Questions

Q1.

(a) Given that *k* is a constant, find

$$\int\left(\frac{4}{x^3}+kx\right)\mathrm{d}x$$

simplifying your answer.

$$\int 4x^{-3} + kx \quad dx$$

$$= \frac{4x^{-2}}{-2} + \frac{kx^2}{2} + c$$

$$= \frac{-2}{x^2} + \frac{k}{2}x^2 + c$$

(3)

(b) Hence find the value of *k* such that

$$\int_{0.5}^{2}\left(\frac{4}{x^3}+kx\right)\mathrm{d}x = 8$$

$$\left[-\frac{2}{x^2} + \frac{k}{2}x^2\right]_{0.5}^{2} = 8$$

$$\left(-\frac{2}{2^2} + \frac{k}{2}(2)^2\right) - \left(\frac{-2}{0.5^2} + \frac{k}{2}(0.5)^2\right) = 8$$

$$-\frac{1}{2} + 2k - \left(-8 + \frac{1}{8}k\right) = 8$$

$$-\frac{1}{2} + 2k + 8 - \frac{1}{8}k = 8$$

$$\frac{15}{8}k = \frac{1}{2}$$

$$k = \frac{4}{15}$$

(3)

(Total for question = 6 marks)

Q2.

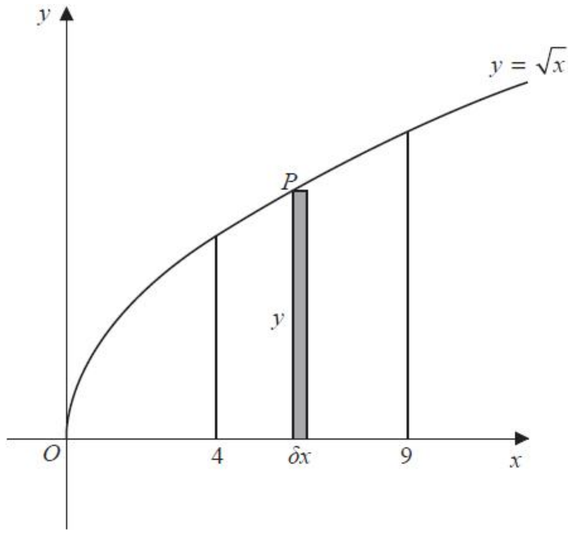

Figure 3

Figure 3 shows a sketch of the curve with equation $y = \sqrt{x}$

The point P(x, y) lies on the curve.

The rectangle, shown shaded on Figure 3, has height y and width δx.

Calculate

$$\lim_{\delta x \to 0} \sum_{x=4}^{9} \sqrt{x}\, \delta x$$

$$\lim_{\delta x \to 0} \sum_{x=4}^{9} \sqrt{x}\, \delta x = \int_4^9 \sqrt{x}\, dx$$

$$= \int_4^9 x^{\frac{1}{2}}\, dx$$

$$= \left[\frac{x^{3/2}}{3/2} \right]_4^9$$

$$= \left[\frac{2}{3} x^{3/2} \right]_4^9$$

$$= \left(\frac{2}{3}(9)^{3/2} \right) - \left(\frac{2}{3}(4)^{3/2} \right)$$

$$= \frac{38}{3}$$

(Total for question = 3 marks)

Q3.

(a) Express $\frac{5}{(x-1)(3x+2)}$ in partial fractions.

$$\frac{5}{(x-1)(3x+2)} = \frac{A}{x-1} + \frac{B}{3x+2}$$

$$5 = A(3x+2) + B(x-1)$$

$x = 1 \quad 5 = 5A \quad 1 = A \qquad x = -\frac{2}{3} \quad 5 = -\frac{5}{3}B \quad -3 = B$

$$\frac{1}{x-1} - \frac{3}{3x+2}$$

(3)

(b) Hence find $\int \frac{5}{(x-1)(3x+2)}\,\mathrm{d}x$, where $x > 1$.

$$\int \frac{1}{x-1} - \frac{3}{3x+2}\,dx$$

$$= \ln|x-1| - \frac{3\ln|3x+2|}{3} + c$$

$$= \ln|x-1| - \ln|3x+2| + c \qquad \text{let } c = \ln k$$

$$= \ln\left|\frac{x-1}{3x+2}\right| + \ln k$$

(3)

(c) Find the particular solution of the differential equation

$(x - 1)(3x + 2)\frac{\mathrm{d}y}{\mathrm{d}x} = 5y, \qquad x > 1,$

for which $y = 8$ at $x = 2$. Give your answer in the form $y = \mathrm{f}(x)$.

$$(x-1)(3x+2)\frac{dy}{dx} = 5y$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{5}{(x-1)(3x+2)}$$

$$\int \frac{1}{y}\,dy = \int \frac{5}{(x-1)(3x+2)}\,dx$$

$$\ln|y| = \ln\left|\frac{x-1}{3x+2}\right| + \ln k$$

$$\ln|y| = \ln\left|\frac{k(x-1)}{3x+2}\right|$$

$$y = \frac{k(x-1)}{3x+2}$$

$$8 = \frac{k(2-1)}{3(2)+2}$$

$$64 = k$$

$$y = \frac{64(x-1)}{3x+2}$$

(6)

(Total 12 marks)

Q4.

(a) Express $\lim_{\delta x \to 0} \sum_{x=2.1}^{6.3} \frac{2}{x} \delta x$ as an integral.

$$\int_{2.1}^{6.3} \frac{2}{x} \, dx$$

(1)

(b) Hence show that

$$\lim_{\delta x \to 0} \sum_{x=2.1}^{6.3} \frac{2}{x} \delta x = \ln k$$

where k is a constant to be found.

$$\int_{2.1}^{6.3} \frac{2}{x} \, dx = \left[2 \ln|x| \right]_{2.1}^{6.3}$$

$$= 2 \ln 6.3 - 2 \ln 2.1$$

$$= 2 \left(\ln \left(\frac{6.3}{2.1} \right) \right)$$

$$= 2 \ln 3$$

$$= \ln 9$$

(2)

(Total for question = 3 marks)

Q5.

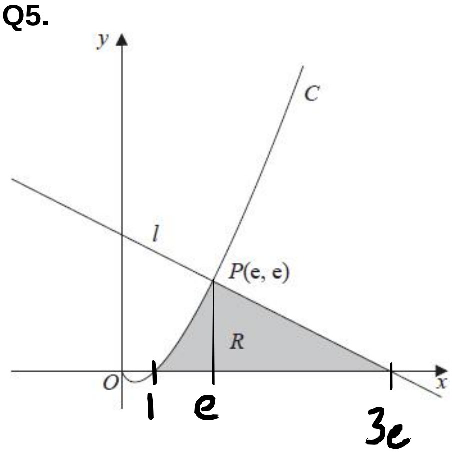

Figure 2

$y = x\ln x$ when $y = 0$

$0 = x\ln x$

$x = 0$ or $\ln x = 0$

$x = 1$

Figure 2 shows a sketch of part of the curve C with equation $y = x \ln x$, $\quad x > 0$

The line l is the normal to C at the point $P(\mathrm{e}, \mathrm{e})$

The region R, shown shaded in Figure 2, is bounded by the curve C, the line l and the x-axis.

Show that the exact area of R is $A\mathrm{e}^2 + B$ where A and B are rational numbers to be found.

Equation of l

(Product rule)

$y = x\ln x$

$\frac{dy}{dx} = \ln x + x \times \frac{1}{x}$

$= \ln x + 1$

gradient of tangent $= 2$

gradient of normal $= -\frac{1}{2}$

(Normal at P)

$y - e = -\frac{1}{2}(x - e)$

when $y = 0$

$-e = -\frac{1}{2}(x - e)$ $(\times -2)$

$2e = x - e$

$3e = x$

Area under curve

$\int_1^e x\ln x \, dx$

(Parts)

$u = \ln x \qquad \frac{dv}{dx} = x$

$\frac{du}{dx} = \frac{1}{x} \qquad v = \frac{x^2}{2}$

$= \left[\frac{x^2}{2}\ln x\right]_1^e - \int_1^e \frac{1}{2}x \, dx$

$= \frac{e^2}{2} - \left[\frac{\frac{1}{2}x^2}{2}\right]_1^e$

$= \frac{e^2}{2} - \left(\frac{1}{4}e^2 - \frac{1}{4}\right) = \frac{1}{4}e^2 + \frac{1}{4}$

Area of triangle

$\frac{1}{2} \times 2e \times e = e^2$

Total area $= e^2 + \frac{1}{4}e^2 + \frac{1}{4}$

$= \frac{5}{4}e^2 + \frac{1}{4}$ //

(10)

(Total for question = 10 marks)

Q6.
In this question you must show all stages of your working.

Solutions relying on calculator technology are not acceptable.

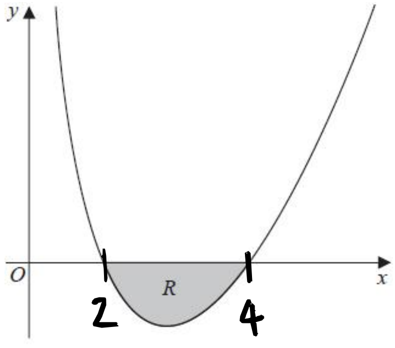

Figure 3

$y = 0$

$\dfrac{(x-2)(x-4)}{4\sqrt{x}} = 0$

$(x-2)(x-4) = 0$

$x = 2, 4$

Figure 3 shows a sketch of part of a curve with equation

$$y = \frac{(x-2)(x-4)}{4\sqrt{x}} \qquad x > 0$$

The region *R*, shown shaded in Figure 3, is bounded by the curve and the *x*-axis.

Find the exact area of *R*, writing your answer in the form $a\sqrt{2} + b$, where *a* and *b* are constants to be found.

$$\int_2^4 \frac{(x-2)(x-4)}{4\sqrt{x}}\, dx$$

$$= \int_2^4 \frac{x^2 - 6x + 8}{4x^{1/2}}\, dx$$

$$= \int_2^4 \frac{1}{4}x^{3/2} - \frac{3}{2}x^{1/2} + 2x^{-1/2}\, dx$$

$$= \left[\frac{\frac{1}{4}x^{5/2}}{5/2} - \frac{\frac{3}{2}x^{3/2}}{3/2} + \frac{2x^{1/2}}{1/2} \right]_2^4$$

$$= \left[\frac{1}{10}x^{5/2} - x^{3/2} + 4x^{1/2} \right]_2^4$$

$$= \left(\frac{1}{10}(4)^{5/2} - (4)^{3/2} + 4(4)^{1/2}\right) - \left(\frac{1}{10}(2)^{5/2} - 2^{3/2} + 4(2)^{1/2}\right)$$

$$= \frac{16}{5} - \frac{12\sqrt{2}}{5}$$

$$\text{Area} = \frac{12\sqrt{2}}{5} - \frac{16}{5}$$

(Total for question = 6 marks)

Q7.

The rate of decay of the mass of a particular substance is modelled by the differential equation

$$\frac{dx}{dt} = -\frac{5}{2}x, \qquad t \geqslant 0$$

where x is the mass of the substance measured in grams and t is the time measured in days.

Given that $x = 60$ when $t = 0$,

(a) solve the differential equation, giving x in terms of t. You should show all steps in your working and give your answer in its simplest form.

$$\frac{dx}{dt} = -\frac{5}{2}x$$

$$\int \frac{1}{x}\,dx = \int -\frac{5}{2}\,dt$$

$$\ln|x| = \frac{-5}{2}t + c$$

when $x = 60$, $t = 0$: $\ln|60| = c$

$$\ln x = -\frac{5}{2}t + \ln(60)$$

$$x = e^{-5/2 t + \ln(60)}$$

$$x = e^{-5/2 t} \times e^{\ln 60}$$

$$x = 60e^{-5/2 t}$$

(4)

(b) Find the time taken for the mass of the substance to decay from 60 grams to 20 grams.

Give your answer to the nearest minute.

when $x = 60$ $t = 0$

when $x = 20$

$$20 = 60e^{-5/2 t}$$

$$\frac{1}{3} = e^{-5/2 t}$$

$$\ln\frac{1}{3} = -\frac{5}{2}t$$

$$-\frac{2}{5}\ln\frac{1}{3} = t$$

$$t = 0.439$$

0.439 days

$0.439\ldots \times 24 \times 60$

$= 632.8$ mins

$= 633$ mins

(3)

(Total for question = 7 marks)

Q8.

(i) Given that $y > 0$, find

$$\int \frac{3y-4}{y(3y+2)}\,dy$$

$$\frac{3y-4}{y(3y+2)} = \frac{A}{y} + \frac{B}{3y+2}$$

$$3y-4 = A(3y+2) + By$$

$y=0$ $\quad -4 = 2A$ $\quad -2 = A$

$y = -\frac{2}{3}$ $\quad -2 = -\frac{2}{3}B$ $\quad B = 3$

$$\int -\frac{2}{y} + \frac{3}{3y+2}\,dy$$

$$= -2\ln|y| + \frac{3\ln|3y+2|}{3} + c$$

$$= -2\ln|y| + \ln|3y+2| + c$$

(6)

(ii) (a) Use the substitution $x = 4\sin^2\theta$ to show that

$$\int_0^3 \sqrt{\left(\frac{x}{4-x}\right)}\,dx = \lambda \int_0^{\frac{\pi}{3}} \sin^2\theta\,d\theta$$

where λ is a constant to be determined.

$$\int_0^{\frac{\pi}{3}} \frac{\sqrt{4\sin^2\theta}}{\sqrt{4-4\sin^2\theta}} \times 8\sin\theta\cos\theta\,d\theta$$

$$= \int_0^{\frac{\pi}{3}} \frac{2\sin\theta}{\sqrt{4(1-\sin^2\theta)}} \times 8\sin\theta\cos\theta\,d\theta$$

$$= \int_0^{\frac{\pi}{3}} \frac{16\sin^2\theta\cos\theta}{2\cos\theta}\,d\theta$$

$$= 8\int_0^{\frac{\pi}{3}} \sin^2\theta\,d\theta$$

$x = 4\sin^2\theta$

$\frac{dx}{d\theta} = 4 \times 2\sin\theta\cos\theta$

$\frac{dx}{d\theta} = 8\sin\theta\cos\theta$

$dx = 8\sin\theta\cos\theta\,d\theta$

If $x = 3$ $\quad 3 = 4\sin^2\theta$ $\quad \sin\theta = \sqrt{\frac{3}{4}}$ $\quad \theta = \frac{\pi}{3}$

$x = 0$ $\quad 0 = 4\sin^2\theta$ $\quad \sin\theta = 0$ $\quad \theta = 0$

(5)

(b) Hence use integration to find

$$\int_0^3 \sqrt{\left(\frac{x}{4-x}\right)}\,dx$$

giving your answer in the form $a\pi + b$, where a and b are exact constants.

$$8\int_0^{\frac{\pi}{3}} \sin^2\theta \, d\theta$$

$$= \int_0^{\pi/3} 8\sin^2\theta \, d\theta$$

$$= \int_0^{\pi/3} 4 - 4\cos 2\theta \, d\theta$$

$$= \left[4\theta - \frac{4\sin 2\theta}{2}\right]_0^{\pi/3}$$

$$= \left[4\theta - 2\sin 2\theta\right]_0^{\pi/3}$$

$$= \left(4\left(\frac{\pi}{3}\right) - 2\sin\left(\frac{2\pi}{3}\right)\right) - (0)$$

$$= \frac{4}{3}\pi - \sqrt{3}$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$\cos 2\theta = 1 - \sin^2\theta - \sin^2\theta$$

$$\cos 2\theta = 1 - 2\sin^2\theta$$

$$\sin^2\theta = \frac{1 - \cos 2\theta}{2}$$

(4)

(Total for question = 15 marks)

Q9.

(a) Find

$$\int 10x(x^{\frac{1}{2}} - 2)\,dx$$

giving each term in its simplest form.

$$\int 10x^{3/2} - 20x \; dx$$

$$= \frac{10x^{5/2}}{5/2} - \frac{20x^2}{2} + c$$

$$= 4x^{5/2} - 10x^2 + c$$

(4)

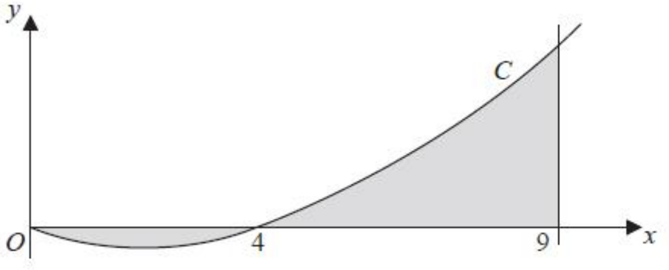

Figure 2

Figure 2 shows a sketch of part of the curve *C* with equation

$$y = 10x(x^{\frac{1}{2}} - 2), \qquad x \geqslant 0$$

The curve *C* starts at the origin and crosses the *x*-axis at the point (4, 0).

The area, shown shaded in Figure 2, consists of two finite regions and is bounded by the curve *C*, the *x*-axis and the line $x = 9$

(b) Use your answer from part (a) to find the total area of the shaded regions.

$$\int_0^4 10x^{3/2} - 20x \, dx$$

$$= \left[4x^{5/2} - 10x^2 \right]_0^4$$

$$= \left(4(4)^{5/2} - 10(4)^2\right) - (0)$$

$$= -32$$

$$\int_4^9 10x^{3/2} - 20x \, dx$$

$$= \left[4x^{5/2} - 10x^2 \right]_4^9$$

$$= \left(4(9)^{5/2} - 10(9)^2\right) - (-32)$$

$$= 162 + 32$$

$$= 194$$

(5)

(Total for question = 9 marks)

$$\text{Total} = 194 + 32$$
$$= 226$$

Q10.

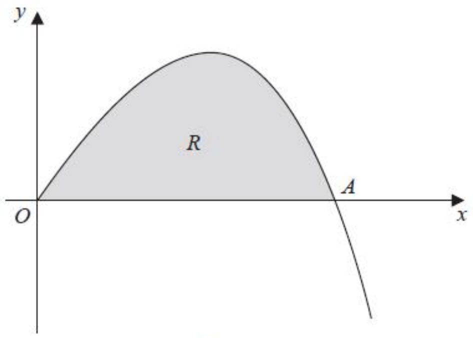

Figure 1

Figure 1 shows a sketch of part of the curve with equation $y = 4x - xe^{\frac{1}{2}x}, \; x \geqslant 0$

The curve meets the *x*-axis at the origin *O* and cuts the *x*-axis at the point *A*.

(a) Find, in terms of ln2, the *x* coordinate of the point *A*.

$y = 0$

$4x - xe^{1/2 x} = 0$

$x(4 - e^{1/2 x}) = 0$

$x = 0 \qquad 4 - e^{1/2 x} = 0$

$4 = e^{1/2 x}$

$\ln 4 = \frac{1}{2}x$

$x = 2\ln 4$

$x = 2\ln(2^2)$

$x = 4\ln 2$

(2)

(b) Find

$$\int xe^{\frac{1}{2}x}\,dx$$

$u = x \qquad \frac{dv}{dx} = e^{1/2 x}$

$\frac{du}{dx} = 1 \qquad v = \frac{e^{1/2 x}}{1/2}$

$v = 2e^{1/2 x}$

$= 2xe^{1/2 x} - \int 2e^{1/2 x}\,dx$

$= 2xe^{1/2 x} - \frac{2e^{1/2 x}}{1/2} + c$

$= 2xe^{1/2 x} - 4e^{1/2 x} + c$

(3)

The finite region R, shown shaded in Figure 1, is bounded by the x-axis and the curve with equation

$$y = 4x - xe^{\frac{1}{2}x}, \ x \geqslant 0$$

(c) Find, by integration, the exact value for the area of R.
Give your answer in terms of ln2

$$\int_0^{4\ln 2} 4x - xe^{\frac{1}{2}x} \, dx$$

$$= \left[\frac{4x^2}{2} - \left(2xe^{\frac{1}{2}x} - 4e^{\frac{1}{2}x}\right)\right]_0^{4\ln 2}$$

$$= \left[2x^2 - 2xe^{\frac{1}{2}x} + 4e^{\frac{1}{2}x}\right]_0^{4\ln 2}$$

(3)

(Total for question = 8 marks)

$$= \left(2(4\ln 2)^2 - 2(4\ln 2)e^{\frac{1}{2}(4\ln 2)} + 4e^{\frac{1}{2}(4\ln 2)}\right) - \left(4e^0\right)$$

$$= 2 \times 16(\ln 2)^2 - 8\ln 2 e^{\ln 4} + 4e^{\ln 4} - 4$$

$$= 32(\ln 2)^2 - 32\ln 2 + 16 - 4$$

$$= 32(\ln 2)^2 - 32\ln 2 + 12$$

Q11.

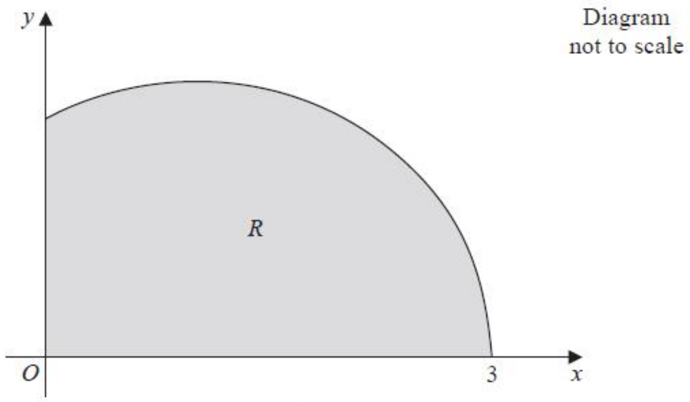

Figure 2

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$
$$\cos 2\theta = \cos^2\theta - (1-\cos^2\theta)$$
$$\cos 2\theta = 2\cos^2\theta - 1$$
$$\frac{\cos 2\theta + 1}{2} = \cos^2\theta$$

Figure 2 shows a sketch of the curve with equation $y = \sqrt{(3-x)(x+1)}, \quad 0 \leqslant x \leqslant 3$

The finite region *R*, shown shaded in Figure 2, is bounded by the curve, the *x*-axis, and the *y*-axis.

(a) Use the substitution $x = 1 + 2\sin\theta$ to show that

$$\int_0^3 \sqrt{(3-x)(x+1)}\,dx = k\int_{-\frac{\pi}{6}}^{\frac{\pi}{2}} \cos^2\theta\,d\theta$$

where *k* is a constant to be determined.

$$x = 1 + 2\sin\theta$$
$$\frac{dx}{d\theta} = 2\cos\theta$$
$$dx = 2\cos\theta\,d\theta$$
$$x = 3 \quad 3 = 1 + 2\sin\theta$$
$$1 = \sin\theta$$
$$\theta = \frac{\pi}{2}$$
$$x = 0 \quad 0 = 1 + 2\sin\theta$$
$$-\frac{1}{2} = \sin\theta$$
$$\theta = -\frac{\pi}{6}$$

$$\int_{-\pi/6}^{\pi/2} \sqrt{(3-(1+2\sin\theta))(1+2\sin\theta+1)} \times 2\cos\theta\,d\theta$$
$$= \int_{-\pi/6}^{\pi/2} \sqrt{(2-2\sin\theta)(2+2\sin\theta)} \times 2\cos\theta\,d\theta$$
$$= \int_{-\pi/6}^{\pi/2} \sqrt{4(1-\sin^2\theta)} \times 2\cos\theta\,d\theta$$
$$= 4\int_{-\frac{\pi}{6}}^{\pi/2} \cos^2\theta\,d\theta$$

(5)

(b) Hence find, by integration, the exact area of *R*.

$$= 4\int_{-\pi/6}^{\pi/2} \frac{\cos 2\theta}{2} + \frac{1}{2}\,d\theta$$
$$= \int_{-\frac{\pi}{6}}^{\frac{\pi}{2}} 2\cos 2\theta + 2\,d\theta$$
$$= \left[\sin 2\theta + 2\theta\right]_{-\pi/6}^{\pi/2}$$
$$= \left(\sin\pi + \pi\right) - \left(\sin\left(-\frac{\pi}{3}\right) - \frac{2\pi}{6}\right)$$
$$= \pi + \frac{\sqrt{3}}{2} + \frac{1}{3}\pi$$
$$= \frac{4}{3}\pi + \frac{\sqrt{3}}{2}$$

(3)

(Total for question = 8 marks)

Q12.

$$\frac{dy}{dx} = 6x^{-\frac{1}{2}} + x\sqrt{x}, \quad x > 0$$

Given that $y = 37$ at $x = 4$, find y in terms of x, giving each term in its simplest form.

$$\frac{dy}{dx} = 6x^{-1/2} + x\sqrt{x}$$

$$\frac{dy}{dx} = 6x^{-1/2} + x^{3/2}$$

$$y = \frac{6x^{1/2}}{1/2} + \frac{x^{5/2}}{5/2} + c$$

$$y = 12x^{1/2} + \frac{2}{5}x^{5/2} + c$$

$y = 37$

$x = 4$

$$37 = 12(4)^{1/2} + \frac{2}{5}(4)^{5/2} + c$$

$$37 = 36.8 + c$$

$$c = \frac{1}{5}$$

$$y = 12x^{1/2} + \frac{2}{5}x^{5/2} + \frac{1}{5}$$

(7)

(Total 7 marks)

Q13.

Using the substitution $u = 2 + \sqrt{(2x + 1)}$, or other suitable substitutions, find the exact value of

$$\int_0^4 \frac{1}{2+\sqrt{(2x+1)}}\,dx$$

giving your answer in the form $A + 2\ln B$, where A is an integer and B is a positive constant.

$$u = 2 + \sqrt{2x+1}$$
$$u = 2 + (2x+1)^{\frac{1}{2}}$$
$$\frac{du}{dx} = \frac{1}{2}(2x+1)^{-\frac{1}{2}} \times 2$$
$$\frac{du}{dx} = (2x+1)^{-\frac{1}{2}}$$
$$du = \frac{1}{(2x+1)^{\frac{1}{2}}}\,dx$$
$$(2x+1)^{\frac{1}{2}}\,du = dx$$

$x = 4$: $u = 2 + \sqrt{9}$, $u = 5$

$x = 0$: $u = 2 + \sqrt{1}$, $u = 3$

$$\int_3^5 \frac{1}{u} \times (2x+1)^{\frac{1}{2}}\,du$$
$$= \int_3^5 \frac{1}{u} \times (u-2)\,du$$
$$= \int_3^5 1 - \frac{2}{u}\,du$$
$$= \Big[u - 2\ln u\Big]_3^5$$
$$= (5 - 2\ln 5) - (3 - 2\ln 3)$$
$$= 5 - 2\ln 5 - 3 + 2\ln 3$$
$$= 2 + 2(\ln 3 - \ln 5)$$
$$= 2 + 2\ln\left(\frac{3}{5}\right)$$

(8)

(Total 8 marks)

Q14.

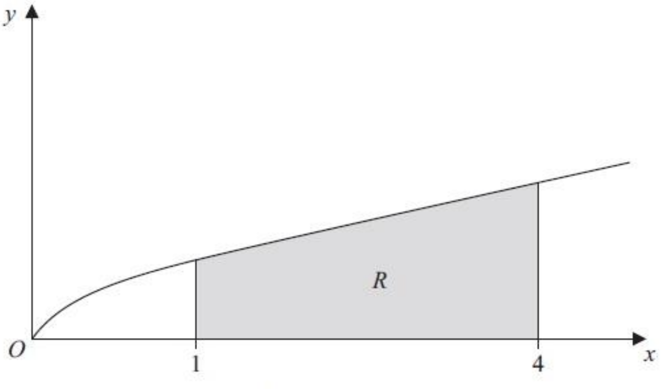

Figure 1

Figure 1 shows a sketch of part of the curve with equation $y = \frac{x}{1+\sqrt{x}}$. The finite region R, shown shaded in Figure 1, is bounded by the curve, the x-axis, the line with equation $x = 1$ and the line with equation $x = 4$.

(a) Complete the table with the value of y corresponding to $x = 3$, giving your answer to 4 decimal places.

x	1	2	3	4
y	0.5	0.8284	1.0981	1.3333

(1)

(b) Use the trapezium rule, with all the values of y in the completed table, to obtain an estimate of the area of the region R, giving your answer to 3 decimal places.

number of strips = 3

$h = \frac{4-1}{3} = 1$

$$\int_1^4 \frac{x}{1+\sqrt{x}} \approx \frac{1}{2}(1)\left(0.5 + 1.3333 + 2(0.8284 + 1.0981)\right)$$

$$= 2.843$$

(3)

(c) Use the substitution $u = 1 + \sqrt{x}$, to find, by integrating, the exact area of R.

$$y = \frac{x}{1+\sqrt{x}}$$

$$\int_1^4 \frac{x}{1+\sqrt{x}}\,dx$$

$$= \int_2^3 \frac{(u-1)^2}{u} \times 2(u-1)\,du$$

$$= \int_2^3 \frac{2(u-1)^3}{u}\,du$$

$$= \int_2^3 \frac{2(u^3-3u^2+3u-1)}{u}\,du$$

$$= \int_2^3 2u^2 - 6u + 6 - \frac{2}{u}\,du$$

$$= \left[\frac{2u^3}{3} - \frac{6u^2}{2} + 6u - 2\ln u\right]_2^3$$

$$= \left(\frac{2\times3^3}{3} - \frac{6\times3^2}{2} + 6\times3 - 2\ln 3\right) - \left(\frac{2\times2^3}{3} - \frac{6\times2^2}{2} + 6\times2 - 2\ln 2\right)$$

$$= (9 - 2\ln 3) - \left(\frac{16}{3} - 2\ln 2\right)$$

$$= \frac{11}{3} - 2\ln 3 + 2\ln 2$$

$$= \frac{11}{3} + 2\ln\left(\frac{2}{3}\right)$$

$$u = 1 + \sqrt{x}$$

$$u = 1 + x^{1/2}$$

$$\frac{du}{dx} = \frac{1}{2}x^{-1/2}$$

$$du = \frac{1}{2\sqrt{x}}\,dx$$

$$2\sqrt{x}\,du = dx$$

$$2(u-1)\,du = dx$$

$x = 4$ $\quad u = 1 + \sqrt{4} = 3$

$x = 1$ $\quad u = 1 + \sqrt{1} = 2$

(8)

(Total 12 marks)

Q15.

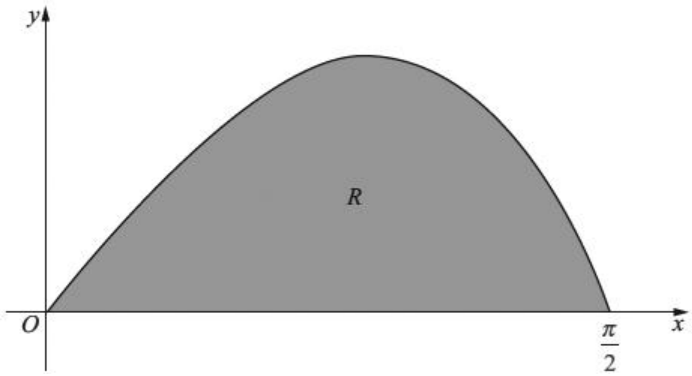

Figure 3

Figure 3 shows a sketch of the curve with equation $y = \frac{2\sin 2x}{(1+\cos x)}, \ 0 \leqslant x \leqslant \frac{\pi}{2}$.

The finite region R, shown shaded in Figure 3, is bounded by the curve and the x-axis.

The table below shows corresponding values of x and y for $y = \frac{2\sin 2x}{(1+\cos x)}$

x	0	$\frac{\pi}{8}$	$\frac{\pi}{4}$	$\frac{3\pi}{8}$	$\frac{\pi}{2}$
y	0	0.73508	1.17157	1.02280	0

(a) Complete the table above giving the missing value of y to 5 decimal places.

(1)

(b) Use the trapezium rule, with all the values of y in the completed table, to obtain an estimate for the area of R, giving
your answer to 4 decimal places.

$$h = \frac{\frac{\pi}{2} - 0}{4} = \frac{\pi}{8}$$

$$\int_0^{\frac{\pi}{2}} \frac{2\sin 2x}{1+\cos x}\,dx \approx \frac{1}{2}\left(\frac{\pi}{8}\right)\left(0+0+2(0.73508+1.17157+1.02280)\right)$$

$$= 1.1504$$

(3)

(c) Using the substitution $u = 1 + \cos x$, or otherwise, show that

$$\int \frac{2\sin 2x}{(1+\cos x)}\,dx = 4\ln(1+\cos x) - 4\cos x + k$$

where k is a constant.

$$\int \frac{4\sin x\cos x}{u} \times \frac{-1}{\sin x}\,du$$

$u = 1 + \cos x$

$\frac{du}{dx} = -\sin x$

$du = -\sin x\,dx$

$\frac{-1}{\sin x}\,du = dx$

$\sin 2x = 2\sin x\cos x$

$$= \int \frac{-4\cos x}{u}\,du$$

$$= \int \frac{-4(u-1)}{u}\,du$$

$$= \int -4 + \frac{4}{u}\,du$$

$$= -4u + 4\ln u + c$$

$$= -4(1+\cos x) + 4\ln(1+\cos x) + c$$

$$= -4 - 4\cos x + 4\ln(1+\cos x) + c$$

let $k = c - 4$

$$= 4\ln(1+\cos x) - 4\cos x + k$$

(5)

(d) Hence calculate the error of the estimate in part (b), giving your answer to 2 significant figures.

$$\int_0^{\frac{\pi}{2}} \frac{2\sin 2x}{1+\cos x}\,dx$$

$$= \Big[4\ln(1+\cos x) - 4\cos x\Big]_0^{\frac{\pi}{2}}$$

$$= 4\ln(1) - (4\ln(2) - 4)$$

$$= 4 - 4\ln 2 \approx 1.2274$$

1.1504

$(4 - 4\ln 2) - (1.1504)$

$= 0.077$

(3)

(Total 12 marks)

Q16.

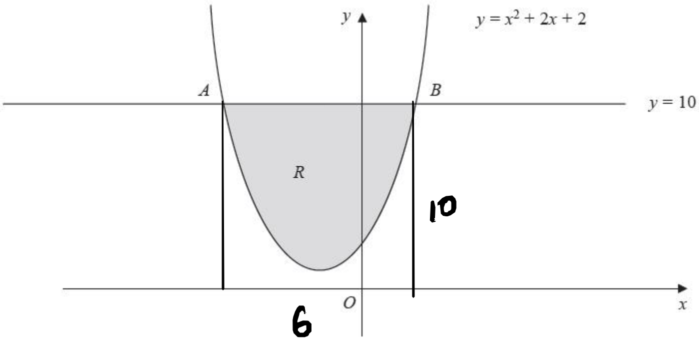

Figure 1

The line with equation $y = 10$ cuts the curve with equation $y = x^2 + 2x + 2$ at the points A and B as shown in Figure 1. The figure is not drawn to scale.

(a) Find by calculation the x-coordinate of A and the x-coordinate of B.

$10 = x^2 + 2x + 2$

$0 = x^2 + 2x - 8$

$0 = (x+4)(x-2)$

$x = -4, 2$

$A = -4$

$B = 2$ //

(2)

The shaded region R is bounded by the line with equation $y = 10$ and the curve as shown in Figure 1.

(b) Use calculus to find the exact area of R.

$$\int_{-4}^{2} x^2 + 2x + 2 \, dx$$

$$= \left[\frac{x^3}{3} + \frac{2x^2}{2} + 2x \right]_{-4}^{2}$$

$$= \left[\frac{x^3}{3} + x^2 + 2x \right]_{-4}^{2}$$

$$= \left(\frac{2^3}{3} + 2^2 + 2(2) \right) - \left(\frac{(-4)^3}{3} + (-4)^2 + 2(-4) \right)$$

$$= \frac{32}{3} - \left(-\frac{40}{3} \right)$$

$$= 24$$

$6 \times 10 = 60$

Area of $R = 60 - 24$

$= 36$ //

(7)

(Total 9 marks)

Q17.

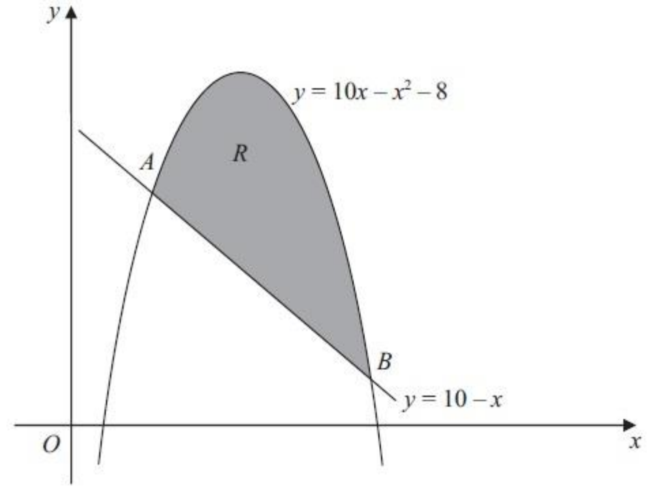

Figure 2

Figure 2 shows the line with equation $y = 10 - x$ and the curve with equation $y = 10x - x^2 - 8$

The line and the curve intersect at the points A and B, and O is the origin.

(a) Calculate the coordinates of A and the coordinates of B.

$y = 10 - x \qquad y = 10x - x^2 - 8$

$10 - x = 10x - x^2 - 8$

$x^2 - 11x + 18 = 0$

$(x-9)(x-2) = 0$

$x = 2, 9$

$x = 2$
$y = 10 - 2$
$= 8$

$x = 9$
$y = 10 - 9$
$= 1$

$A = (2, 8)$

$B = (9, 1)$ //

(5)

The shaded area R is bounded by the line and the curve, as shown in Figure 2.

(b) Calculate the exact area of R.

$\int_2^9 10x - x^2 - 8 - (10 - x)\, dx$

$= \int_2^9 11x - x^2 - 18\, dx$

$= \left[\frac{11x^2}{2} - \frac{x^3}{3} - 18x \right]_2^9$

$= \left(\frac{11(9)^2}{2} - \frac{9^3}{3} - 18(9) \right) - \left(\frac{11(2)^2}{2} - \frac{2^3}{3} - 18(2) \right)$

$= \frac{81}{2} - \left(-\frac{50}{3}\right) = \frac{343}{6}$ //

(7)

(Total 12 marks)