Mark Scheme

Q1.

Question Number	Scheme		Marks
	$u_{n+2} = 6u_{n+1} - 9u_n$, $n \geqslant 1$, $u_1 = 6$, $u_2 = 27$; $u_n = 3^n(n+1)$		
(i)	$n = 1$; $u_1 = 3(2) = 6$	Check that $u_1 = 6$ and $u_2 = 27$	B1
	$n = 2$; $u_2 = 3^2(2+1) = 27$		
	So u_n is true when $n = 1$ and $n = 2$.		
	Assume that $u_k = 3^k(k+1)$ and $u_{k+1} = 3^{k+1}(k+2)$ are true.	Could assume for $n = k, n = k-1$ and show for $n = k+1$	
	Then $u_{k+2} = 6u_{k+1} - 9u_k$		
	$= 6(3^{k+1})(k+2) - 9(3^k)(k+1)$	Substituting u_k and u_{k+1} into $u_{k+2} = 6u_{k+1} - 9u_k$	M1
		Correct expression	A1
	$= 2(3^{k+2})(k+2) - (3^{k+2})(k+1)$	Achieves an expression in 3^{k+2}	M1
	$= (3^{k+2})(2k+4-k-1)$		
	$= (3^{k+2})(k+3)$		
	$= (3^{k+2})(k+2+1)$	$(3^{k+2})(k+2+1)$ or $(3^{k+2})(k+3)$	A1
	If the result is **true for $n = k$ and $n = k+1$** then it is now **true for $n = k+2$**. As it is **true for $n = 1$ and $n = 2$** then it is **true for all n** $(\in \mathbb{Z}^+)$.	Correct conclusion seen at the end. Condone true for $n = 1$ and $n = 2$ seen anywhere. This should be compatible with assumptions.	A1 cso
			[6]

Question	Scheme	Notes	Marks
(ii)	$f(n) = 3^{3n-2} + 2^{3n+1}$ is divisible by 19		
	In all ways, first M is for applying $f(k+1)$ with at least 1 power correct. The second M is dependent on at least one accuracy being awarded and making $f(k+1)$ the subject and the final A is correct solution only.		
(ii) **Way 1**	$f(1) = 3^1 + 2^4 = 19$ {which is divisible by 19}.	Shows $f(1) = 19$	B1
	$\{\therefore f(n)$ is divisible by 19 when $n = 1\}$ {Assume that for $n = k$, $f(k) = 3^{3k-2} + 2^{3k+1}$ is divisible by 19 for $k \in \mathbb{Z}^+$.}		
	$f(k+1) - f(k) = 3^{3(k+1)-2} + 2^{3(k+1)+1} - (3^{3k-2} + 2^{3k+1})$	Applies $f(k+1)$ with at least 1 power correct	M1
	$f(k+1) - f(k) = 27(3^{3k-2}) + 8(2^{3k+1}) - (3^{3k-2} + 2^{3k+1})$ $f(k+1) - f(k) = 26(3^{3k-2}) + 7(2^{3k+1})$ $= 7(3^{3k-2} + 2^{3k+1}) + 19(3^{3k-2})$ **or** $= 26(3^{3k-2} + 2^{3k+1}) - 19(2^{3k+1})$ $= 7f(k) + 19(3^{3k-2})$ **or** $= 26f(k) - 19(2^{3k+1})$	Either $7(3^{3k-2} + 2^{3k+1})$ or $7f(k)$; $19(3^{3k-2})$ or $26(3^{3k-2} + 2^{3k+1})$ or $26f(k)$; $-19(2^{3k+1})$	A1; A1
	$\therefore f(k+1) = 8f(k) + 19(3^{3k-2})$ **or** $f(k+1) = 27f(k) - 19(2^{3k+1})$	Dependent on at least one of the previous accuracy marks being awarded. Makes Applies $f(k+1)$ with at least 1 power correct the subject	dM1
	$\{\therefore f(k+1) = 8f(k) + 19(3^{3k-2})$ is divisible by 19 as both $8f(k)$ and $19(3^{3k-2})$ are both divisible by 19}		
	If the result is **true for $n = k$**, then it is now **true for $n = k + 1$**. As the result has shown to be **true for $n = 1$**, then the result is **true for all n** $(\in \mathbb{Z}^+)$.	Correct conclusion seen at the end. Condone true for n = 1 stated earlier.	A1 cso **[6]**
(ii) **Way 2**	$f(1) = 3^1 + 2^4 = 19$ {which is divisible by 19}.	Shows $f(1) = 19$	B1
	$\{\therefore f(n)$ is divisible by 19 when $n = 1\}$ Assume that for $n = k$, $f(k) = 3^{3k-2} + 2^{3k+1}$ is divisible by 19 for $k \in \mathbb{Z}^+$.		
	$f(k+1) = 3^{3(k+1)-2} + 2^{3(k+1)+1}$	Applies $f(k+1)$ with at least 1 power correct	M1
	$f(k+1) = 27(3^{3k-2}) + 8(2^{3k+1})$ $= 8(3^{3k-2} + 2^{3k+1}) + 19(3^{3k-2})$ **or** $= 27(3^{3k-2} + 2^{3k+1}) - 19(2^{3k+1})$	Either $8(3^{3k-2} + 2^{3k+1})$ or $8f(k)$; $19(3^{3k-2})$ or $27(3^{3k-2} + 2^{3k+1})$ or $27f(k)$; $-19(2^{3k+1})$	A1; A1
	$\therefore f(k+1) = 8f(k) + 19(3^{3k-2})$ **or** $f(k+1) = 27f(k) - 19(2^{3k+1})$	**Dependent on at least one of the previous accuracy marks being awarded.**	dM1
	$\{\therefore f(k+1) = 8f(k) + 19(3^{3k-2})$ is divisible by 19 as both $8f(k)$ and $19(3^{3k-2})$ are both divisible by 19}		
	If the result is **true for $n = k$**, then it is now **true for $n = k + 1$**. As the result has shown to be **true for $n = 1$**, then the result is **true for all n** $(\in \mathbb{Z}^+)$.	Correct conclusion seen at the end. Condone true for n = 1 stated earlier.	A1 cso **[6]**

(ii) Way 3	$f(n) = 3^{3n-2} + 2^{3n+1}$ is divisible by 19 $f(1) = 3^1 + 2^4 = 19$ {which is divisible by 19}. {$\therefore f(n)$ is divisible by 19 when $n = 1$ } Assume that for $n = k$, $f(k) = 3^{3k-2} + 2^{3k+1}$ is divisible by 19 for $k \in \mathbb{Z}^+$.	Shows $f(1) = 19$	B1
	$f(k+1) - \alpha f(k) = 3^{3(k+1)-2} + 2^{3(k+1)+1} - \alpha(3^{3k-2} + 2^{3k+1})$	Applies $f(k+1)$ with at least 1 power correct	M1
	$f(k+1) - \alpha f(k) = (27-\alpha)(3^{3k-2}) + (8-\alpha)2^{3k+1}$ $= (8-\alpha)(3^{3k-2} + 2^{3k+1}) + 19(3^{3k-2})$ **or** $= (27-\alpha)(3^{3k-2} + 2^{3k+1}) - 19(2^{3k+1})$	$(8-\alpha)(3^{3k-2} + 2^{3k+1})$ or $(8-\alpha)f(k); 19(3^{3k-2})$ NB choosing $\alpha = 8$ makes first term disappear. $(27-\alpha)(3^{3k-2} + 2^{3k+1})$ or $(27-\alpha)f(k); -19(2^{3k+1})$ NB choosing $\alpha = 27$ makes first term disappear.	A1; A1
	$\therefore f(k+1) = 8f(k) + 19(3^{3k-2})$ **or** $f(k+1) = 27f(k) - 19(2^{3k+1})$	**Dependent on at least one of the previous accuracy marks being awarded.** Makes $f(k+1)$ the subject.	dM1
	{$\therefore f(k+1) = 27f(k) - 19(2^{3k+1})$ is divisible by 19 as both $27f(k)$ and $19(2^{3k+1})$ are both divisible by 19}		
	If the result is **true for $n = k$**, then it is now **true for $n = k+1$**. As the result has shown to be **true for $n = 1$**, then the result is true **for all n** $(\in \mathbb{Z}^+)$.	Correct conclusion seen at the end. Condone true for n = 1 stated earlier.	A1 cso
			[6] **12**
	Question Notes		
(ii)	Accept use of $f(k) = 3^{3k-2} + 2^{3k+1} = 19m$ o.e. and award method and accuracy as above.		

(**Q08 6667/01, June 2017**)

Q2.

Question Number	Scheme	Notes	Marks
	$f(n) = 8^n - 2^n$ is divisible by 6.		
	$f(1) = 8^1 - 2^1 = 6$,	Shows that $f(1) = 6$	B1
	Assume that for $n = k$, $f(k) = 8^k - 2^k$ is divisible by 6.		
	$f(k+1) - f(k) = 8^{k+1} - 2^{k+1} - \left(8^k - 2^k\right)$	Attempt $f(k+1) - f(k)$	M1
	$= 8^k(8-1) + 2^k(1-2) = 7\times 8^k - 2^k$		
	$= 6\times 8^k + 8^k - 2^k \left(= 6\times 8^k + f(k)\right)$	M1: Attempt $f(k+1) - f(k)$ as a multiple of 6 A1: rhs a correct multiple of 6	M1A1
	$f(k+1) = 6\times 8^k + 2f(k)$	Completes to $f(k+1) =$ a multiple of 6	A1
	If the result is **true for** $n = k$, then it is now **true for** $n = k+1$. As the result has been shown to be **true for** $n = 1$, then the result is **true for all** n $(\in \mathbb{Z}^+.)$		A1cso
		Do not award final A if n defined incorrctly e.g. 'n is an integer' award A0	
			(6)
			Total 6
Way 2	$f(1) = 8^1 - 2^1 = 6$,	Shows that $f(1) = 6$	B1
	Assume that for $n = k$, $f(k) = 8^k - 2^k$ is divisible by 6.		
	$f(k+1) = 8^{k+1} - 2^{k+1} = 8\left(8^k - 2^k + 2^k\right) - 2.2^k$	Attempts $f(k+1)$ in terms of 2^k and 8^k	M1
	$f(k+1) = 8^{k+1} - 2^{k+1} = 8\left(f(k) + 2^k\right) - 2.2^k$	M1:Attempts $f(k+1)$ in terms of $f(k)$ A1: rhs correct and a multiple of 6	M1A1

	$f(k+1)=8f(k)+6.2^k$	Completes to $f(k+1)$ = a multiple of 6	A1
	If the result is **true for** $n=k$, then it is now **true for** $n=k+1$. As the result has been shown to be **true for** $n=1$, then the result is **true for all** n $(\in \square^{+}.)$		A1cso
Way 3	$f(1)=8^1-2^1=6$,	Shows that $f(1)=6$	B1
	Assume that for $n=k$, $f(k)=8^k-2^k$ is divisible by 6.		
	$f(k+1)-8f(k)=8^{k+1}-2^{k+1}-8.8^k+8.2^k$	Attempt $f(k+1)-8f(k)$	M1
		Any multiple m replacing 8 award M1	
	$f(k+1)-8f(k)=8^{k+1}-8^{k+1}+8.2^k-2.2^k=6.2^k$	M1: Attempt $f(k+1)-f(k)$ as a multiple of 6 A1: rhs a correct multiple of 6	M1A1
	$f(k+1)=8f(k)+6.2^k$	Completes to $f(k+1)$ = a multiple of 6	A1
		General Form for multiple m $f(k+1)=6.8^k+(2-m)(8^k-2^k)$	
	If the result is **true for** $n=k$, then it is now **true for** $n=k+1$. As the result has been shown to be **true for** $n=1$, then the result is **true for all** n $(\in \square^{+}.)$		A1cso

(**Q07 6667/01, June 2014**)

Q3.

Question Number	Scheme	Notes	Marks
	$\mathrm{f}(n) = 2^{2n-1} + 3^{2n-1}$ is divisible by 5.		
	$\mathrm{f}(1) = 2^1 + 3^1 = 5$,	Shows that $\mathrm{f}(1) = 5$.	B1
	Assume that for $n = k$, $\mathrm{f}(k) = 2^{2k-1} + 3^{2k-1}$ is divisible by 5 for $k \in ¢^+$.		
	$\mathrm{f}(k+1) - \mathrm{f}(k) = 2^{2(k+1)-1} + 3^{2(k+1)-1} - \left(2^{2k-1} + 3^{2k-1}\right)$	M1: Attempts $\mathrm{f}(k+1) - \mathrm{f}(k)$. A1: Correct expression for $\underline{\mathrm{f}(k+1)}$ (Can be unsimplified)	M1A1
	$= 2^{2k+1} + 3^{2k+1} - 2^{2k-1} - 3^{2k-1}$		
	$= 2^{2k-1+2} + 3^{2k-1+2} - 2^{2k-1} - 3^{2k-1}$		
	$= 4\left(2^{2k-1}\right) + 9\left(3^{2k-1}\right) - 2^{2k-1} - 3^{2k-1}$	Achieves an expression in 2^{2k-1} and 3^{2k-1}	M1
	$= 3\left(2^{2k-1}\right) + 8\left(3^{2k-1}\right)$		
	$= 3\left(2^{2k-1}\right) + 3\left(3^{2k-1}\right) + 5\left(3^{2k-1}\right)$		
	$= 3\mathrm{f}(k) + 5\left(3^{2k-1}\right)$		
	$\therefore \mathrm{f}(k+1) = 4\mathrm{f}(k) + 5\left(3^{2k-1}\right)$ or $4(2^{2k-1} + 3^{2k-1}) + 5\left(3^{2k-1}\right)$	Where $\mathrm{f}(k+1)$ is correct and is clearly a multiple of 5.	A1
	If the result is true for $n = k$, then it is now true for $n = k+1$. As the result has shown to be true for $n = 1$, then the result is true for all n.	Correct conclusion at the end, at least as given, and all previous marks scored.	A1 cso
			[6]
			6 marks
	All methods should complete to $\mathrm{f}(k+1) = ...$ where $\mathrm{f}(k+1)$ is clearly shown to be divisible by 5 to enable the final 2 marks to be available.		
Note that there are many different ways of proving this result by induction.			

(**Q07 6667/01, June 2012**)

Q4.

Question Number	Scheme	Notes	Marks
(a)	$n=1$; LHS $=\begin{pmatrix}3 & 0\\6 & 1\end{pmatrix}^1=\begin{pmatrix}3 & 0\\6 & 1\end{pmatrix}$ RHS $=\begin{pmatrix}3^1 & 0\\3(3^1-1) & 1\end{pmatrix}=\begin{pmatrix}3 & 0\\6 & 1\end{pmatrix}$ As LHS = RHS, the matrix result is true for $n=1$.	Check to see that the result is true for $n=1$.	B1
	Assume that the matrix equation is true for $n=k$, ie. $\begin{pmatrix}3 & 0\\6 & 1\end{pmatrix}^k=\begin{pmatrix}3^k & 0\\3(3^k-1) & 1\end{pmatrix}$		
	With $n=k+1$ the matrix equation becomes $\begin{pmatrix}3 & 0\\6 & 1\end{pmatrix}^{k+1}=\begin{pmatrix}3 & 0\\6 & 1\end{pmatrix}^k\begin{pmatrix}3 & 0\\6 & 1\end{pmatrix}$		
	$=\begin{pmatrix}3^k & 0\\3(3^k-1) & 1\end{pmatrix}\begin{pmatrix}3 & 0\\6 & 1\end{pmatrix}$ or $\begin{pmatrix}3 & 0\\6 & 1\end{pmatrix}\begin{pmatrix}3^k & 0\\3(3^k-1) & 1\end{pmatrix}$	$\begin{pmatrix}3^k & 0\\3(3^k-1) & 1\end{pmatrix}$ by $\begin{pmatrix}3 & 0\\6 & 1\end{pmatrix}$	M1
	$=\begin{pmatrix}3^{k+1}+0 & 0+0\\9(3^k-1)+6 & 0+1\end{pmatrix}$ or $\begin{pmatrix}3^{k+1}+0 & 0+0\\6.3^k+3(3^k-1) & 0+1\end{pmatrix}$	Correct unsimplified matrix with no errors seen.	A1
	$=\begin{pmatrix}3^{k+1} & 0\\9(3^k)-3 & 1\end{pmatrix}$		
	$=\begin{pmatrix}3^{k+1} & 0\\3\left(3(3^k)-1\right) & 1\end{pmatrix}$	Manipulates so that $k\to k+1$ on at least one term.	dM1
	$=\begin{pmatrix}3^{k+1} & 0\\3(3^{k+1}-1) & 1\end{pmatrix}$	Correct result with no errors seen with some working between this and the previous A1	A1
	If the result is true for $n=k$ (1) then it is now true for $n=k+1$. (2) As the result has shown to be true for $n=1$, (3) then the result is true for all n. (4) **All 4 aspects need to be mentioned at some point for the last A1.**	Correct conclusion with all previous marks earned	A1 cso
			(6)

Question Number	Scheme	Notes	Marks
(b)	$f(1) = 7^{2-1} + 5 = 7 + 5 = 12,$	Shows that $f(1) = 12$.	B1
	$\{$which is divisible by 12$\}$. $\{\therefore f(n)$ is divisible by 12 when $n = 1.\}$		
	Assume that for $n = k$, $f(k) = 7^{2k-1} + 5$ is divisible by 12 for $k \in ¢^{+}$.		
	So, $f(k+1) = 7^{2(k+1)-1} + 5$	Correct unsimplified expression for $f(k+1)$.	B1
	giving, $f(k+1) = 7^{2k+1} + 5$		
	$\therefore f(k+1) - f(k) = \left(7^{2k+1} + 5\right) - \left(7^{2k-1} + 5\right)$	Applies $f(k+1) - f(k)$. No simplification is necessary and condone missing brackets.	M1
	$= 7^{2k+1} - 7^{2k-1}$		
	$= 7^{2k-1}\left(7^2 - 1\right)$	Attempting to isolate 7^{2k-1}	M1
	$= 48\left(7^{2k-1}\right)$	$48\left(7^{2k-1}\right)$	A1cso
	$\therefore f(k+1) = f(k) + 48\left(7^{2k-1}\right)$, which is divisible by 12 as both $f(k)$ and $48\left(7^{2k-1}\right)$ are both divisible by 12.(1) If the result is true for $n = k$, (2) then it is now true for $n = k+1$. (3) As the result has shown to be true for $n = 1$,(4) then the result is true for all n. (5). **All 5 aspects need to be mentioned at some point for the last A1.**	Correct conclusion with no incorrect work. Don't condone missing brackets.	A1 cso
			(6)
	There are other ways of proving this by induction. See appendix for 3 alternatives. If you are in any doubt consult your team leader and/or use the review system.		
			12

(**Q07 6667/01, June 2011**)

Online Maths Teaching

Q5.

Question Number	Scheme		Marks
	$u_{n+1} = 4u_n + 2$, $u_1 = 2$ and $u_n = \frac{2}{3}(4^n - 1)$		
	$n = 1$; $u_1 = \frac{2}{3}(4^1 - 1) = \frac{2}{3}(3) = 2$ So u_n is true when $n = 1$.	Check that $u_n = \frac{2}{3}(4^n - 1)$ yields 2 when $n = 1$.	B1
	Assume that for $n = k$ that, $u_k = \frac{2}{3}(4^k - 1)$ is true for $k \in \mathbb{Z}^+$.		
	Then $u_{k+1} = 4u_k + 2$		
	$= 4\left(\frac{2}{3}(4^k - 1)\right) + 2$	Substituting $u_k = \frac{2}{3}(4^k - 1)$ into $u_{n+1} = 4u_n + 2$.	M1
	$= \frac{8}{3}(4)^k - \frac{8}{3} + 2$	An attempt to multiply out the brackets by 4 or $\frac{8}{3}$	M1
	$= \frac{2}{3}(4)(4)^k - \frac{2}{3}$		
	$= \frac{2}{3}4^{k+1} - \frac{2}{3}$		
	$= \frac{2}{3}(4^{k+1} - 1)$	$\frac{2}{3}(4^{k+1} - 1)$	A1
	Therefore, the general statement, $u_n = \frac{2}{3}(4^n - 1)$ is true when $n = k+1$. (As u_n is true for $n = 1$,) then u_n is true for all positive integers by mathematical induction	Require 'True when n=1', 'Assume true when n=k' and 'True when $n = k+1$ ' then true for all n o.e.	A1
			(5) **[5]**

(**Q08 6667/01, Jan 2011**)

Q6.

Question Number	Scheme	Marks
	$n=1$: $\quad 1^2 = \frac{1}{3}\times 1\times 1\times 3$	B1
	(Hence result is true for $n=1$.)	
	$\sum_{r=1}^{k+1}(2r-1)^2 = \sum_{r=1}^{k}(2r-1)^2 + (2k+1)^2$	
	$= \frac{1}{3}k(2k-1)(2k+1) + (2k+1)^2$, by induction hypothesis	M1
	$= \frac{1}{3}(2k+1)(2k^2-k+6k+3)$	
	$= \frac{1}{3}(2k+1)(2k^2+5k+3)$	
	$= \frac{1}{3}(2k+1)(2k+3)(k+1)$	M1 A1
	$= \frac{1}{3}(k+1)[2(k+1)-1][2(k+1)+1]$	
	(Hence, if result is true for $n=k$, then it is true for $n=k+1$.)	
	By Mathematical Induction, above implies the result is true for all $n \in \mathbb{Z}^+$. ✱ cso	A1 (5) [5]

(**Q03 6676/01, June 2007**)

Q7.

Question Number	Scheme		Marks
	(a) $f(k+1)-f(k)=3^{4k+4}+2^{4k+6}-3^{4k}-2^{4k+2}$		
	$=3^{4k}(3^4-1)+2^{4k+2}(2^4-1)$	M1	
	$=3^{4k}\times 80+2^{4k+2}\times 15$ can be implied	A1	
	$=3^{4k-1}\times 240+2^{4k+2}\times 15=15(16\times 3^{4k-1}+2^{4k+2})$	M1	
	Hence $15 \mid f(k+1)-f(k)$ ✱ cso	A1	(4)
	Note: $f(k+1)-f(k)$ is divisible by 240 and other appropriate multiples of 15 lead to the required result.		
	(b) $n=1$: $f(1)=3^4+2^6=145=5\times 29 \Rightarrow 5 \mid f(1)$	B1	
	(Hence result is true for $n=1$.)		
	From (a) $f(k+1)-f(k)=15\lambda$, say. By induction hypothesis $f(k)=5\mu$, say.		
	$f(k+1)=f(k)+15\lambda=5(\mu+3\lambda) \Rightarrow 5 \mid f(k+1)$	M1	
	(Hence, if result is true for $n=k$, then it is true for $n=k+1$.)		
	By Mathematical Induction, above implies the result is true for all $n\in\mathbb{Z}^+$. ✱ Accept equivalent arguments cso	A1	(3)
	(c) $f(1)=145=5\times 29$ is not divisible by 15, so result is not true for all $\mathbb{Z}^+$.	B1	(1)
	Note: There is no integer for which $f(n)$ is divisible by 15 and any specific example should be accepted.		[8]

(**Q04 6676/01, June 2007**)

Q8.

Question Number	Scheme	Marks
	When $n=1$, LHS $=\frac{1}{1\times 2}=\frac{1}{2}$, RHS $=\frac{1}{1+1}=\frac{1}{2}$. So LHS = RHS and result true for $n=1$	B1
	Assume true for $n=k$; $\sum_{r=1}^{k}\frac{1}{r(r+1)}=\frac{k}{k+1}$ and so $\sum_{r=1}^{k+1}\frac{1}{r(r+1)}=\frac{k}{k+1}+\frac{1}{(k+1)(k+2)}$	M1
	$\sum_{r=1}^{k+1}\frac{1}{r(r+1)}=\frac{k(k+2)+1}{(k+1)(k+2)}=\frac{k^2+2k+1}{(k+1)(k+2)}=\frac{(k+1)^2}{(k+1)(k+2)}=\frac{k+1}{k+2}$	M1 A1
	and so result is true for $n=k+1$ (and by induction true for $n\in\mathbf{Z}^+$)	B1 **[5]**

(**Q03 6667/01, Jan 2009**)

Q9.

Question Number	Scheme	Marks
	At $n=1$, $u_n=5\times 6^0+1=6$ and so result true for $n=1$	B1
	Assume true for $n=k$; $u_k=5\times 6^{k-1}+1$,and so $u_{k+1}=6(5\times 6^{k-1}+1)-5$	M1, A1
	$\therefore u_{k+1}=5\times 6^k+6-5$ $\quad\therefore u_{k+1}=5\times 6^k+1$	A1
	and so result is true for $n=k+1$ and by induction true for $n\geq 1$	B1 **[5]**

(**Q04 6667/01, Jan 2009**)

Q10.

Question Number	Scheme	Marks
Q (a)	$f(1) = 5 + 8 + 3 = 16$, (which is divisible by 4). ($\therefore$ True for $n = 1$).	B1
	Using the formula to write down f(k + 1), $f(k+1) = 5^{k+1} + 8(k+1) + 3$	M1 A1
	$f(k+1) - f(k) = 5^{k+1} + 8(k+1) + 3 - 5^k - 8k - 3$	M1
	$= 5(5^k) + 8k + 8 + 3 - 5^k - 8k - 3 = 4(5^k) + 8$	A1
	$f(k+1) = 4(5^k + 2) + f(k)$, **which is divisible by 4**	A1ft
	$\therefore$ True for $n = k+1$ **if** true for $n = k$. True for $n = 1$, $\therefore$ true for all n.	A1cso (7)
(b)	For $n = 1$, $\begin{pmatrix} 2n+1 & -2n \\ 2n & 1-2n \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix}^1$ ($\therefore$ True for $n = 1$.)	B1
	$\begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix}^{k+1} = \begin{pmatrix} 2k+1 & -2k \\ 2k & 1-2k \end{pmatrix}\begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} 2k+3 & -2k-2 \\ 2k+2 & -2k-1 \end{pmatrix}$	M1 A1 A1
	$= \begin{pmatrix} 2(k+1)+1 & -2(k+1) \\ 2(k+1) & 1-2(k+1) \end{pmatrix}$	M1 A1
	$\therefore$ True for $n = k+1$ **if** true for $n = k$. **True for $n = 1$, $\therefore$ true for all n**	A1 cso (7) [14]
(a) Alternative for 2nd M:	$f(k+1) = 5(5^k) + 8k + 8 + 3$ M1 $= 4(5^k) + 8 + (5^k + 8k + 3)$ A1 or $= 5(5^k + 8k + 3) - 32k - 4$ $= 4(5^k + 2) + f(k)$, or $= 5f(k) - 4(8k + 1)$ which is divisible by 4 A1 (or similar methods)	
Notes	(a) B1 Correct values of 16 or 4 for $n = 1$ or for $n = 0$ (Accept "is a multiple of ") M1 Using the formula to write down f(k + 1) A1 Correct expression of f(k+1) (or for f(n +1) M1 Start method to connect f(k+1) with f(k) as shown A1 correct working toward multiples of 4. A1 ft result including f(k +1) as subject, A1cso conclusion (b) B1 correct statement for $n = 1$ or $n = 0$ First M1: Set up product of two appropriate matrices – product can be either way round A1 A0 for one or two slips in simplified result A1 A1 all correct simplified A0 A0 more than two slips M1: States in terms of (k + 1) A1 Correct statement A1 for induction conclusion	
Part (b) Alternative	May write $\begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix}^{k+1} = \begin{pmatrix} 2k+3 & -2k-2 \\ 2k+2 & -2k-1 \end{pmatrix}$. Then may or may not complete the proof. This can be awarded the second M (substituting $k + 1$)and following A (simplification) in part (b). The first three marks are awarded as before . Concluding that they have reached the same matrix and therefore a result will then be part of final A1 cso but also need other statements as in the first method.	

(**Q07 6667/01, June 2009**)

Q11.

Question Number	Scheme	Marks
	For $n=1$: $u_1=2,\ u_1=5^0+1=2$	B1
	Assume true for $n=k$:	
	$u_{k+1}=5u_k-4=5(5^{k-1}+1)-4=5^k+5-4=5^k+1$	M1 A1
	$\therefore$ True for $n=k+1$ if true for $n=k$.	
	True for $n=1$,	
	$\therefore$ true for all n.	A1 cso
		[4]
	Notes Accept $u_1=1+1=2$ or above B1 $5(5^{k-1}+1)-4$ seen award M1 5^k+1 or $5^{(k+1)-1}+1$ award first A1 All three elements stated somewhere in the solution award final A1	

(**Q03 6667/01, Jan 2010**)

Q12.

Question Number	Scheme		Marks
	(a) LHS = $f(k+1) = 2^{k+1} + 6^{k+1}$	OR RHS = $= 6f(k) - 4(2^k) = 6(2^k + 6^k) - 4(2^k)$	M1
	$= 2(2^k) + 6(6^k)$	$= 2(2^k) + 6(6^k)$	A1
	$= 6(2^k + 6^k) - 4(2^k) = 6f(k) - 4(2^k)$	$= 2^{k+1} + 6^{k+1} = f(k+1)$ (*)	A1 (3)
	OR $f(k+1) - 6f(k) = 2^{k+1} + 6^{k+1} - 6(2^k + 6^k)$		M1
	$= (2-6)(2^k) = -4.2^k$, and so $f(k+1) = 6f(k) - 4(2^k)$		A1, A1 (3)
	(b) $n = 1$: $f(1) = 2^1 + 6^1 = 8$, **which is divisible** by 8		B1
	Either Assume f(*k*) divisible by 8 and try to use $f(k+1) = 6f(k) - 4(2^k)$	**Or** Assume f(*k*) divisible by 8 and try to use $f(k+1) - f(k)$ or $f(k+1) + f(k)$ including factorising $6^k = 2^k 3^k$	M1
	Show $4(2^k) = 4 \times 2(2^{k-1}) = 8(2^{k-1})$ or $8(\frac{1}{2}2^k)$ Or valid statement	$= 2^3 2^{k-3}(1 + 5.3^k)$ or $= 2^3 2^{k-3}(3 + 7.3^k)$ o.e.	A1
	Deduction that result is implied for $n = k+1$ and so is true for positive integers by induction (may include $n = 1$ true here)	Deduction that result is implied for $n = k+1$ and so is true for positive integers by induction (must include explanation of why $n = 2$ is also true here)	A1cso (4) **7 marks**
	Notes		

Notes

(a) M1: for substitution into LHS (or RHS) or $f(k+1) - 6f(k)$

A1: for correct split of the two separate powers cao

A1: for completion of proof with no error or ambiguity (needs (for example) to start with one side of equation and reach the other or show that each side separately is $2(2^k) + 6(6^k)$ and **conclude** LHS = RHS)

(b) B1: for substitution of $n = 1$ and **stating** "true for $n = 1$" or "divisible by 8" or tick. (This statement may appear in the concluding statement of the proof)

M1: Assume f(*k*) divisible by 8 and consider $f(k+1) = 6f(k) - 4(2^k)$ or equivalent expression that could lead to proof – not merely $f(k+1) - f(k)$ unless deduce that 2 is a factor of 6 (see right hand scheme above).

A1: Indicates each term divisible by 8 **OR** takes out factor 8 or 2^3

A1: Induction statement . Statement $n = 1$ here could contribute to B1 mark earlier.

NB: $f(k+1) - f(k) = 2^{k+1} - 2^k + 6^{k+1} - 6^k = 2^k + 5.6^k$ **only** is M0 A0 A0

(b) **"Otherwise" methods**

Could use: $f(k+1) = 2f(k) + 4(6^k)$ or $f(k+2) = 36f(k) - 32(6^k)$ or $f(k+2) = 4f(k) + 32(2^k)$ in a similar way to given expression and Left hand mark scheme is applied.

Special Case: Otherwise Proof **not involving induction**: This can only be awarded the B1 for checking $n = 1$.

(**Q06 6667/01, June 2010**)

Q13.

Question Number	Scheme	Marks
(i)	If $n = 1$, $\begin{pmatrix} 1 & 0 \\ -1 & 5 \end{pmatrix}^1 = \begin{pmatrix} 1 & 0 \\ -\frac{1}{4}(5^1 - 1) & 5^1 \end{pmatrix}$ so **true for $n = 1$**	B1
	Assume result true for $n = k$	
	$\begin{pmatrix} 1 & 0 \\ -1 & 5 \end{pmatrix}^{k+1} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{4}(5^k - 1) & 5^k \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1 & 5 \end{pmatrix}$ or $\begin{pmatrix} 1 & 0 \\ -1 & 5 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -\frac{1}{4}(5^k - 1) & 5^k \end{pmatrix}$	M1
	$\begin{pmatrix} 1 & 0 \\ -1 & 5 \end{pmatrix}^{k+1} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{4}(5^k - 1) - 5^k & 5 \times 5^k \end{pmatrix}$ or $\begin{pmatrix} 1 & 0 \\ -1 - 5.\frac{1}{4}(5^k - 1) & 5 \times 5^k \end{pmatrix}$	M1 A1
	$= \begin{pmatrix} 1 & 0 \\ -\dfrac{1}{4}5^k + \dfrac{1}{4} - 5^k & 5^{k+1} \end{pmatrix}$ or $\begin{pmatrix} 1 & 0 \\ -1 - \frac{1}{4}5^{k+1} + \dfrac{5}{4} & 5^{k+1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{4}(5^{k+1} - 1) & 5^{k+1} \end{pmatrix}$	A1
	True for $n = k + 1$ if **true** for $n = k$, (and **true** for $n = 1$) so **true** by induction for all $n \in \mathbb{Z}^+$.	A1cso (6)
(ii)	If $n = 1$, $\sum_{r=1}^{n}(2r-1)^2 = 1$ and $\dfrac{1}{3}n(4n^2 - 1) = 1$, so **true for $n = 1$**.	B1
	Assume result true for $n = k$ so $\sum_{r=1}^{k+1}(2r-1)^2 = \dfrac{1}{3}k(4k^2 - 1) + (2(k+1) - 1)^2$	M1
	$= \sum_{r=1}^{k+1}(2r-1)^2 = \dfrac{1}{3}(2k+1)\{(2k^2 - k) + (3(2k+1))\}$	M1 A1
	$= \dfrac{1}{3}(2k+1)\{(2k^2 + 5k + 3)\}$ or $\dfrac{1}{3}(k+1)(4k^2 + 8k + 3)$ or $\dfrac{1}{3}((2k+3)(2k^2 + 3k + 1)\}$	
	$= \dfrac{1}{3}(k+1)(2k+1)(2k+3)$ $= \dfrac{1}{3}(k+1)(4(k+1)^2 - 1)$	dA1
	True for $n = k + 1$ if **true** for $n = k$, (and **true** for $n = 1$) so **true** by induction for all $n \in \mathbb{Z}^+$	A1cso (6)
		12 marks

Notes

(i) B1: Checks $n = 1$ on both sides and states true for $n = 1$ seen anywhere.
M1: Assumes true for $n = k$ and indicates intention to multiply power k by power 1 either way around.
M1: Multiplies matrices. Condone one slip. A1: Correct unsimplified matrix
A1: Intermediate step required cao
A1: cso Makes correct induction statement including at least statements in bold.
Statement **true** for $n = \mathbf{1}$ here could contribute to B1 mark earlier.
(ii) B1: Checks $n = 1$ on both sides and states true for $n = 1$ seen anywhere.
M1: Assumes true for $n = k$ and adds $(k+1)^{\text{th}}$ term to sum of k terms. Accept $4(k+1)^2 - 4(k+1)+1$ or $(2k+1)^2$ for $(k+1)^{\text{th}}$ term. M1: Factorises out a linear factor of the three possible - usually $2k+1$
A1: Correct expression with one linear and one quadratic factor.
dA1: Need to see $\frac{1}{3}(k+1)(4(k+1)^2-1)$ somewhere dependent upon previous A1.
Accept assumption plus $(k+1)^{\text{th}}$ term and $\frac{1}{3}(k+1)(4(k+1)^2-1)$ both leading to $\frac{1}{3}(4k^3+12k^2+11k+3)$ then award for expressions seen as above.
A1: cso Makes correct complete induction statement including at least statements in bold. Statement true for $n = 1$ here could contribute to B1 mark earlier.

(**Q05 6667/01, June 2015**)

Q14.

Question	Scheme	Marks	AOs
	$n=1,\ \sum_{r=1}^{1}\frac{1}{(2r-1)(2r+1)}=\frac{1}{1\times 3}=\frac{1}{3}$ and $\frac{n}{2n+1}=\frac{1}{2\times 1+1}=\frac{1}{3}$ (true for n=1)	B1	2.2a
	Assume general statement is true for $n=k$. So assume $\sum_{r=1}^{k}\frac{1}{(2r-1)(2r+1)}=\frac{k}{2k+1}$ is true.	M1	2.4
	$\left(\sum_{r=1}^{k+1}\frac{1}{(2r-1)(2r+1)}=\right)$ " $\frac{k}{2k+1}$ " $+\frac{1}{(2k+1)(2k+3)}$	M1	2.1
	$=\frac{k(2k+3)+1}{(2k+1)(2k+3)}$	dM1	1.1b
	$=\frac{2k^2+3k+1}{(2k+1)(2k+3)}=\frac{(2k+1)(k+1)}{(2k+1)(2k+3)}=\frac{(k+1)}{2(k+1)+1}$ or $\frac{k+1}{2k+3}$	A1	1.1b
	As $\sum_{r=1}^{k+1}\frac{1}{(2r-1)(2r+1)}=\frac{(k+1)}{2(k+1)+1}$ then the general result is true for $n=k+1$ As the general result has been shown to be true for $n=1$, and true for $n=k$ implies true for $n=k+1$, so the result is true for all $n\in\mathbb{N}$	A1cso	2.4
		(6)	
			(6 marks)

Notes		
	B1	Substitutes $n=1$ into both sides of the statement to show they are equal. As a minimum expect to see $\frac{1}{1\times 3}$ and $\frac{1}{2+1}$ for the substitutions. (No need to state true for $n = 1$ for this mark.)
	M1	Assumes (general result) true for $n=k$. (Assume (true for) $n = k$ is sufficient – note that this may be recovered in their conclusion if they say e.g. if true for $n = k$ then ... etc.)
	M1	Attempts to add $(k+1)$th term to their sum of k terms. Must be adding the $(k+1)$th term but allow slips with the sum.
	dM1	Depends on previous M. Combines their two fractions over a correct common denominator for their fractions, which may be $(2k+1)^2(2k+3)$ (allow a slip in the numerator).
	A1	Correct algebraic work leading to $\frac{(k+1)}{2(k+1)+1}$ or $\frac{k+1}{2k+3}$
	A1	cso Depends on all except the B mark being scored (but must have an attempt to show the $n = 1$ case). Demonstrates the expression is the correct for $n = k + 1$ (both sides must have been seen somewhere) and gives a correct induction statement with **all** three underlined statements (or equivalents) seen at some stage during their solution (so true for $n = 1$ may be seen at the start). For demonstrating the correct expression, accept giving in the form $\frac{(k+1)}{2(k+1)+1}$, or reaching $\frac{k+1}{2k+3}$ and stating "which is the correct form with $n = k + 1$" or similar – but some indication is needed. Note: if mixed variables are used in working (r's and k's mixed up) then withhold the final A. Note: If n is used throughout instead of k allow all marks if earned.

(**Q03 8FM0/01, June 2019**)

Q15.

Question	Scheme	Marks	AOs
	Way 1: $\mathbf{f}(k+1)-\mathbf{f}(k)$ When $n=1$, $2^{n+2}+3^{2n+1}=2^3+3^3=35$ Shows the statement is true for $n=1$, allow 5(7)	B1	2.2a
	Assume true for $n=k$, so $2^{k+2}+3^{2k+1}$ is divisible by 7	M1	2.4
	$f(k+1)-f(k)=2^{k+3}+3^{2k+3}-\left(2^{k+2}+3^{2k+1}\right)$	M1	2.1
	$=2\times2^{k+2}+9\times3^{2k+1}-2^{k+2}-3^{2k+1}$ $=2^{k+2}+8\times3^{2k+1}$ $=f(k)+7\times3^{2k+1}$ or $8f(k)-7\times2^{k+2}$	A1	1.1b
	$f(k+1)=2f(k)+7\times3^{2k+1}$ or $9f(k)-7\times2^{k+2}$	A1	1.1b
	If true for $n=k$ **then true for** $n=k+1$ and as it is **true for** $n=1$ the statement is **true for all** (positive integers) n	A1	2.4
		(6)	
	Way 2: $\mathbf{f}(k+1)$ When $n=1$, $2^{n+2}+3^{2n+1}=2^3+3^3=35$ So the statement is true for $n=1$	B1	2.2a
	Assume true for $n=k$, so $2^{k+2}+3^{2k+1}$ is divisible by 7	M1	2.4
	$f(k+1)=2^{(k+1)+2}+3^{2(k+1)+1}$	M1	2.1
	$f(k+1)=2^{k+3}+3^{2k+3}=2\times2^{k+2}+9\times3^{2k+1}$ $=2\left(2^{k+2}+3^{2k+1}\right)+7\times3^{2k+1}$ $=2f(k)+7\times3^{2k+1}$ or $9f(k)-7\times2^{k+2}$	A1 A1	1.1b 1.1b
	If true for $n=k$ **then true for** $n=k+1$ and as it is **true for** $n=1$ the statement is **true for all** (positive integers) n	A1	2.4
		(6)	

	Way 3: $\mathbf{f}(k+1)-m\,\mathbf{f}(k)$ When $n=1$, $2^{n+2}+3^{2n+1}=2^3+3^3=35$ So the statement is true for $n=1$	B1	2.2a
	Assume true for $n=k$, so $2^{k+2}+3^{2k+1}$ is divisible by 7	M1	2.4
	$\mathrm{f}(k+1)-m\mathrm{f}(k)=2^{k+3}+3^{2k+3}-m\left(2^{k+2}+3^{2k+1}\right)$	M1	2.1
	$=2\times 2^{k+2}+9\times 3^{2k+1}-m\times 2^{k+2}-m\times 3^{2k+1}$ $=(2-m)2^{k+2}+9\times 3^{2k+1}-m\times 3^{2k+1}$ $=(2-m)\left(2^{k+2}+3^{2k+1}\right)+7\times 3^{2k+1}$	A1	1.1b
	$\mathrm{f}(k+1)=(2-m)\left(2^{k+2}+3^{2k+1}\right)+7\times 3^{2k+1}+m\mathrm{f}(k)$	A1	1.1b
	If true for $n=k$ then true for $n=k+1$ and as it is **true for $n=1$** the statement is **true for all** (positive integers) n	A1	2.4
		(6)	
			(6 marks)

Notes:

Way 1: $\mathbf{f}(k+1)-\mathbf{f}(k)$

B1: Shows that f(1) = 35 and concludes or shows divisible by 7. This may be seen in the final statement.

M1: Makes a statement that assumes the result is true for some value of n

M1: Attempts $f(k+1)-f(k)$

A1: Achieves a correct expression for $f(k+1)-f(k)$ in terms of $f(k)$

A1: Reaches a correct expression for $f(k+1)$ in terms of $f(k)$

A1: Correct conclusion. This mark is dependent on all previous marks, look out for scoring B1 in this statement. It is gained by conveying the ideas of **all four bold** points either at the end of their solution or as a narrative in their solution.

Way 2: $\mathbf{f}(k+1)$

B1: Shows that f(1) = 35 and concludes divisible by 7

M1: Makes a statement that assumes the result is true for some value of n

M1: Attempts $f(k+1)$

A1: Correctly obtains either $2f(k)$ **or** $7\times3^{2k+1}$ or either $9f(k)$ or $-7\times2^{k+2}$

A1: Reaches a correct expression for $f(k+1)$ in terms of $f(k)$

A1: Correct conclusion. This mark is dependent on all previous marks. It is gained by conveying the ideas of **all four bold** points either at the end of their solution or as a narrative in their solution.

Way 3: $\mathbf{f}(k+1)-m\,\mathbf{f}(k)$

B1: Shows that f(1) = 35 and concludes divisible by 7

M1: Makes a statement that assumes the result is true for some value of n

M1: Attempts $f(k+1)-mf(k)$

A1: Achieves a correct expression for $f(k+1)-mf(k)$ in terms of $f(k)$

A1: Reaches a correct expression for $f(k+1)$ in terms of $f(k)$

A1: Correct conclusion. This mark is dependent on all previous marks. It is gained by conveying the ideas of **all four bold** points either at the end of their solution or as a narrative in their solution.

(**Q08 8FM0/01, Oct 2020**)

Q16.

Question	Scheme	Marks	AOs
(i)	$n=1,\ lhs=\frac{1}{1(2)}=\frac{1}{2}\quad rhs=\frac{1}{1+1}=\frac{1}{2}$ So the result is true for $n=1$	B1	2.2a
	Assume true for $n=k$ so $\sum_{r=1}^{k}\frac{1}{r(r+1)}=\frac{k}{k+1}$	M1	2.4
	$\sum_{r=1}^{k+1}\frac{1}{r(r+1)}=\frac{k}{k+1}+\frac{1}{(k+1)(k+2)}=\frac{k(k+2)}{(k+1)(k+2)}+\frac{1}{(k+1)(k+2)}$	M1	2.1
	$=\frac{k(k+2)+1}{(k+1)(k+2)}=\frac{k^2+2k+1}{(k+1)(k+2)}=\frac{(k+1)^2}{(k+1)(k+2)}=\frac{\underline{k+1}}{\underline{k+1}+1}$ cso	A1	1.1b
	If true for $n=k$ then it has been shown **true for** $n=k+1$ and as it is **true for** $n=1$, the statement is **true for all positive integers n.**	A1	2.4
		(5)	
(ii)	**Way 1: f $(k+1)$** $n=1,\ 3^{2n+4}-2^{2n}=3^6-2^2=729-4=725$ So the result is true for $n=1$ as 725 is divisible by 5	B1	2.2a
	Assume true for $n=k$ so is $3^{2k+4}-2^{2k}$ divisible by 5	M1	2.4
	$\mathrm{f}(k+1)=3^{2(k+1)+4}-2^{2(k+1)}=3^{2k+6}-2^{2k+2}$ Look for $A\times3^{2k+4}-A\times2^{2k}+B\times2^{2k}$ or $A\times3^{2k+4}-A\times2^{2k}+B\times3^{2k+4}$ $9\times3^{2k+4}-9\times2^{2k}+5\times2^{2k}$ or $4\times3^{2k+4}-4\times2^{2k}+5\times3^{2k+4}$	M1	2.1
	$=9\mathrm{f}(k)+5\times2^{2k}$ or $=4\mathrm{f}(k)+5\times3^{2k+4}$	A1	1.1b
	If true for $n=k$ then it has been shown **true for** $n=k+1$ and as it is **true for** $n=1$, the statement is **true for all positive integers n.**	A1	2.4
		(5)	

	Way 2: $\mathbf{f}(k+1)$- $\mathbf{f}(k)$ $n=1,\ \ 3^{2n+4}-2^{2n}=3^6-2^2=729-4=725$ So the result is true for $n=1$ as 725 is divisible by 5	B1	2.2a
	Assume true for $n=k$ so $3^{2k+4}-2^{2k}$ is divisible by 5	M1	2.4
	$\mathrm{f}(k+1)-\mathrm{f}(k)=3^{2(k+1)+4}-2^{2(k+1)}-3^{2k+4}+2^{2k}$ Look for $A\times3^{2k+4}-A\times2^{2k}+B\times2^{2k}$ or $A\times3^{2k+4}-A\times2^{2k}+B\times3^{2k+4}$ $=8\times3^{2k+4}-8\times2^{2k}+5\times2^{2k}$ or $3\times3^{2k+4}-3\times2^{2k}+5\times3^{2k+4}$	M1	2.1
	$\mathrm{f}(k+1)=9\mathrm{f}(k)+5\times2^{2k}$	A1	1.1b

	or $\mathrm{f}(k+1) = 4\mathrm{f}(k) + 5 \times 3^{2k+4}$		
	If true for $n = k$ then it has been shown **true for $n = k + 1$** and as it is **true for $n = 1$**, the statement is **true for all positive integers n.**	A1	2.4
		(5)	
	Way 3: $\mathrm{f}(k) = 5M$ $n=1,\ \ 3^{2n+4} - 2^{2n} = 3^6 - 2^2 = 729 - 4 = 725$ So the result is true for $n = 1$ as 725 is divisible by 5	B1	2.2a
	Assume true for $n = k$ so $3^{2k+4} - 2^{2k} = 5M$ is divisible by 5	M1	2.4
	$\mathrm{f}(k+1) = 3^{2(k+1)+4} - 2^{2(k+1)} \left(= 3^{2k+6} - 2^{2k+2}\right)$ $\mathrm{f}(k+1) = 3^2 \times 3^{2k+4} - 2^2 \times 2^{2k} = 3^2 \times \left(5M + 2^{2k}\right) - 2^2 \times 2^{2k}$ OR $\mathrm{f}(k+1) = 3^2 \times 3^{2k+4} - 2^2 \times 2^{2k} = 3^2 \times 3^{2k+4} - 2^2 \times \left(3^{2k+4} - 5M\right)$	M1	2.1
	$\mathrm{f}(k+1) = 45M + 5 \times 2^{2k}$ OR $\mathrm{f}(k+1) = 5 \times 3^{2k+4} + 20M$ o.e.	A1	1.1b
	If true for $n = k$ then it has been shown **true for $n = k + 1$** and as it is **true for $n = 1$**, the statement is **true for all positive integers n.**	A1	2.4
		(5)	
	Way 4: $\mathrm{f}(k+1) - m\,\mathrm{f}(k)$ $n=1,\ \ 3^{2n+4} - 2^{2n} = 3^6 - 2^2 = 729 - 4 = 725$ So the result is true for $n = 1$ as 725 is divisible by 5	B1	2.2a
	Assume true for $n = k$ so $3^{2k+4} - 2^{2k}$ is divisible by 5	M1	2.4
	$\mathrm{f}(k+1) - m\mathrm{f}(k) = 3^{2k+6} - 2^{2k+2} - m\left(3^{2k+4} - 2^{2k}\right)$ $3^2 \times 3^{2k+4} - 2^2 \times 2^{2k} - m \times 3^{2k+4} + m \times 2^{2k}$ $(9-m) \times 3^{2k+4} - 4 \times 2^{2k} + m \times 2^{2k}$ $(9-m) \times \left(3^{2k+4} - 2^{2k}\right) + 5 \times 2^{2k}$	M1	2.1
	$\mathrm{f}(k+1) = (9-m) \times \mathrm{f}(k) + 5 \times 2^{2k} + m\mathrm{f}(k)$ $\mathrm{f}(k+1) = (9-m) \times \left(3^{2k+4} - 2^{2k}\right) + 5 \times 2^{2k} + m\mathrm{f}(k)$ Note if $m = 4$ leads to $5 \times \left(3^{2k+4}\right)$	A1	1.1b
	If true for $n = k$ then it has been shown **true for $n = k + 1$** and as it is **true for $n = 1$**, the statement is **true for all positive integers n.**	A1	2.4
		(5)	
			(10 marks)

Notes
(i) Must be using induction to score any marks

B1: Shows that the result holds for $n = 1$. Minimum LHS $= \frac{1}{1(2)}$ RHS $= \frac{1}{1+1}$

M1: Makes a statement that assumes the result is true for some value of n, say k

M1: Attempts to add the next term and makes progress by attempting a common denominator.

A1: Achieves a correct expression in terms of $k + 1$, with no errors seen cso. Alternatively finds $\sum_{r=1}^{k+1} \frac{1}{r(r+1)} = \frac{k+1}{k+2}$ and reaches the same expression using induction.

A1: Correct conclusion. This mark is dependent on all previous marks. It is gained by conveying the ideas of **all four bold** points either at the end of their solution or as a narrative in their solution. **Not dependent on B1 as long as attempted**

(ii)

Way 1: $\mathbf{f}(k+1)$

B1: Shows that the result holds for $n = 1$

M1: Makes a statement that assumes the result is true for some value of n, say k

M1: Attempts $f(k + 1)$ and attempts to express in terms of $f(k)$

A1: Achieves a correct expression in terms of $f(k)$

A1: Correct conclusion. This mark is dependent on all previous marks. It is gained by conveying the ideas of **all four bold** points either at the end of their solution or as a narrative in their solution. **Not dependent on B1 as long as attempted**

Way 2: $\mathbf{f}(k+1)$- $\mathbf{f}(k)$

B1: Shows that the result holds for $n = 1$

M1: Makes a statement that assumes the result is true for some value of n, say k

M1: Attempts $f(k + 1) - f(k)$ and attempts to express in terms of $f(k)$

A1: Achieves a correct expression for $f(k + 1)$ in terms of $f(k)$

A1: Correct conclusion. This mark is dependent on all previous marks. It is gained by conveying the ideas of **all four bold** points either at the end of their solution or as a narrative in their solution. **Not dependent on B1 as long as attempted**

Way 3: $\mathbf{f}(k)= 5M$

B1: Shows that the result holds for $n = 1$

M1: Makes a statement that assumes the result is true for some value of n, say k

M1: Attempts $f(k + 1)$ and writes in terms of $5M$.

A1: Achieves a correct expression for $f(k + 1)$ in terms of M and 2^{2k} or M and 3^{2k+4}

A1: Correct conclusion. This mark is dependent on all previous marks. It is gained by conveying the ideas of **all four bold** points either at the end of their solution or as a narrative in their solution. **Not dependent on B1 as long as attempted**

Way 4: $\mathbf{f}(k+1)$- $m\mathbf{f}(k)$

B1: Shows that the result holds for $n = 1$

M1: Makes a statement that assumes the result is true for some value of n, say k

M1: Attempts $f(k + 1) - m\,f(k)$ and attempts to express in terms of $f(k)$

A1: Achieves a correct expression for $f(k + 1)$ in terms of $f(k)$

A1: Correct conclusion. This mark is dependent on all previous marks. It is gained by conveying the ideas of **all four bold** points either at the end of their solution or as a narrative in their solution. **Not dependent on B1 as long as attempted**

Note conclusion may be in terms of $f(1), f(k), f(k+1)$

(**Q07 8FM0/01, June 2024**)

Q17.

Question	Scheme	Marks	AOs
(a)	$n=1,\ \text{lhs}=1(2)(3)=6,\quad \text{rhs}=\frac{1}{2}(1)(2)^2(3)=6$ (true for $n=1$)	B1	2.2a
	Assume true for $n=k$ so $\sum_{r=1}^{k} r(r+1)(2r+1)=\frac{1}{2}k(k+1)^2(k+2)$	M1	2.4
	$\sum_{r=1}^{k+1} r(r+1)(2r+1)=\frac{1}{2}k(k+1)^2(k+2)+(k+1)(k+2)(2k+3)$	M1	2.1
	$=\frac{1}{2}(k+1)(k+2)\left[k(k+1)+2(2k+3)\right]$	dM1	1.1b
	$=\frac{1}{2}(k+1)(k+2)\left[k^2+5k+6\right]=\frac{1}{2}(k+1)(k+2)(k+2)(k+3)$ Shows that $=\frac{1}{2}(\underline{k+1})(\underline{k+1}+1)^2(\underline{k+1}+2)$ Alternatively shows that $\sum_{r=1}^{k+1} r(r+1)(2r+1)=\frac{1}{2}(k+1)(k+1+1)^2(k+1+2)$ $=\frac{1}{2}(k+1)(k+2)^2(k+3)$ Compares with their summation and concludes true for $n=k+1$, may be seen in the conclusion.	A1	1.1b
	If the statement is **true for** $n = k$ **then** it has been shown **true for** $n = k + 1$ and as it is **true for** $n = 1$, the statement is **true** for all **positive integers** n.	A1	2.4
		(6)	
(b)	$\sum_{r=n}^{2n} r(r+1)(2r+1)=\frac{1}{2}(2n)(2n+1)^2(2n+2)-\frac{1}{2}(n-1)n^2(n+1)$	M1	3.1a
	$=\frac{1}{2}n(n+1)\left[4(2n+1)^2-n(n-1)\right]$	M1	1.1b
	$=\frac{1}{2}n(n+1)(15n^2+17n+4)$ $=\frac{1}{2}n(n+1)(3n+1)(5n+4)$	A1	1.1b
		(3)	
			(9 marks)

Notes
(a) Note ePen B1 M1 M1 A1 A1 A1 B1: Substitutes $n = 1$ into both sides to show that they are both equal to 6. (There is no need to state true for $n = 1$ for this mark) M1: Makes a statement that assumes the result is true for some value of n, say k M1: Adds the $(k + 1)$th term to the assumed result dM1: Dependent on previous M, factorises out $\frac{1}{2}(k+1)(k+2)$ A1: Reaches a correct the required expression no errors and shows that this is the correct sum for $n = k + 1$ A1: Depends on all except **B** mark being scored (must have been some attempt to show true for $n = 1$). Correct conclusion conveying all the points in bold. (b) M1: Realises that $\sum_{r=1}^{2n} r(r+1)(2r+1) - \sum_{r=1}^{n-1} r(r+1)(2r+1)$ is required and uses the result from part (a) to obtain the required sum in terms of n M1: Attempts to factorise by $\frac{1}{2}n(n+1)$ A1: Correct expression or correct values

(**Q08 8FM0/01, Oct 2021**)

Q18.

Question	Scheme	Marks	AOs
	$n=1$ LHS $=1$ RHS $=\frac{1}{4}(1)^2(2)^2=1$	B1	2.2a
	Assume true for $n=k$ or $\sum_{r=1}^{k} r^3=\frac{1}{4}k^2(k+1)^2$	M1	2.5
	$\sum_{r=1}^{k+1} r^3=\frac{1}{4}k^2(k+1)^2+(k+1)^3$ Or $(k+1)^3=\sum_{r=1}^{k+1} r^3-\frac{1}{4}k^2(k+1)^2$ or $(k+1)^3=\frac{1}{4}(k+1)^2(k+1)^2-\frac{1}{4}k^2(k+1)^2$	M1	2.1
	$\frac{1}{4}(k+1)^2\left[k^2+4(k+1)\right]$ Or Multiples out $\frac{1}{4}k^2(k+1)^2+(k+1)^3$ and $\frac{1}{4}(k+1)^2(k+2)^2$ to form a quartic **for each** Or Multiplies out or factorises out $\frac{1}{4}(k+1)^2(k+1)^2-\frac{1}{4}k^2(k+1)^2$	M1	1.1b

	$\frac{1}{4}(k+1)^2\left[k^2+4k+4\right]=\frac{1}{4}(k+1)^2(k+2)^2$ leading to $\frac{1}{4}(k+1)^2(\{k+1\}+1)^2$ Or $k^4+6k^3+13k^2+12k+4=k^4+6k^3+13k^2+12k+4$ And draws conclusion true Or Shows $\frac{1}{4}(k+1)^2(k+1)^2-\frac{1}{4}k^2(k+1)^2=(k+1)^3$ by either multiplying out both sides to reach the same cubic, and draws a conclusion or by factorisation	A1	1.1b
	If true for $n=k$ then true for $n=k+1$, and as also true for $n=1$, so the result is true for all n	A1	2.4
		(6)	
			(6 marks)

Note
B1: Shows the statement is true for $n=1$. Minimum requirement $1=\frac{1}{4}(1)(4)$ **M1**: Makes the inductive assumption, **assume** true $n=k$. This may appear in the conclusion. **M1**: A correct statement for $n=k+1$ **M1**: Factorises out $(k+1)^2$ or multiples out $\frac{1}{4}k^2(k+1)^2+(k+1)^3$ and $\frac{1}{4}(k+1)^2(k+2)^2$ to form a quartic for each **A1**: Achieves $\frac{1}{4}(k+1)^2(\{k+1\}+1)^2$ with no omissions or errors. Shows both are sides are $k^4+6k^3+13k^2+12k+4$ no omissions or errors. and draws a minimal conclusion. Shows $\frac{1}{4}(k+1)^2(k+1)^2-\frac{1}{4}k^2(k+1)^2=(k+1)^3$ may expand both sides and draw a minimal conclusion or factorises, no omissions or errors. **Note**: they may state aiming for $\frac{1}{4}(k+1)^2(k+2)^2$ and then achieves this which is fine for A1 **A1**: Makes appropriate concluding sentence covering the points indicated in scheme. Dependent on all previous marks except B1. The statement true for $n=1$ could appear at the start of the solution

(**Q06 8FM0/01, June 2025**)

Q19.

Question	Scheme	Marks	AOs
(i)	$n=1, \begin{pmatrix}5 & -8\\ 2 & -3\end{pmatrix}^1 = \begin{pmatrix}5 & -8\\ 2 & -3\end{pmatrix}, \quad \begin{pmatrix}4\times1+1 & -8(1)\\ 2\times1 & 1-4(1)\end{pmatrix} = \begin{pmatrix}5 & -8\\ 2 & -3\end{pmatrix}$ So the result is true for $n = 1$	B1	2.2a
	Assume true for $n = k$ so $\begin{pmatrix}5 & -8\\ 2 & -3\end{pmatrix}^k = \begin{pmatrix}4k+1 & -8k\\ 2k & 1-4k\end{pmatrix}$	M1	2.4
	$\begin{pmatrix}5 & -8\\ 2 & -3\end{pmatrix}^{k+1} = \begin{pmatrix}4k+1 & -8k\\ 2k & 1-4k\end{pmatrix}\begin{pmatrix}5 & -8\\ 2 & -3\end{pmatrix}$ or $\begin{pmatrix}5 & -8\\ 2 & -3\end{pmatrix}^{k+1} = \begin{pmatrix}5 & -8\\ 2 & -3\end{pmatrix}\begin{pmatrix}4k+1 & -8k\\ 2k & 1-4k\end{pmatrix}$	M1	1.1b
	$\begin{pmatrix}4k+1 & -8k\\ 2k & 1-4k\end{pmatrix}\begin{pmatrix}5 & -8\\ 2 & -3\end{pmatrix} = \begin{pmatrix}5(4k+1)-16k & -8(4k+1)+24k\\ 10k+2(1-4k) & -16k-3(1-4k)\end{pmatrix}$ or $\begin{pmatrix}5 & -8\\ 2 & -3\end{pmatrix}\begin{pmatrix}4k+1 & -8k\\ 2k & 1-4k\end{pmatrix} = \begin{pmatrix}5(4k+1)-16k & -40k-8(1-4k)\\ 2(1+4k)-6k & -16k-3(1-4k)\end{pmatrix}$	A1	1.1b
	$= \begin{pmatrix}4(k+1)+1 & -8(k+1)\\ 2(k+1) & 1-4(k+1)\end{pmatrix}$	A1	2.1
	<u>If true for $n = k$</u> then <u>true for $n = k + 1$</u>, <u>true for $n = 1$</u> so <u>true for all (positive integers) n</u> (Allow "for all values")	A1	2.4
		(6)	

(ii) Way 1	$f(k+1) - f(k)$		
	When $n = 1$, $4^{n+1} + 5^{2n-1} = 16 + 5 = 21$ so the statement is true for $n = 1$	B1	2.2a
	Assume true for $n = k$ so $4^{k+1} + 5^{2k-1}$ is divisible by 21	M1	2.4
	$f(k+1) - f(k) = 4^{k+2} + 5^{2k+1} - 4^{k+1} - 5^{2k-1}$	M1	2.1
	$= 4\times4^{k+1} + 25\times5^{2k-1} - 4^{k+1} - 5^{2k-1}$		
	$= 3f(k) + 21\times5^{2k-1}$ or e.g. $= 24f(k) - 21\times4^{k+1}$	A1	1.1b
	$f(k+1) = 4f(k) + 21\times5^{2k-1}$ or e.g. $f(k+1) = 25f(k) - 21\times4^{k+1}$	A1	1.1b
	<u>If true for $n = k$</u> then <u>true for $n = k + 1$</u>, <u>true for $n = 1$</u> so <u>true for all (positive integers) n</u> (Allow "for all values")	A1	2.4
		(6)	

(ii) Way 2	f(k + 1)		
	When $n = 1$, $4^{n+1} + 5^{2n-1} = 16 + 5 = 21$ so the statement is true for $n = 1$	B1	2.2a
	Assume true for $n = k$ so $4^{k+1} + 5^{2k-1}$ is divisible by 21	M1	2.4
	$f(k+1) = 4^{k+1+1} + 5^{2(k+1)-1}$	M1	2.1
	$f(k+1) = 4 \times 4^{k+1} + 5^{2k+1} = 4 \times 4^{k+1} + 4 \times 5^{2k-1} + 25 \times 5^{2k-1} - 4 \times 5^{2k-1}$ $f(k+1) = 4f(k) + 21 \times 5^{2k-1}$	A1 A1	1.1b 1.1b
	<u>If true for $n = k$</u> then <u>true for $n = k + 1$</u>, <u>true for $n = 1$</u> so <u>true for all (positive integers) n</u> (Allow "for all values")	A1	2.4
		(6)	
(ii) Way 3	f(k + 1) − mf(k)		
	When $n = 1$, $4^{n+1} + 5^{2n-1} = 16 + 5 = 21$ so the statement is true for $n = 1$	B1	2.2a
	Assume true for $n = k$ so $4^{k+1} + 5^{2k-1}$ is divisible by 21	M1	2.4
	$f(k+1) - mf(k) = 4^{k+2} + 5^{2k+1} - m(4^{k+1} + 5^{2k-1})$	M1	2.1
	$= (4-m)4^{k+1} + 5^{2k+1} - m \times 5^{2k-1}$ $= (4-m)(4^{k+1} + 5^{2k-1}) + 21 \times 5^{2k-1}$	A1	1.1b
	$= (4-m)(4^{k+1} + 5^{2k-1}) + 21 \times 5^{2k-1} + mf(k)$	A1	1.1b
	<u>If true for $n = k$</u> then <u>true for $n = k + 1$</u>, <u>true for $n = 1$</u> so <u>true for all (positive integers) n</u> (Allow "for all values")	A1	2.4
		(6)	
(ii) Way 4	f(k) = 21M		
	When $n = 1$, $4^{n+1} + 5^{2n-1} = 16 + 5 = 21$ so the statement is true for $n = 1$	B1	2.2a
	Assume true for $n = k$ so $4^{k+1} + 5^{2k-1} = 21M$	M1	2.4
	$f(k+1) = 4^{k+1+1} + 5^{2(k+1)-1}$	M1	2.1
	$f(k+1) = 4 \times 4^{k+1} + 5^{2k+1} = 4(21M - 5^{2k-1}) + 5^{2k+1}$ $f(k+1) = 84M + 21 \times 5^{2k-1}$	A1 A1	1.1b 1.1b
	<u>If true for $n = k$</u> then <u>true for $n = k + 1$</u>, <u>true for $n = 1$</u> so <u>true for all (positive integers) n</u> (Allow "for all values")	A1	2.4
		(6)	
			(12 marks)

Notes
(i) B1: Shows that the result holds for $n = 1$. Must see **substitution** into the rhs. The minimum would be: $\begin{pmatrix} 4+1 & -8 \\ 2 & 1-4 \end{pmatrix} = \begin{pmatrix} 5 & -8 \\ 2 & -3 \end{pmatrix}$. M1: Makes a statement that assumes the result is true for some value of n (Assume (true for) $n = k$ is sufficient – note that this may be recovered in their conclusion if they say e.g. if true for $n = k$ then … etc.) M1: Sets up a correct multiplication statement either way round A1: Achieves a correct un-simplified matrix A1: Reaches a correct simplified matrix with no errors **and the correct un-simplified matrix seen previously.** Note that the simplified result may be proved by equivalence. A1: Correct conclusion. This mark is dependent on all previous marks apart from the B mark. It is gained by conveying the ideas of **all** four underlined points **either** at the end of their solution **or** as a narrative in their solution. (ii) **Way 1** B1: Shows that f(1) = 21 M1: Makes a statement that assumes the result is true for some value of n (Assume (true for) $n = k$ is sufficient – note that this may be recovered in their conclusion if they say e.g. if true for $n = k$ then … etc.) M1: Attempts $f(k + 1) – f(k)$ or equivalent work A1: Achieves a correct expression for $f(k + 1) – f(k)$ in terms of $f(k)$ A1: Reaches a correct expression for $f(k + 1)$ in terms of $f(k)$ A1: Correct conclusion. This mark is dependent on all previous marks apart from the B mark. It is gained by conveying the ideas of **all** four underlined points **either** at the end of their solution **or** as a narrative in their solution.

Way 2

B1: Shows that f(1) = 21

M1: Makes a statement that assumes the result is true for some value of n (Assume (true for) $n = k$ is sufficient – note that this may be recovered in their conclusion if they say e.g. if true for $n = k$ then … etc.)

M1: Attempts $f(k + 1)$

A1: Correctly obtains $4f(k)$ **or** $21 \times 5^{2k-1}$

A1: Reaches a correct expression for $f(k + 1)$ in terms of $f(k)$

A1: Correct conclusion. This mark is dependent on all previous marks apart from the B mark. It is gained by conveying the ideas of **all** four underlined points **either** at the end of their solution **or** as a narrative in their solution.

Way 3

B1: Shows that f(1) = 21

M1: Makes a statement that assumes the result is true for some value of n (Assume (true for) $n = k$ is sufficient – note that this may be recovered in their conclusion if they say e.g. if true for $n = k$ then … etc.)

M1: Attempts $f(k + 1) - mf(k)$

A1: Achieves a correct expression for $f(k + 1) - mf(k)$ in terms of $f(k)$

A1: Reaches a correct expression for $f(k + 1)$ in terms of $f(k)$

A1: Correct conclusion. This mark is dependent on all previous marks apart from the B mark. It is gained by conveying the ideas of **all** four underlined points **either** at the end of their solution **or** as a narrative in their solution.

Way 4

B1: Shows that f(1) = 21

M1: Makes a statement that assumes the result is true for some value of n (Assume (true for) $n = k$ is sufficient – note that this may be recovered in their conclusion if they say e.g. if true for $n = k$ then … etc.)

M1: Attempts $f(k + 1)$

A1: Correctly obtains $84M$ **or** $21 \times 5^{2k-1}$

A1: Reaches a correct expression for $f(k + 1)$ in terms of M and 5^{2k-1}

A1: Correct conclusion. This mark is dependent on all previous marks apart from the B mark. It is gained by conveying the ideas of **all** four underlined points **either** at the end of their solution **or** as a narrative in their solution.

(**Q08 8FM0/01, June 2018**)

Q20.

Question Number	Scheme	Notes	Marks
(a)	$n=1, \text{LHS}=1^3=1, \text{RHS}=\frac{1}{4}\times1^2\times2^2=1$	Shows **both** LHS = 1 **and** RHS = 1	B1
	Assume true for n = k		
	When n = k + 1 $\sum_{r=1}^{k+1} r^3 = \frac{1}{4}k^2(k+1)^2+(k+1)^3$	Adds $(k+1)^3$ to the given result	M1
	$=\frac{1}{4}(k+1)^2[k^2+4(k+1)]$	Attempt to factorise out $\frac{1}{4}(k+1)^2$	dM1
		Correct expression with $\frac{1}{4}(k+1)^2$ factorised out.	A1
	$=\frac{1}{4}(k+1)^2(k+2)^2$ Must see 4 things: true for n = 1, assumption true for n = k, said true for n = k + 1 and therefore true for all n	Fully complete proof with no errors and comment. All the previous marks must have been scored.	A1cso
	See extra notes for alternative approaches		(5)
(b)	$\sum(r^3-2)=\sum r^3-\sum 2$	Attempt two sums	M1
	$\sum r^3-\sum 2n$ is **M0**		
	$=\frac{1}{4}n^2(n+1)^2-2n$	Correct expression	A1
	$=\frac{n}{4}(n^3+2n^2+n-8)$ *	Completion to printed answer with no errors seen.	A1
			(3)
(c)	$\sum_{r=20}^{r=50}(r^3-2)=\frac{50}{4}\times130042-\frac{19}{4}\times7592$ $(=1625525-36062)$	Attempt $S_{50}-S_{20}$ or $S_{50}-S_{19}$ and substitutes into a correct expression at least once.	M1
		Correct numerical expression (unsimplified)	A1
	= 1 589 463	cao	A1
			(3)
(c) Way 2	$\sum_{r=20}^{r=50}(r^3-2)=\sum_{r=20}^{r=50}r^3-\sum_{r=20}^{r=50}(2)=\frac{50^2}{4}\times51^2-\frac{19^2}{4}\times20^2-2\times31$	M1 for ($S_{50}-S_{20}$ or $S_{50}-S_{19}$ for cubes) – (2x30 or 2x31)	**Total 11**
		A1 correct numerical expression	
	=1 589 463	A1	

(**Q05 6667/01, Jan 2012**)

Online Maths Teaching

Q21.

Question Number	Scheme		Marks
(a)	$\sum_{r=1}^{n} r(2r-1) = \frac{1}{6}n(n+1)(4n-1)$		
	$n=1;\quad \text{LHS} = \sum_{r=1}^{1} r(2r-1) = 1$ $\text{RHS} = \frac{1}{6}(1)(2)(3) = 1$ As LHS = RHS, the summation formula is true for $n = 1$.	$\frac{1}{6}(1)(2)(3) = 1$ seen	B1
	Assume that the summation formula is true for $n = k$.		
	ie. $\sum_{r=1}^{k} r(2r-1) = \frac{1}{6}k(k+1)(4k-1)$.		
	With $n = k+1$ terms the summation formula becomes:		
	$\sum_{r=1}^{k+1} r(2r-1) = \underline{\frac{1}{6}k(k+1)(4k-1) + (k+1)(2(k+1)-1)}$	$S_{k+1} = S_k + u_{k+1}$ with $S_k = \frac{1}{6}k(k+1)(4k-1)$.	M1
	$= \frac{1}{6}k(k+1)(4k-1) + (k+1)(2k+1)$		
	$= \frac{1}{6}(k+1)\big(k(4k-1) + 6(2k+1)\big)$	Factorise by $\frac{1}{6}(k+1)$	dM1
	$= \frac{1}{6}(k+1)\big(4k^2 + 11k + 6\big)$	$\big(4k^2 + 11k + 6\big)$ or equivalent quadratic seen	A1
	$= \frac{1}{6}(k+1)(k+2)(4k+3)$		
	$= \frac{1}{6}(k+1)(k+1+1)(4(k+1)-1)$	Correct completion to S_{k+1} in terms of $k+1$ dependent on both Ms.	dM1
	If the summation formula is <u>true for $n = k$</u>, then it is shown to be <u>true for $n = k+1$</u>. As the result is <u>true for $n = 1$</u>, it is now also <u>true for all n</u> and $n \in \mathbb{Z}^+$ by mathematical induction.	Conclusion with all 4 underlined elements that can be seen anywhere in the solution	A1 cso
			[6]

Question Number	Scheme		Marks
(b)	$\sum_{r=n+1}^{3n} r(2r-1) = S_{3n} - S_n$		
	$= \frac{1}{6}.3n(3n+1)(12n-1) - \frac{1}{6}n(n+1)(4n-1)$	Use of $S_{3n} - S_n$ or $S_{3n} - S_{n+1}$ with the result from (a) used at least once.	M1
		Correct un-simplified expression.	A1
	$= \frac{1}{6}n\{3(3n+1)(12n-1) - (n+1)(4n-1)\}$		
	$= \frac{1}{6}n\{3(36n^2+9n-1) - (4n^2+3n-1)\}$	Factorises out $\frac{1}{6}n$ or $\frac{1}{3}n$ and an attempt to open up the brackets.	dM1
	$= \frac{1}{6}n\{108n^2+27n-3-4n^2-3n+1\}$		
	$= \frac{1}{6}n\{104n^2+24n-2\}$		
	$= \frac{1}{3}n(52n^2+12n-1)$	$= \frac{1}{3}n(52n^2+12n-1)$	A1
	$\{a=52, b=12, c=-1\}$		[4]
			10

(**Q07 6667/01/R, June 2013**)

Q22.

Question Number	Scheme	Marks
	(a) $\sum_{r=1}^{1} r^3 = 1^3 = 1$ and $\frac{1}{4}\times 1^2 \times 2^2 = 1$	B1
	Assume true for $n = k$:	
	$\sum_{r=1}^{k+1} r^3 = \frac{1}{4}k^2(k+1)^2 + (k+1)^3$	B1
	$\frac{1}{4}(k+1)^2\left[k^2+4(k+1)\right] = \frac{1}{4}(k+1)^2(k+2)^2$	M1 A1
	$\therefore$ True for $n = k+1$ if true for $n = k$. True for $n = 1$, $\therefore$ true for all n.	A1cso (5)
	(b) $\sum r^3 + 3\sum r + \sum 2 = \frac{1}{4}n^2(n+1)^2 + 3\left(\frac{1}{2}n(n+1)\right), +2n$	B1, B1
	$= \frac{1}{4}n\left[n(n+1)^2 + 6(n+1) + 8\right]$	M1
	$= \frac{1}{4}n\left[n^3 + 2n^2 + 7n + 14\right] = \frac{1}{4}n(n+2)(n^2+7)$ (*)	A1 A1cso (5)
	(c) $\sum_{15}^{25} = \sum_{1}^{25} - \sum_{1}^{14}$ with attempt to sub in answer to part (b)	M1
	$= \frac{1}{4}(25\times 27\times 632) - \frac{1}{4}(14\times 16\times 203) = 106650 - 11368 = 95282$	A1 (2) [12]
	Notes (a) Correct method to identify $(k+1)^2$ as a factor award M1 $\frac{1}{4}(k+1)^2(k+2)^2$ award first A1 All three elements stated somewhere in the solution award final A1 (b) Attempt to factorise by n for M1 $\frac{1}{4}$ and $n^3+2n^2+7n+14$ for first A1 (c) no working 0/2	

(**Q06 6667/01, Jan 2010**)

Q23.

Question Number	Scheme	Marks
	(a) If $n=1$, $\sum_{r=1}^{n} r^2 = 1$ and $\frac{1}{6}n(n+1)(2n+1) = \frac{1}{6}\times 1\times 2\times 3 = 1$, so true for $n = 1$.	B1
	Assume result true for $n = k$	M1
	$\sum_{r=1}^{k+1} r^2 = \frac{1}{6}k(k+1)(2k+1)+(k+1)^2$	M1
	$=\frac{1}{6}(k+1)(2k^2+7k+6)$ or $=\frac{1}{6}(k+2)(2k^2+5k+3)$ or $=\frac{1}{6}(2k+3)(k^2+3k+2)$	A1
	$=\frac{1}{6}(k+1)(k+2)(2k+3)$ $=\frac{1}{6}(k+1)(\{k+1\}+1)(2\{k+1\}+1)$ or equivalent	dM1
	True for $n = k+1$ if true for $n = k$, (and true for $n = 1$) so true by induction for all n.	A1cso (6)
	Alternative for (a) After first three marks B M M1 as earlier :	B1M1M1
	May state RHS $=\frac{1}{6}(k+1)(\{k+1\}+1)(2\{k+1\}+1) = \frac{1}{6}(k+1)(k+2)(2k+3)$ for third M1	dM1
	Expands to $\frac{1}{6}(k+1)(2k^2+7k+6)$ and show equal to $\sum_{r=1}^{k+1} r^2 = \frac{1}{6}k(k+1)(2k+1)+(k+1)^2$ for A1	A1
	So true for $n = k+1$ if true for $n = k$, and true for $n = 1$, so true by induction for all n.	A1cso (6)
	(b) $\sum_{r=1}^{n}(r^2+5r+6) = \sum_{r=1}^{n} r^2 + 5\sum_{r=1}^{n} r + \quad (\sum_{r=1}^{n} 6)$	M1
	$\frac{1}{6}n(n+1)(2n+1)+\frac{5}{2}n(n+1), \quad +6n$	A1, B1
	$=\frac{1}{6}n[(n+1)(2n+1)+15(n+1)+36]$	M1
	$=\frac{1}{6}n[2n^2+18n+52] = \frac{1}{3}n(n^2+9n+26)$ or $a = 9$, $b = 26$	A1 (5)
	(c) $\sum_{r=n+1}^{2n}(r+2)(r+3) = \frac{1}{3}2n(4n^2+18n+26) - \frac{1}{3}n(n^2+9n+26)$	M1 A1ft
	$\frac{1}{3}n(8n^2+36n+52-n^2-9n-26) = \frac{1}{3}n(7n^2+27n+26)$ (*)	A1cso (3) **14 marks**
	Notes: (a) B1: Checks $n = 1$ on both sides and states true for $n = 1$ here or in conclusion M1: **Assumes true** for $n = k$ (should use one of these **two** words) M1: Adds (k+1)th term to sum of k terms A1: Correct work to support proof M1: Deduces $\frac{1}{6}n(n+1)(2n+1)$ with $n = k+1$ A1: Makes induction statement. Statement true for $n = 1$ here could contribute to B1 mark earlier	

	Question Notes continued: (b) M1: Expands and splits (but allow 6 rather than sigma 6 for this mark) A1: first two terms correct B1: for $6n$ M1: Take out factor $n/6$ or $n/3$ correctly – no errors factorising A1: for correct factorised cubic or for identifying a and b (c) M1: Try to use $\sum_{1}^{2n}(r+2)(r+3)-\sum_{1}^{n}(r+2)(r+3)$ with previous result used **at least once** A1ft Two correct expressions for their a and b values A1: Completely correct work to printed answer

(**Q07 6667/01, June 2010**)

Q24.

Question	Scheme	Marks	AOs
(a)	$n=1,\ \sum_{r=1}^{1} r^2 = 1$ and $\frac{1}{6}n(n+1)(2n+1) = \frac{1}{6}(1)(2)(3) = 1$	B1	2.2a
	Assume general statement is true for $n = k$. So assume $\sum_{r=1}^{k} r^2 = \frac{1}{6}k(k+1)(2k+1)$ is true.	M1	2.4
	$\sum_{r=1}^{k+1} r^2 = \frac{1}{6}k(k+1)(2k+1) + (k+1)^2$	M1	2.1
	$= \frac{1}{6}(k+1)(2k^2+7k+6)$	A1	1.1b
	$= \frac{1}{6}(k+1)(k+2)(2k+3) = \frac{1}{6}(k+1)(\{k+1\}+1)(2\{k+1\}+1)$	A1	1.1b
	Then the general result is true for $n = k+1$. As the general result has been shown to be true for $n = 1$, then the general result is true for all $n \in \mathbb{Z}^+$.	A1	2.4
		(6)	
(b)	$\sum_{r=1}^{n} r(r+6)(r-6) = \sum_{r=1}^{n} (r^3 - 36r)$		
	$= \frac{1}{4}n^2(n+1)^2 - \frac{36}{2}n(n+1)$	M1 A1	2.1 1.1b
	$= \frac{1}{4}n(n+1)\left[n(n+1) - 72\right]$	M1	1.1b
	$= \frac{1}{4}n(n+1)(n-8)(n+9)$ * **cso**	A1*	1.1b
		(4)	
(c)	$\frac{1}{4}n(n+1)(n-8)(n+9) = \frac{17}{6}n(n+1)(2n+1)$	M1	1.1b
	$\frac{1}{4}(n-8)(n+9) = \frac{17}{6}(2n+1)$	M1	1.1b
	$3n^2 - 65n - 250 = 0$	A1	1.1b
	$(3n+10)(n-25) = 0$	M1	1.1b
	(As n must be a positive integer,) $n = 25$	A1	2.3
		(5)	
		(15 marks)	

	Question Notes	
(a)	B1	Checks $n=1$ works for both sides of the general statement.
	M1	Assumes (general result) true for $n=k$.
	M1	Attempts to add $(k+1)$th term to the sum of k terms.
	A1	Correct algebraic work leading to either $\frac{1}{6}(k+1)(2k^2+7k+6)$ or $\frac{1}{6}(k+2)(2k^2+5k+3)$ or $\frac{1}{6}(2k+3)(k^2+3k+2)$
	A1	Correct algebraic work leading to $\frac{1}{6}(k+1)(\{k+1\}+1)(2\{k+1\}+1)$
	A1	cso leading to a correct induction statement conveying **all** three underlined points.
(b)	M1	Substitutes at least one of the standard formulae into their expanded expression.
	A1	Correct expression.
	M1	Depends on previous M mark. Attempt to factorise at least $n(n+1)$ having used both standard formulae correctly.
	A1*	Obtains $\frac{1}{4}n(n+1)(n-8)(n+9)$ by cso.
(c)	M1	Sets their part (a) answer equal to $\frac{17}{6}n(n+1)(2n+1)$
	M1	Cancels out $n(n+1)$ from both sides of their equation.
	A1	$3n^2-65n-250=0$
	M1	A valid method for solving a 3 term quadratic equation.
	A1	**Only one solution** of $n=25$

(**Q06 8FM0/01, Specimen papers**)