## Exam Questions – Complex Numbers (Chapter 1 and 2)

**Q1.**

$$z = \frac{4}{1+i}$$

Find, in the form $a + ib$ where $a, b \in \mathbb{R}$

(a) $z$

$$\frac{4}{1+i} \times \frac{1-i}{1-i} = \frac{4-4i}{1-i^2}$$

$$= \frac{4-4i}{1+1}$$

$$= \frac{4-4i}{2} = 2-2i$$

(2)

(b) $z^2$

$$(2-2i)(2-2i)$$

$$= 4 - 4i - 4i + 4i^2$$

$$= 4 - 8i - 4$$

$$= -8i$$

(2)

Given that $z$ is a complex root of the quadratic equation $x^2 + px + q = 0$, where $p$ and $q$ are real integers,

(c) find the value of $p$ and the value of $q$.

$$(z - (2-2i))(z - (2+2i))$$

If $2-2i$ is a root $2+2i$ is a root

$$= (z - 2 + 2i)(z - 2 - 2i)$$

$$= z^2 - 2z \cancel{-2zi} - 2z + 4 \cancel{+4i} \cancel{+2iz} \cancel{-4i} - 4i^2$$

$$= z^2 - 4z + 4 + 4$$

$$= z^2 - 4z + 8$$

(3)

$p = -4$

$q = 8$

**(Total for question = 7 marks)**

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**Q2.**

$$f(x) = 9x^3 - 33x^2 - 55x - 25$$

Given that $x = 5$ is a solution of the equation $f(x) = 0$, use an algebraic method to solve $f(x) = 0$ completely.

if $x=5$ is a root then $(x-5)$

$f(x) = 9x^3 - 33x^2 - 55x - 25$

$= (x-5)(9x^2 + 12x + 5)$

if $f(x)=0$

$9x^2 + 12x + 5 = 0$ $\qquad (x-5)=0$

$\qquad\qquad x = 5$

$x = \dfrac{-12 \pm \sqrt{12^2 - 4 \times 9 \times 5}}{2 \times 9}$

$= \dfrac{-12 \pm \sqrt{-36}}{18}$ $\qquad x = -\dfrac{2}{3} \pm \dfrac{1}{3}i$

(5)

**(Total for question = 5 marks)**

**Q3.**

Given that 4 and 2i − 3 are roots of the equation

$$x^3 + ax^2 + bx - 52 = 0$$

where *a* and *b* are real constants,

(a) write down the third root of the equation,

$-2i - 3$

(1)

(b) find the value of *a* and the value of *b*.

$(x - (2i-3))(x - (-2i-3))$

$= (x - 2i + 3)(x + 2i + 3)$

$= x^2 + 2xi + 3x - 2xi - 4i^2 - 6i + 3x + 6i + 9$

$= x^2 + 6x + 13$

$(x^2 + 6x + 13)(x - 4)$

$= x^3 - 4x^2 + 6x^2 - 24x + 13x - 52$

$= x^3 + 2x^2 - 11x - 52$

$a = 2$

$b = -11$

(5)

**(Total for question = 6 marks)**

**Q4.**

The complex numbers $z$ and $w$ are given by

$$z = 8 + 3\text{i}, \quad w = -2\text{i}$$

Express in the form $a + b\text{i}$, where $a$ and $b$ are real constants,

(a) $z - w$,

$(8+3i) - (-2i)$

$= 8 + 3i + 2i$

$= 8 + 5i$

**(1)**

(b) $zw$.

$(8 + 3i)(-2i)$

$= -16i - 6i^2$

$= 6 - 16i$

**(2)**

**(Total 3 marks)**

**Q5.**

The complex number $w$ is given by

$$w = 10 - 5\text{i}$$

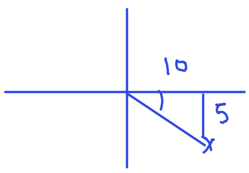
(a) Find $|w|$.

$\sqrt{10^2 + 5^2} = \sqrt{125}$

$= 5\sqrt{5}$

**(1)**

(b) Find arg $w$, giving your answer in radians to 2 decimal places.




$\tan^{-1}\left(\frac{5}{10}\right) = 0.4636$

$\arg w = -0.46$

**(2)**

The complex numbers $z$ and $w$ satisfy the equation

$$(2 + \text{i})(z + 3\text{i}) = w$$

(c) Use algebra to find $z$, giving your answer in the form $a + b\text{i}$, where $a$ and $b$ are real numbers.

$(2+i)(z+3i) = 10 - 5i$

$z + 3i = \frac{10-5i}{2+i} \times \frac{2-i}{2-i}$

$z + 3i = \frac{20 - 10i - 10i + 5i^2}{4 - i^2}$

$z + 3i = \frac{15 - 20i}{5}$

$z + 3i = 3 - 4i$

$z = 3 - 7i$

**(4)**

Given that

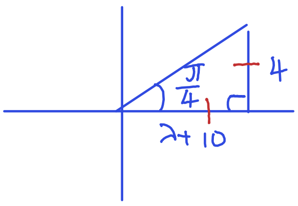
$$\arg(\lambda + 9\text{i} + w) = \frac{\pi}{4}$$

where $\lambda$ is a real constant,

(d) find the value of $\lambda$.

$\arg(\lambda + 9i + 10 - 5i) = \frac{\pi}{4}$

$\arg((\lambda + 10) + 4i) = \frac{\pi}{4}$




$\lambda + 10 = 4$

$\lambda = -6$

**(2)**

**(Total 9 marks)**

**Q6.**

Given that $x = \frac{1}{2}$ is a root of the equation

$$2x^3 - 9x^2 + kx - 13 = 0, \quad k \in \mathbb{R}$$

find

(a) the value of $k$,

$x = \frac{1}{2}$ is a root $(2x - 1)$ is a factor

$2\left(\frac{1}{2}\right)^3 - 9\left(\frac{1}{2}\right)^2 + k\left(\frac{1}{2}\right) - 13 = 0$

$\frac{1}{4} - \frac{9}{4} + \frac{k}{2} - 13 = 0$

$\frac{k}{2} = 15$

$k = 30$

**(3)**

(a) the other 2 roots of the equation.

$(2x - 1)(x^2 - 4x + 13)$

$x^2 - 4x + 13 = 0$

$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4 \times 1 \times 13}}{2 \times 1}$

$= \frac{4 \pm 6i}{2} = 2 \pm 3i$

**(4)**

**(Total 7 marks)**

**Q7.**

$z = \dfrac{50}{3+4\text{i}}$

Find, in the form $a$ + i $b$ where $a,b \in$ R,

(a) $z$,

$\dfrac{50}{3+4i} \times \dfrac{3-4i}{3-4i}$

$= \dfrac{150 - 200i}{9 + 16} = 6 - 8i$

**(2)**

(b) $z^2$ .

$(6-8i)(6-8i)$

$= 36 - 96i + 64i^2$

$= -28 - 96i$

**(2)**

Find

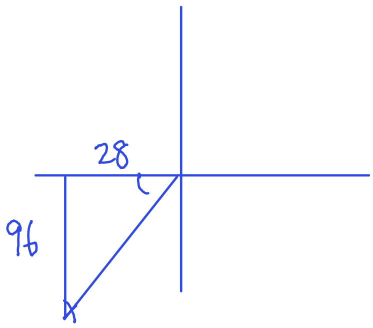
(c) $|z|$,

$\sqrt{6^2 + (-8)^2}$

$= 10$

**(2)**

(d) arg $z^2$ , giving your answer in degrees to 1 decimal place.




$\tan^{-1}\left(\dfrac{96}{28}\right) = 73.7398°$

$180 - 73.7379 = 106.3°$

$\arg z^2 = -106.3°$

**(2)**

**(Total 8 marks)**

**Q8.**
The roots of the equation

$$z^3 - 8z^2 + 22z - 20 = 0$$

are $z_1$, $z_2$ and $z_3$.

(a) Given that $z_1 = 3 + i$, find $z_2$ and $z_3$.

$z_1 = 3 + i$

$z_2 = 3 - i$

$(z - 3 - i)(z - 3 + i)$

$= z^2 - 3z - 3z + 9 - i^2$

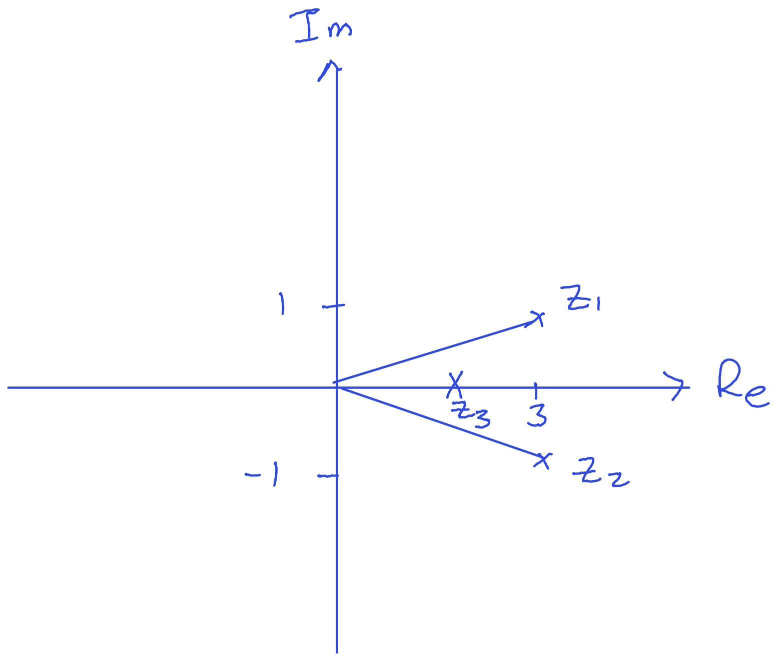
$= z^2 - 6z + 10$

$(z - 2)(z^2 - 6z + 10) = 0$

$z_1 = 3 + i \qquad z_2 = 3 - i \qquad z_3 = 2$

**(4)**

(b) Show, on a single Argand diagram, the points representing $z_1$, $z_2$ and $z_3$.



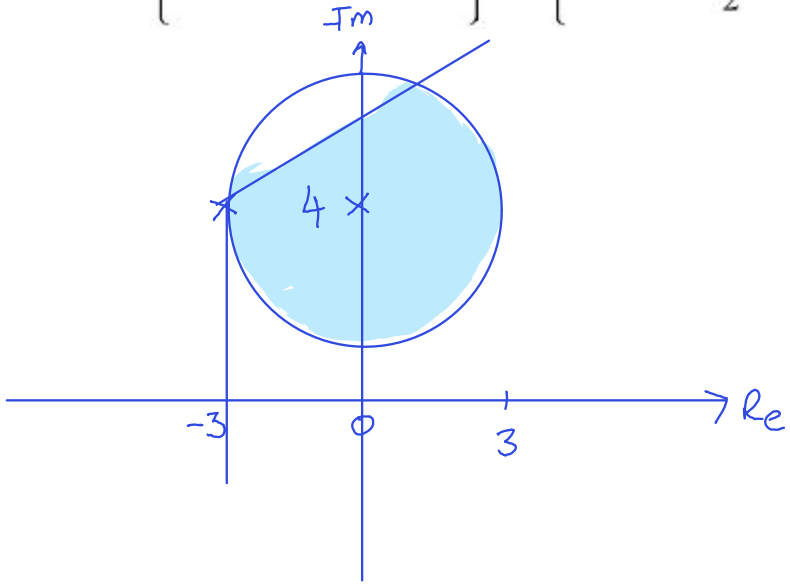

**(2)**

**(Total 6 marks)**

**Q9.**

(a) Shade on an Argand diagram the set of points

$$\left\{ z \in \mathbb{C} : |z-4\mathrm{i}| \leqslant 3 \right\} \cap \left\{ z \in \mathbb{C} : -\frac{\pi}{2} < \arg(z+3-4\mathrm{i}) \leqslant \frac{\pi}{4} \right\}$$




$\arg(z - (-3+4i))$

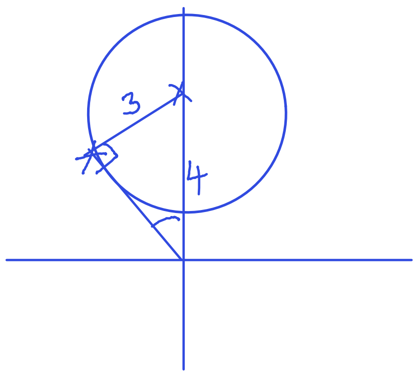
**(6)**

The complex number $w$ satisfies

$$|w-4\mathrm{i}| = 3$$

(b) Find the maximum value of arg $w$ in the interval ( $-\pi$, $\pi$ ].

Give your answer in radians correct to 2 decimal places.




$\sin^{-1}\left(\frac{3}{4}\right) + \frac{\pi}{2}$

$= 2.42$

**(2)**

**(Total for question = 8 marks)**

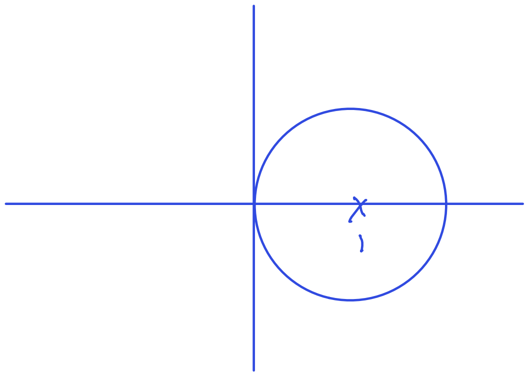
**Q10.**

A complex number $z$ is represented by the point $P$ in the complex plane.

Given that $z$ satisfies

$$|z-1|=1$$

(a) sketch on an Argand diagram the locus of $P$ as $z$ varies.

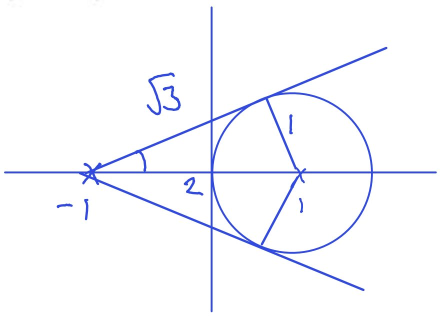



**(2)**

Given that $z$ also satisfies

$$\arg(z+1)=\theta$$

(b) determine the possible values of $\theta$ such that the locus $\arg(z+1)=\theta$ is a tangent to the locus $|z-1|=1$



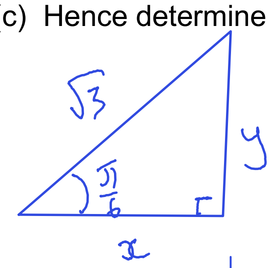

$\arg(z-(-1))=\theta$

$\sin^{-1}\left(\frac{1}{2}\right)=\frac{\pi}{6}$

$\theta=\frac{\pi}{6}$ or $-\frac{\pi}{6}$

**(3)**

(c) Hence determine the exact possible complex numbers $z$.



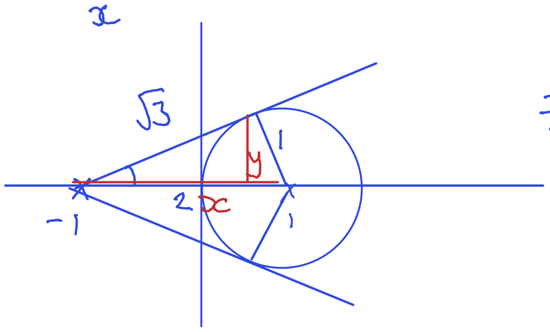

$\cos\left(\frac{\pi}{6}\right)=\frac{x}{\sqrt{3}}$

$x=\frac{3}{2}$

$\sin\left(\frac{\pi}{6}\right)=\frac{y}{\sqrt{3}}$

$y=\frac{\sqrt{3}}{2}$

**(3)**



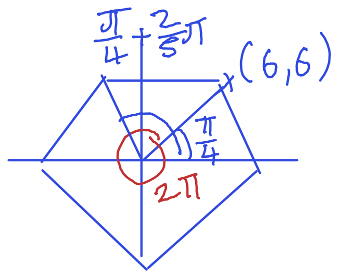

$z=\frac{1}{2}\pm\frac{\sqrt{3}}{2}i$

**(Total for question = 8 marks)**

**Q11.**

(i) The point *P* is one vertex of a regular pentagon in an Argand diagram. The centre of the pentagon is at the origin.

Given that *P* represents the complex number 6 + 6i, determine the complex numbers that represent the other vertices of the pentagon, giving your answers in the form $re^{i\theta}$




$\frac{2\pi}{5} = \frac{8\pi}{20}$

$|z| = \sqrt{6^2+6^2}$
$= 6\sqrt{2}$

$z = 6\sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right)$

or

$z = 6\sqrt{2}e^{i\pi/4}$

$z = 6\sqrt{2}e^{i\frac{13}{20}\pi}$

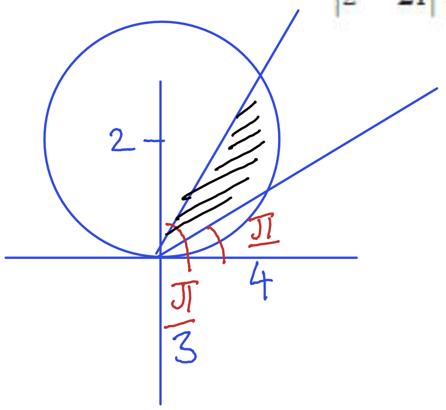
$z = 6\sqrt{2}e^{i\frac{21}{20}\pi}$ $\quad z = 6\sqrt{2}e^{i\frac{37}{20}\pi}$

$z = 6\sqrt{2}e^{i\frac{29\pi}{20}}$

**(5)**

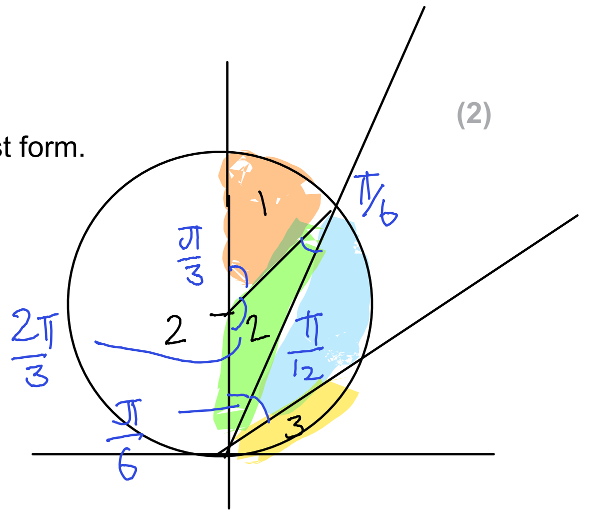
(ii) (a) On a single Argand diagram, shade the region, *R*, that satisfies both

$$|z - 2i| \leqslant 2 \qquad \text{and} \qquad \frac{1}{4}\pi \leqslant \arg z \leqslant \frac{1}{3}\pi$$




**(2)**

(b) Determine the exact area of *R*, giving your answer in simplest form.




Half circle

$\frac{\pi r^2}{2} = \frac{\pi \times 2^2}{2} = 2\pi$

① $\frac{\frac{\pi}{3}}{2\pi} \times \pi \times 2^2 = \frac{1}{6}\pi \times 4$
$= \frac{2}{3}\pi$

② $\frac{1}{2} \times 2 \times 2 \times \sin\left(\frac{2\pi}{3}\right) = \sqrt{3}$

③ $\frac{\frac{\pi}{2}}{2\pi} \times \pi \times 2^2 - \frac{1}{2} \times 2 \times 2 \times \sin\frac{\pi}{2} = \pi - 2$

Total $2\pi - \frac{2}{3}\pi - \sqrt{3} - \pi + 2$
$= \frac{\pi}{3} - \sqrt{3} + 2$

**(4)**

**(Total for question = 11 marks)**

**Q12.**

Given that on an Argand diagram the locus of points defined by $|z + 5 - 12\text{i}| = 10$ is a circle,

$|z - (-5 + 12i)| = 10$

(a) write down,

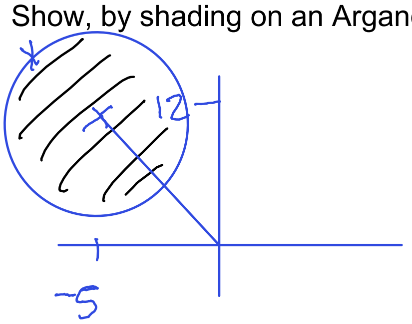
(i) the coordinates of the centre of this circle,
(ii) the radius of this circle.

i) $(-5, 12)$
ii) $10$

**(2)**

(b) Show, by shading on an Argand diagram, the set of points defined by

$$|z + 5 - 12\text{i}| \leqslant 10$$




**(1)**

(c) For the set of points defined in part (b), determine the maximum value of $|z|$

$\sqrt{5^2 + 12^2} + r$

$= 13 + 10$

$= 23$

**(3)**

The set of points $A$ is defined by

$$A = \{z : 0 \leqslant \arg(z + 5 - 20\text{i}) \leqslant \pi\} \cap \{z : |z + 5 - 12\text{i}| \leqslant 10\}$$

(d) Determine the area of the region defined by $A$, giving your answer to 3 significant figures.

$0 \leq \arg(z - (-5 + 20i)) \leq \pi$

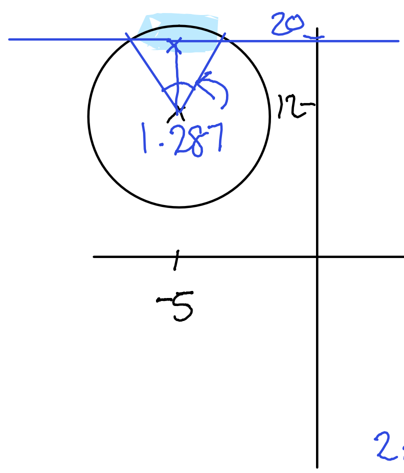
$(x+5)^2 + (y-12)^2 = 100$

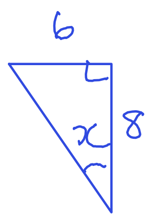
$y = 20 \quad (x+5)^2 + 8^2 = 100$

$(x+5)^2 = 36$

$x = \pm 6 - 5$

$x = -11, 1$




$\tan^{-1}\left(\frac{6}{8}\right) = x$

$x = 0.6435$

$2x = 1.287$

**(4)**

Area of sector − triangle

$\frac{1.287}{2\pi} \times \pi \times 10^2 - \frac{1}{2} \times 10 \times 10 \times \sin 1.287$

$= 16.4$

**(Total for question = 14 marks)**

**Q13.**

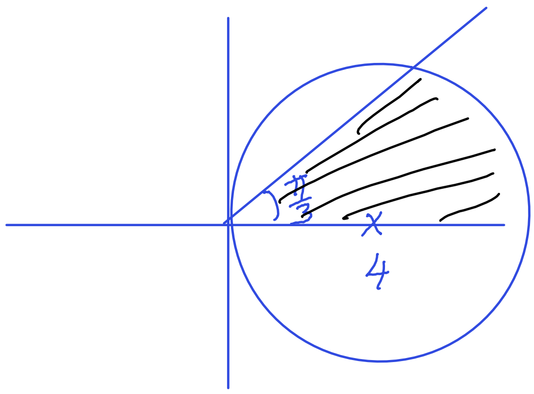
The locus *C* is given by

$$|z - 4| = 4$$

The locus *D* is given by

$$\arg z = \frac{\pi}{3}$$

(a) Sketch, on the same Argand diagram, the locus *C* and the locus *D*




**(4)**

The set of points *A* is defined by

$$A = \{z \in \mathbb{C} : |z - 4| \leqslant 4\} \cap \left\{z \in \mathbb{C} : 0 \leqslant \arg z \leqslant \frac{\pi}{3}\right\}$$

(b) Show, by shading on your Argand diagram, the set of points *A*

Above

**(1)**

(c) Find the area of the region defined by *A*, giving your answer in the form $p\pi + q\sqrt{3}$ where *p* and *q* are constants to be determined.

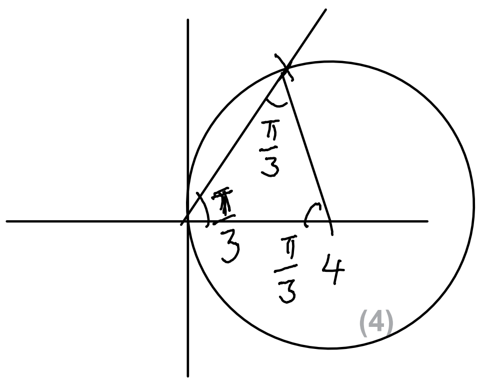
Area of segment = Sector − triangle

$$\frac{\frac{\pi}{3}}{2\pi} \times \pi \times 4^2 - \frac{1}{2} \times 4 \times 4 \times \sin\frac{\pi}{3}$$

$$= \frac{8}{3}\pi - 4\sqrt{3}$$

Shaded area = half circle − segment

$$\frac{\pi \times 4^2}{2} - \left(\frac{8}{3}\pi - 4\sqrt{3}\right) = \frac{16}{3}\pi + 4\sqrt{3}$$




**(4)**

**(Total for question = 9 marks)**