambda = 2$$

$$r = \begin{pmatrix} 3 \\ 4 \\ -2 \end{pmatrix} + 2\begin{pmatrix} 5 \\ 12 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 3+10 \\ 4+24 \\ -2+2 \end{pmatrix} = \begin{pmatrix} 13 \\ 28 \\ 0 \end{pmatrix} //$$

(2)

Determine, according to the model,

(c) the acute angle between pipes P_1 and P_2, giving your answer in degrees to 3 significant figures,

$$\frac{\begin{pmatrix} 5 \\ 12 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 4 \\ -2 \end{pmatrix}}{\sqrt{5^2+12^2+1^2}\sqrt{2^2+4^2+2^2}} = \cos\theta$$

$$\frac{10+48-2}{\sqrt{170}\sqrt{24}} = \cos\theta$$

$$\theta = 28.8^\circ //$$

(3)

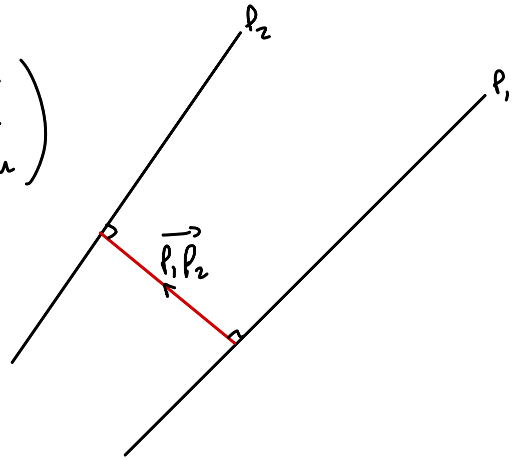
(d) the shortest distance between pipes P_1 and P_2

P_1: $r = \begin{pmatrix} 3 \\ 4 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 5 \\ 12 \\ 1 \end{pmatrix}$

$= \begin{pmatrix} 3 + 5\lambda \\ 4 + 12\lambda \\ -2 + \lambda \end{pmatrix}$

P_2: $r = \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 4 \\ -2 \end{pmatrix}$

$= \begin{pmatrix} 1 + 2\mu \\ 3 + 4\mu \\ -1 - 2\mu \end{pmatrix}$



$|\overrightarrow{P_1P_2}| = \begin{pmatrix} 1 + 2\mu - (3 + 5\lambda) \\ 3 + 4\mu - (4 + 12\lambda) \\ -1 - 2\mu - (-2 + \lambda) \end{pmatrix}$

$= \begin{pmatrix} -2 + 2\mu - 5\lambda \\ -1 + 4\mu - 12\lambda \\ 1 - 2\mu - \lambda \end{pmatrix}$

$\begin{pmatrix} -2 + 2\mu - 5\lambda \\ -1 + 4\mu - 12\lambda \\ 1 - 2\mu - \lambda \end{pmatrix} \cdot \begin{pmatrix} 5 \\ 12 \\ 1 \end{pmatrix} = 0$

$-10 + 10\mu - 25\lambda - 12 + 48\mu - 144\lambda + 1 - 2\mu - \lambda = 0$

$56\mu - 170\lambda - 21 = 0$

$56\mu - 170\lambda = 21$

$\begin{pmatrix} -2 + 2\mu - 5\lambda \\ -1 + 4\mu - 12\lambda \\ 1 - 2\mu - \lambda \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 4 \\ -2 \end{pmatrix} = 0$

$-4 + 4\mu - 10\lambda - 4 + 16\mu - 48\lambda - 2 + 4\mu + 2\lambda = 0$

$24\mu - 56\lambda - 10 = 0$

$24\mu - 56\lambda = 10$

$\mu = \frac{131}{236} \quad \lambda = \frac{7}{118}$

(5)

(Total for question = 12 marks)

$|\overrightarrow{P_1P_2}| = \left\| \begin{pmatrix} -2 + 2\mu - 5\lambda \\ -1 + 4\mu - 12\lambda \\ 1 - 2\mu - \lambda \end{pmatrix} \right\|$

$= \left\| \begin{pmatrix} -\frac{70}{59} \\ \frac{30}{59} \\ -\frac{10}{59} \end{pmatrix} \right\|$

$= \sqrt{\left(\frac{-70}{59}\right)^2 + \left(\frac{30}{59}\right)^2 + \left(\frac{-10}{59}\right)^2}$

$= 1.3\text{m}$ //

Q10.

A gas company maintains a straight pipeline that passes under a mountain.

The pipeline is modelled as a straight line and one side of the mountain is modelled as a plane.

There are accessways from a control centre to two access points on the pipeline.

Modelling the control centre as the origin O, the two access points on the pipeline have coordinates $P(-300, 400, -150)$ and $Q(300, 300, -50)$, where the units are metres.

(a) Find a vector equation for the line PQ, giving your answer in the form $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$,

where λ is a scalar parameter.

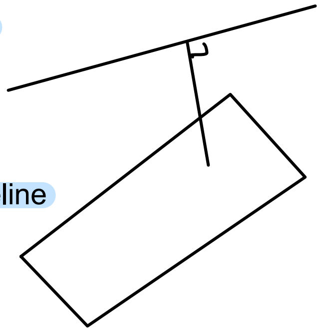
$$\overrightarrow{PQ} = \begin{pmatrix} 300+300 \\ 300-400 \\ -50+150 \end{pmatrix} = \begin{pmatrix} 600 \\ -100 \\ 100 \end{pmatrix}$$

$$r = \begin{pmatrix} 300 \\ 300 \\ -50 \end{pmatrix} + \lambda \begin{pmatrix} 600 \\ -100 \\ 100 \end{pmatrix} //$$

(2)

The equation of the plane modelling the side of the mountain is $2x + 3y - 5z = 300$

The company wants to create a new accessway from this side of the mountain to the pipeline.



The accessway will consist of a tunnel of shortest possible length between the pipeline and the point $M(100, k, 100)$ on this side of the mountain, where k is a constant.

(b) Using the model, find

(i) the coordinates of the point at which this tunnel will meet the pipeline,
(ii) the length of this tunnel.

Shortest distance between line and $(100, k, 100)$ – on plane which will be perpendicular to line.

$x = 100$
$y = k$
$z = 100$

$2(100) + 3k - 5(100) = 300$

$200 + 3k - 500 = 300$

$3k = 600$

$k = 200$

$$\begin{pmatrix} 100 \\ 200 \\ 100 \end{pmatrix}$$

Point on line

$$\begin{pmatrix} 300+600\lambda \\ 300-100\lambda \\ -50+100\lambda \end{pmatrix}$$

line to $M\begin{pmatrix} 100 \\ 200 \\ 100 \end{pmatrix}$

$$\overrightarrow{RM}: \begin{pmatrix} 200+600\lambda \\ 100-100\lambda \\ -150+100\lambda \end{pmatrix}$$

$$\begin{pmatrix} 200+600\lambda \\ 100-100\lambda \\ -150+100\lambda \end{pmatrix} \cdot \begin{pmatrix} 600 \\ -100 \\ 100 \end{pmatrix} = 0$$

$120000 + 360000\lambda - 10000 + 10000\lambda - 15000 + 10000\lambda = 0$

$380000\lambda = -95000$

$\lambda = -\frac{1}{4}$

i) $$\begin{pmatrix} 300+600(-\frac{1}{4}) \\ 300-100(-\frac{1}{4}) \\ -50+100(-\frac{1}{4}) \end{pmatrix} = \begin{pmatrix} 150 \\ 325 \\ -75 \end{pmatrix} //$$

ii) $$\overrightarrow{RM} = \begin{pmatrix} 50 \\ 125 \\ 175 \end{pmatrix}$$

(7)

$|\overrightarrow{RM}| = \sqrt{50^2+125^2+175^2}$

$= 221\text{m} //$

It is only practical to construct the new accessway if it will be significantly shorter than both of the existing accessways, OP and OQ.

(c) Determine whether the company should build the new accessway.

$$|\overrightarrow{OP}| = \sqrt{(-300)^2 + 400^2 + (-150)^2}$$
$$= 522m$$

$$|\overrightarrow{OQ}| = \sqrt{(300)^2 + (300)^2 + (-50)^2}$$
$$= 427m$$

New tunnel is considerably shorter, almost half, so likely will be built.

(2)

(d) Suggest one limitation of the model.

It is unlikely the side of the mountain is a flat plane. The shortest length of pipe may not be possible due to rock type.

(1)

(Total for question = 12 marks)