Questions

Q1.

Show, using the formulae for $\sum_{r=1}^{n} r$ and $\sum_{r=1}^{n} r^2$, that

$$\sum_{r=1}^{n} 3(2r - 1)^2 = n(2n + 1)(2n - 1)$$

for all positive integers n.

$$\sum_{r=1}^{n} 3(4r^2 - 4r + 1)$$

$$= \sum_{r=1}^{n} 12r^2 - 12r + 3$$

$$= 12\sum_{r=1}^{n} r^2 - 12\sum_{r=1}^{n} r + 3\sum_{r=1}^{n} 1$$

$$= 12\left(\frac{1}{6}n(n+1)(2n+1)\right) - 12\left(\frac{n}{2}(n+1)\right) + 3n$$

$$= 2n(n+1)(2n+1) - 6n(n+1) + 3n$$

$$= n\left[2(n+1)(2n+1) - 6(n+1) + 3\right]$$

$$= n\left[4n^2 + 6n + 2 - 6n - 6 + 3\right]$$

$$= n\left[4n^2 - 1\right]$$

$$= n(2n+1)(2n-1)$$

(5)

(Total 5 marks)

Q2.

(a) Show, using the formulae for $\sum r$ and $\sum r^2$, that

$$\sum_{r=1}^{n}(6r^2 + 4r - 1) = n(n+2)(2n+1)$$

$$6\sum_{r=1}^{n} r^2 + 4\sum_{r=1}^{n} r - \sum_{r=1}^{n} 1$$

$$= 6\left(\tfrac{1}{6}n(n+1)(2n+1)\right) + 4\left(\tfrac{n}{2}(n+1)\right) - n$$

$$= n(n+1)(2n+1) + 2n(n+1) - n$$

$$= n\left[(n+1)(2n+1) + 2(n+1) - 1\right]$$

$$= n\left[2n^2 + 3n + 1 + 2n + 2 - 1\right]$$

$$= n\left[2n^2 + 5n + 2\right]$$

$$= n(2n+1)(n+2)$$

(5)

(b) Hence, or otherwise, find the value of

$$\sum_{r=11}^{n=20}(6r^2 + 4r - 1)$$

$$\sum_{r=1}^{20}(6r^2 + 4r - 1) - \sum_{r=1}^{10}(6r^2 + 4r - 1)$$

$$20(2(20)+1)(20+2) - 10(2(10)+1)(10+2)$$

(2)

(Total 7 marks)

$$= 20 \times 41 \times 22 - 10 \times 21 \times 12$$

$$= 15520$$

Q3.

(a) Use the standard results for $\sum_{r=1}^{n} r$ and $\sum_{r=1}^{n} r^3$ to show that

$$\sum_{r=1}^{n} r(r^2 - 3) = \frac{1}{4}n(n+1)(n+3)(n-2)$$

$$\sum_{r=1}^{n} (r^3 - 3r)$$

$$= \sum_{r=1}^{n} r^3 - 3\sum_{r=1}^{n}$$

$$= \frac{1}{4}n^2(n+1)^2 - 3\left(\frac{n}{2}(n+1)\right)$$

$$= \frac{1}{4}n^2(n+1)^2 - \frac{3n}{2}(n+1)$$

$$= \frac{1}{4}n(n+1)\left[n(n+1) - 6\right]$$

$$= \frac{1}{4}n(n+1)(n^2+n-6)$$

$$= \frac{1}{4}n(n+1)(n+3)(n-2)$$

(5)

(b) Calculate the value of

$$\sum_{r=10}^{50} r(r^2 - 3)$$

$$\sum_{r=1}^{50} r(r^2-3) - \sum_{r=1}^{9} r(r^2-3)$$

$$= \frac{1}{4}(50)(50+1)(50+3)(50-2) - \frac{1}{4}(9)(9+1)(9+3)(9-2)$$

$$= \frac{1}{4}(50)(51)(53)(48) - \frac{1}{4}(9)(10)(12)(7)$$

(3)

(Total 8 marks)

$$= 1619910$$

Q4.

(a) Using the formulae for $\sum_{r=1}^{n} r$ and $\sum_{r=1}^{n} r^2$, show that

$$\sum_{r=1}^{n}(r+1)(r+4)=\frac{n}{3}(n+4)(n+5)$$

for all positive integers n.

$$\sum_{r=1}^{n} r^2+5r+4$$

$$=\sum_{r=1}^{n} r^2+5\sum_{r=1}^{n} r+4\sum_{r=1}^{n} 1$$

$$=\frac{1}{6}n(n+1)(2n+1)+5\left[\frac{n}{2}(n+1)\right]+4n$$

$$=\frac{n}{6}\left[(n+1)(2n+1)+15(n+1)+24\right]$$

$$=\frac{n}{6}\left[2n^2+3n+1+15n+39\right]$$

$$=\frac{n}{6}\left[2n^2+18n+40\right]$$

$$=\frac{n}{3}\left[n^2+9n+20\right]=\frac{n}{3}(n+4)(n+5)$$

(5)

(b) Hence show that

$$\sum_{r=n+1}^{2n}(r+1)(r+4)=\frac{n}{3}(n+1)(an+b)$$

where a and b are integers to be found.

$$\sum_{r=1}^{2n}(r+1)(r+4)-\sum_{r=1}^{n}(r+1)(r+4)$$

$$=\frac{(2n)}{3}(2n+4)(2n+5)-\frac{n}{3}(n+4)(n+5)$$

$$=\frac{n}{3}\left[2(4n^2+18n+20)-(n^2+9n+20)\right]$$

(3)

(Total for question = 8 marks)

$$=\frac{n}{3}\left[7n^2+27n+20\right]$$

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$$=\frac{n}{3}(n+1)(7n+20)$$

Q5.

(a) Use the results for $\sum_{r=1}^{n} r$ and $\sum_{r=1}^{n} r^2$ to show that

$$\sum_{r=1}^{n}(2r-1)^2 = \frac{1}{3}n(2n+1)(2n-1)$$

for all positive integers *n*.

$$\sum_{r=1}^{n}(4r^2-4r+1)$$

$$= 4\sum_{r=1}^{n} r^2 - 4\sum_{r=1}^{n} r + \sum_{r=1}^{n} 1$$

$$= 4\left(\frac{1}{6}n(n+1)(2n+1)\right) - 4\left(\frac{n}{2}(n+1)\right) + n$$

$$= \frac{1}{6}n\left[4(n+1)(2n+1) - 12(n+1) + 6\right]$$

$$= \frac{1}{6}n\left[8n^2 + 12n + 4 - 12n - 6\right]$$

$$= \frac{1}{6}n\left[8n^2-2\right] = \frac{1}{3}n\left[4n^2-1\right]$$

$$= \frac{1}{3}n(2n+1)(2n-1)$$

(6)

(b) Hence show that

$$\sum_{r=n+1}^{3n}(2r-1)^2 = \frac{2}{3}n\left(an^2+b\right)$$

where *a* and *b* are integers to be found.

$$\sum_{r=1}^{3n}(2r-1)^2 - \sum_{r=1}^{n}(2r-1)^2$$

$$= \frac{1}{3}(3n)(2(3n)+1)(2(3n)-1) - \frac{1}{3}n(2n+1)(2n-1)$$

$$= \frac{1}{3}n\left[3(36n^2-1) - (4n^2-1)\right]$$

$$= \frac{1}{3}n\left[108n^2 - 3 - 4n^2 + 1\right]$$

$$= \frac{1}{3}n(104n^2-2) = \frac{2}{3}n(52n^2-1)$$

(4)

(Total 10 marks)

Q6.

(a) Use the standard results for summations to show that for all positive integers n

$$\sum_{r=1}^{n}(5r-2)^2 = \frac{1}{6}n(an^2+bn+c)$$

where a, b and c are integers to be determined.

$$\sum_{r=1}^{n}(25r^2 - 20r + 4)$$

$$= 25\sum_{r=1}^{n} r^2 - 20\sum_{r=1}^{n} r + 4\sum_{r=1}^{n} 1$$

$$= 25\left(\frac{1}{6}n(n+1)(2n+1)\right) - 20\left(\frac{n}{2}(n+1)\right) + 4n$$

$$= \frac{1}{6}n\left(25(n+1)(2n+1) - 60(n+1) + 24\right)$$

$$= \frac{1}{6}n\left(50n^2 + 75n + 25 - 60n - 60 + 24\right)$$

(5)

$$= \frac{1}{6}n\left(50n^2 + 15n - 11\right)$$

(b) Hence determine the value of k for which

$$\sum_{r=1}^{k}(5r-2)^2 = 94k^2$$

$$94k^2 = \frac{1}{6}k\left(50k^2 + 15k - 11\right)$$

$$564k^2 = 50k^3 + 15k^2 - 11k$$

$$0 = 50k^3 - 549k^2 - 11k$$

$$0 = k(50k^2 - 549k - 11)$$

$$0 = k(50k+1)(k-11)$$

$\cancel{k=0}$ $\quad \cancel{k = -\frac{1}{50}}$ $\quad k = 11$

(4)

(Total for question = 9 marks)

So $k = 11$

Q7.

(a) Use the standard results for summations to show that, for all positive integers *n*,

$$\sum_{r=1}^{n} r^2(r+1) = \frac{1}{12}n(n+1)(n+2)(an+b)$$

where *a* and *b* are integers to be determined.

$$\sum_{r=1}^{n} r^3 + r^2 = \frac{1}{4}n^2(n+1)^2 + \frac{1}{6}n(n+1)(2n+1)$$

$$= \frac{1}{12}n(n+1)\left[3n(n+1) + 2(2n+1)\right]$$

$$= \frac{1}{12}n(n+1)(3n^2+3n+4n+2)$$

$$= \frac{1}{12}n(n+1)(3n^2+7n+2)$$

$$= \frac{1}{12}n(n+1)(n+2)(3n+1)$$

(4)

(b) Hence show that, for all positive integers *k*,

$$\sum_{r=k+1}^{3k} r^2(r+1) = \frac{1}{3}k(3k+1)(Ak^2+Bk+C)$$

where *A*, *B* and *C* are integers to be determined.

$$\sum_{r=k+1}^{3k} r^2(r+1)$$

$$= \sum_{r=1}^{3k} r^2(r+1) - \sum_{r=1}^{k} r^2(r+1)$$

$$= \frac{1}{12}(3k)(3k+1)(3k+2)(9k+1) - \frac{1}{12}k(k+1)(k+2)(3k+1)$$

$$= \frac{1}{12}k(3k+1)\left[3(27k^2+21k+2) - (k^2+3k+2)\right]$$

$$= \frac{1}{12}k(3k+1)\left[80k^2+60k+4\right] = \frac{1}{3}k(3k+1)(20k^2+15k+1)$$

(3)

(c) Hence, using algebra and making your method clear, determine the value of *k* for which

$$25\sum_{r=k+1}^{3k} r^2(r+1) = 192k^3(3k+1)$$

$$\frac{25}{3}k(3k+1)(20k^2+15k+1) = 192k^3(3k+1)$$

$$576k^3(3k+1) - 25k(3k+1)(20k^2+15k+1) = 0$$

$$k(3k+1)\left[576k^2 - 25(20k^2+15k+1)\right] = 0$$

(3)

$$k(3k+1)\left[76k^2 - 375k - 25\right] = 0$$

$$k(3k+1)(76k+5)(k-5) = 0$$

$$\underline{k=5}$$

(Total for question = 10 marks)